| | |
|---|---|
| Title | Generically rational polynomials |
| Author(s) | Sugie, Tohru; Miyanishi, Masayoshi |
| Citation | Osaka Journal of Mathematics. 1980, 17(2), p. 339-362 |
| Version Type | VoR |
| URL | https://doi.org/10.18910/4925 |
| rights | |
| Note | |

Miyanishi, M. and Sugie, T.
**Osaka J. Math.**
**17** (1980), 339-362

# GENERICALLY RATIONAL POLYNOMIALS

Dedicated to Professor Y. Nakai on his sixtieth birthday

MASAYOSHI MIYANISHI*) AND TOHRU SUGIE

(Received February 13, 1979)

**Introduction.** Let $k$ be an algebraically closed field of characteristic zero. Let $k[x,y]$ be a polynomial ring of two variables and let $\boldsymbol{A}_k^2=\mathrm{Spec}(k[x,y])$. Embed $\boldsymbol{A}_k^2$ into the projective plane $\boldsymbol{P}_k^2$ as the complement of a line $l_\infty$. Let $f\in k[x,y]$ be an irreducible polynomial, let $F_\alpha$ be the curve on $\boldsymbol{A}_k^2$ defined by $f=\alpha$ for every $\alpha\in k$ and let $C_\alpha$ be the closure of $F_\alpha$ in $\boldsymbol{P}_k^2$. Then the set $\Lambda(f):=\{C_\alpha;\ \alpha\in k\cup(\infty)\}$ is a linear pencil on $\boldsymbol{P}_k^2$ defined by $f$, where $C_\infty=dl_\infty$, $d$ being the degree of $f$. The set $\Lambda_0(f):=\{F_\alpha;\ \alpha\in k\}$ is called *the linear pencil on $\boldsymbol{A}_k^2$ defined by $f$*. The polynomial $f$ is called *generically rational* when the general members of $\Lambda(f)$ (or $\Lambda_0(f)$) are irreducible rational curves. Since the algebraic function field $k(x,y)$ of one variable over the subfield $k(f)$ then has genus 0, Tsen's theorem says that $f$ is generically rational if and only if $f$ is a field generator in the sense of Russell [9, 10], i.e., there is an element $g\in k(x,y)$ such that $k(x,y)=k(f,g)$. If $f$ is a generically rational polynomial, we can associate with $f$ a non-negative integer $n$, where $n+1$ is the number of places at infinity of a general member $F_\alpha$ of $\Lambda_0(f)$, i.e., the number of places of $F_\alpha$ whose centers lie outside $\boldsymbol{A}_k^2$.

If $d\geqq 1$, the pencil $\Lambda(f)$ has base points situated outside $\boldsymbol{A}_k^2$. Let $\varphi\colon W\rightarrow\boldsymbol{P}_k^2$ be the shortest succession of quadratic transformations with centers at the base points (including infinitely near base points) of $\Lambda(f)$ such that the proper transform $\Lambda'$ of $\Lambda(f)$ by $\varphi$ has no base points. Then the linear pencil $\Lambda'$ defines a morphism $\rho\colon W\rightarrow\boldsymbol{P}_k^1$, whose general fibers are the proper transforms of general members of $\Lambda(f)$; thence they are nonsingular rational curves by virtue of Bertini's theorem. Moreover, $W$ contains in a canonical way an open subset isomorphic to $\boldsymbol{A}_k^2$. A generically rational polynomial $f$ is said to be *of simple type* if the morphism $\rho$ has $n+1$ cross-sections contained in the boundary set $W-\boldsymbol{A}_k^2$ (cf. Definition 1.8, below).

If $n=0$, a generically rational polynomial $f$ is sent to one of the coordinates $x$, $y$ of $\boldsymbol{A}_k^2$ by a biregular automorphism of $\boldsymbol{A}_k^2$ (cf. Abhyankar-Moh's theorem [1, 4]); hence $f$ is of simple type. If $n=1$, a generically rational polynomial is always of simple type (cf. Theorem 2.3, below). However, if $n>1$, a generi-

*) Supported by Grant-in-Aid for Scientific Research

caily rational polynomial is not necessarily of simple type, as clarified by Saito [13]. In effect, Saito determined in [12, 13] the standard forms of generically rational polynomials for $n=1$, 2 by analytic methods.

One of the purposes of the present article is to determine the standard forms of generically rational polynomials of simple type for arbitrary $n\geqq 0$ by means of purely algebraic methods (cf. the results in §§2, 3).

The notations which we use frequently in this article are the following:

Let $R$ be a $k$-algebra. Then $R^*$ is the multiplicative group consisting of invertible elements of $R$; $R^*$ is called the unit group of $R$; if $R$ is a finitely generated, normal $k$-algebra then $R^*/k^*$ is a finitely generated abelian group (cf. [3]). For elements $x, y$ of $R^*$, we denote $x\sim y$ if $xy^{-1}\in k^*$.

Let $V$ be a nonsingular projective surface. Let $D_1$, $D_2$ be divisors on $V$. Then $(D_1\cdot D_2)$ is the intersection number (or multiplicity) of $D_1$, $D_2$ on $V$. Let $C$ be a curve on $V$ and let $P$ be a point on $C$. Then $\text{mult}_P C$ is the multiplicity of $C$ at $P$. If $D_1$ and $D_2$ are divisors, locally effective at $P$, then $i(D_1, D_2; P)$ is the local intersection multiplicity of $D_1$, $D_2$ at $P$. For an effective divisor $D$, we denote by $|D|$ the underlying reduced curve.(*)

Let $\varphi: W\rightarrow V$ be a birational morphism of nonsingular rational surfaces. If $D$ is a divisor on $V$ then $\varphi^*(D)$ (or $\varphi'(D)$, resp.) denotes the total transform (or the proper transform, resp.) of $D$ by $\varphi$. Similarly, if $\Lambda$ is a linear pencil on $V$ then $\varphi'(\Lambda)$ denotes the proper transform of $\Lambda$ by $\varphi$.

Let $\rho: V\rightarrow B$ be a surjective morphism from a nonsingular projective surface onto a nonsingular curve, whose general fibers are nonsingular rational curves; we call $\rho$ a ***P***$^1$*-fibration*. An irreducible curve $\Gamma$ on $V$ is called a *crcss-section* (or *quasi-section*, resp.) if $(\Gamma\cdot C)=1$ (or $(\Gamma\cdot C)\geqq 1$, resp.) for a general fiber $C$ of $\rho$.

The ground field $k$ is always assumed to be an algebraically closed field of characteristic zero. The affine space (or the projective space, resp.) of dimension $n$ defined over $k$ is denoted by $\boldsymbol{A}^n_k$ (or $\boldsymbol{P}^n_k$, resp.).

## 1. Standard compactifications of $\boldsymbol{A}^2_k$

**1.1. Lemma** (cf. Gizatullin [2]). *Let $\varphi: V\rightarrow B$ be a surjective morphism from a nonsingular projective surface $V$ onto a nonsingular complete curve $B$ such that almost all fibers are isomorphic to $\boldsymbol{P}^1_k$. Let $F=n_1C_1+\cdots+n_rC_r$ be a singular fiber of $\varphi$, where $C_i$ is an irreducible curve, $C_i\neq C_j$ if $i\neq j$, and $n_i>0$. Then we have*:

(1) *The greatest common divisor $(n_1, \cdots, n_r)$ of $n_1, \cdots, n_r$ is* 1; $\text{Supp}(F)=\bigcup_{i=1}^{r} C_i$ *is connected.*

(*) The readers are warned of not confusing $|D|$ with the complete linear system determined by $D$. In the present article, we do not use the symbol $|D|$ to signify the latter meaning.

(2) *For* $1 \leqq i \leqq r$, $C_i$ *is isomorphic to* $\boldsymbol{P}_k^1$ *and* $(C_i^2)<0$.

(3) *For* $i \neq j$, $(C_i \cdot C_j)=0$ *or* 1.

(4) *For three distinct indices* $i, j$ *and* $l$, $C_i \cap C_j \cap C_l = \phi$.

(5) *One of* $C_i$*'s, say* $C_1$, *is an exceptional component, i.e., an exceptional curve of the first kind. If* $\tau: V \to V_1$ *is the contraction of* $C_1$, *then* $\varphi$ *factors as* $\varphi: V \xrightarrow{\tau} V_1 \xrightarrow{\varphi_1} B$, *where* $\varphi_1: V_1 \to B$ *is a fibration by* $\boldsymbol{P}^1$.

(6) *If one of* $n_i$*'s, say* $n_1$, *equals* 1 *then there is an exceptional component among* $C_i$*'s with* $2 \leqq i \leqq r$.

**1.2.** A generalization of Lemma 1.1 is the following

**Lemma.** *Let* $V$ *be a nonsingular projective surface and let* $\Lambda$ *be an irreducible linear pencil on* $V$ *such that general members of* $\Lambda$ *are rational curves. Let* $\mathfrak{B}$ *be the set of points of* $V$ *which are base points of* $\Lambda$. *Let* $F=n_1C_1+\cdots+n_rC_r$ *be a reducible member of* $\Lambda$ *such that* $r \geqq 2$, *where* $C_i$ *is an irreducible component,* $C_i \neq C_j$ *if* $i \neq j$, *and* $n_i>0$. *Then the following assertions hold true*:

(1) *If* $C_i \cap \mathfrak{B} = \phi$ *then* $C_i$ *is isomorphic to* $\boldsymbol{P}_k^1$ *and* $(C_i^2)<0$.

(2) *If* $C_i \cap C_j \neq \phi$ *for* $i \neq j$ *and* $C_i \cap C_j \cap \mathfrak{B} = \phi$ *then* $C_i \cap C_j$ *consists of a single point where* $C_i$ *and* $C_j$ *intersect each other transversally.*

(3) *For three distinct indices* $i, j, l$, *if* $C_i \cap C_j \cap C_l \cap \mathfrak{B} = \phi$ *then* $C_i \cap C_j \cap C_l = \phi$.

(4) *Assume that* $(C_i^2)<0$ *whenever* $C_i \cap \mathfrak{B} \neq \phi$. *Then the set* $S := \{C_i; C_i$ *is an irreducible component of* $F$ *such that* $C_i \cap \mathfrak{B} = \phi\}$ *is nonempty, and there is an exceptional component in the set* $S$.

(5) *With the same assumption as in* (4) *above, if a component of* $S$, *say* $C_1$, *has multiplicity* $n_1=1$ *then there exists an exceptional component in* $S$ *other than* $C_1$.

Proof. Let $\rho: \tilde{V} \to V$ be the shortest succession of quadratic transformations with centers at base points (including infinitely near base points) of $\Lambda$ such that the proper transform $\tilde{\Lambda}$ of $\Lambda$ by $\rho$ has no base points. Then, by Bertini's theorem, general members of $\tilde{\Lambda}$ are isomorphic to $\boldsymbol{P}_k^1$. The assertions (1), (2) and (3) are then apparently true by Lemma 1.1. We shall prove the assertions (4) and (5), assuming that $\mathfrak{B} \neq \phi$. Let $P \in \mathfrak{B}$. Set $P_0 := P$, and let $P_1, \cdots, P_{s-1}$ exhaust infinitely near base points of $\Lambda$ such that $P_i$ is an infinitely near point of $P_{i-1}$ of order one for $1 \leqq i \leqq s-1$. For $1 \leqq i \leqq s$, let $\sigma_i: V_i \to V_{i-1}$ be the quadratic transformation of $V_{i-1}$ with center at $P_{i-1}$, where $V_0 := V$, and let $\sigma = \sigma_1 \cdot \cdots \cdot \sigma_s$. Then $\sigma$ factors $\rho$, i.e., $\rho = \sigma \cdot \bar{\rho}$. Let $E_i := (\sigma_{i+1} \cdot \cdots \cdot \sigma_s)'(\sigma_i^{-1}(P_{i-1}))$ for $1 \leqq i < s$ and let $E_s' := \sigma_s^{-1}(P_{s-1})$. Let $E_i := \bar{\rho}'(E_i')$ for $1 \leqq i \leqq s$. It is clear that $E_i' \cong E_i$ and $(E_i'^2) \geqq (E_i^2)$ for $1 \leqq i \leqq s$, and that $(E_i^2) < -1$ for $1 \leqq i < s$ and $(E_s^2) = -1$. Moreover, $E_s$ is not contained in any member of $\tilde{\Lambda}$; indeed, if otherwise, $\tilde{\Lambda}$ would have yet a base point on $E_s$, which contradicts the choice of points $P_1, \cdots, P_{s-1}$. The member $\tilde{F}$ of $\tilde{\Lambda}$ corresponding to $F$ of $\Lambda$

may contain some (not necessarily all) of $E_1, \cdots, E_{s-1}$. After the above process made for every point of $\mathfrak{B}$ and every sequence of infinitely near base points $\{P_0, P_1, \cdots, P_{s-1}\}$ as above, we know that if we write $\tilde{F}=(n_1\tilde{C}_1+\cdots+n_r\tilde{C}_r)+(m_1D_1+\cdots+m_tD_t)$ with $\tilde{C}_i=\rho'(C_i)$ for $1\leqq i\leqq r$ then we have;

1° if $C_i\in S$ then $\tilde{C}_i\cong C_i$ and $(\tilde{C}_i^2)=(C_i^2)$,

2° if $C_i\notin S$ then $(\tilde{C}_i^2)\leqq -2$,

3° $(D_i^2)\leqq -2$ for $1\leqq i\leqq t$.

