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Osaka University
NOTE ON THE LEFSCHETZ FIXED POINT THEOREM

Dedicated to Professor A. Komatu on his 60th birthday

MINORU NAKAOKA

(Received November 28, 1968)

1. Introduction

Let $V$ be an open set of the $n$-dimensional euclidean space $\mathbb{R}^n$, and $f: V \to \mathbb{R}^n$ be a continuous map such that the fixed point set $F = \{x \in V \mid f(x) = x\}$ is compact. If $i: V \subset \mathbb{R}^n$, then $i-f$ maps $(V, V-F)$ to $(\mathbb{R}^n, \mathbb{R}^n-0)$. Considering the homomorphism of the integral homology groups induced by $i-f$, A. Dold [2] defines the fixed point index $I_f \in \mathbb{Z}$ by

$$(i-f)^* \mu_0 = I_f \mu_0,$$

where $\mu_0 \in H_n(\mathbb{R}^n, \mathbb{R}^n-0; \mathbb{Z})$ is an orientation of $\mathbb{R}^n$ and $\mu_0 \in H_n(V, V-F; \mathbb{Z})$ is the 'fundamental' class corresponding to the orientation $\mu_0$. With this definition, he proves the following Lefschetz fixed point theorem:

**Theorem A.** Let $V$ be an open set of $\mathbb{R}^n$, and $f: V \to V$ be a continuous map such that $f(V)$ is contained in a compact set $K \subset V$. Then the fixed point index $I_f$ of $f$ and the Lefschetz number of $(f \mid K)^*: H_*(K; \mathbb{Q}) \to H_*(K; \mathbb{Q})$ are both defined and they agree, where $\mathbb{Q}$ is the field of rational numbers.

Precisely, he proves the theorem in which $V$ is replaced by a euclidean neighborhood retract $Y$. However this generalization follows directly from the above one, because he defines the fixed point index of $f: Y \to Y$ to be that of the composite $i \circ f \circ r: V \to V$, where $i: Y \to V$, $r: V \to Y (r \circ i = id)$ is a euclidean neighborhood retraction.

On the other hand, R. Brown [1] shows the Lefschetz fixed point theorem for a compact orientable $n$-dimensional topological manifold $M$ (see also [3]). Taking an orientation of $M$, let $\mu \in H_n(M; \mathbb{Z})$ and $U \in H^n(M \times M, M \times M - d(M); \mathbb{Z})$ denote the corresponding fundamental class and Thom class respectively, where $d(M)$ is the diagonal of $M \times M$. Denote by $U' \in H^n(M \times M; \mathbb{Z})$ the image of $U$ under the natural homomorphism. Then the theorem of Brown is as follows:

**Theorem B.** Let $M$ be a compact orientable $n$-dimensional topological
Let \( f : M \to M \) be a continuous map. Define \( \hat{f} : M \to M \times M \) by \( \hat{f}(x) = (f(x), x) \) for \( x \in M \). Then the Kronecker product \((\hat{f})^* U', \mu\) is equal to the Lefschetz number of \( f_* : H_*(M; \mathbb{Q}) \to H_*(M; \mathbb{Q}) \).

The purpose of this note is to prove a theorem which contains Theorem A and B as corollaries.

Let \( M \) be an orientable \( n \)-dimensional topological manifold which is not necessarily compact, and \( f : M \to M \) be a continuous map such that the fixed point set \( F \) of \( f \) is compact. Take an orientation of \( M \). Then the Thom class \( U \in H^n(M \times M, M \times M - d(M); \mathbb{Z}) \) and the fundamental class \( \mu_F \in H_*(M, M - F; \mathbb{Z}) \) are well-defined. Considering \( \hat{f} : (M, M - F) \to (M \times M, M \times M - d(M)) \), we define the fixed point index \( I(f) \) by

\[
I(f) = \langle U, \hat{f}_* \mu_F \rangle \in \mathbb{Z}.
\]

Then our theorem is stated as follows:

**Theorem C.** Let \( M \) be an orientable \( n \)-dimensional topological manifold, and \( f : M \to M \) be a continuous map such that \( f(M) \) is contained in a compact set \( K \subset M \). Then the fixed point index \( I(f) \) of \( f \) and the Lefschetz number of \((f|K)_* : H_*(K; \mathbb{Q}) \to H_*(K; \mathbb{Q})\) are both defined and they agree.

