<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>Note on the Lefschetz fixed point theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Nakaoka, Minoru</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 1969, 6(1), p. 135-142</td>
</tr>
<tr>
<td><strong>Version Type</strong></td>
<td>VoR</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/4923">https://doi.org/10.18910/4923</a></td>
</tr>
<tr>
<td><strong>rights</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Note</strong></td>
<td></td>
</tr>
</tbody>
</table>
1. Introduction

Let $V$ be an open set of the $n$-dimensional euclidean space $\mathbb{R}^n$, and $f: V \rightarrow \mathbb{R}^n$ be a continuous map such that the fixed point set $F = \{x \in V | f(x) = x\}$ is compact. If $i: V \subset \mathbb{R}^n$, then $i - f$ maps $(V, V - F)$ to $(\mathbb{R}^n, \mathbb{R}^n - 0)$. Considering the homomorphism of the integral homology groups induced by $i - f$, A. Dold [2] defines the fixed point index $I_f \in \mathbb{Z}$ by

$$(i - f)^* \mu_F = I_f \mu_o,$$

where $\mu_o \in H_n(\mathbb{R}^n, \mathbb{R}^n - 0; \mathbb{Z})$ is an orientation of $\mathbb{R}^n$ and $\mu_F \in H_n(V, V - F; \mathbb{Z})$ is the 'fundamental' class corresponding to the orientation $\mu_o$. With this definition, he proves the following Lefschetz fixed point theorem:

**Theorem A.** Let $V$ be an open set of $\mathbb{R}^n$, and $f: V \rightarrow V$ be a continuous map such that $f(V)$ is contained in a compact set $K \subset V$. Then the fixed point index $I_f$ of $f$ and the Lefschetz number of $(f|K)_*: H_*(K; \mathbb{Q}) \rightarrow H_*(K; \mathbb{Q})$ are both defined and they agree, where $\mathbb{Q}$ is the field of rational numbers.

Precisely, he proves the theorem in which $V$ is replaced by a euclidean neighborhood retract $Y$. However this generalization follows directly from the above one, because he defines the fixed point index of $f: Y \rightarrow Y$ to be that of the composite $i \circ f \circ r: V \rightarrow V$, where $i: Y \rightarrow V$, $r: V \rightarrow Y (r \circ i = id)$ is a euclidean neighborhood retraction.

On the other hand, R. Brown [1] shows the Lefschetz fixed point theorem for a compact orientable $n$-dimensional topological manifold $M$ (see also [3]). Taking an orientation of $M$, let $\mu \in H_*(M; \mathbb{Z})$ and $U \in H^*(M \times M, M \times M - d(M); \mathbb{Z})$ denote the corresponding fundamental class and Thom class respectively, where $d(M)$ is the diagonal of $M \times M$. Denote by $U' \in H^*(M \times M; \mathbb{Z})$ the image of $U$ under the natural homomorphism. Then the theorem of Brown is as follows:

**Theorem B.** Let $M$ be a compact orientable $n$-dimensional topological
manifold, and \( f: M \to M \) be a continuous map. Define \( \hat{f}: M \to M \times M \) by \( \hat{f}(x) = (f(x), x) \) for \( x \in M \). Then the Kronecker product \( \langle \hat{f}^* U', \mu \rangle \) is equal to the Lefschetz number of \( f_* : H_*(M; \mathbb{Q}) \to H_*(M; \mathbb{Q}) \).

The purpose of this note is to prove a theorem which contains Theorem A and B as corollaries.

Let \( M \) be an orientable \( n \)-dimensional topological manifold which is not necessarily compact, and \( f: M \to M \) be a continuous map such that the fixed point set \( F \) of \( f \) is compact. Take an orientation of \( M \). Then the Thom class \( U \in H^n(M \times M, M \times M - d(M); \mathbb{Z}) \) and the fundamental class \( \mu_F \in H_*(M, M - F; \mathbb{Z}) \) are well-defined. Considering \( \hat{f}: (M, M - F) \to (M \times M, M \times M - d(M)) \), we define the fixed point index \( I(f) \) by

\[
I(f) = \langle U, \hat{f}^* \mu_F \rangle \in \mathbb{Z}.
\]

Then our theorem is stated as follows:

**Theorem C.** Let \( M \) be an orientable \( n \)-dimensional topological manifold, and \( f: M \to M \) be a continuous map such that \( f(M) \) is contained in a compact set \( K \subset M \). Then the fixed point index \( I(f) \) of \( f \) and the Lefschetz number of \( (f|K)_* : H_*(K; \mathbb{Q}) \to H_*(K; \mathbb{Q}) \) are both defined and they agree.

