Nakaoka, M.
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NOTE ON THE LEFSCHETZ FIXED POINT THEOREM

Dedicated to Professor A. Komatu on his 60th birthday

Minoru NAKAOKA

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1. Introduction

Let V be an open set of the n-dimensional euclidean space $\boldsymbol{R}^n$, and $f\colon V\to\boldsymbol{R}^n$ be a continuous map such that the fixed point set $F=\{x\in V\,|\,f(x)=x\}$ is compact. If $i\colon V\subset\boldsymbol{R}^n$, then $i-f$ maps $(V, V-F)$ to $(\boldsymbol{R}^n, \boldsymbol{R}^n-0)$. Considering the homomorphism of the integral homology groups induced by $i-f$, A. Dold [2] defines the *fixed point index* $I_f\in\boldsymbol{Z}$ by

$$(i-f)_*\mu_F^V = I_f\mu_0\,,$$

where $\mu_0\in H_n(\boldsymbol{R}^n, \boldsymbol{R}^n-0;\boldsymbol{Z})$ is an orientation of $\boldsymbol{R}^n$ and $\mu_F^V\in H_n(V, V-F;\ \boldsymbol{Z})$ is the 'fundamental' class corresponding to the orientation μ_0. With this definition, he proves the following Lefschetz fixed point theorem:

Theorem A. *Let V be an open set of $\boldsymbol{R}^n$, and $f\colon V\to V$ be a continuous map such that $f(V)$ is contained in a compact set $K\subset V$. Then the fixed point index I_f of f and the Lefschetz number of $(f\,|\,K)_*\colon H_*(K;\ \boldsymbol{Q})\to H_*(K;\ \boldsymbol{Q})$ are both defined and they agree, where $\boldsymbol{Q}$ is the field of rational numbers.*

Precisely, he proves the theorem in which V is replaced by a euclidean neighborhood retract Y. However this generalization follows directly from the above one, because he defines the fixed point index of $f\colon Y\to Y$ to be that of the composite $i\circ f\circ r\colon V\to V$, where $i\colon Y\to V$, $r\colon V\to Y$ $(r\circ i=id)$ is a euclidean neighborhood retraction.

On the other hand, R. Brown [1] shows the Lefschetz fixed point theorem for a compact orientable n-dimensional topological manifold M (see also [3]). Taking an orientation of M, let $\mu\in H_n(M;\boldsymbol{Z})$ and $U\in H^n(M\times M, M\times M-d(M);\ \boldsymbol{Z})$ denote the corresponding fundamental class and Thom class respectively, where $d(M)$ is the diagonal of $M\times M$. Denote by $U'\in H^n(M\times M;\ \boldsymbol{Z})$ the image of U under the natural homomorphism. Then the theorem of Brown is as follows:

Theorem B. *Let M be a compact orientable n-dimensional topological* *manifold, and* $f: M \to M$ *be a continuous map. Define* $\hat{f}: M \to M \times M$ *by* $\hat{f}(x) = (f(x), x)$ *for* $x \in M$. *Then the Kronecker product* $\langle \hat{f}^* U', \mu \rangle$ *is equal to the Lefschetz number of* $f_*: H_*(M; \boldsymbol{Q}) \to H_*(M; \boldsymbol{Q})$.

The purpose of this note is to prove a theorem which contains Theorem A and B as corollaries.

Let M be an orientable n-dimensional topological manifold which is not necessarily compact, and $f: M \to M$ be a continuous map such that the fixed point set F of f is compact. Take an orientation of M. Then the Thom class $U \in H^n(M \times M, M \times M - d(M); \boldsymbol{Z})$ and the fundamental class $\mu_F \in H_n(M, M-F; \boldsymbol{Z})$ are well-defined. Considering $\hat{f}: (M, M-F) \to (M \times M, M \times M - d(M))$, we define the *fixed point index* $I(f)$ by

$$I(f) = \langle U, \hat{f}_* \mu_F \rangle \in \boldsymbol{Z}.$$

Then our theorem is stated as follows:

Theorem C. *Let* M *be an orientable n-dimensional topological manifold, and* $f: M \to M$ *be a continuous map such that* $f(M)$ *is contained in a compact set* $K \subset M$. *Then the fixed point index* $I(f)$ *of* f *and the Lefschetz number of* $(f \mid K)_*: H_*(K; \boldsymbol{Q}) \to H_*(K; \boldsymbol{Q})$ *are both defined and they agree.*

Our proof of this theorem is different from that of Theorem A due to Dold. Therefore this paper gives another proof of Theorem A.