Our proof of this theorem is different from that of Theorem A due to Dold. Therefore this paper gives another proof of Theorem A.

The method we use to prove Theorem C is essentially the one due to J. Milnor [4] and is the one employed by Brown to prove Theorem B.

2. A fundamental lemma

Let \( M \) be an \( n \)-dimensional topological manifold, and \( d : M \to M \times M \) be the diagonal map. Let \( K \) be a compact subset of \( M \).

**Lemma 1.** There are an open neighborhood \( W \) of \( d(K) \) in \( K \times M \) and a retraction \( r : W \to d(K) \) such that the diagram

\[
\begin{array}{ccc}
K \times M & \to & K \\
\downarrow d(K) & & \downarrow l \\
W & \rightarrow & W \\
\uparrow r & & \uparrow r
\end{array}
\]

is homotopy commutative, where \( k \) and \( l \) are the inclusion maps.

**Proof.** For \( r > 0 \), let

\[
O_r = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1^2 + \cdots + x_n^2 < r \}.
\]

It is easily seen that there exists a finite set \( \{V_1, \ldots, V_s\} \) of coordinate neighbor-
hoods of $M$ such that

$$
\bigcup_{i=1}^s h_i^{-1}(0_i) \supseteq K,
$$

where $h_i : V_i \approx \mathbb{R}^n$ is a homeomorphism.

Put

$$
V'_i = h_i^{-1}(O_i), \quad V''_i = h_i^{-1}(O_2),
$$

$$
V' = \bigcup_{i=1}^s V'_i, \quad V'' = \bigcup_{i=1}^s V''_i.
$$

The space $V''/V'' - V''_i$ obtained from the closure $V''$ by identifying $V'' - V''_i$ to one point is homeomorphic with the $n$-sphere $S^n$. Therefore a homeomorphism $f$ of $V''$ into $S^n \times \cdots \times S^n$ ($s$ times) is defined by

$$
f(x) = (f_1 p_1(x), \ldots, f_s p_s(x)) \quad (x \in V''),
$$

where $p_1 : V'' \to V''/V'' - V''_i$ is the projection and $f_i : V''/V'' - V''_i \approx \mathbb{R}^n$ is a homeomorphism. Since $V' \subset V''$ and $S^n \times \cdots \times S^n \subset \mathbb{R}^m$ ($m=(n+1)s$), we can regard $V'$ as a closed subset of $\mathbb{R}^m$. Since each $V_i$ is an ANR, so is $V = \bigcup_{i=1}^s V_i$.

Consequently, the inclusion map $V' \subset V$ has an extension $g : Q \to V$, where $Q$ is a neighborhood of $V'$ in $\mathbb{R}^m$. It is obvious that there exists $\varepsilon > 0$ such that if $x, y \in V'$ and the distance from $x$ to $y$ in $\mathbb{R}^m$ is smaller than $\varepsilon$ then

$$
(1-t)x + ty \in Q \quad \text{for any } t \in [0, 1].
$$

Put

$$
W = \{(x, y) \in K \times V' | d(x, y) < \varepsilon\},
$$

and define $r : W \to d(K)$ by $r(x, y) = (x, x)$.

We can now define a homotopy $f_t : W \to K \times M$ of $k \circ r$ to $l$ by

$$
f_t(x, y) = (x, g((1-t)x + ty)) \quad \text{q.e.d.}
$$

Let $R$ be a fixed principal ideal domain, and we shall take coefficients of homology and cohomology from $R$. Consider the cup product

$$
\smile : H^*(K \times (M, M - K)) \otimes H^*(K \times M) \to H^*(K \times (M, M - K)).
$$

**Lemma 2.** For $\alpha \in H^*(M)$ and $\gamma \in H^*(K \times M, K \times M - d(K))$ we have

$$
j^* \gamma \circ p_1^* i^* \alpha = j^* \gamma \circ p_2^* \alpha,
$$

where $p_1 : K \times M \to K, p_2 : K \times M \to M$ are the projections and $i : K \to M, j : K \times (M, M - K) \to (K \times M, K \times M - d(K))$ are the inclusion maps.
Proof. By Lemma 1 and the naturality of the cup product, we have a commutative diagram

\[
\begin{array}{c}
H^*(d(K)) \\
\downarrow k^* \\
H^*(d(K)) \\
\end{array} \quad \begin{array}{c}
H^*(K \times M) \xrightarrow{\gamma} H^*(K \times M, K \times M - d(K)) \\
\downarrow l^* \\
H^*(W) \xrightarrow{l^*} H^*(W, W - d(K)).
\end{array}
\]