Our proof of this theorem is different from that of Theorem A due to Dold. Therefore this paper gives another proof of Theorem A.

The method we use to prove Theorem C is essentially the one due to J. Milnor [4] and is the one employed by Brown to prove Theorem B.

2. A fundamental lemma

Let \( M \) be an \( n \)-dimensional topological manifold, and \( d: M \to M \times M \) be the diagonal map. Let \( K \) be a compact subset of \( M \).

**Lemma 1.** There are an open neighborhood \( W \) of \( d(K) \) in \( K \times M \) and a retraction \( r: W \to d(K) \) such that the diagram

\[
\begin{array}{ccc}
K \times M & \xrightarrow{\hat{f}} & K \\
\downarrow r & & \downarrow l \\
W & \xrightarrow{d} & K \\
\end{array}
\]

is homotopy commutative, where \( k \) and \( l \) are the inclusion maps.

Proof. For \( r > 0 \), let

\[
O_r = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n | x_1^2 + \cdots + x_n^2 < r \}.
\]

It is easily seen that there exists a finite set \( \{ V_1, \ldots, V_s \} \) of coordinate neighbor-
hoods of \( M \) such that
\[
\bigcup_{i=1}^{s} h_i^{-1}(O_i) \supseteq K,
\]
where \( h_i : V_i \approx \mathbb{R}^n \) is a homeomorphism.

Put
\[
V_i' = h_i^{-1}(O_i), \quad V_i'' = h_i^{-1}(O_i),
\]
\[
V' = \bigcup_{i=1}^{s} V_i', \quad V'' = \bigcup_{i=1}^{s} V_i''.
\]
The space \( V''/V'' - V'' \) obtained from the closure \( V'' \) by identifying \( V'' - V'' \) to one point is homeomorphic with the \( n \)-sphere \( S^n \). Therefore a homeomorphism \( f \) of \( V'' \) into \( S^n \times \cdots \times S^n \) (\( s \) times) is defined by
\[
f(x) = (f_1 p_1(x), \ldots, f_s p_s(x)) \quad (x \in V''),
\]
where \( p_1 : V'' \to V''/V'' - V'' \) is the projection and \( f_i : V''/V'' - V'' \approx S^n \) is a homeomorphism. Since \( V' \subseteq V'' \) and \( S^n \times \cdots \times S^n \subseteq \mathbb{R}^m \) (\( m = (n+1)s \)), we can regard \( V' \) as a closed subset of \( \mathbb{R}^m \). Since each \( V_i \) is an ANR, so is \( V = \bigcup_{i=1}^{s} V_i \).

Consequently, the inclusion map \( V' \subseteq V \) has an extension \( g : Q \to V \), where \( Q \) is a neighborhood of \( V' \) in \( \mathbb{R}^m \). It is obvious that there exists \( \varepsilon > 0 \) such that if \( x, y \in V' \) and the distance from \( x \) to \( y \) in \( \mathbb{R}^m \) is smaller than \( \varepsilon \) then \((1-t)x + ty \in Q\) for any \( t \in [0, 1] \). Put
\[
W = \{(x, y) \in K \times V' | d(x, y) < \varepsilon \},
\]
and define \( r : W \to d(K) \) by \( r(x, y) = (x, x) \).

