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GENERICALLY RATIONAL POLYNOMIALS

Dedicated to Professor Y. Nakai on his sixtieth birthday

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Introduction. Let k be an algebraically closed field of characteristic zero. Let $k[x, y]$ be a polynomial ring of two variables and let $\boldsymbol{A}_k^2 = \mathrm{Spec}(k[x, y])$. Embed $\boldsymbol{A}_k^2$ into the projective plane $\boldsymbol{P}_k^2$ as the complement of a line l_∞. Let $f \in k[x, y]$ be an irreducible polynomial, let F_α be the curve on $\boldsymbol{A}_k^2$ defined by $f = \alpha$ for every $\alpha \in k$ and let C_α be the closure of F_α in $\boldsymbol{P}_k^2$. Then the set $\Lambda(f) := \{C_\alpha; \alpha \in k \cup (\infty)\}$ is a linear pencil on $\boldsymbol{P}_k^2$ defined by f, where $C_\infty = dl_\infty$, d being the degree of f. The set $\Lambda_0(f) := \{F_\alpha; \alpha \in k\}$ is called *the linear pencil on $\boldsymbol{A}_k^2$ defined by f*. The polynomial f is called *generically rational* when the general members of $\Lambda(f)$ (or $\Lambda_0(f)$) are irreducible rational curves. Since the algebraic function field $k(x, y)$ of one variable over the subfield $k(f)$ then has genus 0, Tsen's theorem says that f is generically rational if and only if f is a field generator in the sense of Russell [9, 10], i.e., there is an element $g \in k(x, y)$ such that $k(x, y) = k(f, g)$. If f is a generically rational polynomial, we can associate with f a non-negative integer n, where $n+1$ is the number of places at infinity of a general member F_α of $\Lambda_0(f)$, i.e., the number of places of F_α whose centers lie outside $\boldsymbol{A}_k^2$.

If $d \geqq 1$, the pencil $\Lambda(f)$ has base points situated outside $\boldsymbol{A}_k^2$. Let $\varphi: W \to \boldsymbol{P}_k^2$ be the shortest succession of quadratic transformations with centers at the base points (including infinitely near base points) of $\Lambda(f)$ such that the proper transform Λ' of $\Lambda(f)$ by φ has no base points. Then the linear pencil Λ' defines a morphism $\rho: W \to \boldsymbol{P}_k^1$, whose general fibers are the proper transforms of general members of $\Lambda(f)$; thence they are nonsingular rational curves by virtue of Bertini's theorem. Moreover, W contains in a canonical way an open subset isomorphic to $\boldsymbol{A}_k^2$. A generically rational polynomial f is said to be *of simple type* if the morphism ρ has $n+1$ cross-sections contained in the boundary set $W - \boldsymbol{A}_k^2$ (cf. Definition 1.8, below).

If $n=0$, a generically rational polynomial f is sent to one of the coordinates x, y of $\boldsymbol{A}_k^2$ by a biregular automorphism of $\boldsymbol{A}_k^2$ (cf. Abhyankar-Moh's theorem [1, 4]); hence f is of simple type. If $n=1$, a generically rational polynomial is always of simple type (cf. Theorem 2.3, below). However, if $n>1$, a generi-

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caily rational polynomial is not necessarily of simple type, as clarified by Saito [13]. In effect, Saito determined in [12, 13] the standard forms of generically rational polynomials for $n=1, 2$ by analytic methods.

One of the purposes of the present article is to determine the standard forms of generically rational polynomials of simple type for arbitrary $n\geqq 0$ by means of purely algebraic methods (cf. the results in §§2, 3).

The notations which we use frequently in this article are the following:

Let R be a k-algebra. Then R^* is the multiplicative group consisting of invertible elements of R; R^* is called the unit group of R; if R is a finitely generated, normal k-algebra then R^*/k^* is a finitely generated abelian group (cf. [3]). For elements x, y of R^*, we denote $x\sim y$ if $xy^{-1}\in k^*$.

