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NOTE ON THE LEFSCHETZ FIXED POINT THEOREM

Dedicated to Professor A. Komatu on his 60th birthday

MINORU NAKAOKA

(Received November 28, 1968)

1. Introduction

Let $V$ be an open set of the $n$-dimensional euclidean space $\mathbb{R}^n$, and $f: V \to \mathbb{R}^n$ be a continuous map such that the fixed point set $F = \{x \in V \mid f(x) = x\}$ is compact. If $i: V \subseteq \mathbb{R}^n$, then $i - f$ maps $(V, V - F)$ to $(\mathbb{R}^n, \mathbb{R}^n - 0)$. Considering the homomorphism of the integral homology groups induced by $i - f$, A. Dold [2] defines the fixed point index $I_f \in \mathbb{Z}$ by

$$(i - f)_* \mu_0^V = I_f \mu_0,$$

where $\mu_0 \in H_n(\mathbb{R}^n, \mathbb{R}^n - 0; \mathbb{Z})$ is an orientation of $\mathbb{R}^n$ and $\mu_0^V \in H_n(V, V - F; \mathbb{Z})$ is the 'fundamental' class corresponding to the orientation $\mu_0$. With this definition, he proves the following Lefschetz fixed point theorem:

**Theorem A.** Let $V$ be an open set of $\mathbb{R}^n$, and $f: V \to V$ be a continuous map such that $f(V)$ is contained in a compact set $K \subseteq V$. Then the fixed point index $I_f$ of $f$ and the Lefschetz number of $(f|_K)_*: H_*(K; \mathbb{Q}) \to H_*(K; \mathbb{Q})$ are both defined and they agree, where $\mathbb{Q}$ is the field of rational numbers.

Precisely, he proves the theorem in which $V$ is replaced by a euclidean neighborhood retract $Y$. However this generalization follows directly from the above one, because he defines the fixed point index of $f: Y \to Y$ to be that of the composite $i \circ f \circ r: V \to V$, where $i: Y \to V$, $r: V \to Y (r \circ i = id)$ is a euclidean neighborhood retraction.

On the other hand, R. Brown [1] shows the Lefschetz fixed point theorem for a compact orientable $n$-dimensional topological manifold $M$ (see also [3]). Taking an orientation of $M$, let $\mu \in H_*(M; \mathbb{Z})$ and $U \in H^n(M \times M, M \times M - d(M); \mathbb{Z})$ denote the corresponding fundamental class and Thom class respectively, where $d(M)$ is the diagonal of $M \times M$. Denote by $U' \in H^*(M \times M; \mathbb{Z})$ the image of $U$ under the natural homomorphism. Then the theorem of Brown is as follows:

**Theorem B.** Let $M$ be a compact orientable $n$-dimensional topological
manifold, and \( f : M \to M \) be a continuous map. Define \( f : M \to M \times M \) by \( f(x) = (f(x), x) \) for \( x \in M \). Then the Kronecker product \( \langle f^* U', \mu \rangle \) is equal to the Lefschetz number of \( f_* : H_*(M; \mathbb{Q}) \to H_*(M; \mathbb{Q}) \).

The purpose of this note is to prove a theorem which contains Theorem A and B as corollaries.

Let \( M \) be an orientable \( n \)-dimensional topological manifold which is not necessarily compact, and \( f : M \to M \) be a continuous map such that the fixed point set \( F \) of \( f \) is compact. Take an orientation of \( M \). Then the Thom class \( U \in H^n(M \times M, M \times M - d(M); \mathbb{Z}) \) and the fundamental class \( \mu_F \in H_*(M, M - F; \mathbb{Z}) \) are well-defined. Considering \( \hat{f} : (M, M - F) \to (M \times M, M \times M - d(M)) \), we define the fixed point index \( I(f) \) by

\[
I(f) = \langle U, \hat{f}^* \mu_F \rangle \in \mathbb{Z}.
\]

Then our theorem is stated as follows:

**Theorem C.** Let \( M \) be an orientable \( n \)-dimensional topological manifold, and \( f : M \to M \) be a continuous map such that \( f(M) \) is contained in a compact set \( K \subset M \). Then the fixed point index \( I(f) \) of \( f \) and the Lefschetz number of \((f | K)_* : H_*(K; \mathbb{Q}) \to H_*(K; \mathbb{Q})\) are both defined and they agree.