Then the assertions (4) and (5) follow from the assertions (5) and (6) of Lemma 1.1. Q.E.D.

**1.3.** Let $V$ be a nonsingular projective surface containing an open subset $U$, which is isomorphic to $A_k^2$. Since $U$ is affine, the boundary set $V-U$ is connected and purely of codimension 1. Write $V-U=\bigcup_{i=1}^{r} C_i$, where $C_i$ is an irreducible component. We assume that the boundary curve $V-U$ has only normal crossings as singularities. Then we have the following

**Lemma** (cf. Ramanujam [8]). *Let $V$ and $C_i$'s be as above. Then the following assertions hold true*:

(1) *For every $i$, $C_i$ is isomorphic to $P_k^1$.*

(2) *For $i\neq j$, $(C_i\cdot C_j)=0$ or 1.*

(3) *For three distinct indices $i, j$ and $l$, $C_i\cap C_j\cap C_l=\phi$.*

(4) *There is no circular chain (or loop) $\{C_{i_1}, \cdots, C_{i_s}\}$ such that $(C_{i_j}\cdot C_{i_{j+1}})=1$ for $1\leqq j<s$ and $(C_{i_s}\cdot C_{i_1})=1$.*

$V$ is then called a *normal compactification* of $A_k^2$. $V$ is called a *minimal* normal compactification if the following additional condition is satisfied:

(5) *If an irreducible component, say $C_1$, of $V-U$ is an exceptional curve of the first kind, then at least three other components of $V-U$ meet $C_1$.*

**1.4.** Let $V$ be a normal compactification of $A_k^2$. Then the dual graph of the boundary curve $V-U$ is defined by assigning a vertex ○ to each irreducible component and by connecting two vertices by an edge (like ○—○) if two corresponding components meet each other. The condition (4) of Lemma 1.3 says that the dual graph of $V-U$ is a *tree*. The dual graph is said to be *linear* if it is a tree and each vertex has at most two branches. Then we have the following.

**Lemma** (cf. Ramanujam [8]). *Let $V$ be a minimal normal compactification of $A_k^2$. Then the dual graph of the boundary curve is linear.*

The dual graphs of all possible minimal normal compactifications of $A_k^2$ were classified by Morrow [7]. An algebraic proof of the above lemma of Ramanujam and also of Morrow's result (even over the ground field of positive characteristic) was given by S. Mori [6]. However, we use this result only in one place (cf. 3.11) in this article.

**1.5.** Let $k[x, y]$ be a polynomial ring in two variables $x$, $y$ and let $f$ be an irreducible polynomial of degree $d>0$ in $k[x, y]$. Let $\Lambda(f)$ be the linear pencil on $\boldsymbol{P}_k^2$ defined by $f$. Then $\Lambda(f)$ has base points on the line $l_\infty$ at infinity. Set $V_0 := \boldsymbol{P}_k^2$. Let $\varphi: \tilde{V} \to V_0$ be the shortest succession of quadratic transformations with centers at base points (including infinitely near base points) of $\Lambda(f)$ such that the proper transform $\tilde{\Lambda}$ of $\Lambda(f)$ by $\varphi$ has no base points. Let $\tilde{\rho}: \tilde{V} \to \boldsymbol{P}_k^1$ be the surjective morphism defined by $\tilde{\Lambda}$. Then each irreducible (exceptional) curve arising in the process $\varphi$ of quadratic transformations is either a quasi-section of $\tilde{\rho}$ or contained in a fiber of $\tilde{\rho}$. Let $\tilde{\Gamma}_1, \cdots, \tilde{\Gamma}_p$ exhaust all irreducible (exceptional) curves arising in the process $\varphi$, which are quasi-sections of $\tilde{\rho}$. Then it is clear that every $\tilde{\Gamma}_i$ is isomorphic to $\boldsymbol{P}_k^1$.

Note that $\tilde{V}$ contains in a canonical way an open subset $\tilde{U}$ which is isomorphic to $\boldsymbol{A}_k^2$ and that $\tilde{V}$ is a normal compactification of $\tilde{U} \cong \boldsymbol{A}_k^2$. Let $\psi: \tilde{V} \to V$ be a contraction of all possible exceptional (contractible) components of $\tilde{V} - \tilde{U}$, which are contained in the fibers of $\tilde{\rho}$. Then $V$ is a nonsingular projective surface containing an open subset $U$, which is isomorphic to $\boldsymbol{A}_k^2$. However, $V$ is not necessarily a normal compactification of $\boldsymbol{A}_k^2$ (cf. [5; Example 2.4.4, p. 122]). Moreover, there exists a surjective morphism $\rho: V \to \boldsymbol{P}_k^1$ such that $\tilde{\rho} = \rho \cdot \psi$. We set $\Gamma_i = \psi(\tilde{\Gamma}_i)$ for $1 \leqq i \leqq p$ and denote by $S_\infty$ the fiber of $\rho$ which corresponds to the member $dl_\infty$ of $\Lambda(f)$. If we identify $U$ with $\mathrm{Spec}(k[x, y])$, there exists an inhomogeneous coordinate $u$ of $\boldsymbol{P}_k^1$ such that the point $\rho(S_\infty)$ is given by $u = \infty$ and $S_\alpha \cap U = F_\alpha$ for each $\alpha \in k$, where $S_\alpha$ is the fiber of $\rho$ lying over the point $u = \alpha$.

**1.6.** Assume that $f$ is generically rational, i.e., general members of $\Lambda(f)$ are rational curves. Then, with the above notations, $S_\infty$ is isomorphic to $\boldsymbol{P}_k^1$. Let $S_1, \cdots, S_r$ be all fibers of $\rho$ such that $F_i := S_i \cap U$ is a reducible curve for $1 \leqq i \leqq r$. Let $m_i$ be the number of irreducible components of $F_i$ for $1 \leqq i \leqq r$. On the other hand, $\Gamma_1, \cdots, \Gamma_p$ exhaust all quasi-sections of $\rho$ which are contained in $V - U$, while they are not necessarily nonsingular. Let $\delta_j$ be the degree of the morphism $\rho|_{\Gamma_j}: \Gamma_j \to \boldsymbol{P}_k^1$. Note that $\psi|_{\tilde{\Gamma}_j}: \tilde{\Gamma}_j \to \Gamma_j$ is the desingularization of $\Gamma_j$. For $1 \leqq j \leqq p$, let $\nu_j = \sum_Q (e_Q - 1)$, where $e_Q$ is the ramification index of $\tilde{\rho}|_{\tilde{\Gamma}_j}$ at a point $Q \in \tilde{\Gamma}_j$ and the summation ranges over all points $Q$ of $\tilde{\Gamma}_j$ such that $\tilde{\rho}(Q) \neq \rho(S_\infty)$. Then we have the following.

**Lemma** (cf. Saito [13], Suzuki [15]). *With the assumptions and the notations as above, we have*:

(1) $p + r - 1 = \sum_{i=1}^{r} m_i$; (2) $n + 1 = p + \sum_{j=1}^{p} \nu_j$, *where* $n+1$ *is the number of*

*places at infinity of a general member $F_\alpha$ of $\Lambda_0(f)$* (cf. *Introduction.*)

Proof. (1) We note that $S_1, \cdots, S_r$ are all singular fibers of $\rho$. Indeed, suppose that $S$ is a singular fiber of $\rho$ other than $S_1, \cdots, S_r$. Then $S$ has only one irreducible component $C$ which meets $U$. Let $\mu$ be the multiplicity of $C$ in $S$. Since $\mu C \cap U = S \cap U$ is defined by an equation $f=\alpha$ with some $\alpha \in k$, we have $f-\alpha=\lambda g^\mu$ for some irreducible polynomial $g \in k[x, y]$ and $\lambda \in k^*$. The assertion (6) of Lemma 1.1 implies that $\mu>1$. Then $f$ is reducible, which contradicts the choice of $f$. Let $S$ be an irreducible fiber of $\rho$. Then we have

$$\chi(V) = \chi(S)\cdot\chi(\boldsymbol{P}^1_k) + \sum_{i=1}^{r}(\chi(S_i)-\chi(S)),$$

where $\chi(\ )$ denotes the Euler number (cf. Šafarevič [11; p. 58]). Let $m_i'$ be the number of irreducible components of $S_i$. Since the dual graph of $S_i$ is a tree, it is easy to show that $\chi(S_i)=2m_i'-(m_i'-1)=m_i'+1$. On the other hand, it is easy to see from construction of $V$ that the Picard group $\mathrm{Pic}(V)$ is a free abelian group of rank $1+p+\sum_{i=1}^{r}(m_i'-m_i)$. Since $V$ is rational, $\chi(V)$ is then equal to $2+1+p+\sum_{i=1}^{r}(m_i'-m_i)$. Thus, from the above equality, we obtain

$$3+p+\sum_{i=1}^{r}(m_i'-m_i) = 4+\sum_{i=1}^{r}(m_i'-1),$$

whence follows the first equality.
(2) For each $j$ $(1 \leqq j \leqq p)$, $S_\infty \cap \Gamma_j$ consists of a single point, which is also a one-place point. For, if otherwise, $\tilde{V}-\tilde{U}$ would contain a circular chain of irreducible components, which is a contradiction by the assertion of Lemma 1.3. Then there is only one point on $\tilde{\Gamma}_j$ lying over the point $\rho(S_\infty)$, where the ramification index of $\tilde{\rho}|_{\tilde{\Gamma}_j}$ equals $\delta_j$. By Hurwitz's formula, we obtain $\delta_j-1=\nu_j$ for every $j$ $(1 \leqq j \leqq p)$. On the other hand, note that if $S$ is a general fiber of $\rho$, $S$ has $n+1$ distinct points outside of $S \cap U$, among which $\delta_j$ points lie on $\Gamma_j$ for each $j$ $(1 \leqq j \leqq p)$. Therefore, we have the second equality. Q.E.D.

**1.7. Lemma.** *Let $f$ be a generically rational polynomial in $k[x, y]$. Then, with the notations as above, the following conditions are equivalent*:

(1) $n=0$;

(2) $p=1$;

(3) $r=0$, i.e., *the curve $F_\alpha$ on $\boldsymbol{A}^2_k$ defined by $f=\alpha$ is irreducible for every $\alpha \in k$*;

(4) *$f$ is sent to one of coordinates $x$, $y$ by a biregular automorphism of $\boldsymbol{A}^2_k :=$* $\mathrm{Spec}(k[x, y])$.

Proof. Note that $p>0$. Then the implication (1)$\Rightarrow$(2) is clear. Suppose

$p=1$. Then $r=0$ by the first equality of Lemma 1.6, because $m_i \geqq 2$ for every $i$. This implies the assertion (3) (cf. the proof of the first equality of Lemma 1.6). Suppose the condition (3) is satisfied. Then $V$ is a relatively minimal rational ruled surface, whence $V$ is a Hirzebruch surface $F_a = \mathrm{Proj}(\mathcal{O}_{P^1} \otimes \mathcal{O}_{P^1}(a))$ with $a \geqq 0$. Let $M$ be the minimal section of $F_a$, i.e., $M$ is the cross-section of $\rho$ with $(M^2) = -a$; when $a=0$ we take a cross-section of $\rho$ as $M$. Since $\mathrm{Pic}(V)$ is a free abelian group generated by $M$ and the fiber $S_\infty$ of $\rho$, the unique quasi-section $\Gamma$ of $\rho$ contained in $V-U$ is linearly equivalent to a divisor of the form $\alpha M + \beta S_\infty$, where $\alpha$ and $\beta$ are non-negative integers such that $\beta \geqq \alpha a$. Since $U \cong A_k^2$ and $V-U = \Gamma \cup S_\infty$, we know that $\Gamma$ and $S_\infty$ also generate $\mathrm{Pic}(V)$. Then $\alpha=1$, and $\Gamma$ is a cross-section. This implies that $S \cap U$ is isomorphic to $A_k^1$ for every fiber $S$ of $\rho$ other than $S_\infty$. Hence the curve $F_0$ defined by $f=0$ is isomorphic to $A_k^1$. Now, the assertion (4) follows from Abhyankar-Moh's theorem (cf. Abhyankar-Moh [1], Miyanishi [4]). The implication (4)$\Rightarrow$(1) is clear. Q.E.D.