The method we use to prove Theorem C is essentially the one due to J. Milnor [4] and is the one employed by Brown to prove Theorem B.

2. A fundamental lemma

Let M be an n-dimensional topological manifold, and $d: M \to M \times M$ be the diagonal map. Let K be a compact subset of M.

Lemma 1. *There are an open neighborhood* W *of* $d(K)$ *in* $K \times M$ *and a retraction* $r: W \to d(K)$ *such that the diagram*

$$\begin{tikzcd} & K \times M \\ d(K) \arrow[ur, "k"] & \\ & W \arrow[uu, "l"'] \arrow[ul, "r"] \end{tikzcd}$$

is homotopy commutative, where k *and* l *are the inclusion maps.*

Proof. For $r > 0$, let

$$O_r = \{(x_1, \cdots, x_n) \in \boldsymbol{R}^n \mid x_1^2 + \cdots + x_n^2 < r\}.$$

It is easily seen that there exists a finite set $\{V_1, \cdots, V_s\}$ of coordinate neighborhoods of M such that

$$\bigcup_{i=1}^{s} h_i^{-1}(O_1) \supset K,$$

where $h_i : V_i \approx \boldsymbol{R}^n$ is a homeomorphism.

Put

$$V_i' = h_i^{-1}(O_1), \quad V_i'' = h_i^{-1}(O_2),$$

$$V' = \bigcup_{i=1}^{s} V_i', \quad V'' = \bigcup_{i=1}^{s} V_i''.$$

The space $\bar{V}''/\bar{V}''-V_i''$ obtained from the closure $\bar{V}''$ by identifying $\bar{V}''-V_i''$ to one point is homeomorphic with the n-sphere S^n. Therefore a homeomorphism f of V'' into $S^n \times \cdots \times S^n$ (s times) is defined by

$$f(x) = (f_1 p_1(x), \cdots, f_s p_s(x)) \qquad (x \in V''),$$

where $p_i : \bar{V}'' \to \bar{V}''/\bar{V}''-V_i''$ is the projection and $f_i : \bar{V}''/\bar{V}''-V_i'' \approx S^n$ is a homeomorphism. Since $\bar{V}' \subset V''$ and $S^n \times \cdots \times S^n \subset \boldsymbol{R}^m$ ($m=(n+1)s$), we can regard $\bar{V}'$ as a closed subset of $\boldsymbol{R}^m$. Since each V_i is an ANR, so is $V = \bigcup_{i=1}^{s} V_i$.

Consequently, the inclusion map $\bar{V}' \subset V$ has an extension $g : Q \to V$, where Q is a neighborhood of $\bar{V}'$ in $\boldsymbol{R}^m$. It is obvious that there exists $\varepsilon > 0$ such that if $x, y \in \bar{V}'$ and the distance from x to y in $\boldsymbol{R}^m$ is smaller than ε then $(1-t)x+ty \in Q$ for any $t \in [0, 1]$. Put

$$W = \{(x, y) \in K \times V' \mid d(x, y) < \varepsilon\},$$

and define $r : W \to d(K)$ by $r(x, y) = (x, x)$.

We can now define a homotopy $f_t : W \to K \times M$ of $k \circ r$ to l by

$$f_t(x, y) = (x, g((1-t)x+ty)). \qquad \text{q.e.d.}$$

Let R be a fixed principal ideal domain, and we shall take coefficients of homology and cohomology from R. Consider the cup product

$$\smile : H^*(K \times (M, M-K)) \otimes H^*(K \times M) \to H^*(K \times (M, M-K)).$$

Lemma 2. *For* $\alpha \in H^*(M)$ *and* $\gamma \in H^*(K \times M, K \times M - d(K))$ *we have*

$$j^*\gamma \smile p_1^* i^* \alpha = j^*\gamma \smile p_2^* \alpha,$$

where $p_1 : K \times M \to K$, $p_2 : K \times M \to M$ *are the projections and* $i : K \to M$, $j : K \times (M, M-K) \to (K \times M, K \times M - d(K))$ *are the inclusion maps.*

Proof. By Lemma 1 and the naturality of the cup product, we have a commutative diagram

$$\begin{tikzcd} & H^*(K\times M) \arrow[ll, "k^*"'] \arrow[r, "\gamma\smile"] \arrow[d, "l^*"] & H^*(K\times M, K\times M-d(K)) \arrow[d, "l^*"] \\ H^*(d(K)) & H^*(W) \arrow[ll, "r^*"] \arrow[r, "l^*\gamma\smile"] & H^*(W, W-d(K)) . \end{tikzcd}$$