Our proof of this theorem is different from that of Theorem A due to Dold. Therefore this paper gives another proof of Theorem A.

The method we use to prove Theorem C is essentially the one due to J. Milnor [4] and is the one employed by Brown to prove Theorem B.

2. A fundamental lemma

Let \( M \) be an \( n \)-dimensional topological manifold, and \( d : M \to M \times M \) be the diagonal map. Let \( K \) be a compact subset of \( M \).

**Lemma 1.** There are an open neighborhood \( W \) of \( d(K) \) in \( K \times M \) and a retraction \( r : W \to d(K) \) such that the diagram

\[
\begin{array}{ccc}
K \times M & \xrightarrow{k} & K \\
d(K) & \xleftarrow{r} & W \\
\end{array}
\]

is homotopy commutative, where \( k \) and \( l \) are the inclusion maps.

**Proof.** For \( r > 0 \), let

\[
O_r = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1^2 + \cdots + x_n^2 < r \}.
\]

It is easily seen that there exists a finite set \( \{V_1, \ldots, V_s\} \) of coordinate neighbor-
hoods of $M$ such that
\[ \bigcup_{i=1}^s h_i^{-1}(O_i) \supseteq K, \]
where $h_i : V_i \cong \mathbb{R}^n$ is a homeomorphism.

Put
\[ V'_i = h_i^{-1}(O_i), \quad V''_i = h_i^{-1}(O_2), \]
\[ V' = \bigcup_{i=1}^s V'_i, \quad V'' = \bigcup_{i=1}^s V''_i. \]

The space $V''/V'' - V''_i$ obtained from the closure $V''$ by identifying $V'' - V''_i$ to one point is homeomorphic with the $n$-sphere $S^n$. Therefore a homeomorphism $f$ of $V''$ into $S^n \times \cdots \times S^n$ ($s$ times) is defined by
\[ f(x) = (f_1 p_1(x), \ldots, f_s p_s(x)) \quad (x \in V''). \]
where $p_i : V'' \to V''/V'' - V''_i$ is the projection and $f_i : V''/V'' - V''_i \cong S^n$ is a homeomorphism. Since $V' \subseteq V''$ and $S^n \times \cdots \times S^n \subseteq \mathbb{R}^m (m = (n+1)s)$, we can regard $V'$ as a closed subset of $\mathbb{R}^m$. Since each $V_i$ is an ANR, so is $V = \bigcup_{i=1}^s V_i$.

Consequently, the inclusion map $V' \subseteq V$ has an extension $g : Q \to V$, where $Q$ is a neighborhood of $V'$ in $\mathbb{R}^m$. It is obvious that there exists $\varepsilon > 0$ such that if $x, y \in V'$ and the distance from $x$ to $y$ in $\mathbb{R}^m$ is smaller than $\varepsilon$ then $(1-t)x + ty \in Q$ for any $t \in [0, 1]$. Put
\[ W = \{ (x, y) \in K \times V' \mid d(x, y) < \varepsilon \}, \]
and define $r : W \to d(K)$ by $r(x, y) = (x, x)$.

We can now define a homotopy $f_t : W \to K \times M$ of $k \circ r$ to $l$ by
\[ f_t(x, y) = (x, g((1-t)x + ty)) \quad \text{q.e.d.} \]

Let $R$ be a fixed principal ideal domain, and we shall take coefficients of homology and cohomology from $R$. Consider the cup product
\[ \cup : H^*(K \times (M, M-K)) \otimes H^*(K \times M) \to H^*(K \times (M, M-K)). \]

**Lemma 2.** For $\alpha \in H^*(M)$ and $\gamma \in H^*(K \times (M, K \times M - d(K))$ we have
\[ j^* \gamma \cup p_1^* i^* \alpha = j^* \gamma \cup p_2^* \alpha, \]
where $p_1 : K \times M \to K, p_2 : K \times M \to M$ are the projections and $i : K \to M, j : K \times (M, M-K) \to (K \times M, K \times M - d(K))$ are the inclusion maps.
Proof. By Lemma 1 and the naturality of the cup product, we have a commutative diagram

\[
\begin{array}{cccc}
H^*(d(K)) & \xrightarrow{k^*} & H^*(K \times M) & \xrightarrow{\gamma} & H^*(K \times M, K \times M - d(K)) \\
\downarrow{l^*} & & \uparrow{l^*} & & \uparrow{l^*} \\
H^*(W) & \xrightarrow{l^*} & H^*(W, W - d(K)) & \xrightarrow{l^*} & H^*(W, W - d(K))
\end{array}
\]