If we define \( p : d(K) \to K \) by \( p(x, x) = x (x \in K) \), then it holds that \( p_1 \circ k = p \) and \( p_2 \circ k = i \circ p \). Therefore it follows that

\[
l^*(\gamma \circ \rho_i \alpha) = l^* \gamma \circ r^* k^* \rho_i \alpha
\]

\[
= l^* \gamma \circ r^* p_1^* \rho_i \alpha = l^* \gamma \circ r^* k^* p_1^* \alpha
\]

\[
= l^* (\gamma \circ p_1^* \alpha).
\]

Since \( l^* : H^*(K \times M, K \times M - d(K)) \cong H^*(W, W - d(K)) \) is an excision isomorphism, we obtain

\[
\gamma \circ p_1^* i^* \alpha = \gamma \circ p_1^* \alpha.
\]

This, together with the naturality of the cup product, implies the desired result. q.e.d.

For topological pairs \((X, A)\) and \((Y, B)\), consider the slant product

\[
\lambda : H^*((X, A) \times (Y, B)) \otimes H_*(Y, B) \to H^*(X, A).
\]

The following relations hold between the cup, cap and slant products: For \( \gamma \in H^*((X, A) \times (Y, B)), \alpha \in H^*(X), \beta \in H^*(Y) \) and \( b \in H_*(Y, B) \), we have

\[
(1) \quad \alpha \cap (\gamma/b) = (p_1^* \alpha \cap \gamma)/b,
\]

\[
\gamma/(\beta \cap b) = (\gamma \cap p_2^* \beta)/b.
\]

in \( H^*(X, A) \), where \( p_1 : X \times Y \to X \) and \( p_2 : X \times Y \to Y \) are the projections (see [5]).

By an orientation \( \mu \) over \( R \) of an \( n \)-dimensional topological manifold \( M \) we mean a function which assigns to each \( x \in M \) a generator \( \mu_x \) of \( H_*(M, M - x) \) which "varies continuously" with \( x \), in the following sense. For each \( x \) there exist a neighborhood \( N \) and an element \( \mu_N \in H_*(M, M - N) \) such that the image of \( \mu_N \) in \( H_*(M, M - y) \) under the natural homomorphism is \( \mu_y \) for each \( y \in N \).

If an orientation over \( R \) of the manifold \( M \) exists, \( M \) is called orientable over \( R \).

Assume that \( M \) is orientable over \( R \) and an orientation \( \mu \) of \( M \) is given. Then it is known that, for each compact subset \( K \) of \( M \), there is a unique element \( \mu_K \in H_*(M, M - K) \) whose image in \( H_*(M, M - x) \) under the natural homomorphism is \( \mu_x \) for any \( x \in K \) (see [3]). It is also known that there exists a unique
element $U \in H^n(M \times M, M \times M - d(M))$ such that

$$\langle l^*_x U, \mu_x \rangle = 1$$

for any $x \in M$, where $l_x : (M, M - x) \to (M \times M, M \times M - d(M))$ is a continuous map sending $x' \in M$ to $(x, x') \in M \times M$ (see [3], [5]). Denote by $U_K \in H^n(K \times (M, M - K))$ the image of $U$ under the natural homomorphism.

A simple calculation shows

\[(2)\quad U_K/\mu_K = 1.\]

We shall now prove the following fundamental lemma.

**Lemma 3.** The diagram

\[
\begin{array}{ccc}
H^q(M) & \xrightarrow{(-1)^q i^*} & H^q(K) \\
\downarrow{\mu_K} & & \downarrow{U_K/|} \\
H_{n-q}(M, M-K) & & \\
\end{array}
\]

is commutative, where $i : K \subset M$.

**Proof.** For $\alpha \in H^q(M)$, we obtain by (1), (2) and Lemma 2

\[
U_K/(\alpha \smile \mu_K) = (U_K \smile p^*_K \alpha)/\mu_K \\
= (U_K \smile p^*_K i^* \alpha)/\mu_K = (-1)^q (p^*_K i^* \alpha \smile U_K)/\mu_K \\
= (-1)^q i^* \alpha \smile (U_K/\mu_K) = (-1)^q i^* \alpha,
\]

which proves the desired result. q.e.d.