We can now define a homotopy \( f_t : W \to K \times M \) of \( k \circ r \) to \( l \) by
\[
f_t(x, y) = (x, g((1-t)x + ty)).
\]

Let \( R \) be a fixed principal ideal domain, and we shall take coefficients of homology and cohomology from \( R \). Consider the cup product
\[
\cup : H^*(K \times (M, M - K)) \otimes H^*(K \times M) \to H^*(K \times (M, M - K)).
\]

**Lemma 2.** For \( \alpha \in H^*(M) \) and \( \gamma \in H^*(K \times (M, M - d(K))) \) we have
\[
j^* \gamma \cup p_1^* i^* \alpha = j^* \gamma \cup p_2^* \alpha,
\]
where \( p_1 : K \times M \to K, p_2 : K \times M \to M \) are the projections and \( i : K \to M, j : K \times (M, M - K) \to (K \times M, K \times M - d(K)) \) are the inclusion maps.
Proof. By Lemma 1 and the naturality of the cup product, we have a commutative diagram

\[
\begin{array}{ccc}
H^*(d(K)) & \xrightarrow{k^*} & H^*(K \times M) \\
\downarrow & & \downarrow \\
H^*(W) & \xrightarrow{l^*} & H^*(W, W-d(K))
\end{array}
\]

If we define \( p : d(K) \to K \) by \( p(x, x) = x (x \in K) \), then it holds that \( p_1 \circ k = p \) and \( p_2 \circ k = i \circ p \). Therefore it follows that

\[
l^*(\gamma \cup p_i^* \alpha) = l^* \gamma \cup r^* k^* p_i^* \alpha \\
= l^* \gamma \cup r^* p^* i^* \alpha = l^* \gamma \cup r^* k^* p_2^* \alpha \\
= l^* (\gamma \cup p_2^* \alpha)
\]

Since \( l^* : H^*(K \times M, K \times M - d(K)) \cong H^*(W, W - d(K)) \) is an excision isomorphism, we obtain

\[
\gamma \cup p_2^* \alpha = \gamma \cup p_2^* \alpha
\]

This, together with the naturality of the cup product, implies the desired result.

q.e.d.

For topological pairs \((X, A)\) and \((Y, B)\), consider the slant product

\[
\gamma : H^*((X, A) \times (Y, B)) \otimes H_*(Y, B) \to H^*(X, A)
\]

The following relations hold between the cup, cap and slant products: For \( \gamma \in H^*((X, A) \times (Y, B)), \alpha \in H^*(X), \beta \in H^*(Y) \) and \( b \in H_*(Y, B) \), we have

\[
(1) \quad \alpha \cup (\gamma/b) = (p_1^* \alpha \cup \gamma)/b, \\
\gamma/(\beta \cup b) = (\gamma \cup p_2^* \beta)/b
\]

in \( H^*(X, A) \), where \( p_1 : X \times Y \to X \) and \( p_2 : X \times Y \to Y \) are the projections (see [5]).

By an orientation \( \mu \) over \( R \) of an \( n \)-dimensional topological manifold \( M \) we mean a function which assigns to each \( x \in M \) a generator \( \mu_x \) of \( H_*(M, M-x) \) which "varies continuously" with \( x \), in the following sense. For each \( x \) there exist a neighborhood \( N \) and an element \( \mu_N \in H_*(M, M-N) \) such that the image of \( \mu_N \) in \( H_*(M, M-y) \) under the natural homomorphism is \( \mu_y \) for each \( y \in N \).

If an orientation over \( R \) of the manifold \( M \) exists, \( M \) is called orientable over \( R \).

Assume that \( M \) is orientable over \( R \) and an orientation \( \mu \) of \( M \) is given. Then it is known that, for each compact subset \( K \) of \( M \), there is a unique element \( \mu_K \in H_*(M, M-K) \) whose image in \( H_*(M, M-x) \) under the natural homomorphism is \( \mu_x \) for any \( x \in K \) (see [3]). It is also known that there exists a unique
element $U \in H^n(M \times M, M \times M - d(M))$ such that

$$\langle i^* U, \mu_x \rangle = 1$$

for any $x \in M$, where $l_x : (M, M - x) \to (M \times M, M \times M - d(M))$ is a continuous map sending $x' \in M$ to $(x, x') \in M \times M$ (see [3], [5]). Denote by $U_K \in H^n(K \times (M, M - K))$ the image of $U$ under the natural homomorphism.

A simple calculation shows

(2) $U_K/\mu_K = 1$.

We shall now prove the following fundamental lemma.

**Lemma 3.** The diagram

$$
\begin{array}{ccc}
H^q(M) & \xrightarrow{(-1)^q i^*} & H^q(K) \\
\downarrow \mu_K & & \downarrow U_K/|
\\
H_{n-q}(M, M-K) & &
\end{array}
$$

is commutative, where $i : K \subset M$.

