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K-GROUPS OF *EIII* AND *FII*

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1. Let G be a compact, connected, simply-connected Lie group and K a closed connected subgroup of G of maximal rank. As is well known [3], the complex K -group of G/K is isomorphic to $R(K) \otimes_{R(G)} Z$ and it is a free abelian group with rank equal to the quotient of the order of the Weyl group of G by the order of the Weyl group of K . Here $R(G)$ is the complex representation ring of G . The purpose of this paper is to determine an additive structure of the complex K -groups of symmetric spaces $EIII = E_6/Spin(10) \cdot SO(2)$ and $FII = F_4/Spin(9)$. To simplify the notation we write x for the element $x \otimes 1$ of $R(K) \otimes_{R(G)} Z$ in the following.

Let Δ^+ and Δ^- be the half-spin representations of $Spin(10)$, and let ρ and t be the canonical non-trivial 10- and 1-dimensional complex representations of $Spin(10)$ and $SO(2)$ respectively. Then

$$R(Spin(10) \times SO(2)) = Z[\lambda^1 \rho, \lambda^2 \rho, \lambda^3 \rho, \Delta^+, \Delta^-, t, t^{-1}]$$

and $R(Spin(10) \cdot SO(2))$ is isomorphic to the subalgebra of $R(Spin(10) \times SO(2))$ generated by the representations of $Spin(10) \times SO(2)$ which are trivial on $Spin(10) \cap SO(2) = Z_4$ (See [5] and [2, I], Prop. 2.1). Furthermore then our result is stated as follows.

Theorem.

$$K^*(EIII) \cong Z\{x^i, x^j w, x^j w^2, x^k v \mid 0 \leq i \leq 8, 0 \leq j \leq 4, 0 \leq k \leq 3\}$$

where

$$\begin{aligned} x &= t^4 - 1, \\ w &= (t^2 \rho - 10) - x^3 + 2x^2 - 5x, \\ v &= 45xw^3 + 26x^5w^2 \end{aligned}$$

and $Z\{a, b, c, \dots\}$ is the free abelian group generated by the set $\{a, b, c, \dots\}$.

Besides we have (2.1) of Section 2 concerning a ring structure of $K^*(EIII)$. Now, recently Steinberg [4] gave a general formula of a free basis over $R(G)$ for an $R(G)$ -module $R(K)$ (by restriction).

2. Using the notation in Table V of [1] we denote by ρ_1 and ρ_2 the 27-dimensional representations of E_6 with the highest weights $\frac{1}{3}(4\alpha_1+3\alpha_2+5\alpha_3+6\alpha_4+4\alpha_5+2\alpha_6)$ and $\frac{1}{3}(2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6)$ respectively and by Ad the adjoint representation of E_6 .

Lemma. *Let $i^*: R(E_6) \rightarrow R(\text{Spin}(10) \cdot \text{SO}(2))$ be the restriction induced by the natural inclusion $i: \text{Spin}(10) \cdot \text{SO}(2) \rightarrow E_6$. Then we have*

- (i) $i^*(Ad) = \lambda^2\rho + t^3\Delta^+ + t^{-3}\Delta^- + 1$
- (ii) $i^*(\rho_1) = t^4 + t\Delta^- + t^{-2}\rho$
- (iii) $i^*(\rho_2) = t^{-4} + t^{-1}\Delta^+ + t^2\rho$
- (iv) $i^*(\lambda^2\rho_1) = t^5\Delta^- + t^2\lambda^3\rho + t^2\rho + t^{-1}\Delta^-\rho + t^{-4}\lambda^2\rho$
- (v) $i^*(\lambda^2\rho_2) = t^{-5}\Delta^+ + t^{-2}\lambda^3\rho + t^{-2}\rho + t\Delta^+\rho + t^4\lambda^2\rho$
- (vi) $i^*(\lambda^3\rho_1) = i^*(\lambda^3\rho_2) = t^6\lambda^3\rho + t^{-6}\lambda^3\rho + t^3\Delta^+\lambda^2\rho + t^{-3}\Delta^-\lambda^2\rho + \rho\lambda^3\rho + \lambda^2\rho$.

Proof. (i)-(iii) are verified by observing the restriction of all weights of ρ_1 , ρ_2 and Ad to $\text{Spin}(10) \cdot \text{SO}(2)$. Here this reduction is based on the formula given in Section 1 of [5], and the weights of ρ_1 , ρ_2 and Ad are listed in Section 5.

Consider the exterior powers of the formulas (ii) and (iii) then we can easily check (iv)-(vi) since $\lambda^2\Delta^\pm = \lambda^3\rho$, $\lambda^2(\lambda^2\rho) + \lambda^4\rho = \rho\lambda^3\rho$ and $\lambda^3\Delta^- + \rho\Delta^- = \Delta^+\lambda^2\rho$.
q.e.d.