If we define \( p : d(K) \to K \) by \( p(x, x) = x \ (x \in K) \), then it holds that \( p_1 \circ k = p \) and \( p_2 \circ k = i \circ p \). Therefore it follows that

\[
l^*(\gamma \circ p_1^* \alpha) = l^* \gamma \circ r^* k^* p_1^* \alpha = l^* \gamma \circ r^* k^* p_2^* \alpha \]

Since \( l^* : H^*(K \times M, K \times M - d(K)) \cong H^*(W, W - d(K)) \) is an excision isomorphism, we obtain

\[
\gamma \circ p_2^* \alpha = \gamma \circ p_2^* \alpha .
\]

This, together with the naturality of the cup product, implies the desired result. q.e.d.

For topological pairs \((X, A)\) and \((Y, B)\), consider the slant product

\[
/ : H^*((X, A) \times (Y, B)) \otimes H_*(Y, B) \to H_*(X, A) .
\]

The following relations hold between the cup, cap and slant products: For \( \gamma \in H^*((X, A) \times (Y, B)), \alpha \in H_*(X), \beta \in H_*(Y) \) and \( b \in H_*(Y, B) \), we have

\[
(1) \quad \alpha \sim (\gamma/b) = (p_1^* \alpha \sim \gamma)/b ,
\]

\[
(\gamma/\beta \sim b) = (\gamma \sim p_2^* \beta)/b
\]

in \( H^*(X, A) \), where \( p_1 : X \times Y \to X \) and \( p_2 : X \times Y \to Y \) are the projections (see [5]).

By an orientation \( \mu \) over \( R \) of an \( n \)-dimensional topological manifold \( M \) we mean a function which assigns to each \( x \in M \) a generator \( \mu_x \) of \( H_n(M, M - x) \) which “varies continuously” with \( x \), in the following sense. For each \( x \) there exist a neighborhood \( N \) and an element \( \mu_N \in H_n(M, M - N) \) such that the image of \( \mu_N \) in \( H_n(M, M - y) \) under the natural homomorphism is \( \mu_y \) for each \( y \in N \).

If an orientation over \( R \) of the manifold \( M \) exists, \( M \) is called orientable over \( R \).

Assume that \( M \) is orientable over \( R \) and an orientation \( \mu \) of \( M \) is given. Then it is known that, for each compact subset \( K \) of \( M \), there is a unique element \( \mu_K \in H_n(M, M - K) \) whose image in \( H_n(M, M - x) \) under the natural homomorphism is \( \mu_x \) for any \( x \in K \) (see [3]). It is also known that there exists a unique
element \( U \in H^*(M \times M, M \times M - d(M)) \) such that

\[ \langle l^*_x U, \mu_x \rangle = 1 \]

for any \( x \in M \), where \( l_x : (M, M - x) \to (M \times M, M \times M - d(M)) \) is a continuous map sending \( x' \in M \) to \((x, x') \in M \times M\) (see [3], [5]). Denote by \( U_K \in H^*(K \times (M, M - K)) \) the image of \( U \) under the natural homomorphism.

A simple calculation shows

\[ (2) \quad U_K|\mu_K = 1. \]

We shall now prove the following fundamental lemma.

**Lemma 3.** The diagram

\[
\begin{array}{ccc}
H^q(M) & \xrightarrow{(-1)^q i^*} & H^q(K) \\
\downarrow \mu_K & & \downarrow U_K| \\
H_{n-q}(M, M-K) & & 
\end{array}
\]

is commutative, where \( i : K \subset M \).