If we define $p : d(K)\to K$ by $p(x, x)=x\ (x\in K)$, then it holds that $p_1\circ k=p$ and $p_2\circ k=i\circ p$. Therefore it follows that

$$\begin{aligned} l^*(\gamma\smile p_1^*i^*\alpha) &= l^*\gamma\smile r^*k^*p_1^*i^*\alpha \\ &= l^*\gamma\smile r^*p^*i^*\alpha = l^*\gamma\smile r^*k^*p_2^*\alpha \\ &= l^*(\gamma\smile p_2^*\alpha) . \end{aligned}$$

Since $l^* : H^*(K\times M, K\times M-d(K))\cong H^*(W, W-d(K))$ is an excision isomorphism, we obtain

$$\gamma\smile p_2^*i^*\alpha = \gamma\smile p_2^*\alpha .$$

This, together with the naturality of the cup product, implies the desired result.
q.e.d.

For topological pairs (X, A) and (Y, B), consider the slant product

$$/ : H^*((X, A)\times(Y, B))\otimes H_*(Y, B)\to H^*(X, A) .$$

The following relations hold between the cup, cap and slant products: For $\gamma\in H^*((X\ A)\times(Y, B))$, $\alpha\in H^*(X)$, $\beta\in H^*(Y)$ and $b\in H_*(Y, B)$, we have

$$(1)\qquad \begin{aligned} \alpha\smile(\gamma/b) &= (p_1^*\alpha\smile\gamma)/b , \\ \gamma/(\beta\frown b) &= (\gamma\smile p_2^*\beta)/b \end{aligned}$$

in $H^*(X, A)$, where $p_1 : X\times Y\to X$ and $p_2 : X\times Y\to Y$ are the projections (see [5]).

By an *orientation* μ over R of an n-dimensional topological manifold M we mean a function which assigns to each $x\in M$ a generator μ_x of $H_n(M, M-x)$ which "varies continuously" with x, in the following sense. For each x there exist a neighborhood N and an element $\mu_N\in H_n(M, M-N)$ such that the image of μ_N in $H_n(M, M-y)$ under the natural homomorphism is μ_y for each $y\in N$.

If an orientation over R of the manifold M exists, M is called *orientable* over R.

Assume that M is orientable over R and an orientation μ of M is given. Then it is known that, for each compact subset K of M, there is a unique element $\mu_K\in H_n(M, M-K)$ whose image in $H_n(M, M-x)$ under the natural homomorphism is μ_x for any $x\in K$ (see [3]). It is also known that there exists a unique element $U\in H^n(M\times M, M\times M-d(M))$ such that

$$\langle l_x^* U, \mu_x\rangle = 1$$

for any $x\in M$, where $l_x:(M, M-x)\to(M\times M, M\times M-d(M))$ is a continuous map sending $x'\in M$ to $(x, x')\in M\times M$ (see [3], [5]). Denote by $U_K\in H^n(K\times(M, M-K))$ the image of U under the natural homomorphism.

A simple calculation shows

$$U_K/\mu_K = 1\,. \tag{2}$$

We shall now prove the following fundamental lemma.

Lemma 3. *The diagram*

$$\begin{tikzcd} H^q(M) \arrow[rr, "(-1)^{nq} i^*"] \arrow[dr, "\frown \mu_K"'] & & H^q(K) \\ & H_{n-q}(M, M-K) \arrow[ur, "U_K/"'] & \end{tikzcd}$$

is commutative, where $i:K\subset M$.

Proof. For $\alpha\in H^q(M)$, we obtain by (1), (2) and Lemma 2

$$\begin{aligned} U_K/(\alpha\frown\mu_K) &= (U_K\smile p_2^*\alpha)/\mu_K \\ &= (U_K\smile p_1^* i^*\alpha)/\mu_K = (-1)^{nq}(p_1^* i^*\alpha\smile U_K)/\mu_K \\ &= (-1)^{nq} i^*\alpha\smile(U_K/\mu_K) = (-1)^{nq} i^*\alpha\,, \end{aligned}$$

which proves the desired result. q.e.d.

3. Lefschetz fixed point theorem

Let M be an n-dimensional topological manifold which is orientable over R. Let V be an open set of M, and K be a compact subset of V. Given an orientation μ of M, we shall denote by $\mu_K^V\in H_n(V, V-K)$ the element corresponding to μ_K under the excision isomorphism $H_n(V, V-K)\cong H_n(M, M-K)$.