Let V be a nonsingular projective surface. Let D_1, D_2 be divisors on V. Then $(D_1\cdot D_2)$ is the intersection number (or multiplicity) of D_1, D_2 on V. Let C be a curve on V and let P be a point on C. Then $\mathrm{mult}_P C$ is the multiplicity of C at P. If D_1 and D_2 are divisors, locally effective at P, then $i(D_1, D_2; P)$ is the local intersection multiplicity of D_1, D_2 at P. For an effective divisor D, we denote by $|D|$ the underlying reduced curve.(*)

Let $\varphi: W\rightarrow V$ be a birational morphism of nonsingular rational surfaces. If D is a divisor on V then $\varphi^*(D)$ (or $\varphi'(D)$, resp.) denotes the total transform (or the proper transform, resp.) of D by φ. Similarly, if Λ is a linear pencil on V then $\varphi'(\Lambda)$ denotes the proper transform of Λ by φ.

Let $\rho: V\rightarrow B$ be a surjective morphism from a nonsingular projective surface onto a nonsingular curve, whose general fibers are nonsingular rational curves; we call ρ a $\boldsymbol{P}^1$-*fibration*. An irreducible curve Γ on V is called a *cross-section* (or *quasi-section*, resp.) if $(\Gamma\cdot C)=1$ (or $(\Gamma\cdot C)\geqq 1$, resp.) for a general fiber C of ρ.

The ground field k is always assumed to be an algebraically closed field of characteristic zero. The affine space (or the projective space, resp.) of dimension n defined over k is denoted by $\boldsymbol{A}^n_k$ (or $\boldsymbol{P}^n_k$, resp.).

1. Standard compactifications of $\boldsymbol{A}^2_k$

1.1. Lemma (cf. Gizatullin [2]). *Let $\varphi: V\rightarrow B$ be a surjective morphism from a nonsingular projective surface V onto a nonsingular complete curve B such that almost all fibers are isomorphic to $\boldsymbol{P}^1_k$. Let $F=n_1C_1+\cdots+n_rC_r$ be a singular fiber of φ, where C_i is an irreducible curve, $C_i\neq C_j$ if $i\neq j$, and $n_i>0$. Then we have*:

(1) *The greatest common divisor $(n_1, \cdots, n_r)$ of $n_1, \cdots, n_r$ is* 1; $\mathrm{Supp}(F)=\bigcup_{i=1}^{r} C_i$ *is connected.*

(*) The readers are warned of not confusing $|D|$ with the complete linear system determined by D. In the present article, we do not use the symbol $|D|$ to signify the latter meaning.

(2) *For* $1 \leqq i \leqq r$, C_i *is isomorphic to* $\boldsymbol{P}_k^1$ *and* $(C_i^2)<0$.

(3) *For* $i \neq j$, $(C_i \cdot C_j)=0$ *or* 1.

(4) *For three distinct indices* i, j *and* l, $C_i \cap C_j \cap C_l = \phi$.

(5) *One of* C_i*'s, say* C_1, *is an exceptional component, i.e., an exceptional curve of the first kind. If* $\tau: V \to V_1$ *is the contraction of* C_1, *then* φ *factors as* $\varphi: V \xrightarrow{\tau} V_1 \xrightarrow{\varphi_1} B$, *where* $\varphi_1: V_1 \to B$ *is a fibration by* $\boldsymbol{P}^1$.

(6) *If one of* n_i*'s, say* n_1, *equals* 1 *then there is an exceptional component among* C_i*'s with* $2 \leqq i \leqq r$.

