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Minami, H.
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# K-GROUPS OF *EIII* AND *FII*

HARUO MINAMI

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**1.** Let $G$ be a compact, connected, simply-connected Lie group and $K$ a closed connected subgroup of $G$ of maximal rank. As is well known [3], the complex $K$-group of $G/K$ is isomorphic to $R(K)\underset{R(G)}{\otimes}Z$ and it is a free abelian group with rank equal to the quotient of the order of the Weyl group of $G$ by the order of the Weyl group of $K$. Here $R(G)$ is the complex representation ring of $G$. The purpose of this paper is to determine an additive structure of the complex $K$-groups of symmetric spaces $EIII=E_6/Spin(10)\cdot SO(2)$ and $FII=F_4/Spin(9)$. To simplify the notation we write $x$ for the element $x\otimes 1$ of $R(K)\underset{R(G)}{\otimes}Z$ in the following.

Let $\Delta^+$ and $\Delta^-$ be the half-spin representations of $Spin(10)$, and let $\rho$ and $t$ be the canonical non-trivial 10- and 1-dimensional complex representations of $Spin(10)$ and $SO(2)$ respectively. Then

$$R(Spin(10)\times SO(2)) = Z[\lambda^1\rho, \lambda^2\rho, \lambda^3\rho, \Delta^+, \Delta^-, t, t^{-1}]$$

and $R(Spin(10)\cdot SO(2))$ is isomorphic to the subalgebra of $R(Spin(10)\times SO(2))$ generated by the representations of $Spin(10)\times SO(2)$ which are trivial on $Spin(10)\cap SO(2)=Z_4$ (See [5] and [2, I], Prop. 2.1). Furthermore then our result is stated as follows.

**Theorem.**

$$K^*(EIII)\cong Z\{x^i, x^iw, x^jw^2, x^kv \mid 0\leq i\leq 8,\ 0\leq j\leq 4,\ 0\leq k\leq 3\}$$

*where*

$$\begin{aligned} x &= t^4-1\,,\\ w &= (t^2\rho-10)-x^3+2x^2-5x\,,\\ v &= 45xw^3+26x^5w^2 \end{aligned}$$

*and* $Z\{a, b, c, \cdots\}$ *is the free abelian group generated by the set* $\{a, b, c, \cdots\}$.

Besides we have (2.1) of Section 2 concerning a ring structure of $K^*(EIII)$. Now, recently Steinberg [4] gave a general formula of a free basis over $R(G)$ for an $R(G)$-module $R(K)$ (by restriction).

2. Using the notation in Table V of [1] we denote by $\rho_1$ and $\rho_2$ the 27-dimensional representations of $E_6$ with the highest weights $\frac{1}{3}(4\alpha_1+3\alpha_2+5\alpha_3+6\alpha_4+4\alpha_5+2\alpha_6)$ and $\frac{1}{3}(2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6)$ respectively and by $Ad$ the adjoint representation of $E_6$.

**Lemma.** *Let $i^*$: $R(E_6)\to R(Spin(10)\cdot SO(2))$ be the restriction induced by the natural inclusion $i$: $Spin(10)\cdot SO(2)\to E_6$. Then we have*

(i) $i^*(Ad)=\lambda^2\rho+t^3\Delta^+ +t^{-3}\Delta^- +1$
(ii) $i^*(\rho_1)=t^4+t\Delta^- +t^{-2}\rho$
(iii) $i^*(\rho_2)=t^{-4}+t^{-1}\Delta^+ +t^2\rho$
(iv) $i^*(\lambda^2\rho_1)=t^5\Delta^- +t^2\lambda^3\rho+t^2\rho+t^{-1}\Delta^-\rho+t^{-4}\lambda^2\rho$
(v) $i^*(\lambda^2\rho_2)=t^{-5}\Delta^+ +t^{-2}\lambda^3\rho+t^{-2}\rho+t\Delta^+\rho+t^4\lambda^2\rho$
(vi) $i^*(\lambda^3\rho_1)=i^*(\lambda^3\rho_2)=t^6\lambda^3\rho+t^{-6}\lambda^3\rho+t^3\Delta^+\lambda^2\rho+t^{-3}\Delta^-\lambda^2\rho+\rho\lambda^3\rho+\lambda^2\rho$.

Proof. (i)-(iii) are verified by observing the restriction of all weights of $\rho_1$, $\rho_2$ and $Ad$ to $Spin(10)\cdot SO(2)$. Here this reduction is based on the formula given in Section 1 of [5], and the weights of $\rho_1$, $\rho_2$ and $Ad$ are listed in Section 5.