**Proof.** For \( \alpha \in H^q(M) \), we obtain by (1), (2) and Lemma 2

\[
U_K|((\alpha \lhd \mu_K) = (U_K \circ p^*_K \alpha)|\mu_K \\
= (U_K \circ p^*_K i^* \alpha)|\mu_K = (-1)^q (p^*_K i^* \alpha \lhd U_K)|\mu_K \\
= (-1)^q i^* \alpha \lhd (U_K|\mu_K) = (-1)^q i^* \alpha ,
\]

which proves the desired result. q.e.d.

3. **Lefschetz fixed point theorem**

Let \( M \) be an \( n \)-dimensional topological manifold which is orientable over \( R \). Let \( V \) be an open set of \( M \), and \( K \) be a compact subset of \( V \). Given an orientation \( \mu \) of \( M \), we shall denote by \( \mu_K \in H_*(V, V - K) \) the element corresponding to \( \mu_K \) under the excision isomorphism \( H_*(V, V - K) \cong H_*(M, M - K) \).

If \( f : V \to M \) is a continuous map such that the fixed point set \( F \) is compact, then we call

\[ I(f) = \langle U, f_* \mu \rangle \in R \]

the **fixed point index** of \( f \), where \( f : (V, V - F) \to (M \times M, M \times M - d(M)) \) is a continuous map given by \( f(x) = (f(x), x) (x \in V) \). It follows that \( I(f) \) is independent of the choice of orientation.

For a compact set \( K \) such that \( F \subset K \subset M \), we have
where $f_* : H_*(V, V-K) \to H_*(M \times M, M \times M - d(M))$. This follows from that $\mu_F^V$ is the image of $\mu_F^V$ under the natural homomorphism.

**Lemma 4.** In the case $M = \mathbb{R}^n$, we have

$$(i-f)_* \mu_F^V = I(f) \mu_0,$$

where $i : (V, V-F) \to (\mathbb{R}^n, \mathbb{R}^n - 0)$ is a continuous map sending $x \in V$ to $x - f(x) \in \mathbb{R}^n$.

Proof. Define $\Delta : (\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^n - d(\mathbb{R}^n)) \to (\mathbb{R}^n, \mathbb{R}^n - 0)$ by $\Delta(x, y) = y - x$ ($x, y \in \mathbb{R}^n$). Then, for $l_o : (\mathbb{R}^n, \mathbb{R}^n - 0) \to (\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^n - d(\mathbb{R}^n))$, we have $\Delta \circ l_o = id$. Denote by $\mu_\phi \in H^*(\mathbb{R}^n, \mathbb{R}^n - 0)$ the dual to $\mu_0 \in H_*(\mathbb{R}^n, \mathbb{R}^n - 0)$. Since $\langle l_o^* \Delta^* \mu_\phi, \mu_0 \rangle = 1$, we have

$$\Delta^* \mu_\phi = U \in H^*(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^n - d(\mathbb{R}^n)).$$

Since $\Delta \circ f = i - f$, we obtain

$$I(f) = \langle \Delta^* \mu_\phi, f^* \mu_F^V \rangle = \langle \mu_\phi, (i-f)_* \mu_F^V \rangle,$$

which shows the desired result. q.e.d.

Let $N$ be a graded module over a field $R$, and $\varphi : N \to N$ be an endomorphism of degree 0 which factors through a finitely generated graded module. Taking a homogeneous basis $\{a_\lambda\}$ of $N$, put

$$\varphi(a_\lambda) = \sum \mu r_{\lambda \mu} a_\mu \quad (r_{\lambda \mu} \in R).$$

Then it follows that $r_{\lambda \lambda}$ is zero except a finite number of $\lambda$, and that

$$\Lambda(\varphi) = \sum \lambda (-1)^{deg a_\lambda} r_{\lambda \lambda} \in R$$

is independent of the choice of $\{a_\lambda\}$ (see [2]). $\Lambda(\varphi)$ is called the Lefschetz number of $\varphi$.

**Theorem D.** Let $M$ be an $n$-dimensional topological manifold which is orientable over a field $R$, and let $f : M \to M$ be a continuous map such that $f(M)$ is contained in a compact set $K \subseteq M$. Then the fixed point index $I(f)$ of $f$ and the Lefschetz number $\Lambda((f | K)_*)$ of the homomorphism $(f | K)_* : H_*(K) \to H_*(K)$ of homology with coefficients in $R$ are both defined and they agree.