**1.8.** DEFINITION. *Let $f$ be a generically rational polynomial in $k[x, y]$. $f$ is said to be of simple type if the equality $p=n+1$ holds.*

The second equality of Lemma 1.6 shows that $f$ is of simple type if and only if $\Gamma_1, \cdots, \Gamma_p$ are all cross-sections of $\rho$.

**Lemma.** *Let $f$ be a generically rational polynomial in $k[x, y]$ such that a general member of $\Lambda_0(f)$ has two places at infinity,* i.e., $n=1$. *Then $f$ is of simple type. Moreover, $\Lambda_0(f)$ has only one reducible fiber, which has two irreducible components.*

Proof. By Lemma 1.7, we have $p \geqq 2$. Then, by the second equality of Lemma 1.6, we must have $p=2$, whence $f$ is of simple type. Then the first equality of Lemma 1.6 implies that $r=1$ and $m_1=2$. Q.E.D.

**1.9.** Let $f$ be a generically rational polynomial in $k[x, y]$. We assume in this paragraph that $f$ is of simple type, i.e., $p=n+1$. Then every $\Gamma_j (1 \leqq j \leqq p)$ is a nonsingular rational curve. We assume, furthermore, that $p \geqq 2$, i.e., $n \geqq 1$. Since $U$ is isomorphic to $A_k^2$, two distinct $\Gamma_j$ and $\Gamma_l$ do not meet each other on any fiber $S$ of $\rho$ other than $S_\infty$; indeed, if otherwise, $\tilde{V}-\tilde{U}$ would contain a circular chain of irreducible components. Note also that $S_1, \cdots, S_r$ exhaust all singular fibers of $\rho$.

Let $\Delta$ be an irreducible component of $S_i - S_i \cap U$ (if it exists at all) for some $i (1 \leqq i \leqq r)$. If there is a sequence of irreducible components $\{\Delta_1, \cdots, \Delta_t\}$ of $S_i - S_i \cap U$ such that $\Delta_1 = \Delta$, $\Delta_l \cap \Delta_{l+1} \neq \phi$ for $1 \leqq l < t$ and $\Delta_t \cap \Gamma_j \neq \phi$ for some $j$ $(1 \leqq j \leqq n+1)$, we say that $\Delta$ is connected to $\Gamma_j$. Since $V-U$ is a connected curve, every irreducible component $\Delta$ of $S_i - S_i \cap U$ is connected to some $\Gamma_j$ $(1 \leqq j \leqq n+1)$, while $\Delta$ is not connected to two distinct $\Gamma_j$ and $\Gamma_{j'}$; indeed, if otherwise, $\tilde{V}-\tilde{U}$ would contain a circular chain of irreducible components. For each pair $(i,j)$ with $1\leqq i\leqq r$ and $1\leqq j\leqq n+1$, let $E_{ij}$ be the union of all irreducible components $\Delta$ of $S_i-S_i\cap U$, which are connected to $\Gamma_j$. Then $E_{ij}$ is a connected curve if it is not empty.

Set $P_\infty:=\rho(S_\infty)$ and $P_i:=\rho(S_i)$ for $1\leqq i\leqq r$. Then the points $P_\infty$ and $P_i$'s are determined by $u=\infty$ and $u=c_i(1\leqq i\leqq r)$, respectively, with respect to the inhomogeneous coordinate $u$ of $\boldsymbol{P}_k^1$ (cf. the paragraph 1.5). Set $Z:=\boldsymbol{P}_k^1-\{P_\infty, P_1, \cdots, P_r\}$. Then $\rho^{-1}(Z)$ with the morphism $\rho\colon \rho^{-1}(Z)\to Z$ is a trivial $\boldsymbol{P}^1$-bundle over $Z$, i.e., there exists a $Z$-isomorphism $\eta\colon \rho^{-1}(Z)\cong Z\times\boldsymbol{P}_k^1$. We may assume that there exist points $Q_1, \cdots, Q_{n+1}$ on $\boldsymbol{P}_k^1$ such that $\eta^{-1}(Z\times\{Q_j\})=\Gamma_j\cap\rho^{-1}(Z)$ for $1\leqq j\leqq n+1$. Now, choose an inhomogeneous coordinate $v$ on $\boldsymbol{P}_k^1$ (=the fiber of $\rho$) such that $Q_{n+1}$ is defined by $v=\infty$ and $Q_j$ is defined by $v=d_j$ for $1\leqq j\leqq n$; in the sequel, we set $Q_\infty:=Q_{n+1}$ and $\Gamma_\infty:=\Gamma_{n+1}$. Let $p_2\colon Z\times\boldsymbol{P}_k^1\to\boldsymbol{P}_k^1$ be the second projection. Then the morphism $p_2\cdot\eta\colon\rho^{-1}(Z)\to\boldsymbol{P}_k^1$ (or equivalently saying, the inclusion of the subfield $k(v)$ into $k(x,y)$) defines a linear pencil $L$ on $V$ without fixed components such that $\Gamma_1, \cdots, \Gamma_{n+1}$ are contained in distinct members $\Xi_1, \cdots, \Xi_n$, $\Xi_\infty:=\Xi_{n+1}$ of $L$, respectively.

For each $i\,(1\leqq i\leqq r)$, let $S_{i1}, \cdots, S_{im_i}$ exhaust all irreducible components of $S_i$ such that $S_{ia}\cap U\neq\phi$ for $1\leqq a\leqq m_i$. Set $F_{ia}:=S_{ia}\cap U$ and let $f_{ia}$ be an irreducible polynomial in $k[x,y]$ such that $F_{ia}$ is defined by $f_{ia}=0$, where $1\leqq a\leqq m_i$. Then, for each $i\,(1\leqq i\leqq r)$, we have

$$u-c_i=f-c_i=\lambda_i\cdot\prod_{a=1}^{m_i}(f_{ia})^{\alpha_{ia}},$$

where $\lambda_i\in k^*$ and $\alpha_{ia}$ is a positive integer. Then we have:

**1.9.1.** $k[x, y, (\prod_{i=1}^{r}\prod_{a=1}^{m_i}f_{ia})^{-1}]=k[u, v, \prod_{i=1}^{r}(u-c_i)^{-1}, \prod_{j=1}^{n}(v-d_j)^{-1}]$.

**1.9.2.** *If $n\geqq 2$, the linear pencil $L$ has no base points which lie outside $S_\infty$.*

Proof. Suppose that $Q$ is a base point of $L$ with $Q\notin S_\infty$. Then it is clear that $Q\in S_i$ for some $i\,(1\leqq i\leqq r)$. Note that, for every $j\,(1\leqq j\leqq n+1)$, the underlying curve of $\Xi_j$ is contained in the union of $\Gamma_j$ and the underlying curves of $S_i$'s $(1\leqq i\leqq r)$. Since $n+1\geqq 3$, this implies that either there are at least two components of $S_i$ intersecting one of the cross-sections $\Gamma_j$'s, or there are at least three components of $S_i$ passing through the point $Q$, which contradicts the assertion (4) of Lemma 1.1.

**1.9.3.** *After a suitable modification*(*) *of $V$ with centers at points of $S_\infty$, we may*

(*) a succession of quadratic transformations with centers at points and contractions of exceptional curves of the first kind.

*assume that $L$ has no base points on $S_\infty$.*

Proof. If $S_\infty$ is not contained in any member of $L$ then $L$ has no base points on $S_\infty$. Suppose that $L$ has a base point $Q$ on $S_\infty$. Let $s:=i(\Xi, \Xi'; Q)$ for distinct general members $\Xi$ and $\Xi'$ of $L$. Set $Q_0:=Q$, and let $Q_1, \cdots, Q_{s-1}$ be infinitely near points of $Q_0$, which lie on (the proper transforms of) $\Xi$, such that $Q_i$ is an infinitely near point of $Q_{i-1}$ of order one for $1 \leqq i < s$. Let $\sigma_i$: $V_i \to V_{i-1}$ be the quadratic transformation of $V_{i-1}$ with center at $Q_{i-1}$, where $V_0:=V$, and let $\sigma=\sigma_1 \cdot \cdots \cdot \sigma_s$. Let $E_i:=(\sigma_{i+1} \cdot \cdots \cdot \sigma_s)'(\sigma_i^{-1}(Q_{i-1}))$ for $1 \leqq i < s$, and let $E_s:=\sigma_s^{-1}(Q_{s-1})$. Let $S'_\infty:=\sigma'(S_\infty)$, let $\Gamma'_j:=\sigma'(\Gamma_j)$ for $1 \leqq j \leqq n+1$, and let $L':=\sigma'(L)$. Then we have the following dual graph of $\sigma^{-1}(S_\infty)$:


Since $n+1 \geqq 2$, some member $\Xi_j$ $(1 \leqq j \leqq n+1)$ does not contain $S_\infty$. Moreover, since $(\Gamma_j \cdot S_\infty)=1$, we know that $Q_0, Q_1, \cdots, Q_{s-1}$ exhaust all base points (including infinitely near base points) of $L$ centered on $S_\infty$. Therefore, $L'$ has no base points centered on $S_\infty$. Thus $E_s$ is a cross-section of $L'$, and $S'_\infty \cup E_1 \cup \cdots \cup E_{s-1}$ is contained in some member of $L'$. Let $\bar{\sigma}$: $V_s \to \bar{V}$ be the contraction of $S'_\infty, E_1, \cdots, E_{s-1}$. Now, replace $V$ by $\bar{V}$. Then the above modification $\bar{\sigma} \cdot \sigma^{-1}$: $V \to \bar{V}$ changes only $S_\infty$. Q.E.D.

**1.9.4.** *Assume that $r \geqq 2$. Then, for every $j$ $(1 \leqq j \leqq n+1)$, at most one of $E_{1j}, \cdots, E_{rj}$ is a nonempty set.*

Proof. Suppose that (at least) two of $E_{1j}, \cdots, E_{rj}$, say $E_{1j}$ and $E_{2j}$, are nonempty sets. By 1.9.3, we may assume that $L$ has no base points on $S_\infty$. Then $V$ is a normal compactification of $U$. Note that $(S_\infty^2)=0$ and no exceptional components are contained in $\bigcup_{i=1}^{r} E_{ij}$. By contracting (possible) exceptional components in $V-U$, we would obtain a minimal normal compactification of $\boldsymbol{A}_k^2$, for which the dual graph of the boundary curve is not linear. This contradicts Lemma 1.4. Q.E.D.

**1.9.5.** *For distinct pairs $(i, j)$ and $(i', j')$, we have $E_{ij} \cap E_{i'j'}=\phi$. Furthermore, we have $S_i - S_i \cap U = \bigcup_{j=1}^{n+1} E_{ij}$ for every $i$ $(1 \leqq i \leqq r)$.*

Proof. Clear from the construction and the above arguments.

**1,9.6.** Assume that $n \geqq 2$. With $V$ modified so that $L$ has no base points (cf. 1.9.3), $L$ defines a $\boldsymbol{P}^1$-fibration $\tau$: $V \to \boldsymbol{P}_k^1$ such that $S_\infty$ is a cross-section of $\tau$.