By Lemma we see that

(2.1) $R(\text{Spin}(10) \cdot \text{SO}(2)) \otimes_{R(E_6)} Z (\cong K^*(EIII))$ is multiplicatively generated by two elements x and w with relations

$$(2.2) \quad (x^3 + 3x^2 + 3x)w^2 + (x^{12} - x^{11} + x^{10} - x^9 + x^8 - x^7 + 3x^6)w \\ + 2(x^{16} - x^{15} + x^{14} - x^{13} + x^{12} - x^{11} + x^{10}) - x^9 = 0$$

and

$$w^3 - (2x^8 - 2x^7 + 2x^6 - x^5 + 7x^4 + 5x^3 + 18x^2 + 15x)w^2 \\ - \{8(x^{12} - x^{11} + x^{10} - x^9) + 10x^8 - 2x^7 + 15x^6\}w - 8(x^{16} - x^{15} + x^{14} - x^{13} + x^{12}) \\ + 7x^{11} - 9x^{10} + 5x^9 = 0.$$

When we calculate (2.2), note that

$$(2.3) \quad x^{17} = 0$$

since $EIII$ is a differentiable manifold of dimension 32, and $x+1$ is invertible.

It follows from (2.2) that

$$(2.4) \quad \{x^9(x^5 + 15x^4 + 78x^3 + 182x^2 + 195x + 78)w + 13x^7 + 53x^6 + 84x^5 + 45x^4\} = 0.$$

3. Proof of Theorem. By (2.3) and (2.4) we have inductively

$$(3.1) \quad 624x^9w = 109x^{16} - 154x^{15} + 228x^{14} - 360x^{13}$$

and so by this formula we have

$$(3.2) \quad x^9w^2 = 0.$$

Then we get

$$(3.3) \quad x^9 = \{14(x^8 + x^7 + x^6 + x^5 + x^4) + 13x^3 + 9x^2 + 3x\}w^2 \\ + \{5(x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7) + 3x^6\}w$$

by (3.1), (3.2) and the first formula of (2.2), and moreover

$$(3.4) \quad xx^3 = (12x^8 + 16x^7 + 12x^6 + 15x^5)w^2 + (x^{12} + 9x^{11} + x^{10} + 9x^9)w$$

by (3.1)-(3.3) and the secondary formula of (2.2).

It follows that $26x^{12}w = -15x^{16}$ and $x^{16} = 3x^8w^2$ from (3.1) and (3.3) respectively. Therefore $26x^{12}w + 45x^8w^2 = 0$ and so we see that

$$x^8w^2 = 26x^3v \quad \text{and} \quad x^{12}w = -45x^3v$$

using the equality $x^4w^3 = 15x^8w^2 + 9x^{12}w$ obtained by (3.4). The analogous arguments show inductively

$$(3.5) \quad x^5w^2 = -45 \cdot 53944550x^3v + 45 \cdot 104903x^2v - 45 \cdot 246xv + 26v$$

and

$$x^9w = 4196254501x^3v - 5 \cdot 1631629x^2v + 19131xv - 45v.$$

Consequently we have Theorem after a slight consideration because $x^4v = v^2 = v = 0$ and the rank of $K^*(EIII)$ is equal to 27.

4. Denote by $j: F_4 \rightarrow E_6$ the canonical imbedding of F_4 in E_6 . Then $j^*: R(E_6) \rightarrow R(F_4)$ is surjective and particularly

$$j^*(\rho_1) = j^*(\rho_2) = \rho' + 1 \quad \text{and} \quad j^*(Ad) = Ad' + \rho'$$

(See (6.7) and (6.8) of [2.I]) where ρ' is the irreducible representation of F_4 with the highest weight $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ using the notation in Table VIII of [1] and Ad' is the adjoint representation of F_4 . Therefore Lemma implies the following

Corollary 1 (cf. [6], Theorem 15.1).

$$\begin{cases} k^*(\rho') = 1 + \rho + \Delta \\ k^*(\lambda^2\rho') = \rho + 2\lambda^2\rho + \lambda^3\rho + \Delta + \Delta\rho \\ k^*(\lambda^3\rho') = 2\lambda^2\rho + 2\lambda^3\rho - \Delta + \rho\lambda^2\rho + \rho\lambda^3\rho + \Delta\rho + 2\Delta\lambda^2\rho \\ k^*(Ad') = \lambda^2\rho + \Delta \end{cases}$$

where $k^*: R(F_4) \rightarrow R(\text{Spin}(9))$ is the restriction induced by the natural inclusion $k: \text{Spin}(9) \rightarrow F_4$, and Δ is the spin representation of $\text{Spin}(9)$ and ρ is the canonical non-trivial 9-dimensional representation of $\text{Spin}(9)$.