Proof. The fixed point set $F$ of $f$ is a closed subset of $K$, and hence is compact. Therefore $I(f)$ is defined.

From Lemma 3 it follows that the diagram
is commutative. It is obvious from the definition of the cap product that the
image of the homomorphism $\cup$ is finitely generated. Therefore $(f[K])^*$
factors through a finitely generated module, and hence $\Lambda((f[K])^*)$ is defined.

Let $\{\alpha_\lambda\}$, $\{\beta_\mu\}$ and $\{\rho_\nu\}$ be homogeneous bases of $H^*(M)$, $H^*(M, M-K)$
and $H^*(K)$ respectively, and put

$$ f^*(\rho_\nu) = \sum_\lambda m_{\nu\lambda} \alpha_\lambda, $$

$$ U_K = \sum_{\nu, \mu} c_{\nu\mu} \rho_\nu \times \beta_\mu, $$

$$ \langle \beta_\mu \cup \alpha_\lambda, \mu_K \rangle = y_{\nu\lambda}. $$

Then it follows from the above commutative diagram that

$$ (-1)^{n \deg p_v} (f[K])^* \rho_\nu = U_K/(f^* \rho_\nu \cup \mu_K) $$

$$ = \sum_{\nu, \mu} (c_{\nu\mu} \rho_\nu \times \beta_\mu) (f^* \rho_\nu \cup \mu_K) $$

$$ = \sum_{\nu, \mu} c_{\nu\mu} \langle \beta_\mu, f^* \rho_\nu \cup \mu_K \rangle \rho_\nu $$

$$ = \sum_{\nu, \mu} c_{\nu\mu} m_{\nu\lambda} \langle \beta_\mu \cup \alpha_\lambda, \mu_K \rangle \rho_\nu $$

$$ = \sum_{\nu, \lambda} c_{\nu\mu} m_{\nu\lambda} y_{\nu\lambda} \rho_\nu. $$

Therefore we have

$$ \Delta((f[K])^*) = \sum_{\lambda, \mu, \nu} (-1)^{n+1} \deg p_v c_{\nu\mu} m_{\nu\lambda} y_{\nu\lambda}. $$

The diagram

$$ H^*(M \times M, M \times M-d(M)) \xrightarrow{f^*} H^*(M, M-K) $$

$$ H^*(K \times (M, M-K)) \xrightarrow{(f \times id)^*} H^*(M \times (M, M-K)) $$

is commutative, where $i^*$ is the natural homomorphism. Therefore it follows
from (3) that

$$ I(f) = \langle U, f^* \mu_K \rangle = \langle f^* U, \mu_K \rangle $$

$$ = \langle d^* (f \times \text{id})^* U_K, \mu_K \rangle $$

$$ = \sum_{\mu} c_{\nu\mu} \langle d^* (f^* \rho_\nu \times \beta_\mu), \mu_K \rangle. $$
= ∑ c_{νμ} f^{*} ρ_{ν} \circ β_{μ}, \mu_{k}
= ∑ c_{νμ} m_{νλ} \langle α_{λ} \circ β_{μ}, \mu_{k}\rangle
= ∑ (-1)^{(n-1) deg} c_{νμ} m_{νλ} \langle β_{μ} \circ α_{λ}, \mu_{k}\rangle
= ∑ (-1)^{(n-1) deg} c_{νμ} m_{νλ} y_{μλ}.

Consequently we obtain \( I(f) = \Lambda((f \mid K)^{*}) \). Since \( \Lambda((f \mid K)^{*}) = \Lambda((f \mid K)_{\mu}) \) is obvious, we have the desired result. q.e.d.

A topological manifold which is orientable (over \( \mathbb{Z} \)) is orientable over \( \mathbb{Q} \), and \( I(f) \) for \( R=\mathbb{Z} \) coincides with \( I(f) \) for \( R=\mathbb{Q} \). Therefore Theorem D implies Theorem C.

Lemma 4 shows that \( I(f) \) coincides with \( I_{f} \) due to Dold when \( M=\mathbb{R}^{n} \). Therefore Theorem C implies Theorem A. It is clear that Theorem C implies Theorem B.

OSAKA UNIVERSITY

Bibliography