Consider the exterior powers of the formulas (ii) and (iii) then we can aesily check (iv)-(vi) since $\lambda^2\Delta^\pm=\lambda^3\rho$, $\lambda^2(\lambda^2\rho)+\lambda^4\rho=\rho\lambda^3\rho$ and $\lambda^3\Delta^- +\rho\Delta^-=\Delta^+\lambda^2\rho$. q.e.d.

By Lemma we see that

(2.1) $R(Spin(10)\cdot SO(2))\underset{R(E_6)}{\otimes} Z(\cong K^*(EIII))$ *is multiplicatively generated by two elements $x$ and $w$ with relations*

$$(2.2)\quad (x^3+3x^2+3x)w^2+(x^{12}-x^{11}+x^{10}-x^9+x^8-x^7+3x^6)w +2(x^{16}-x^{15}+x^{14}-x^{13}+x^{12}-x^{11}+x^{10})-x^9=0$$

*and*

$$w^3-(2x^8-2x^7+2x^6-x^5+7x^4+5x^3+18x^2+15x)w^2 -\{8(x^{12}-x^{11}+x^{10}-x^9)+10x^8-2x^7+15x^6\}w-8(x^{16}-x^{15}+x^{14}-x^{13}+x^{12}) +7x^{11}-9x^{10}+5x^9=0\,.$$

When we calculate (2.2), note that

$$(2.3)\qquad x^{17}=0$$

since $EIII$ is a differentiable manifold of dimension 32, and $x+1$ is invertible.

It follows from (2.2) that

$$(2.4)\quad \{x^9(x^5+15x^4+78x^3+182x^2+195x+78)w+13x^7+53x^6+84x^5+45x^4\}=0\,.$$

**3.** Proof of Theorem. By (2.3) and (2.4) we have inductively

$$624x^9w = 109x^{16}-154x^{15}+228x^{14}-360x^{13} \tag{3.1}$$

and so by this formula we have

$$x^9w^2 = 0\,. \tag{3.2}$$

Then we get

$$\begin{aligned} x^9 = {} & \{14(x^8+x^7+x^6+x^5+x^4)+13x^3+9x^2+3x\}w^2 \\ & +\{5(x^{12}+x^{11}+x^{10}+x^9+x^8+x^7)+3x^6\}w \end{aligned} \tag{3.3}$$

by (3.1), (3.2) and the first formula of (2.2), and moreover

$$xw^3 = (12x^8+16x^7+12x^6+15x^5)w^2+(x^{12}+9x^{11}+x^{10}+9x^9)w \tag{3.4}$$

by (3.1)-(3.3) and the secondary formula of (2.2).

It follows that $26x^{12}w=-15x^{16}$ and $x^{16}=3x^8w^2$ from (3.1) and (3.3) respectively. Therefore $26x^{12}w+45x^8w^2=0$ and so we see that

$$x^8w^2 = 26x^3v \quad \text{and} \quad x^{12}w = -45x^3v$$

using the equality $x^4w^3=15x^8w^2+9x^{12}w$ obtained by (3.4). The analogous arguments show inductively

$$x^5w^2 = -45\cdot 53944550x^3v+45\cdot 104903x^2v-45\cdot 246xv+26v \tag{3.5}$$

and

$$x^9w = 4196254501x^3v-5\cdot 1631629x^2v+19131xv-45v\,.$$

Consequently we have Theorem after a slight consideration because $x^4v=v^2=vw=0$ and the rank of $K^*(EIII)$ is equal to 27.

**4.** Denote by $j$: $F_4\to E_6$ the canonical imbedding of $F_4$ in $E_6$. Then $j^*$: $R(E_6)\to R(F_4)$ is surjective and particularly

$$j^*(\rho_1) = j^*(\rho_2) = \rho'+1 \quad \text{and} \quad j^*(Ad) = Ad'+\rho'$$

(See (6.7) and (6.8) of [2.I]) where $\rho'$ is the irreducible representation of $F_4$ with the highest weight $\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4$ using the notation in Table VIII of [1] and $Ad'$ is the adjoint representation of $F_4$. Therefore Lemma implies the following

**Corollary 1** (cf. [6], Theorem 15.1).

$$\begin{cases} k^*(\rho') = 1+\rho+\Delta \\ k^*(\lambda^2\rho') = \rho+2\lambda^2\rho+\lambda^3\rho+\Delta+\Delta\rho \\ k^*(\lambda^3\rho') = 2\lambda^2\rho+2\lambda^3\rho-\Delta+\rho\lambda^2\rho+\rho\lambda^3\rho+\Delta\rho+2\Delta\lambda^2\rho \\ k^*(Ad') = \lambda^2\rho+\Delta \end{cases}$$

*where* $k^*: R(F_4) \to R(Spin(9))$ *is the restriction induced by the natural inclusion* $k: Spin(9) \to F_4$, *and* $\Delta$ *is the spin representation of* $Spin(9)$ *and* $\rho$ *is the canonical non-trivial* 9*-dimensional representaiton of* $Spin(9)$.