**1.9.7.** Assume that $n\geqq 1$. After the modification of $V$ so that $L$ has no base points on $S_\infty$, we call $V$ *a standard compactification of* $\boldsymbol{A}_k^2$ *with respect to* a generically rational polynomial $f$ of simple type. Then we have the following

**Lemma.** *Let $f$ be a generically rational polynomial of simple type in $k[x, y]$ with $n\geqq 1$. Let $V$ and $\bar{V}$ be standard compactifications of $\boldsymbol{A}_k^2:=\mathrm{Spec}(k[x, y])$ with respect to $f$. Let $\rho: V\rightarrow \boldsymbol{P}_k^1$ (or $\bar{\rho}: \bar{V}\rightarrow \boldsymbol{P}_k^1$, resp.) be the $\boldsymbol{P}^1$-fibration of $V$ (or $\bar{V}$, resp.) defined by $f$. Then there exists an isomorphism $\theta: V\rightarrow \bar{V}$ such that $\rho=\bar{\rho}\cdot\theta$.*

Proof. $V$ (or $\bar{V}$, resp.) contains an open subset $U$ (or $\bar{U}$, resp.) isomorphic to $\boldsymbol{A}_k^2$. The identity morphism $id.: U\xrightarrow{\sim}\bar{U}$ extends to a birational mapping $\theta: V\rightarrow\bar{V}$ such that $\rho=\bar{\rho}\cdot\theta$. We shall show that $\theta$ is an isomorphism. Let $\mathfrak{V}$ and $\bar{\mathfrak{V}}$ be the generic fibers of $\rho$ and $\bar{\rho}$, respectively. Then both $\mathfrak{V}$ and $\bar{\mathfrak{V}}$ are complete normal models of the algebraic function field $k(x, y)$ of one variable over $\mathfrak{k}=k(f)$. Hence $\mathfrak{V}$ and $\bar{\mathfrak{V}}$ are isomorphic to $\boldsymbol{P}_{\mathfrak{k}}^1$. Let $\Gamma_1, \cdots, \Gamma_{n+1}$ (or $\bar{\Gamma}_1, \cdots, \bar{\Gamma}_{n+1}$, resp.) be the cross-sections of $\rho$ (or $\bar{\rho}$, resp.) contained in $V-U$ (or $\bar{V}-\bar{U}$, resp.). Since $\mathfrak{V}-\{\Gamma_1\cap\mathfrak{V}, \cdots, \Gamma_{n+1}\cap\mathfrak{V}\}$ is identified to $\bar{\mathfrak{V}}-\{\bar{\Gamma}_1\cap\bar{\mathfrak{V}}, \cdots, \bar{\Gamma}_{n+1}\cap\bar{\mathfrak{V}}\}$ under $\theta$, $\mathfrak{V}$ is isomorphic to $\bar{\mathfrak{V}}$ under $\theta$, and we may assume that $\theta(\Gamma_j\cap\mathfrak{V})=\bar{\Gamma}_j\cap\bar{\mathfrak{V}}$ for $1\leqq j\leqq n+1$. Then $\theta: \Gamma_j\rightarrow\bar{\Gamma}_j$ is an isomorphism for $1\leqq j\leqq n+1$. Let $S_1, \cdots, S_r$ (or $\bar{S}_1, \cdots, \bar{S}_r$, resp.) be reducible fibers of $\rho$ (or $\bar{\rho}$, resp.). Define $\bar{E}_{ij}$ $(1\leqq i\leqq r$ and $1\leqq j\leqq n+1)$ for $\bar{V}$ in the same fashion as for $V$; note that $\bigcup_{i=1}^{r}\bigcup_{j=1}^{n+1}\bar{E}_{ij}$ contains no exceptional curves of the first kind. Let $S_\infty$ (or $\bar{S}_\infty$, resp.) be the fiber of $\rho$ (or $\bar{\rho}$, resp.) contained in $V-U$ (or $\bar{V}-\bar{U}$, resp.). If $\theta$ is not biregular, $V-U$ contains an irreducible curve $T$ which becomes an exceptional curve of the first kind after a succession of quadratic transformations with centers at points in $V-U$. Since every component in $\bigcup_{i=1}^{r}\bigcup_{j=1}^{n+1}E_{ij}$ has self-intersection multiplicity $\leqq -2, T$ must be $S_\infty$. However, if $S_\infty$ is contracted after a sequence of quadratic transformations, $\bar{\Gamma}_1, \cdots, \bar{\Gamma}_{n+1}$ meet each other at one point on $\bar{S}_\infty$, which is a contradiction. Hence $\theta$ is biregular. Similarly, $\theta^{-1}$ is biregular. Thus $\theta$ is an isomorphism such that $\rho=\bar{\rho}\cdot\theta$.

Q.E.D.

**1.10.** In the second section we use the following

**Lemma** (cf. Miyanishi [3,5]). *Let $X$ be a nonsingular affine surface defined by an affine k-domain $A$. Assume that the following conditions are satisfied:*

(1) *$A$ is a unique factorization domain and $A^*=k^*$.*

(2) *There exist nonsingular irreducible curves $C_1$ and $C_2$ on $X$ such that $C_1\cap C_2=\{P\}$, and $C_1$ and $C_2$ intersect each other transversally at $P$.*

(3) *$C_1$ (resp. $C_2$) has only one place at infinity.*

(4) *Let* $a_2$ *be a prime element of* $A$ *defining the curve* $C_2$. *Then* $a_2-\alpha$ *is a prime element of* $A$ *for all* $\alpha\in k$.

(5) *There is a nonsingular projective surface* $V$ *containing* $X$ *as an open subset such that the closure* $\bar{C}_2$ *of* $C_2$ *in* $V$ *is nonsingular and* $(a_2)_0$ (*=the zero part of the divisor of* $a_2$)$=\bar{C}_2$. *Then* $X$ *is isomorphic to* $\boldsymbol{A}_k^2$, *and the curves* $C_1$ *and* $C_2$ *are sent to the axes of a suitable coordinate system of* $\boldsymbol{A}_k^2$.

**1.11.** Here, we recall a general result due to Russell [9; Cor. 3.7] on a generically rational polynomial.

**Proposition.** *Let* $f$ *be a generically rational polynomial in* $k[x, y]$. *Then there are at most two points (including infinitely near points) of the curve* $f=0$ *on the line at infinity* $l_\infty$. *In particular, the degree form of* $f$ *has at most two distinct irreducible factors.*

## 2. Generically rational polynomials with $n=1$

In this section, $f$ is a generically rational, irreducible polynomial in $k[x, y]$ with $n=1$. Then $f$ is of simple type and $\Lambda_0(f)$ has only one reducible member consisting of two irreducible components (cf. Lemma 1.8). Let $f=c_1$ be the reducible member of $\Lambda_0(f)$. We use the notations of the previous section with due modifications.

**2.1. Lemma.** *With the notations of* 1.9, *we have one of the following cases*:

(1) *Case* $F_{11}\cap F_{12}\neq\phi$. *Then* $(S_{11}\cdot S_{12})=1$ *and* $F_{1a}\cong\boldsymbol{A}_k^1$ *for* $a=1, 2$.

(2) *Case* $F_{11}\cap F_{12}=\phi$. *Then* $(S_{11}\cdot S_{12})=0$, *and one of* $F_{11}$ *and* $F_{12}$, *say* $F_{11}$, *is isomorphic to* $\boldsymbol{A}_k^1$ *and the other, say* $F_{12}$, *is isomorphic to* $\boldsymbol{A}_*^1$ $(=\boldsymbol{A}_k^1-$(*one point*)).

Proof. (1) Suppose $F_{11}\cap F_{12}\neq\phi$. Then $(S_{11}\cdot S_{12})=1$ (cf. Lemma 1.1, (3)). Since $E_{11}\cap E_{12}=\phi$, we may assume that $(\Gamma_a\cup E_{1a})\cap S_{1a}\neq\phi$ for $a=1, 2$. Since the dual graph of $S_1$ is a tree, it is easy to see that $F_{1a}\cong\boldsymbol{A}_k^1$ for $a=1, 2$.
(2) Suppose $F_{11}\cap F_{12}=\phi$. Then one of $S_{11}$ and $S_{12}$, say $S_{12}$, intersects both $\Gamma_1\cup E_{11}$ and $\Gamma_2\cup E_{12}$, and the other one, say $S_{11}$, intersects only one of $\Gamma_1\cup E_{11}$ and $\Gamma_2\cup E_{12}$. Then $(S_{11}\cdot S_{12})=0$, $F_{11}\cong\boldsymbol{A}_k^1$ and $F_{12}\cong\boldsymbol{A}_*^1$. Q.E.D.

**2.2. Lemma.** *With the notations of* 1.9 *and after a suitable change of coordinates of* $k[x, y]$, *we have one of the following cases*:

(1) $f_{11}=x$ *and* $f_{12}=y$.

(2) $f_{11}=x$ *and* $f_{12}=x^l y+P(x)$, *where* $l>0$ *and* $P(x)\in k[x]$ *with* deg $P(x)<l$ *and* $P(0)\neq0$.

Proof. (1) Suppose $F_{11}\cap F_{12}\neq\phi$. Then, by virtue of Lemmas 1.7 and 1.10, we may change coordinates of $k[x, y]$ so that $f_{11}=x$ and $f_{12}=y$.
(2) Suppose $F_{11}\cap F_{12}=\phi$. Then, by virtue of Abhyankar-Moh's Theorem

(cf. Lemma 1.7), we may assume that $f_{11}=x$. Taking account of 1.9.1, we have only to prove the following assertion:

*Let $g\in k[x,y]$ be an irreducible polynomial such that $g\notin k[x]$ and $g(0,y)\neq 0$. Assume that we have the following identification of rings,*

$$k[x, y, x^{-1}, g^{-1}] = k[u, v, u^{-1}, v^{-1}]\,.$$

*Then $g$ is of the form:*

$$g = \lambda(x^l y+P(x)),\ \textit{where}\ \lambda\in k^*,\ l>0\ \textit{and}\ P(x)\in k[x]$$
$$\textit{with}\ \deg P(x)<l\ \textit{and}\ P(0)\neq 0\,.$$

Indeed, by comparison of the unit groups of both rings, we have: $u\sim x^{\alpha}g^{\beta}$ and $v\sim x^{\gamma}g^{\delta}$, where $\alpha, \beta, \gamma, \delta\in \boldsymbol{Z}$ such that $\alpha\delta-\beta\gamma=\pm 1$. This implies that $g$ satisfies

$$x^a g^b y = G(x, g), \quad \text{where} \quad G(x, g)\in k[x, g]\,.$$

If $b>0$ then we have a relation: $gh(x, y)\in k[x]$ with $h(x, y)\in k[x, y]$, which implies $g\in k[x]$, a contradiction. Hence $b=0$. Write

$$g = A_0(x)y^M+\cdots+A_M(x) \quad \text{and} \quad G(x, g) = B_0(x)g^N+\cdots+B_N(x)\,,$$

where $A_i(x)$, $B_j(x)\in k[x]$ ($0\leqq i\leqq M$ and $0\leqq j\leqq N$) and $A_0(x)B_0(x)\neq 0$. Then we have

$$x^a y = B_0(x)A_0(x)^N y^{NM}+\cdots\cdots\,,$$

whence $M=1$ and $A_0(x)\sim x^l$. Therefore, $g$ is written in the stated form after replacing $y$ by $y+C(x)$. Q.E.D.

**2.3. Theorem** (cf. Saito [12; p. 332], Sugie [14]). *Let $f$ be a generically rational, irreducible polynomial in $k[x,y]$ with $n=1$. Then, after a suitable change of coordinates, $f$ is reduced to either one of the following two forms:*

(1) $f\sim x^{\alpha}y^{\beta}+1$, *where* $\alpha>0$, $\beta>0$ *and* $(\alpha, \beta)=1$.

(2) $f\sim x^{\alpha}(x^l y+P(x))^{\beta}+1$, *where* $\alpha, \beta, l>0$, $(\alpha, \beta)=1$ *and* $P(x)\in k[x]$ *with* $\deg P(x)<l$ *and* $P(0)\neq 0$.

Proof. Clear by Lemma 2.2.

**2.4. Proposition.** *Assume that the curve $C_0$ on $\boldsymbol{P}_k^2$ defined by $f=0$ intersects $l_\infty$ in only one point $P$. Let $d_1:=\mathrm{mult}_P C_0$. Then $d_1<d=$the degree of $f$. Moreover, there exists a birational automorphism $\rho$ of $\boldsymbol{P}_k^2$ such that $\rho$ induces a biregular automorphism on $\boldsymbol{A}_k^2:=\boldsymbol{P}_k^2-l_\infty$ and that the proper transform $C_0'$ of $C_0$ by $\rho$ intersects $l_\infty$ in two points with $(C_0'\cdot l_\infty)\leqq d_1$.*

The existence of an automorphism $\rho$ such that $\rho|_{A_k^2}$ is a biregular automorphism and that the proper transform $\rho'(C_0)$ intersects $l_\infty$ in two points follows from the above theorm. However, the above proposition is proved by a constructive method depending on Lemma 1.2, which allows us to determine more explicitly an automorphism $\rho$ with the stated properties. For more details, the readers will be referred to Miyanishi [5; Chap. II, §6].