1.2. A generalization of Lemma 1.1 is the following

Lemma. *Let* V *be a nonsingular projective surface and let* Λ *be an irreducible linear pencil on* V *such that general members of* Λ *are rational curves. Let* $\mathfrak{B}$ *be the set of points of* V *which are base points of* Λ. *Let* $F=n_1C_1+\cdots+n_rC_r$ *be a reducible member of* Λ *such that* $r \geqq 2$, *where* C_i *is an irreducible component,* $C_i \neq C_j$ *if* $i \neq j$, *and* $n_i>0$. *Then the following assertions hold true*:

(1) *If* $C_i \cap \mathfrak{B}=\phi$ *then* C_i *is isomorphic to* $\boldsymbol{P}_k^1$ *and* $(C_i^2)<0$.

(2) *If* $C_i \cap C_j \neq \phi$ *for* $i \neq j$ *and* $C_i \cap C_j \cap \mathfrak{B}=\phi$ *then* $C_i \cap C_j$ *consists of a single point where* C_i *and* C_j *intersect each other transversally.*

(3) *For three distinct indices* i, j, l, *if* $C_i \cap C_j \cap C_l \cap \mathfrak{B}=\phi$ *then* $C_i \cap C_j \cap C_l = \phi$.

(4) *Assume that* $(C_i^2)<0$ *whenever* $C_i \cap \mathfrak{B} \neq \phi$. *Then the set* $S:=\{C_i;\, C_i$ *is an irreducible component of* F *such that* $C_i \cap \mathfrak{B}=\phi\}$ *is nonempty, and there is an exceptional component in the set* S.

(5) *With the same assumption as in* (4) *above, if a component of* S, *say* C_1, *has multiplicity* $n_1=1$ *then there exists an exceptional component in* S *other than* C_1.

Proof. Let $\rho: \tilde{V} \to V$ be the shortest succession of quadratic transformations with centers at base points (including infinitely near base points) of Λ such that the proper transform $\tilde{\Lambda}$ of Λ by ρ has no base points. Then, by Bertini's theorem, general members of $\tilde{\Lambda}$ are isomorphic to $\boldsymbol{P}_k^1$. The assertions (1), (2) and (3) are then apparently true by Lemma 1.1. We shall prove the assertions (4) and (5), assuming that $\mathfrak{B} \neq \phi$. Let $P \in \mathfrak{B}$. Set $P_0:=P$, and let $P_1, \cdots, P_{s-1}$ exhaust infinitely near base points of Λ such that P_i is an infinitely near point of P_{i-1} of order one for $1 \leqq i \leqq s-1$. For $1 \leqq i \leqq s$, let $\sigma_i: V_i \to V_{i-1}$ be the quadratic transformation of V_{i-1} with center at P_{i-1}, where $V_0:=V$, and let $\sigma=\sigma_1 \cdot \cdots \cdot \sigma_s$. Then σ factors ρ, i.e., $\rho=\sigma \cdot \bar{\rho}$. Let $E_i:=(\sigma_{i+1} \cdot \cdots \cdot \sigma_s)'(\sigma_i^{-1}(P_{i-1}))$ for $1 \leqq i<s$ and let $E_s':=\sigma_s^{-1}(P_{s-1})$. Let $E_i:=\bar{\rho}'(E_i')$ for $1 \leqq i \leqq s$. It is clear that $E_i' \cong E_i$ and $(E_i'^2) \geqq (E_i^2)$ for $1 \leqq i \leqq s$, and that $(E_i^2)<-1$ for $1 \leqq i<s$ and $(E_s^2)=-1$. Moreover, E_s is not contained in any member of $\tilde{\Lambda}$; indeed, if otherwise, $\tilde{\Lambda}$ would have yet a base point on E_s, which contradicts the choice of points $P_1, \cdots, P_{s-1}$. The member $\tilde{F}$ of $\tilde{\Lambda}$ corresponding to F of Λ

may contain some (not necessarily all) of $E_1, \cdots, E_{s-1}$. After the above process made for every point of $\mathfrak{B}$ and every sequence of infinitely near base points $\{P_0, P_1, \cdots, P_{s-1}\}$ as above, we know that if we write $\tilde{F}=(n_1\tilde{C}_1+\cdots+n_r\tilde{C}_r)+(m_1D_1+\cdots+m_tD_t)$ with $\tilde{C}_i=\rho'(C_i)$ for $1\leqq i\leqq r$ then we have;