3. **Lefschetz fixed point theorem**

Let $M$ be an $n$-dimensional topological manifold which is orientable over $R$. Let $V$ be an open set of $M$, and $K$ be a compact subset of $V$. Given an orientation $\mu$ of $M$, we shall denote by $\mu_K \in H_n(V, V - K)$ the element corresponding to $\mu_K$ under the excision isomorphism $H_n(V, V - K) \cong H_n(M, M - K)$.

If $f : V \to M$ is a continuous map such that the fixed point set $F$ is compact, then we call

$$I(f) = \langle U, f_\# \mu_{V} \rangle \in R$$

the **fixed point index** of $f$, where $f : (V, V - F) \to (M \times M, M \times M - d(M))$ is a continuous map given by $f(x) = (f(x), x)(x \in V)$. It follows that $I(f)$ is independent of the choice of orientation.

For a compact set $K$ such that $F \subset K \subset M$, we have
where \( f^* : H_n(V, V - K) \to H_n(M \times M, M \times M - d(M)) \). This follows from that \( \mu^Y_K \) is the image of \( \mu_K \) under the natural homomorphism.

**Lemma 4.** In the case \( M = \mathbb{R}^n \), we have

\[
(i - f)^* \mu^Y_K = I(f) \mu_0,
\]

where \( i - f : (V, V - F) \to (\mathbb{R}^n, \mathbb{R}^n - 0) \) is a continuous map sending \( x \in V \) to \( x - f(x) \in \mathbb{R}^n \).

**Proof.** Define \( \Delta : (\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^n - d(\mathbb{R}^n)) \to (\mathbb{R}^n, \mathbb{R}^n - 0) \) by \( \Delta(x, y) = x - y (x, y \in \mathbb{R}^n) \). Then, for \( l_0 : (\mathbb{R}^n, \mathbb{R}^n - 0) \to (\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^n - d(\mathbb{R}^n)) \), we have \( \Delta \circ l_0 = id \). Denote by \( \mu_0 \in H^n(\mathbb{R}^n, \mathbb{R}^n - 0) \) the dual to \( \mu \in H_n(\mathbb{R}^n, \mathbb{R}^n - 0) \). Since \( \langle l_0^* \Delta^* \mu_0, \mu_0 \rangle = 1 \), we have

\[
\Delta^* \mu_0 = U \in H^n(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^n - d(\mathbb{R}^n)).
\]

Since \( \Delta \circ f = i - f \), we obtain

\[
I(f) = \langle \Delta^* \mu_0, f^* \mu^Y_K \rangle = \langle \mu_0, (i - f)^* \mu^Y_K \rangle,
\]

which shows the desired result. q.e.d.

Let \( N \) be a graded module over a field \( K \), and \( \varphi : N \to N \) be an endomorphism of degree 0 which factors through a finitely generated graded module. Taking a homogeneous basis \( \{ a_\lambda \} \) of \( N \), put

\[
\varphi(a_\lambda) = \sum_\mu r_{\lambda \mu} a_\mu \quad (r_{\lambda \mu} \in K).
\]

Then it follows that \( r_{\lambda \lambda} \) is zero except a finite number of \( \lambda \), and that

\[
\Lambda(\varphi) = \sum_\lambda (-1)^{\deg a_\lambda} r_{\lambda \lambda} \in \mathbb{R}
\]

is independent of the choice of \( \{ a_\lambda \} \) (see [2]). \( \Lambda(\phi) \) is called the Lefschetz number of \( \phi \).

**Theorem D.** Let \( M \) be an \( n \)-dimensional topological manifold which is orientable over a field \( K \), and let \( f : M \to M \) be a continuous map such that \( f(M) \) is contained in a compact set \( K \subset M \). Then the fixed point index \( I(f) \) of \( f \) and the Lefschetz number \( \Lambda((f | K)_*) \) of the homomorphism \( (f | K)_* : H_*(K) \to H_*(K) \) of homology with coefficients in \( K \) are both defined and they agree.

**Proof.** The fixed point set \( F \) of \( f \) is a closed subset of \( K \), and hence is compact. Therefore \( I(f) \) is defined.