**Proof.** For $\alpha \in H^q(M)$, we obtain by (1), (2) and Lemma 2

$$U_K/(\alpha \sim \mu_K) = (U_K \ominus p_1^* \alpha)/\mu_K = (U_K \ominus p_1^* \alpha)/\mu_K = (\alpha \sim (U_K/\mu_K) = (-1)^q (p_1^* i^* \alpha \sim U_K)/\mu_K$$

which proves the desired result. \(\text{q.e.d.}\)

**3. Lefschetz fixed point theorem**

Let $M$ be an $n$-dimensional topological manifold which is orientable over $R$. Let $V$ be an open set of $M$, and $K$ be a compact subset of $V$. Given an orientation $\mu$ of $M$, we shall denote by $\mu_K \in H_* (V, V - K)$ the element corresponding to $\mu_K$ under the excision isomorphism $H_*(V, V - K) \approx H_* (M, M - K)$.

If $f : V \to M$ is a continuous map such that the fixed point set $F$ is compact, then we call

$$I(f) = \langle U, f_* \mu_F \rangle \in R$$

the fixed point index of $f$, where $f : (V, V - F) \to (M \times M, M \times M - d(M))$ is a continuous map given by $f^*(x) = (f(x), x) (x \in V)$. It follows that $I(f)$ is independent of the choice of orientation.

For a compact set $K$ such that $F \subset K \subset M$, we have
where \( f_* : H_*(V, V-K) \to H_*(M \times M, M \times M - d(M)) \). This follows from that \( \mu^K \) is the image of \( \mu^V \) under the natural homomorphism.

**Lemma 4.** In the case \( M = R^n \), we have

\[
(i-f)_* \mu^V = I(f) \mu \quad \mu
\]

where \( i-f : (V, V-F) \to (R^n, R^n - 0) \) is a continuous map sending \( x \in V \) to \( x - f(x) \in R^n \).

**Proof.** Define \( \Delta : (R^n \times R^n, R^n \times R^n - d(R^n)) \to (R^n, R^n - 0) \) by \( \Delta(x, y) = y - x \quad (x, y \in R^n) \). Then, for \( l_o : (R^n, R^n - 0) \to (R^n \times R^n, R^n \times R^n - d(R^n)) \), we have \( \Delta \circ l_o = id \). Denote by \( \mu_0 \in H^*(R^n, R^n - 0) \) the dual to \( \mu \in H_*(R^n, R^n - 0) \). Since \( \langle l_o^* \Delta^* \mu_0, \mu \rangle = 1 \), we have

\[
\Delta^* \mu_0 = U \in H^*(R^n \times R^n, R^n \times R^n - d(R^n))
\]

Since \( \Delta \circ f = i-f \), we obtain

\[
I(f) = \langle \Delta^* \mu_0, f_* \mu \rangle = \langle \mu, (i-f)_* \mu \rangle,
\]

which shows the desired result. q.e.d.

Let \( N \) be a graded module over a field \( R \), and \( \varphi : N \to N \) be an endomorphism of degree 0 which factors through a finitely generated graded module. Taking a homogeneous basis \( \{a_\lambda\} \) of \( N \), put

\[
\varphi(a_\lambda) = \sum_\mu r_{\lambda \mu} a_\mu \quad (r_{\lambda \mu} \in R).
\]

Then it follows that \( r_{\lambda \lambda} \) is zero except a finite number of \( \lambda \), and that

\[
\Lambda(\varphi) = \sum_\lambda (-1)^{\deg a_\lambda} r_{\lambda \lambda} \in R
\]

is independent of the choice of \( \{a_\lambda\} \) (see [2]). \( \Lambda(\phi) \) is called the Lefschetz number of \( \varphi \).

**Theorem D.** Let \( M \) be an \( n \)-dimensional topological manifold which is orientable over a field \( R \), and let \( f : M \to M \) be a continuous map such that \( f(M) \) is contained in a compact set \( K \subset M \). Then the fixed point index \( I(f) \) of \( f \) and the Lefschetz number \( \Lambda((f|K)_*) \) of the homomorphism \( (f|K)_* : H_*(K) \to H_*(K) \) of homology with coefficients in \( R \) are both defined and they agree.