1° if $C_i\in S$ then $\tilde{C}_i\cong C_i$ and $(\tilde{C}_i^2)=(C_i^2)$,

2° if $C_i\notin S$ then $(\tilde{C}_i^2)\leqq -2$,

3° $(D_i^2)\leqq -2$ for $1\leqq i\leqq t$.

Then the assertions (4) and (5) follow from the assertions (5) and (6) of Lemma 1.1. Q.E.D.

1.3. Let V be a nonsingular projective surface containing an open subset U, which is isomorphic to $\boldsymbol{A}_k^2$. Since U is affine, the boundary set $V-U$ is connected and purely of codimension 1. Write $V-U=\bigcup_{i=1}^{r} C_i$, where C_i is an irreducible component. We assume that the boundary curve $V-U$ has only normal crossings as singularities. Then we have the following

Lemma (cf. Ramanujam [8]). *Let V and C_i's be as above. Then the following assertions hold true*:

(1) *For every i, C_i is isomorphic to $\boldsymbol{P}_k^1$.*

(2) *For $i\neq j$, $(C_i\cdot C_j)=0$ or 1.*

(3) *For three distinct indices i, j and l, $C_i\cap C_j\cap C_l=\phi$.*

(4) *There is no circular chain (or loop) $\{C_{i_1}, \cdots, C_{i_s}\}$ such that $(C_{i_j}\cdot C_{i_{j+1}})=1$ for $1\leqq j<s$ and $(C_{i_s}\cdot C_{i_1})=1$.*

V is then called a *normal compactification* of $\boldsymbol{A}_k^2$. V is called a *minimal* normal compactification if the following additional condition is satisfied:

(5) *If an irreducible component, say C_1, of $V-U$ is an exceptional curve of the first kind, then at least three other components of $V-U$ meet C_1.*

1.4. Let V be a normal compactification of $\boldsymbol{A}_k^2$. Then the dual graph of the boundary curve $V-U$ is defined by assigning a vertex ○ to each irreducible component and by connecting two vertices by an edge (like ○—○) if two corresponding components meet each other. The condition (4) of Lemma 1.3 says that the dual graph of $V-U$ is a *tree*. The dual graph is said to be *linear* if it is a tree and each vertex has at most two branches. Then we have the following.

Lemma (cf. Ramanujam [8]). *Let V be a minimal normal compactification of $\boldsymbol{A}_k^2$. Then the dual graph of the boundary curve is linear.*

The dual graphs of all possible minimal normal compactifications of $\boldsymbol{A}_k^2$ were classified by Morrow [7]. An algebraic proof of the above lemma of Ramanujam and also of Morrow's result (even over the ground field of positive characteristic) was given by S. Mori [6]. However, we use this result only in one place (cf. 3.11) in this article.

1.5. Let $k[x, y]$ be a polynomial ring in two variables x, y and let f be an irreducible polynomial of degree $d>0$ in $k[x, y]$. Let $\Lambda(f)$ be the linear pencil on $\boldsymbol{P}_k^2$ defined by f. Then $\Lambda(f)$ has base points on the line l_∞ at infinity. Set $V_0 :=\boldsymbol{P}_k^2$. Let φ: $\tilde{V}\to V_0$ be the shortest succession of quadratic transformations with centers at base points (including infinitely near base points) of $\Lambda(f)$ such that the proper transform $\tilde{\Lambda}$ of $\Lambda(f)$ by φ has no base points. Let $\tilde{\rho}$: $\tilde{V}\to\boldsymbol{P}_k^1$ be the surjective morphism defined by $\tilde{\Lambda}$. Then each irreducible (exceptional) curve arising in the process φ of quadratic transformations is either a quasi-section of $\tilde{\rho}$ or contained in a fiber of $\tilde{\rho}$. Let $\tilde{\Gamma}_1, \cdots, \tilde{\Gamma}_p$ exhaust all irreducible (exceptional) curves arising in the process φ, which are quasi-sections of $\tilde{\rho}$. Then it is clear that every $\tilde{\Gamma}_i$ is isomorphic to $\boldsymbol{P}_k^1$.