Let $l: FII \to EIII$ be the imbedding induced by $j$. Then we see that $l^*: K^*(EIII) \to K^*(FII)$ is surjective and so by the secondary formula of (2.2) or by the direct computation from Corollary 1 $K^*(FII)$ is generated by $l^*(w) = 16 - \Delta$ with relation $(l^*(w))^3 = 0$. Hence we have

**Corollary 2** (cf. [2], Theorem 7.1).

$$K^*(FII) \cong Z[\Delta]/((\Delta - 16)^3)$$

*where* $\Delta$ *is as in Corollary* 1.

**5.** The following tables are obtained by acting the elements of the Weyl group of $E_6$ suitably on the highest weight of each irreducible representation.

Table 1

| The weights of $\rho_1$: | | |
|---|---|---|
| 4 3 5 6 4 2 | 1 0-1 0-2-1 | 1 0 2 3 1 2 |
| 1 3 5 6 4 2 | -2 0-1 0-2-1 | 1 0 2 0 1 2 |
| 1 3 2 6 4 2 | -2 0-1-3-2-1 | 1 0-1 0 1 2 |
| 1 3 2 3 4 2 | -2 0-4-3-2-1 | -2 0-1 0 1 2 |
| 1 3 2 3 1 2 | -2-3-4-3-2-1 | -2 0-1 0 1-1 |
| 1 3 2 3 1-1 | -2-3-4-6-2-1 | -2-3-1-3-2-1 |
| 1 0 2 3 1-1 | -2-3-4-6-5-1 | 1 0-1 0 1-1 |
| 1 0 2 0 1-1 | -2-3-4-6-5-4 | 1 0-1-3-2-1 |
| 1 0 2 0-2-1 | 1 0 2 3 4 2 | 1-3-1-3-2-1 |

Table 2

| The weights of $\rho_2$: | | |
|---|---|---|
| 2 3 4 6 5 4 | -1 0-2 0-1 1 | -1 0 1 0-1 1 |
| 2 3 4 6 5 1 | -1 0-2 0-1-2 | -1 0 1 0-1-2 |
| 2 3 4 6 2 1 | -1-3-2-3-1-2 | -1 0-2-3-1-2 |
| 2 3 4 3 2 1 | -1-3-2-3-4-2 | -1 0-2-3-4-2 |
| 2 3 1 3 2 1 | -1-3-2-6-4-2 | -4-3-5-6-4-2 |
| 2 0 1 3 2 1 | -1-3-5-6-4-2 | -1 3 1 3 2 1 |
| -1 0 1 3 2 1 | 2 0 4 3 2 1 | -1 0-2-3-1 1 |
| -1 0 1 0 2 1 | 2 0 1 0 2 1 | -1-3-2-3-1 1 |
| -1 0-2 0 2 1 | 2 0 1 0-1 1 | 2 0 1 0-1-2 |

Table 3

| The positive roots of $E_6$: | | |
|---|---|---|
| 1 2 2 3 2 1 | 1 1 1 2 1 1 | 0 1 0 1 0 0 |
| 1 1 2 3 2 1 | 1 1 1 1 1 1 | 0 1 0 0 0 0 |
| 1 1 2 2 2 1 | 1 0 1 1 1 1 | 0 1 1 1 1 1 |
| 1 1 2 2 1 1 | 1 0 1 1 1 0 | 0 1 0 1 1 1 |
| 1 1 2 2 1 0 | 0 0 1 1 1 0 | 0 1 0 1 1 0 |
| 1 1 1 2 1 0 | 0 0 1 1 0 0 | 0 0 0 1 1 0 |
| 1 1 1 1 1 0 | 0 0 1 0 0 0 | 0 0 0 1 0 0 |
| 1 1 1 1 0 0 | 0 1 1 2 2 1 | 0 0 1 1 1 1 |
| 1 0 1 1 0 0 | 0 1 1 2 1 1 | 0 0 0 1 1 1 |
| 1 0 1 0 0 0 | 0 1 1 2 1 0 | 0 0 0 0 1 1 |
| 1 0 0 0 0 0 | 0 1 1 1 1 0 | 0 0 0 0 1 0 |
| 1 1 1 2 2 1 | 0 1 1 1 0 0 | 0 0 0 0 0 1 |

where the sequence $m_1 \cdots m_6$ of integers indicates a weight $\frac{1}{3}(m_1\alpha_1 + \cdots + m_6\alpha_6)$ in Tables 1 and 2 and a root $m_1\alpha_1 + \cdots + m_6\alpha_6$ in Table 3 using the notation in page 261 of [1].

OSAKA CITY UNIVERSITY

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