Let $l: FII \rightarrow EIII$ be the imbedding induced by j . Then we see that $l^*: K^*(EIII) \rightarrow K^*(FII)$ is surjective and so by the secondary formula of (2.2) or by the direct computation from Corollary 1 $K^*(FII)$ is generated by $l^*(w) = 16 - \Delta$ with relation $(l^*(w))^3 = 0$. Hence we have

Corollary 2 (cf. [2], Theorem 7.1).

$$K^*(FII) \cong Z[\Delta]/((\Delta - 16)^3)$$

where Δ is as in Corollary 1.

5. The following tables are obtained by acting the elements of the Weyl group of E_6 suitably on the highest weight of each irreducible representation.

Table 1

The weights of ρ_1 :		
4 3 5 6 4 2	1 0-1 0-2-1	1 0 2 3 1 2
1 3 5 6 4 2	-2 0-1 0-2-1	1 0 2 0 1 2
1 3 2 6 4 2	-2 0-1-3-2-1	1 0-1 0 1 2
1 3 2 3 4 2	-2 0-4-3-2-1	-2 0-1 0 1 2
1 3 2 3 1 2	-2-3-4-3-2-1	-2 0-1 0 1-1
1 3 2 3 1-1	-2-3-4-6-2-1	-2-3-1-3-2-1
1 0 2 3 1-1	-2-3-4-6-5-1	1 0-1 0 1-1
1 0 2 0 1-1	-2-3-4-6-5-4	1 0-1-3-2-1
1 0 2 0-2-1	1 0 2 3 4 2	1-3-1-3-2-1

Table 2

The weights of ρ_2 :		
2 3 4 6 5 4	-1 0-2 0-1 1	-1 0 1 0-1 1
2 3 4 6 5 1	-1 0-2 0-1-2	-1 0 1 0-1-2
2 3 4 6 2 1	-1-3-2-3-1-2	-1 0-2-3-1-2
2 3 4 3 2 1	-1-3-2-3-4-2	-1 0-2-3-4-2
2 3 1 3 2 1	-1-3-2-6-4-2	-4-3-5-6-4-2
2 0 1 3 2 1	-1-3-5-6-4-2	-1 3 1 3 2 1
-1 0 1 3 2 1	2 0 4 3 2 1	-1 0-2-3-1 1
-1 0 1 0 2 1	2 0 1 0 2 1	-1-3-2-3-1 1
-1 0-2 0 2 1	2 0 1 0-1 1	2 0 1 0-1-2

Table 3

The positive roots of E_6 :		
1 2 2 3 2 1	1 1 1 2 1 1	0 1 0 1 0 0
1 1 2 3 2 1	1 1 1 1 1 1	0 1 0 0 0 0
1 1 2 2 2 1	1 0 1 1 1 1	0 1 1 1 1 1
1 1 2 2 1 1	1 0 1 1 1 0	0 1 0 1 1 1
1 1 2 2 1 0	0 0 1 1 1 0	0 1 0 1 1 0
1 1 1 2 1 0	0 0 1 1 0 0	0 0 0 1 1 0
1 1 1 1 1 0	0 0 1 0 0 0	0 0 0 1 0 0
1 1 1 1 0 0	0 1 1 2 2 1	0 0 1 1 1 1
1 0 1 1 0 0	0 1 1 2 1 1	0 0 0 1 1 1
1 0 1 0 0 0	0 1 1 2 1 0	0 0 0 0 1 1
1 0 0 0 0 0	0 1 1 1 1 0	0 0 0 0 1 0
1 1 1 2 2 1	0 1 1 1 0 0	0 0 0 0 0 1

where the sequence $m_1 \cdots m_6$ of integers indicates a weight $\frac{1}{3}(m_1\alpha_1 + \cdots + m_6\alpha_6)$ in Tables 1 and 2 and a root $m_1\alpha_1 + \cdots + m_6\alpha_6$ in Table 3 using the notation in page 261 of [1].

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References

- [1] N. Bourbaki: *Groupes et algèbres de Lie* IV-VI, 1968.
- [2] H. Minami: *K-groups of symmetric spaces* I, II, *Osaka J. Math.* **12** (1975), 623–634; **13** (1976), 271–287.
- [3] Harsh V. Pittie: *Homogeneous vector bundles on homogeneous spaces*, *Topology* **11** (1972), 199–203.
- [4] R. Steinberg: *On a theorem of Pittie*, *Topology* **14** (1975), 173–177.
- [5] H.O. Singh Varma: *The topology of EIII and a conjecture of Atiyah and Hirzebruch*, *Nederl. Akad. Wetensch. Indag. Math.* **30** (1968), 67–71.
- [6] I. Yokota: *Exceptional Lie group F_4 and its representation rings*, *J. Fac. Sci. Shinshu Univ.* **3** (1968), 35–60.