Note that $\tilde{V}$ contains in a canonical way an open subset $\tilde{U}$ which is isomorphic to $\boldsymbol{A}_k^2$ and that $\tilde{V}$ is a normal compactification of $\tilde{U}\cong\boldsymbol{A}_k^2$. Let ψ: $\tilde{V}\to V$ be a contraction of all possible exceptional (contractible) components of $\tilde{V}-\tilde{U}$, which are contained in the fibers of $\tilde{\rho}$. Then V is a nonsingular projective surface containing an open subset U, which is isomorphic to $\boldsymbol{A}_k^2$. However, V is not necessarily a normal compactification of $\boldsymbol{A}_k^2$ (cf. [5; Example 2.4.4, p. 122]). Moreover, there exists a surjective morphism ρ: $V\to\boldsymbol{P}_k^1$ such that $\tilde{\rho}=\rho\cdot\psi$. We set $\Gamma_i=\psi(\tilde{\Gamma}_i)$ for $1\leqq i\leqq p$ and denote by S_∞ the fiber of ρ which corresponds to the member dl_∞ of $\Lambda(f)$. If we identify U with Spec$(k[x, y])$, there exists an inhomogeneous coordinate u of $\boldsymbol{P}_k^1$ such that the point $\rho(S_\infty)$ is given by $u=\infty$ and $S_\alpha\cap U=F_\alpha$ for each $\alpha\in k$, where S_α is the fiber of ρ lying over the point $u=\alpha$.

1.6. Assume that f is generically rational, i.e., general members of $\Lambda(f)$ are rational curves. Then, with the above notations, S_∞ is isomorphic to $\boldsymbol{P}_k^1$. Let $S_1, \cdots, S_r$ be all fibers of ρ such that $F_i:=S_i\cap U$ is a reducible curve for $1\leqq i\leqq r$. Let m_i be the number of irreducible components of F_i for $1\leqq i\leqq r$. On the other hand, $\Gamma_1, \cdots, \Gamma_p$ exhaust all quasi-sections of ρ which are contained in $V-U$, while they are not necessarily nonsingular. Let δ_j be the degree of the morphism $\rho|_{\Gamma_j}$: $\Gamma_j\to\boldsymbol{P}_k^1$. Note that $\psi|_{\tilde{\Gamma}_j}$: $\tilde{\Gamma}_j\to\Gamma_j$ is the desingularization of Γ_j. For $1\leqq j\leqq p$, let $\nu_j=\sum_Q(e_Q-1)$, where e_Q is the ramification index of $\tilde{\rho}|_{\tilde{\Gamma}_j}$ at a point $Q\in\tilde{\Gamma}_j$ and the summation ranges over all points Q of $\tilde{\Gamma}_j$ such that $\tilde{\rho}(Q)\neq\rho(S_\infty)$. Then we have the following.