If $f:V\to M$ is a continuous map such that the fixed point set F is compact, then we call

$$I(\hat{f}) = \langle U, \hat{f}_*\mu_F^V\rangle\in R$$

the *fixed point index* of f, where $\hat{f}:(V, V-F)\to(M\times M, M\times M-d(M))$ is a continuous map given by $\hat{f}(x)=(f(x), x)$ $(x\in V)$. It follows that $I(f)$ is independent of the choice of orientation.

For a compact set K such that $F\subset K\subset M$, we have

$$(3) \qquad I(f) = \langle U, \hat{f}_* \mu_K^V \rangle ,$$

where $\hat{f}_* : H_n(V, \ V-K) \to H_n(M \times M, \ M \times M - d(M))$. This follows from that μ_F^V is the image of μ_K^V under the natural homomorphism.

Lemma 4. *In the case $M = \boldsymbol{R}^n$, we have*

$$(i-f)_* \mu_F^V = I(f) \mu_0 ,$$

where $i-f : (V, V-F) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - 0)$ is a continuous map sending $x \in V$ to $x - f(x) \in \boldsymbol{R}^n$.

Proof. Define $\Delta : (\boldsymbol{R}^n \times \boldsymbol{R}^n, \boldsymbol{R}^n \times \boldsymbol{R}^n - d(\boldsymbol{R}^n)) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - 0)$ by $\Delta(x, y) = y - x$ $(x, y \in \boldsymbol{R}^n)$. Then, for $l_0 : (\boldsymbol{R}^n, \boldsymbol{R}^n - 0) \to (\boldsymbol{R}^n \times \boldsymbol{R}^n, \boldsymbol{R}^n \times \boldsymbol{R}^n - d(\boldsymbol{R}^n))$, we have $\Delta \circ l_0 = id$. Denote by $\bar{\mu}_0 \in H^n(\boldsymbol{R}^n, \boldsymbol{R}^n - 0)$ the dual to $\mu_0 \in H_n(\boldsymbol{R}^n, \boldsymbol{R}^n - 0)$. Since $\langle l_0^* \Delta^* \bar{\mu}_0, \mu_0 \rangle = 1$, we have

$$\Delta^* \bar{\mu}_0 = U \in H^n(\boldsymbol{R}^n \times \boldsymbol{R}^n, \boldsymbol{R}^n \times \boldsymbol{R}^n - d(\boldsymbol{R}^n)) .$$

Since $\Delta \circ \hat{f} = i - f$, we obtain

$$I(f) = \langle \Delta^* \bar{\mu}_0, \hat{f}_* \mu_F^V \rangle = \langle \bar{\mu}_0, (i-f)_* \mu_F^V \rangle ,$$

which shows the desired result. q.e.d.

Let N be a graded module over a field R, and $\varphi : N \to N$ be an endomorphism of degree 0 which factors through a finitely generated graded module. Taking a homogeneous basis $\{a_\lambda\}$ of N, put

$$\varphi(a_\lambda) = \sum_\mu r_{\lambda\mu} a_\mu \qquad (r_{\lambda\mu} \in R) .$$

Then it follows that $r_{\lambda\lambda}$ is zero except a finite number of λ, and that

$$\Lambda(\varphi) = \sum_\lambda (-1)^{\deg a_\lambda} r_{\lambda\lambda} \in R$$

is independent of the choice of $\{a_\lambda\}$ (see [2]). $\Lambda(\phi)$ is called the *Lefschetz number* of φ.

Theorem D. *Let M be an n-dimensional topological manifold which is orientable over a field R, and let $f : M \to M$ be a continuous map such that $f(M)$ is contained in a compact set $K \subset M$. Then the fixed point index $I(f)$ of f and the Lefschetz number $\Lambda((f|K)_*)$ of the homomorphism $(f|K)_* : H_*(K) \to H_*(K)$ of homology with coefficients in R are both defined and they agree.*

Proof. The fixed point set F of f is a closed subset of K, and hence is compact. Therefore $I(f)$ is defined.

From Lemma 3 it follows that the diagram

$$\begin{array}{ccc} H^q(K) & \xrightarrow{(-1)^{nq}(f\,|\,K)^*} & H^q(K) \\ \big\downarrow f^* & & \big\uparrow U_K/ \\ H^q(M) & \xrightarrow{\frown\,\mu_K} & H_{n-q}(M, M-K) \end{array}$$

is commutative. It is obvious from the definition of the cap product that the image of the homomorphism $\frown \mu_K$ is finitely generated. Therefore $(f\,|\,K)^*$ factors through a finitely generated module, and hence $\Lambda((f\,|\,K)^*)$ is defined.