**2.5.** It is clear that the integers $\alpha$ and $\beta$ in Theorem 2.3 are the multiplicities of components $S_{11}$ and $S_{12}$ of the fiber $S_1$, respectively, in the standard compactification of $\boldsymbol{A}_k^2$ with respect to $f$. On the other hand, Lemma 1.9.7 implies that the standard compactification of $\boldsymbol{A}_k^2$ with respect to $f$ is obtained by assuming that $f$ is written in the form stated in Theorem 2.3. By a straightforward computation, we know that the dual graphs of $E_{11}$ and $E_{12}$ are linear. For details, see [5; ibid.].

## 3. Generically rational polynomials of simple type with $n>1$

In this section, $f$ is a generically rational, irreducible polynomial of simple type with $n>1$. In the paragraphs 3.1~3.4, we consider the case $r=1$; in the paragraphs 3.5~3.8, we consider the case where $r\geqq 2$ and $m_i<n$ for every $i\,(1\leqq i\leqq r)$; in the paragraphs 3.9~3.12, we consider the case where $r=2$, $m_1=n$ and $m_2=2$.

**3.1.** We shall consider the case $r=1$. For convenience's sake, we simplify the notations as follows: $\Sigma:=S_1$; $\Sigma_a:=S_{1a}$, $F_a:=F_{1a}$, $f_a:=f_{1a}$, $\alpha_a:=\alpha_{1a}$ for $1\leqq a\leqq m_1$, where $m_1=p=n+1$; $E_j:=E_{1j}$ for $1\leqq j\leqq n+1$; we may assume that $c_1=1$ (cf. 1.9.) Note that the pencil $L$ on the standard compactification $V$ of $\boldsymbol{A}_k^2$ with respect to $f$ has no base points; hence $L$ defines a $\boldsymbol{P}^1$-fibration $\tau\colon V\to\boldsymbol{P}_k^1$. Let $\Xi$ be a general fiber of $\tau$. Since $(S_\infty\cdot\Xi)=1$, we have $(\Sigma\cdot\Xi)=1$. This implies that there exists an irreducible component $\Delta$ of $\Sigma$ such that $(\Delta\cdot\Xi)=1$, i.e., $\Delta$ is a cross-section of $\tau$, and the other components of $\Sigma$ are contained in the fibers $\Xi_1,\cdots,\Xi_{n+1}$ and possibly one other fiber of $L$, as will be seen below. First, we shall prove the following:

**Lemma.** *With the assumptions and the notations as above, we have one of the following cases:*

(1) $\Delta\cap U\neq\phi$. *We may assume that* $\Delta=\Sigma_{n+1}$. *Then* $\Xi\cap U\cong\boldsymbol{A}_k^1$ *and* $F_a\cong\boldsymbol{A}_k^1$ *for* $1\leqq a\leqq n$; $\Sigma_a\,(1\leqq a\leqq n)$ *belongs to one and only one of* $\Xi_1,\cdots,\Xi_{n+1}$, *while none of* $\Xi_1,\cdots,\Xi_{n+1}$ *contains two of* $\Sigma_a$*'s; thus, after a suitable change of inhomogeneous coordinate* $v$ *of* $\boldsymbol{P}_k^1$ (*=the fiber of* $\rho$) *and a suitable change of indices, we may assume that* $\Sigma_a$ *belongs to* $\Xi_a$ *for* $1\leqq a\leqq n$; $F_{n+1}=\Delta-(\Delta\cap\Xi_{n+1})\cup\bigcup_{j=1}^{n}((\Gamma_j\cup E_j)\cap\Delta)$, *where* $\Delta\cong\boldsymbol{P}_k^1$; $\alpha_{n+1}=1$.

(2) $\Delta\cap U=\phi$. *We may assume that* $\Delta$ *is a component of* $E_{n+1}$. *Then* $\Xi\cap U\cong\boldsymbol{A}_*^1$; $\Sigma_a\,(1\leqq a\leqq n+1)$ *belongs to one and only one of* $\Xi_1,\cdots,\Xi_n$; *none of*

*$\Sigma_a$'s belongs to $\Xi_{n+1}$; thus, after a suitable change of indices, we may sasume that $\Sigma_a$ belongs to $\Xi_a$ for $1 \leqq a \leqq n$ and $\Sigma_{n+1}$ belongs to either one of $\Xi_j$'s $(1 \leqq j \leqq n)$, say $\Xi_1$, or none of them; $F_1 \cong \boldsymbol{A}_*^1$ for $2 \leqq a \leqq n$ and $F_{n+1} \cong \boldsymbol{A}_k^1$. Moreover,* (i) *$F_1 \cong \boldsymbol{A}_*^1$ either if $\Sigma_{n+1}$ is a component of $\Xi_1$ and $\Sigma_1 \cap \Sigma_{n+1} = \phi$ or if $\Sigma_{n+1}$ belongs to none of $\Xi_j$'s $(1 \leqq j \leqq n)$, and* (ii) *$F_1 \cong \boldsymbol{A}_k^1$ if $\Sigma_{n+1}$ is a component of $\Xi_1$ and $\Sigma_1 \cap \Sigma_{n+1} \neq \phi$.*

Proof. (1) Suppose$\Delta \cap U \neq \phi$. As in the statement, we may assume that $\Delta = \Sigma_{n+1}$. Then $\Xi \cap U = \Xi - \Xi \cap S_\infty$, whence $\Sigma \cap U \cong \boldsymbol{A}_k^1$. Since $(\Xi \cdot \Sigma_a) = 0$ for $1 \leqq a \leqq n$, $\Sigma_a$ belongs to the fibers of $L$. Hence we may assume that one of $\Xi_1, \cdots, \Xi_{n+1}$, say $\Xi_{n+1}$, contains none of $\Sigma_a$'s $(1 \leqq a \leqq n)$. Then the inhomogeneous coordinate $v$ of $\boldsymbol{P}_k^1$ (=the fiber of $\rho$) is an element of $k[x, y]$; indeed, the polar divisor of $v$ on $V$ has no irreducible components meeting the open set $U = \mathrm{Spec}(k[x, y])$. Since $\Xi \cap U \cong \boldsymbol{A}_k^1$, we may assume, after a suitable change of coordinates in $k[x, y]$, that $v = x$ (cf. Abhyankar-Moh's Theorem). Then the curve $v = \alpha$ on $U$ is isomorphic to $\boldsymbol{A}_k^1$ for every $\alpha \in k$. This implies that $\Sigma_a$ $(1 \leqq a \leqq n)$ belongs to one and only one of $\Xi_1, \cdots, \Xi_n$, and that none of $\Xi_j$'s $(1 \leqq j \leqq n)$ contains two of $\Sigma_a$'s. Note that $\Gamma_j \cup E_j$ is contained in $\Xi_j$ for $1 \leqq j \leqq n+1$. It is now clear that $F_{n+1} = \Delta - (\Delta \cap \Xi_{n+1}) \cup \bigcup_{j=1}^{n} ((\Gamma_j \cup E_j) \cap \Delta)$, where $(\Gamma_j \cup E_j) \cap \Delta$ is a single point if it is nonempty and $\Delta \cong \boldsymbol{P}_k^1$. Since $(\Xi \cdot \Sigma) = 1$, the multiplicity $\alpha_{n+1}$ of $\Delta$ in $\Sigma$ equals 1.

(2) Suppose $\Delta \cap U = \phi$. After a suitable change of the inhomogeneous coordinate $v$, we may assume that $\Delta$ is a component of $E_{n+1}$. We have $(\Xi \cdot \Sigma_a) = 0$ for $1 \leqq a \leqq n+1$ and $(\Xi_j \cdot \Delta) = 1$ for $1 \leqq j \leqq n$. Hence $\Gamma_j \cup E_j$, which is contained in $\Xi_j$, is connected to $\Delta$ by means of one or more of $\Sigma_a$'s for $1 \leqq j \leqq n$. This implies that $\Xi_{n+1}$ contains at most one of $\Sigma_a$'s and $\Xi_j$ $(1 \leqq j \leqq n)$ contains at most two of $\Sigma_a$'s. Suppose that $\Xi_{n+1}$ contains one of $\Sigma_a$'s, say $\Sigma_{n+1}$. Then $\Xi_j$ $(1 \leqq j \leqq n)$ contains one and only one of $\Sigma_a$'s $(1 \leqq a \leqq n)$. After a suitable change of indices, we may assume that $\Xi_j$ contains $\Sigma_j$ for $1 \leqq j \leqq n$. Then it is easy to see that $v - d_j \sim f_j^{\beta_j} f_{n+1}^{-\gamma}$ for $1 \leqq j \leqq n$, where $\beta_j > 0$ $(1 \leqq j \leqq n)$ and $\gamma > 0$; $\gamma$ is independent of $j$. Since $u - 1 \sim f_1^{\alpha_1} \cdots f_{n+1}^{\alpha_{n+1}}$, the identification of rings in 1.9.1 implies (by computing the unit groups of the rings on both hand sides) that the following matrix $A$ is unimodular,

$$A = \begin{pmatrix} \alpha_1 \cdots\cdots \alpha_n & \alpha_{n+1} \\ \beta_1 & -\gamma \\ \ddots \quad 0 & \vdots \\ \quad \ddots & \vdots \\ 0 \qquad \beta_n & -\gamma \end{pmatrix},$$

However, $\det A = (-1)^n \{\beta_1 \cdots \beta_n \alpha_{n+1} + \gamma \sum_{i=1}^{n} \beta_1 \cdots \beta_{i-1} \alpha_i \beta_{i+1} \cdots \beta_n\}$, whence $A$ is not unimodular because $n \geqq 2$. Therefore, $\Xi_{n+1}$ contains none of $\Sigma_a$'s $(1 \leqq a \leqq n+1)$. Then, after a suitable change of indices, we may assume that $\Sigma_j$ is used to connect $\Gamma_j \cup E_j$ to $\Delta$ for $1 \leqq j \leqq n$. The remaining component $\Sigma_{n+1}$ belongs to one of $\Xi_j$'s $(1 \leqq j \leqq n)$, say $\Xi_1$, or none of them. It is then clear that $F_a \simeq A^1_*$ for $2 \leqq a \leqq n$. If either $\Sigma_{n+1}$ belongs to none of $\Xi_j$'s $(1 \leqq j \leqq n)$ or $\Sigma_{n+1}$ is a component of $\Xi_1$ and $\Sigma_1 \cap \Sigma_{n+1} = \phi$, then $F_1 \simeq A^1_*$. If $\Sigma_{n+1}$ is a component of $\Xi_1$ and $\Sigma_1 \cap \Sigma_{n+1} \neq \phi$, then $F_1 \simeq A^1_k$. In any case, $F_{n+1} \simeq A^1_k$. Since $\Xi \cap U = \Xi - (\Xi \cap S_\infty) \cup (\Xi \cap \Delta)$, it is isomorphic to $A^1_*$. Q.E.D.

**3.2. Lemma.** *With the assumptions and the notations of Lemma* 3.1, *after a suitable change of coordinates of* $k[x, y]$, *we have one of the following four cases*:

(1) *Case* $\Delta \cap U \neq \phi$:

$$f_j \sim x - d_j \ (1 \leqq j \leqq n) \quad \textit{and}$$

$$f_{n+1} \sim y \cdot \prod_{j=1}^{n} (x-d_j)^{\varepsilon_j} + P(x), \quad \varepsilon_j \geqq 0, \quad P(x) \in k[x],$$

*where* $\varepsilon_j = 0$ *whenever* $(\Gamma_j \cup E_j) \cap \Delta = \phi$, *and where* $\varepsilon_j > 0$ *and* $P(d_j) \neq 0$ *whenever* $(\Gamma_j \cup E_j) \cap \Delta \neq \phi$. *Moreover,* $\alpha_{n+1} = 1$.

(2) *Case* $\Delta \cap U = \phi$ *and* $\Sigma_{n+1} \not\subset |\Xi_j|$[(*)] *for* $1 \leqq j \leqq n$:

$$f_{n+1} \sim x \quad \textit{and} \quad f_j \sim x^l(x^t y + P(x)) - d_j \quad \text{for} \quad 1 \leqq j \leqq n,$$

*where* $l > 0$, $t \geqq 0$ *and* $P(x) \in k[x]$; $\deg P(x) < t$ *and* $P(0) \neq 0$ *if* $t > 0$, *and* $P(x) = 0$ *if* $t = 0$. *Moreover,* $\alpha_{n+1} = 1$.