From Lemma 3 it follows that the diagram

\[
\begin{array}{ccc}
U & \xrightarrow{f^*} & U \\
\downarrow & & \downarrow \\
H_*(M \times M, M \times M - d(M)) & \xrightarrow{\mu^Y_K} & H_*(M, M - d(M))
\end{array}
\]
is commutative. It is obvious from the definition of the cap product that the image of the homomorphism $\sim \mu_K$ is finitely generated. Therefore $(f|K)^*$ factors through a finitely generated module, and hence $\Lambda((f|K)^*)$ is defined.

Let $\{\alpha_\lambda\}, \{\beta_\mu\}$ and $\{\rho_\nu\}$ be homogeneous bases of $H^*(M)$, $H^*(M, M-K)$ and $H^*(K)$ respectively, and put

$$f^*(\rho_\nu) = \sum \lambda m_{\lambda \nu} \alpha_\lambda,$$

$$U_K = \sum_{\nu, \mu} c_{\nu \mu} \rho_\nu \times \beta_\mu,$$

$$\langle \beta_\mu \sim \alpha_\lambda, \mu_K \rangle = y_{\mu \lambda}.$$

Then it follows from the above commutative diagram that

$$(-1)^{\deg \rho_\nu}(f|K)^* \rho_\nu = U_K/(f^* \rho_\nu \sim \mu_K)
= \sum_{\nu, \mu} (c_{\nu \mu} \rho_\nu \times \beta_\mu)(f^* \rho_\nu \sim \mu_K)
= \sum_{\nu, \mu} c_{\nu \mu} \langle \beta_\mu, f^* \rho_\nu \sim \mu_K \rangle \rho_\nu
= \sum_{\nu, \mu} c_{\nu \mu} m_{\nu \lambda} \langle \beta_\mu, \alpha_\lambda \sim \mu_K \rangle \rho_\nu
= \sum_{\nu, \mu} c_{\nu \mu} m_{\nu \lambda} \langle \beta_\mu \sim \alpha_\lambda, \mu_K \rangle \rho_\nu
= \sum_{\nu, \mu} c_{\nu \mu} m_{\nu \lambda} y_{\mu \lambda} \rho_\nu.$$

Therefore we have

$$\Delta((f|K)^*) = \sum_{\lambda, \mu} (-1)^{\deg \rho_\nu} c_{\nu \mu} m_{\nu \lambda} y_{\mu \lambda}.$$

The diagram

$$H^*(M \times M, M \times M - d(M)) \xrightarrow{f^*} H^*(M, M-K) \xrightarrow{i^*} H^*(K \times (M, M-K)) \xrightarrow{(f \times id)^*} H^*(M \times (M, M-K))$$

is commutative, where $i^*$ is the natural homomorphism. Therefore it follows from (3) that

$$I(f) = \langle U, f^* \mu_K \rangle = \langle U, \mu_K \rangle$$

$$= \langle d^* (f \times id)^* U_K, \mu_K \rangle$$

$$= \sum_{\mu} c_{\nu \mu} \langle d^* (f^* \rho_\nu \times \beta_\mu), \mu_K \rangle.$$
\[= \sum_{p,v} c_{p,v} \langle f^* p_v \circ \beta_{\mu}, \mu_K \rangle\]
\[= \sum_{\lambda, p,v} c_{p,v} m_{v,\lambda} \langle \alpha_{\lambda} \circ \beta_{\mu}, \mu_K \rangle\]
\[= \sum_{\lambda, p,v} (-1)^{(n-1) \deg p_v} c_{p,v} m_{v,\lambda} \langle \beta_{\mu} \circ \alpha_{\lambda}, \mu_K \rangle\]
\[= \sum_{\lambda, p,v} (-1)^{(n-1) \deg p_v} c_{p,v} m_{v,\lambda} y_{\mu,\lambda}.\]

Consequently we obtain \(I(f) = \Lambda((f|K)^*)\). Since \(\Lambda((f|K)^*) = \Lambda((f|K)_*)\) is obvious, we have the desired result. q.e.d.

A topological manifold which is orientable (over \(\mathbb{Z}\)) is orientable over \(\mathbb{Q}\), and \(I(f)\) for \(R=\mathbb{Z}\) coincides with \(I(f)\) for \(R=\mathbb{Q}\). Therefore Theorem D implies Theorem C.

Lemma 4 shows that \(I(f)\) coincides with \(I_f\) due to Dold when \(M=\mathbb{R}^n\). Therefore Theorem C implies Theorem A. It is clear that Theorem C implies Theorem B.

**Bibliography**