Let $\{\alpha_\lambda\}$ $\{\beta_\mu\}$ and $\{\rho_\nu\}$ be homogeneous bases of $H^*(M)$, $H^*(M, M-K)$ and $H^*(K)$ respectively, and put

$$f^*(\rho_\nu) = \sum_\lambda m_{\nu\lambda}\alpha_\lambda ,$$

$$U_K = \sum_{\nu,\mu} c_{\nu\mu}\rho_\nu \times \beta_\mu ,$$

$$\langle \beta_\mu \smile \alpha_\lambda, \mu_K\rangle = y_{\mu\lambda} .$$

Then it follows from the above commutative diagram that

$$\begin{aligned} (-1)^{n \deg \rho_\nu}(f\,|\,K)^*\rho_\nu &= U_K/(f^*\rho_\nu \frown \mu_K) \\ &= \sum_{\kappa,\mu} (c_{\kappa\mu}\rho_\kappa \times \beta_\mu)/(f^*\rho_\nu \frown \mu_K) \\ &= \sum_{\kappa,\mu} c_{\kappa\mu}\langle \beta_\mu , f^*\rho_\nu \frown \mu_K\rangle\rho_\kappa \\ &= \sum_{\kappa,\lambda,\mu} c_{\kappa\mu}m_{\nu\lambda}\langle \beta_\mu , \alpha_\lambda \frown \mu_K\rangle\rho_\kappa \\ &= \sum_{\kappa,\lambda,\mu} c_{\kappa\mu}m_{\nu\lambda}\langle \beta_\mu \smile \alpha_\lambda , \mu_K\rangle\rho_\kappa \\ &= \sum_{\kappa,\lambda,\mu} c_{\kappa\mu}m_{\nu\lambda}y_{\mu\lambda}\rho_\kappa . \end{aligned}$$

Therefore we have

$$\Delta((f\,|\,K)^*) = \sum_{\lambda,\mu,\nu} (-1)^{(n+1)\deg \rho_\nu} c_{\nu\mu}m_{\nu\lambda}y_{\mu\lambda} .$$

The diagram

$$\begin{array}{ccc} H^*(M\times M, M\times M-d(M)) & \xrightarrow{\hat{f}^*} & H^*(M, M-K) \\ \big\downarrow i^* & & \big\uparrow d^* \\ H^*(K\times(M, M-K)) & \xrightarrow{(f\times id)^*} & H^*(M\times(M, M-K)) \end{array}$$

is commutative, where i^* is the natural homomorphism. Therefore it follows from (3) that

$$\begin{aligned} I(f) &= \langle U, \hat{f}_*\mu_K\rangle = \langle \hat{f}^*U, \mu_K\rangle \\ &= \langle d^*(f\times id)^*U_K, \mu_K\rangle \\ &= \sum_{\mu,\nu} c_{\nu\mu}\langle d^*(f^*\rho_\nu \times \beta_\mu), \mu_K\rangle \end{aligned}$$

$$\begin{aligned}
&=\sum_{\mu,\nu} c_{\nu\mu}\langle f^*\rho_\nu \smile \beta_\mu,\ \mu_K\rangle \\
&=\sum_{\lambda,\mu,\nu} c_{\nu\mu}m_{\nu\lambda}\langle \alpha_\lambda \smile \beta_\mu,\ \mu_K\rangle \\
&=\sum_{\lambda,\mu,\nu} (-1)^{(n-1)\deg\rho_\nu} c_{\nu\mu}m_{\nu\lambda}\langle \beta_\mu \smile \alpha_\lambda,\ \mu_K\rangle \\
&=\sum_{\lambda,\mu,\nu} (-1)^{(n-1)\deg\rho_\nu} c_{\nu\mu}m_{\nu\lambda}y_{\mu\lambda}\,.
\end{aligned}$$

Consequently we obtain $I(f)=\Lambda((f|K)^*)$. Since $\Lambda((f|K)^*)=\Lambda((f|K)_*)$ is obvious, we have the desired result. q.e.d.

A topological manifold which is orientable (over $\boldsymbol{Z}$) is orientable over $\boldsymbol{Q}$, and $I(f)$ for $R=\boldsymbol{Z}$ coincides with $I(f)$ for $R=\boldsymbol{Q}$. Therefore Theorem D implies Theorem C.

Lemma 4 shows that $I(f)$ coincides with I_f due to Dold when $M=\boldsymbol{R}^n$. Therefore Theorem C implies Theorem A. It is clear that Theorem C implies Theorem B.

OSAKA UNIVERSITY

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