(3) *Case* $\Delta \cap U = \phi$, $\Sigma_{n+1} \subset |\Xi_1|$ *and* $\Sigma_1 \cap \Sigma_{n+1} \neq \phi$:

$$f_{n+1} \sim x, \quad f_1 \sim y \quad \textit{and} \quad f_j \sim x^l y + d_1 - d_j \quad \textit{for} \quad 2 \leqq j \leqq n,$$

*where* $l > 0$. *Moreover,* $\alpha_{n+1} - \alpha_1 l = \pm 1$.

(4) *Case* $\Delta \cap U = \phi$, $\Sigma_{n+1} \subset |\Xi_1|$ *and* $\Sigma_1 \cap \Sigma_{n+1} = \phi$:

$$f_{n+1} \sim x, \quad f_1 \sim x^t y + P(x) \quad \textit{and} \quad f_j \sim x^l(x^t y + P(x)) + d_1 - d_j \quad \textit{for} \quad 2 \leqq j \leqq n,$$

*where* $l$, $t > 0$ *and* $P(x) \in k[x]$ *with* $\deg P(x) < t$ *and* $P(0) \neq 0$. *Moreover,* $\alpha_{n+1} - l\alpha_1 = \pm 1$.

Proof. (1) Since none of $\Sigma_a$'s $(1 \leqq a \leqq n+1)$ is contained in $\Xi_{n+1}$ we have $v \in k[x, y]$, and since $\Xi \cap U \simeq A^1_k$, we may assume that $v = x$ (cf. Abhyankar-Moh's Theorem). Then, each fiber $\Xi_j$ $(1 \leqq j \leqq n)$ of $L$ (or $\tau$) has only one component $\Sigma_j$ which intersects $U$, and $\Sigma_j \cap U = F_j$. Since $\Xi_j$ corresponds to the value $v = d_j$, we have $f_j \sim x - d_j$. Now, the curve $F_{n+1}$, which is defined by $f_{n+1} = 0$, is written in the form:

---

(*) For an effective divisor $D$ on $V$, we denote by $|D|$ the underlying reduced curve of D.

$$f_{n+1} = P_0(x)y^N + \cdots P_N(x) ,$$

where $P_0(x), \cdots, P_N(x) \in k[x]$ and $P_0(x) \neq 0$. Since $F_{n+1}$ meets the curve $x=\alpha$ transversally in a single point for $\alpha \in k$ such that $\alpha \neq d_j$ $(1 \leqq j \leqq n)$ or $\alpha = d_j$ with $F_j \cap F_{n+1} \neq \phi$. Hence we know that $N=1$ and $P_0(x) \sim \prod_{j=1}^{n} (x-d_j)^{\varepsilon_j}$, where $\varepsilon_j = 0$ if $F_j \cap F_{n+1} \neq \phi$, i.e., $(\Gamma_j \cup E_j) \cap \Delta = \phi$. If $F_j \cap F_{n+1} = \phi$, i.e., $(\Gamma_j \cup E_j) \cap \Delta \neq \phi$, we have $\varepsilon_j > 0$ and $P_1(d_j) \neq 0$. Thus, we obtain the stated form for $f_{n+1}$. On the other hand, we have:

$$u-1 \sim f_1^{\alpha_1} \cdots f_{n+1}^{\alpha_{n+1}} \quad \text{and} \quad v-d_j \sim f_j^{\beta_j} \quad \text{with} \quad \beta_j > 0 \quad \text{for} \quad 1 \leqq j \leqq n .$$

The identification of rings in 1.9.1 implies that the following matrix $A$ is unimodular,

$$A = \begin{pmatrix} \alpha_1 \cdots \alpha_n & \alpha_{n+1} \\ \beta_1 \qquad & 0 \\ \ddots \quad 0 & \vdots \\ 0 \quad \ddots & \vdots \\ \qquad \beta_n & 0 \end{pmatrix} .$$

Since $\det A = (-1)^n \alpha_{n+1} \beta_1 \cdots \beta_n$, we have $\beta_1 = \cdots = \beta_n = \alpha_{n+1} = 1$.

(2) Since $\Sigma_{n+1} \not\subset |\Xi_j|$ for $1 \leqq j \leqq n$, there is one more reducible fiber $\Xi_0$, other than $\Xi_j$'s $(1 \leqq j \leqq n+1)$, such that

$$\Xi_0 = T + l\Sigma_{n+1} + Z ,$$

where $(T \cdot S_\infty) = 1$, $(T \cdot \Sigma_{n+1}) \leqq 1$, $|Z| \subset E_{n+1}$ if $Z > 0$, and $l > 0$. Then $T_0$ $(:= T \cap U) \cong A_k^1$ if $(T \cdot \Sigma_{n+1}) = 1$ and $T_0 \cong A_*^1$ if $(T \cdot \Sigma_{n+1}) = 0$. Since $\Xi \cap U \cong A_*^1$, we may assume, by virtue of Theorem 2.3 that $f_{n+1} = x$ and $T_0$ is defined by $y=0$ (if $T \cong A_k^1$) or $x^t y + P(x) = 0$ (if $T \cong A_*^1$), where $t > 0$ and $P(x) \in k[x]$ with $\deg P(x) < t$ and $P(0) \neq 0$. Moreover, we may assume that the fiber $\Xi_0$ corresponds to the value $v=0$, where $v \in k[x, y]$ because $|\Xi_{n+1}| \cap U = \phi$. Since $\Xi_j$ corresponds to the value $v = d_j$ and $F_j = |\Xi_j| \cap U$ for $1 \leqq j \leqq n$, we have $f_j \sim x^l(x^t y + P(x)) - d_j$. We obtain $\alpha_{n+1} = 1$ by the same argument as in the case (1).

(3) The fiber $\Xi_1$ is, then, written in the form:

$$\Xi_1 = \Sigma_1 + l\Sigma_{n+1} + Z ,$$

where $(\Sigma_1 \cdot \Delta) = 1$, $(\Sigma_1 \cdot \Sigma_{n+1}) = 1$, $|Z| = E_1 \cup \Gamma_1$, and $l > 0$. Since $F_1 \cong F_{n+1} \cong A_k^1$, we may assume that $f_{n+1} = x$ and $f_1 = y$ (cf. Lemma 1.10). Since $v \in k[x, y]$ and the fiber $\Xi_1$ corresponds to the value $v = d_1$, we may assume that $v = x^l y + d_1$. Then we have $f_j \sim v - d_j = x^l y + d_1 - d_j$ for $2 \leqq j \leqq n$, because $\Sigma_j \subset |\Xi_j|$ and $(\Sigma_j \cdot \Delta) = 1$. On the other hand, we have:

$$u-1 \sim f_1^{\alpha_1} \cdots f_{n+1}^{\alpha_{n+1}} , \quad v-d_1 \sim f_1 f_{n+1}^l$$
$$\text{and} \quad v-d_j \sim f_j \quad \text{for} \quad 2 \leqq j \leqq n .$$

By the same reasoning as in the case (1), we have the following unimodular matrix

$$A=\begin{pmatrix} \alpha_1\cdots\alpha_n & \alpha_{n+1} \\ 1 & l \\ \quad 1 \quad 0 & \\ \quad\ddots & 0 \\ 0 \quad \ddots & \\ \quad\quad 1 & \end{pmatrix}.$$

Since $\det A=(-1)^n(\alpha_{n+1}-\alpha_1 l)$, we have $\alpha_{n+1}-\alpha_1 l=\pm 1$.

(4) The fiber $\Xi_1$ is, then, written in the form:

$$\Xi_1=\Sigma_1+l\Sigma_{n+1}+Z\,,$$

where $(\Xi_1\cdot\Delta)=1$, $(\Sigma_1\cdot\Sigma_{n+1})=0$, $\Sigma_{n+1}\cap|Z|\neq\phi$, $|Z|=E_1\cup\Gamma_1$ and $l>0$. Since $F_{n+1}\cong \boldsymbol{A}^1_k$, we may assume that $f_{n+1}=x$. Since $v\in k[x,y]$ and $\Xi\cap U\cong \boldsymbol{A}^1_*$, $v$ is a generically rational polynomial with $n=1$, and the curve $v=d_1$ is a reducible member of $\Lambda_0(v)$ with two mutually non-intersecting components. By virtue of Lemma 2.2, we may set

$$f_1=x^t y+P(x)\,,\quad t>0\,,\quad P(x)\in k[x]$$
$$\text{with}\quad \deg P(x)<t \quad\text{and}\quad P(0)\neq 0\,.$$

Then we may assume that $v-d_1=f_1 f_{n+1}^l=x^l(x^t y+P(x))$. Hence $f_j\sim v-d_j=x^l(x^t y+P(x))+d_1-d_j$ for $2\leqq j\leqq n$. We obtain $\alpha_{n+1}-l\alpha_1=\pm 1$ by the same argument as in the case (3). Q.E.D.

**3.3. Theorem.** *Let $f$ be a generically rational, irreducible polynomial of simple type in $k[x,y]$ with $n>1$ and only one reducible member in $\Lambda_0(f)$. Then, after a suitable change of coordinates in $k[x,y]$, $f$ is reduced to one of the following four polynomials*:

(1) $f\sim (\prod_{j=1}^{n}(x-d_j)^{\alpha_j})\cdot(y\cdot\prod_{j=1}^{n}(x-d_j)^{\varepsilon_j}+P(x))+1$,

*where $d_1,\cdots,d_n$ are mutually distinct elements in $k$ and $P(x)\in k[x]$; $\alpha_j>0$ and $\varepsilon_j\geqq 0$ for $1\leqq j\leqq n$; $P(d_j)\neq 0$ if $\varepsilon_j>0$.*

(2) $f\sim x\cdot\prod_{j=1}^{n}(x^l(x^t y+P(x))-d_j)^{\alpha_j}+1$,

*where $l>0$, $t\geqq 0$ and $P(x)\in k[x]$; $\deg P(x)<t$ and $P(0)\neq 0$ if $t>0$ and $P(x)=0$ if $t=0$; $\alpha_j$'s and $d_j$'s are as in the case (1).*

(3) $f\sim x^\beta y^{\alpha_1}\cdot\prod_{j=2}^{n}(x^l y-d_j)^{\alpha_j}+1$,

*where $d_2,\cdots,d_n$ are mutually distinct elements in $k^*$; $\beta>0$, $l>0$ and $\alpha_j>0$ for*

$1 \leqq j \leqq n$; $\beta-\alpha_1 l=\pm 1$.

(4) $f \sim x^{\beta} \cdot (x^t y+P(x))^{\alpha_1} \cdot \prod_{j=2}^{n} (x^l (x^t y+P(x))-d_j)^{\alpha_j}+1$,

*where* $t>0$ *and* $P(x) \in k[x]$ *with* $\deg P(x)<t$ *and* $P(0) \neq 0$; $\beta$, $l$, $\alpha_j$*'s and* $d_j$*'s are as in the case* (3).

Proof. Follows easily from Lemma 3.2.

**3.4.** By comparison of the unit groups of two rings, which are connected to each other by the identification of rings as in 1.9.1, we can prove the following:

**Proposition.** *Let* $f_1, \cdots, f_{n+1}$ $(n \geqq 2)$ *be mutually distinct irreducible polynomials in* $k[x, y]$. *Assume that we have the identification of rings,*

$$k[x, y, (f_1 \cdots f_{n+1})^{-1}] = k[u, v, u^{-1}, (\prod_{j=1}^{n} (v-d_j))^{-1}],$$

*where* $u=f_1^{\alpha_1} \cdots f_{n+1}^{\alpha_{n+1}}$ *with* $\alpha_i>0$ $(1 \leqq i \leqq n+1)$, *and* $d_1, \cdots, d_n$ *are mutually distinct elements of* $k$. *Then, after a suitable change of indices and a suitable change of variables* $u$ *and* $v$, *we are reduced to the following case*:

$$v-d_1 \sim f_1^{\beta_1} f_2^{\beta_2} \quad \text{and} \quad v-d_j \sim f_{j+1} \quad \text{for} \quad 2 \leqq j \leqq n,$$

*where* $\beta_1, \beta_2 \geqq 0$ *and* $\alpha_1 \beta_2-\alpha_2 \beta_1=\pm 1$.