**Proof.** The fixed point set \( F \) of \( f \) is a closed subset of \( K \), and hence is compact. Therefore \( I(f) \) is defined.

From Lemma 3 it follows that the diagram

\[
(3) \quad I(f) = \langle U, f_* \mu \rangle,
\]
is commutative. It is obvious from the definition of the cap product that the image of the homomorphism \( \sim \mu_K \) is finitely generated. Therefore \((f | K)^*\) factors through a finitely generated module, and hence \( \Delta((f | K)^*) \) is defined.

Let \( \{\alpha_\lambda\} \) \( \{\beta_\mu\} \) and \( \{\rho_\nu\} \) be homogeneous bases of \( H^*(M) \), \( H^*(M, M-K) \) and \( H^*(K) \) respectively, and put

\[
f^*(\rho_\nu) = \sum_\lambda m_{\nu \lambda} \alpha_\lambda, \quad U_K = \sum_{\nu, \mu} c_{\nu \mu} \rho_\nu \times \beta_\mu, \quad \langle \beta_\mu \sim \alpha_\lambda, \mu_K \rangle = y_{\nu \lambda}.
\]

Then it follows from the above commutative diagram that

\[
(-1)^{n \deg \rho_\nu}(f | K)^* \rho_\nu = U_K/(f^* \rho_\nu \sim \mu_K)
\]

\[
= \sum_{\nu, \mu} (c_{\nu \mu} \rho_\nu \times \beta_\mu)(f^* \rho_\nu \sim \mu_K)
\]

\[
= \sum_{\nu, \mu} c_{\nu \mu} \beta_\mu, f^* \rho_\nu \sim \mu_K \rho_\nu
\]

\[
= \sum_{\nu, \lambda, k} c_{\nu \mu} m_{\nu \alpha} \beta_\mu, \alpha_\lambda \sim \mu_K \rho_\nu
\]

\[
= \sum_{\nu, \lambda, k} c_{\nu \mu} m_{\nu \lambda} \beta_\mu, \mu_K \rho_\nu
\]

\[
= \sum_{\nu, \lambda, k} c_{\nu \mu} m_{\nu \lambda} y_{\nu \lambda} \rho_\nu.
\]

Therefore we have

\[
\Delta((f | K)^*) = \sum_{\lambda, \mu, \nu} (-1)^{n + 1} \deg \rho_\nu c_{\nu \mu} m_{\nu \lambda} y_{\nu \lambda}.
\]

The diagram

\[
\begin{array}{ccc}
H^*(M \times M, M \times M - d(M)) & \xrightarrow{f^*} & H^*(M, M - K) \\
\downarrow i^* & & \uparrow d^* \\
H^*(K \times (M, M - K)) & \xrightarrow{(f \times \text{id})^*} & H^*(M \times (M, M - K))
\end{array}
\]

is commutative, where \( i^* \) is the natural homomorphism. Therefore it follows from (3) that

\[
I(f) = \langle U, \hat{f} \star \mu_K \rangle = \langle \hat{f}^* U, \mu_K \rangle
\]

\[
= \langle d^* (f \times \text{id})^* U_K, \mu_K \rangle
\]

\[
= \sum_{\nu, \mu} c_{\nu \mu} \langle d^* (f^* \rho_\nu \times \beta_\mu), \mu_K \rangle.
\]
Consequently we obtain $I(f) = \Lambda((f|K)\ast)$. Since $\Lambda((f|K)\ast) = \Lambda((f|K)_\ast)$ is obvious, we have the desired result. q.e.d.

A topological manifold which is orientable (over $\mathbb{Z}$) is orientable over $\mathbb{Q}$, and $I(f)$ for $R = \mathbb{Z}$ coincides with $I(f)$ for $R = \mathbb{Q}$. Therefore Theorem D implies Theorem C.

Lemma 4 shows that $I(f)$ coincides with $I_f$ due to Dold when $M = \mathbb{R}^n$. Therefore Theorem C implies Theorem A. It is clear that Theorem C implies Theorem B.

**Bibliography**