**3.5.** Next, we shall consider the case where $r \geqq 2$ and $m_i<n$ for every $i$ $(1 \leqq i \leqq r)$. Here, we retain the notations in 1.9. Since $m_i \geqq 2$, we have $n \geqq 3$.

**Lemma.** *With the assumptions and the notations as above, we have*:

(1) *For every* $i$ $(1 \leqq i \leqq r)$, $E_{ij} \cap \Xi=\phi$ *for* $1 \leqq j \leqq n+1$, *where* $\Xi$ *is a general fiber of the* $\boldsymbol{P}^1$*-fibration* $\tau: V \rightarrow \boldsymbol{P}_k^1$ (cf. 1.9.6), *and there exists an irreducible component* $\Delta_i$ *among* $S_{ia}$*'s* $(1 \leqq a \leqq m_i)$ *such that* $(\Delta_i \cdot \Xi)=1$, i.e., $\Delta_i$ *is a cross-section of* $\tau$; *moreover,* $E_{ij} \cup \Gamma_j \subset |\Xi_j|$ *for* $1 \leqq i \leqq r$ *and* $1 \leqq j \leqq n+1$.

(2) $\Xi \cap U \cong \boldsymbol{A}_k^1$.

(3) *For every* $i$ $(1 \leqq i \leqq r)$, $F_{ia} \cong \boldsymbol{A}_k^1$ *if* $S_{ia} \neq \Delta_i$ $(1 \leqq a \leqq m_i)$ *and* $\Delta_i \cap U= \Delta_i-\bigcup_{j=1}^{n+1} ((\Gamma_j \cup E_{ij}) \cap \Delta_i)$.

(4) *One of* $\Xi_j$*'s* $(1 \leqq j \leqq n+1)$, *say* $\Xi_{n+1}$, *contains none of* $S_{ia}$*'s* $(1 \leqq i \leqq r$ *and* $1 \leqq a \leqq m_i)$. *Then, every* $\Xi_j$ $(1 \leqq j \leqq n)$ *contains one and only one of* $S_{ia}$*'s* $(1 \leqq i \leqq r$ *and* $1 \leqq a \leqq m_i)$ *with multiplicity* 1. *After a suitable change of indices, we may assume that, for every* $i$ $(1 \leqq i \leqq r)$, *we have*:

$$\Delta_i = S_{i1} \quad \text{and} \quad S_{ia} \subset |\Xi_b|,$$

*where* $b:=\gamma(i, a)=(m_1-1)+\cdots+(m_{i-1}-1)+(a-1)$ *for* $2 \leqq a \leqq m_i$.

Proof. (1) Suppose that, for some $i$ $(1\leqq i\leqq r)$, $S_i$ contains an irreducible component $\Delta$ such that $\Delta\cap U=\phi$ and $(\Delta\cdot\Xi)=1$, i.e., $\Delta$ is a cross-section of $\tau$. Suppose, for convenience's sake, that $\Delta\subset E_{i,n+1}$. Then $(\Gamma_j\cup E_{ij})\cap\Xi=\phi$ for $1\leqq j\leqq n$, whence $\Gamma_j\cup E_{ij}\subset|\Xi_j|$. Since $\Delta$ is a cross-section of $\tau$, every $\Gamma_j\cup E_{ij}$ $(1\leqq j\leqq n)$ is connected to $\Delta$ by one or more components of $S_{ia}$'s $(1\leqq a\leqq m_i)$. However, since $m_i<n$, this is impossible. This implies that there exists a component $\Delta_i$ among $S_{ia}$'s $(1\leqq a\leqq m_i)$ such that $(\Delta_i\cdot\Xi)=1$, i.e., $\Delta_i$ is a cross-section. Then $(\Gamma_j\cup E_{ij})\cap\Xi=\phi$ for $1\leqq j\leqq n+1$, whence $(\Gamma_j\cup E_{ij})\subset|\Xi_j|$.

(2) It is now clear that $\Xi\cap U=\Xi-(\Xi\cap S_\infty)\cong A_k^1$.

(3) Suppose that $S_{ia}\neq\Delta_i$ $(1\leqq a\leqq m_i)$. Since $S_{ia}\cap\Xi=\phi$, $S_{ia}$ is a component of some $\Xi_j$ $(1\leqq j\leqq n+1)$. If $S_{ia}\cap\Delta_i\neq\phi$, then $S_{ia}\cap(\Gamma_j\cup E_{ij})\neq\phi$; indeed, if otherwise, $S_{ia}\subset U$, which is a contradiction. Hence $F_{ia}\cong A_k^1$. If $S_{ia}\cap\Delta_i=\phi$ then $E_{ij}\neq\phi$ and $E_{ij}\cap S_{ia}\neq\phi$, whence $F_{ia}\cong A_k^1$. It is now clear that $\Delta_i\cap U=\Delta_i-\bigcup_{j=1}^{n+1}((\Gamma_j\cup E_{ij})\cap\Delta_i)$.

(4) Since $\Delta_i$ $(1\leqq i\leqq r)$ is a cross-section of $\tau$, $\Delta_i$ is not contained in any one of $\Xi_j$'s $(1\leqq j\leqq n)$. The remaining components of $S_{ia}$'s $(1\leqq i\leqq r$ and $1\leqq a\leqq m_i)$ are contained in the union of $\Xi_1,\cdots,\Xi_{n+1}$. Since $\sum_{i=1}^{r}(m_i-1)=n$ (cf. Lemma 1.6), one of $\Xi_j$'s, say $\Xi_{n+1}$, contains none of $S_{ia}$'s. Then the inhomogeneous coordinate $v$ is an element of $k[x,y]$. Since $\Xi\cap U\cong A_k^1$, we may assume that $v=x$ (cf. Abhyankar-Moh's Theorem). Then $|\Xi_j|\cap U$, which corresponds to the value $v=d_j$, is irreducible for $1\leqq j\leqq n$. Hence, every $\Xi_j$ $(1\leqq j\leqq n)$ contains one and only one of $S_{ia}$'s. The remaining assertion is easy to prove.

Q.E.D

**3.6. Lemma.** *With the assumptions and the notations as in Lemma* 3.5, *we have*:

(1) $f_{ia}\sim x-d_b$ *with* $b=\gamma(i,a)$, *for* $1\leqq i\leqq r$ *and* $2\leqq a\leqq m_i$.

(2) $f_{i1}\sim y\cdot\prod_{j=1}^{n}(x-d_j)^{\varepsilon_j}+P_i(x)$ *with* $\varepsilon_j\geqq0$ *and* $P_i(x)\in k[x]$, *where* $\varepsilon_j=0$ *if* $(\Gamma_j\cup E_{ij})\cap\Delta_i=\phi$, *and* $\varepsilon_j>0$ *and* $P_i(d_j)\neq0$ *if* $(\Gamma_j\cup E_{ij})\cap\Delta_i\neq\phi$.

(3) $\alpha_{i1}=1$ *for every* $i$ $(1\leqq i\leqq r)$.

Proof. (1) As in the proof of Lemma 3.5, we may assume that $v=x$. Then, since $S_{ia}\subset|\Xi_b|$ with $b=\gamma(i,a)$ and $S_{ia}=|\Xi_b|\cap U$, we have $f_{ia}\sim x-d_b$.

(2) $f_{i1}$ can be determined by the same argument as used in the proof of Lemma 3.2, the assertion (1). We only note that $\Delta_i$ meets (or does not meet, resp.) the unique component among $S_{ia}$'s contained in $\Xi_j$ if and only if $(\Gamma_j\cup E_{ij})\cap\Delta_i=\phi$ (or $(\Gamma_j\cup E_{ij})\cap\Delta_i\neq\phi$).

(3) We have:

$$u-c_i\sim\prod_{a=1}^{m_i}f_{ia}^{\alpha_{ia}}\quad\text{for}\quad 1\leqq i\leqq r$$

and $v-d_j\sim f_{ia}$ with $j=\gamma(i,a)$, for $1\leqq j\leqq n$.

Let $A_i$ $(1\leqq i\leqq r)$ and $A$ be the following square matrices of size $m_i$ and $n+r$, respectively;

$$A_i=\begin{pmatrix} \alpha_{i1}\cdots\alpha_{im_i} & & \\ 1 & & \\ & \ddots & 0 \\ 0 & \ddots & \\ & & 1 \end{pmatrix} \quad\text{and}\quad A=\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_r \end{pmatrix}.$$

The identification of rings in 1.9.1 implies that $A$ is a unimodular matrix. Hence $\alpha_{i1}=1$ for $1\leqq i\leqq r$. Q.E.D.

**3.7. Theorem.** *Let $f$ be a generically rational, irreducible polynomial of simple type in $k[x,y]$ with $n>1$, $r\geqq 2$ and $n>m_i$ for every $i$ $(1\leqq i\leqq r)$; indeed, $n\geqq 3$. Then, after a suitable change of coordinates in $k[x,y]$, $f$ has one of the following presentations*:

$$f\sim\Big(\prod_{a=2}^{m_i}(x-d_{e(a)})^{\alpha_{ia}}\Big)\cdot\Big(y\cdot\prod_{j=1}^{n}(x-d_j)^{\varepsilon_j}+P_i(x)\Big)+c_i \quad \textit{for} \quad 1\leqq i\leqq r\,,$$

*where $d_1,\cdots,d_n$ are mutually distinct elements of $k$, $c_i\in k^*$; $\alpha_{ia}>0$, $\varepsilon_j\geqq 0$ and $e(a):=\gamma(i,a)$* (cf. *Lemma* 3.5); *$P_i(d_j)\neq 0$ if $\varepsilon_j>0$; $\varepsilon_j>0$ if $j\neq e(a)$ for all* a *$(1\leqq a\leqq m_i)$.*

Proof. Follows easily from Lemmas 3.5 and 3.6.

**3.8.** Finally, we shall consider the case where $r\geqq 2$ and $m_i=n$ for some $i$ $(1\leqq i\leqq r)$. We may assume that $m_1=n$. Lemma 1.6 then implies $r=2$, $m_1=n$ and $m_2=2$. We consider first the case $n\geqq 3$; the case $n=2$ will be treated in the paragraphs 3.11 and 3.12. The $\boldsymbol{P}^1$-fibration $\rho: V\to\boldsymbol{P}_k^1$ has two singular fibers $S_1$ and $S_2$, where $S_1\cap U$ has $n$ irreducible components and $S_2\cap U$ has 2 irreducible components. Here we retain the notations in 1.9. Let $\Delta_i$ $(i=1,2)$ be the irreducible component in $S_i$ such that $(\Delta_i\cdot\Xi)=1$, where $\Delta_2$ must be one of $S_{21}$ and $S_{22}$, say $S_{21}$, (cf. the proof of the assertion (1) in Lemma 3.5).

**Lemma.** *Assume that $n\geqq 3$. With the assumptions and the notations as above, we have one of the following two cases*:

(I) *Case $\Delta_1\cap U\neq\phi$. Then we have the same situation as stated in Lemma* 3.5 *with $r=2$.*

(II) *Case $\Delta_1\cap U=\phi$. We may assume that $\Delta_1\subset E_{1,n+1}$. Then the following assertions hold true*:

(1) *For every $j$ $(1\leqq j\leqq n)$, $\Gamma_j\cup E_{1j}\subset|\Xi_j|$; $S_{1a}$ $(1\leqq a\leqq n)$ belongs to one and only one of $\Xi_j$'s $(1\leqq j\leqq n)$; after a change of indices, we may assume that $S_{1a}$ is contained in $\Xi_a$ with multiplicity* 1 *for $1\leqq a\leqq n$; $F_{1a}\cong\boldsymbol{A}_*^1$ for $1\leqq a\leqq n$.*

(2) *$\Xi\cap U\cong\boldsymbol{A}_*^1$, where $\Xi$ is a general fiber of the $\boldsymbol{P}^1$-fibration $\tau: V\to\boldsymbol{P}_k^1$.*

(3) *$S_{22}$ belongs to one of $\Xi_j$'s $(1\leqq j\leqq n+1)$, and $S_{22}\not\subset|\Xi_{n+1}|$; we may*

*assume that* $S_{22}\subset|\Xi_1|$; $F_{22}\cong A_k^1$ *and* $F_{21}=S_{21}-\bigcup_{j=1}^{n+1}(\Gamma_j\cup E_{2j})\cap S_{21}$.

Proof. (I) If $\Delta_1\cap U\neq\phi$, we have only to follow the arguments in Lemma 3.5.

(II) Assume that $\Delta_1\cap U=\phi$. We may assume that $\Delta_1\subset E_{1,n+1}$. Since $E_{1j}\cap\Xi=\phi$, we have $\Gamma_j\cup E_{1j}\subset|\Xi_j|$ for $1\leqq j\leqq n$. Since $(\Gamma_j\cup E_{1j})\cap\Delta_1=\phi$, $\Gamma_j\cup E_{1j}$ is connected to $\Delta_1$ by one and only one of $S_{1a}$'s for $1\leqq j\leqq n$. After a suitable change of indices a, we may assume that $S_{1a}\subset|\Xi_a|$. Then it is clear that $S_{1a}$ is a component of $\Xi_a$ with multiplicity 1 and that $F_{1a}\cong A_*^1$ for $1\leqq a\leqq n$. This proves the assertion (1).

(2) Since $\Xi\cap U=\Xi-(\Xi\cap S_\infty)\cup(\Xi\cap\Delta_1)$, $\Xi\cap U\cong A_*^1$.

(3) Suppose that $S_{22}$ belongs to none of $\Xi_j$'s $(1\leqq j\leqq n+1)$. Then $S_{22}\subset U$, which is a contradiction. Hence $S_{22}$ belongs to one of $\Xi_j$'s $(1\leqq j\leqq n+1)$. Suppose that $S_{22}\subset|\Xi_{n+1}|$. Then, in the identification of rings in 1.9.1, we have:

$$u-c_1\sim\prod_{a=1}^{n}f_{1a}^{\alpha_{1a}},\qquad u-c_1\sim f_{21}^{\alpha_{21}}f_{22}^{\alpha_{22}}$$
$$v-d_j\sim f_{1j}f_{22}^{-\beta}\quad\text{for}\quad 1\leqq j\leqq n,$$

where $\beta>0$, $\beta$ is independent of $j$. By comparison of the unit groups of the rings of both hand sides in 1.9.1, we know that the following matrix $A$ is unimodular,

$$A=\begin{pmatrix}\alpha_{11}\cdots\alpha_{1n} & 0 & 0\\ 0\ \cdots\ 0 & \alpha_{21} & \alpha_{22}\\ 1 \qquad & 0 & -\beta\\ \ddots\quad 0 & \vdots & \vdots\\ 0\quad\ddots & \vdots & \vdots\\ \qquad 1 & 0 & -\beta\end{pmatrix}.$$

Since $\det A=-\beta\alpha_{21}\cdot(\sum_{i=1}^{n}\alpha_{1i})$ and $n\geqq3$, $A$ is not a unimodular matrix, which is a contradiction. Hence $S_{22}\not\subset|\Xi_{n+1}|$. Then we may assume that $S_{22}\subset|\Xi_1|$. Then $F_{22}\cong A_k^1$ (cf. the proof of the assertion (3) of Lemma 3.5), and $F_{21}=S_{21}-\bigcup_{j=1}^{n+1}(\Gamma_j\cup E_{2j})\cap S_{21}$. Q.E.D.

**3.9. Lemma.** *With the assumptions and the notations as in Lemma* 3.8, *assume that* $\Delta_1\cap U=\phi$. *Then, after a suitable change of coordinates in* $k[x,y]$, *we have*:

(1) $f_{22}\sim x$, $f_{11}\sim x^ly+P(x)$, *and* $f_{1j}\sim x(x^ly+P(x))+d_1-d_j$ *for* $2\leqq j\leqq n$, *where* $l>0$, $P(x)\in k[x]$, $\deg P(x)<l$ *and* $P(0)\neq0$.

(2) $\alpha_{11}=\alpha_{21}=1$.

Proof. Since $F_{22}\cong A_k^1$, we may assume that $f_{22}=x$ (cf. Abhyankar-Moh's Theorem). Since $|\Xi_{n+1}|\cap U=\phi$, the inhomogeneous coordinate $v$ is an element of $k[x,y]$. Since $\Xi\cap U\cong A_*^1$, $v$ is a generically rational polynomial with $n=1$, and the curve $v=d_1$ is the unique reducible member of $\Lambda_0(v)$ with two disjoint components $F_{22}$ and $F_{11}$. Since $(S_{11}\cdot\Delta_1)=1$, the fiber $\Xi_1$ is of the form:

$$\Xi_1 = S_{11}+tS_{22}+Z \quad \text{with} \quad |Z| = \Gamma_1 \cup E_{11}.$$

On the other hand, since $F_{11}\cap F_{22}=\phi$, $f_{11}$ is written in the form:

$$f_{11} = x^l y + P(x),$$

where $l>0$ and $P(x)\in k[x]$ with $\deg P(x)<l$ and $P(0)\neq 0$. Then we have:

$$v-d_j\sim x^t(x^l y+P(x))+d_1-d_j \quad \text{for} \quad 1\leqq j\leqq n,$$
$$v-d_j\sim f_{1j} \quad \text{for} \quad 2\leqq j\leqq n.$$

Since $u-c_1\sim \prod_{a=1}^{n} f_{1a}^{\alpha_{1a}}$ and $u-c_2\sim f_{21}^{\alpha_{21}}f_{22}^{\alpha_{22}}$, we have the following unimodular matrix,

$$A = \begin{pmatrix} \alpha_{11} \cdots \alpha_{1n} & 0 & 0 \\ 0 \cdots\cdots 0 & \alpha_{21} & \alpha_{22} \\ \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & \begin{matrix} 0 \\ \\ \end{matrix} & \begin{matrix} t \\ 0 \\ \end{matrix} \end{pmatrix}.$$

Since $\det A=\alpha_{11}\alpha_{21}t=1$, we have $\alpha_{11}=\alpha_{21}=t=1$. Q.E.D.

**3.10. Theorem.** *Let $f$ be a generically rational, irreducible polynomial of simple type in $k[x, y]$ with $n\geqq 3$, $r=2$, $m_1=n$ and $m_2=2$. Then, after a suitable change of coordinates in $k[x, y]$, $f$ is reduced to a polynomial of one of the following types:*

(I) *$f$ is a polynomial of the type given in Theorem 3.7, where $r=2$, $m_1=n$ and $m_2=2$;*

(II) $f\sim (x^l y+P(x))\cdot \prod_{j=2}^{n} (x(x^l y+P(x))-d_j)^{\alpha_j}+c,$

*where $l>0$, $\alpha_j>0$ $(2\leqq j\leqq n)$, $P(x)\in k[x]$ with $\deg P(x)<l$ and $P(0)\neq 0$, $c\in k^*$, and $d_j$'s $(2\leqq j\leqq n)$ are mutually distinct elements in $k^*$.*

Proof. Follows from Lemmas 3.8 and 3.9.

**3.11.** We shall consider the remaining case: $n=2$ and $m_1=m_2=2$. As in 3.8, let $\Delta_i$ be an irreducible component of $S_i$ such that $(\Delta_i\cdot\Xi)=1$, where $i=1, 2$. We shall prove the following:

**Lemma.** *With the assumptions and the notations as above, we have either $\Delta_1\cup U\neq\phi$ or $\Delta_2\cap U\neq\phi$.*

Proof. Suppose that $\Delta_i\cap U=\phi$ for $i=1, 2$. We may assume that $\Delta_1\subset E_{13}$ and $\Gamma_i\cup E_{1i}\cup S_{1i}\subset|\Xi_i|$ for $i=1, 2$, (cf. the proof of Lemma 3.8); the component $S_{1i}$ has multiplicity 1 in $\Xi_i$ for $i=1, 2$. Suppose $\Delta_2 \subset E_{23}$. Then $E_{13} \neq \phi$ and $E_{23} \neq \phi$, which contradicts the property 1.9.4; this is the only one place where we have to depend on Lemma 1.4. Therefore, $\Delta_2 \not\subset E_{23}$. We may assume that $\Delta_2 \subset E_{22}$, $\Gamma_1 \cup E_{21} \cup S_{21} \subset |\Xi_1|$ and $\Gamma_3 \cup E_{23} \cup S_{22} \subset |\Xi_3|$. Then we have: $u - c_i \sim f_{i1}^{\alpha_{i1}} f_{i2}^{\alpha_{i2}}$ for $i=1, 2$, $v - d_1 \sim f_{11} f_{21} f_{22}^{-\gamma}$ and $v - d_2 \sim f_{12} f_{22}^{-\gamma}$, where $\gamma > 0$. Hence we have the following unimodular matrix,

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 & 0 \\ 0 & 0 & \alpha_{21} & \alpha_{22} \\ 1 & 0 & 1 & -\gamma \\ 0 & 1 & 0 & -\gamma \end{pmatrix}.$$

Since $\det A = -\alpha_{11}\alpha_{22} - \gamma(\alpha_{11}\alpha_{21} + \alpha_{12}\alpha_{21}) < -1$, this is a contradiction. Q.E.D.

Therefore, we have two cases: $\Delta_i \cap U \neq \phi$ for $i=1, 2$; $\Delta_1 \cap U = \phi$ and $\Delta_2 \cap U \neq \phi$. The case $\Delta_1 \cap U \neq \phi$ and $\Delta_2 \cap U = \phi$ is reduced to the second case. In both cases, we are reduced to the same situation as in Lemma 3.8 with $r=2$ and $m_i=2$ $(i=1, 2)$. Therefore, we have:

**3.12. Theorem.** *Theorem* 3.10 *is valid in the case* $n=2$.

**3.13.** According to Russell [10], a polynomial $f$ in $k[x, y]$ is said to be a *good* field generator if there exists a polynomial $g$ in $k[x, y]$ such that $k(x, y) = k(f, g)$; otherwise, $f$ is said to be a *bad* field generator. The observations in §§2, 3 imply the following:

**Theorem.** *Let $f$ be a generically rational polynomial of simple type in $k[x, y]$. Then there exists a generically rational polynomial $g$ with at most two places at infinity such that $k(x, y) = k(f, g)$. In particular, $f$ is a good field generator.*

We note that a field generator is not necessarily good. An example of a bad field generator was given in [10].

---

## References


[1] S.S. Abhyankar and T.T. Moh: *Embeddings of the line in the plane,* J. Reine Angew. Math. **276** (1975), 148–166.

[2] M.H. Gizatullin: *On affine surfaces that can be completed by a nonsingular rational curve,* Izv. Akad. Nauk SSSR, Ser. Mat. **34** (1970), 778–802; Math. USSR-Izvestija **4** (1970), 787–810.

[3] M. Miyanishi: *An algebraic characterization of the affine plane,* J. Math. Kyoto Univ. **15** (1975), 169–184.

[4] M. Miyanishi: *Analytic irreducibility of certain curves on a nonsingular affine rational surface,* Proc. Internat. Sympos. on Algebraic Geometry (Kyoto), Kinokuniya, Tokyo, 1977, 575–587.

[5] M. Miyanishi: Lectures on curves on rational and unirational surfaces, Tata

Institute of Fundamental Research, Bombay, 1978.
[6] S. Mori: *Embeddings of $A^2$ into rational ruled surfaces,* Kōkyuroku **183** (1973), Res. Inst. Math. Analysis, Kyoto Univ., (in Japanese).
[7] J.A. Morrow: *Minimal normal compactifications of $C^2$*, Rice Univ. Studies, 1973, 97–112.
[8] C.P. Ramanujam: *A topological characterization of the affine plane as an algebraic variety,* Ann. of Math. **94** (1971), 69–88.
[9] P. Russell: *Field generators in two variables,* J. Math. Kyoto Univ. **15** (1975), 555–571.
[10] P. Russell: *Good and bad field generators,* J. Math. Kyoto Univ. **17** (1977), 319–331.
[11] I.R. Šafarevič et al.: Algebraic surfaces, Proc. Steklov Inst. Math. **75** (1965).
[12] H. Saito: *Fonctions entières qui se reduisent à certains polynômes.* I, Osaka J. Math. **9** (1972), 293–332.
[13] H. Saito: *Fonctions entières qui se reduisent à certains polynômes.* II, Osaka J. Math. **14** (1977), 649–674.
[14] T. Sugie: *Certain affine plane curves with two places at infinity,* Thesis for Master's degree, Osaka Univ., 1978, (in Japanese).
[15] M. Suzuki: *Propriétés topologiques des polynômes de deux variables complexes et automorphismes algébriques de l'espace $C^2$*, J. Math. Soc. Japan **26** (1974), 241–257.

Masayoshi Miyanishi
Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan

Tohru Sugie
Department of Mathematics
Kyoto University
Kyoto, 606
Japan