Lemma (cf. Saito [13], Suzuki [15]). *With the assumptions and the notations as above, we have:*

(1) $p+r-1=\sum_{i=1}^r m_i$; (2) $n+1=p+\sum_{j=1}^p \nu_j$, *where* $n+1$ *is the number of*

places at infinity of a general member F_α of $\Lambda_0(f)$ (cf. *Introduction.*)

Proof. (1) We note that $S_1, \cdots, S_r$ are all singular fibers of ρ. Indeed, suppose that S is a singular fiber of ρ other than $S_1, \cdots, S_r$. Then S has only one irreducible component C which meets U. Let μ be the multiplicity of C in S. Since $\mu C \cap U = S \cap U$ is defined by an equation $f=\alpha$ with some $\alpha \in k$, we have $f-\alpha=\lambda g^\mu$ for some irreducible polynomial $g \in k[x, y]$ and $\lambda \in k^*$. The assertion (6) of Lemma 1.1 implies that $\mu > 1$. Then f is reducible, which contradicts the choice of f. Let S be an irreducible fiber of ρ. Then we have

$$\chi(V) = \chi(S)\cdot\chi(\boldsymbol{P}^1_k)+\sum_{i=1}^{r}(\chi(S_i)-\chi(S)),$$

where $\chi(\ )$ denotes the Euler number (cf. Šafarevič [11; p. 58]). Let m'_i be the number of irreducible components of S_i. Since the dual graph of S_i is a tree, it is easy to show that $\chi(S_i)=2m'_i-(m'_i-1)=m'_i+1$. On the other hand, it is easy to see from construction of V that the Picard group $\mathrm{Pic}(V)$ is a free abelian group of rank $1+p+\sum_{i=1}^{r}(m'_i-m_i)$. Since V is rational, $\chi(V)$ is then equal to $2+1+p+\sum_{i=1}^{r}(m'_i-m_i)$. Thus, from the above equality, we obtain

$$3+p+\sum_{i=1}^{r}(m'_i-m_i) = 4+\sum_{i=1}^{r}(m'_i-1),$$

whence follows the first equality.

(2) For each j $(1 \leqq j \leqq p)$, $S_\infty \cap \Gamma_j$ consists of a single point, which is also a one-place point. For, if otherwise, $\tilde{V}-\tilde{U}$ would contain a circular chain of irreducible components, which is a contradiction by the assertion of Lemma 1.3. Then there is only one point on $\tilde{\Gamma}_j$ lying over the point $\rho(S_\infty)$, where the ramification index of $\tilde{\rho}|_{\tilde{\Gamma}_j}$ equals δ_j. By Hurwitz's formula, we obtain $\delta_j-1=\nu_j$ for every j $(1 \leqq j \leqq p)$. On the other hand, note that if S is a general fiber of ρ, S has $n+1$ distinct points outside of $S \cap U$, among which δ_j points lie on Γ_j for each j $(1 \leqq j \leqq p)$. Therefore, we have the second equality. Q.E.D.

1.7. Lemma. *Let f be a generically rational polynomial in $k[x, y]$. Then, with the notations as above, the following conditions are equivalent*:

(1) $n=0$;

(2) $p=1$;

(3) $r=0$, i.e., *the curve F_α on $\boldsymbol{A}^2_k$ defined by $f=\alpha$ is irreducible for every $\alpha \in k$*;

(4) *f is sent to one of coordinates x, y by a biregular automorphism of $\boldsymbol{A}^2_k :=$* $\mathrm{Spec}(k[x, y])$.

Proof. Note that $p>0$. Then the implication (1)$\Rightarrow$(2) is clear. Suppose

$p=1$. Then $r=0$ by the first equality of Lemma 1.6, because $m_i \geqq 2$ for every i. This implies the assertion (3) (cf. the proof of the first equality of Lemma 1.6). Suppose the condition (3) is satisfied. Then V is a relatively minimal rational ruled surface, whence V is a Hirzebruch surface $F_a=\mathrm{Proj}(\mathcal{O}_{P^1}\otimes\mathcal{O}_{P^1}(a))$ with $a\geqq 0$. Let M be the minimal section of F_a, i.e., M is the cross-section of ρ with $(M^2)=-a$; when $a=0$ we take a cross-section of ρ as M. Since $\mathrm{Pic}(V)$ is a free abelian group generated by M and the fiber S_∞ of ρ, the unique quasi-section Γ of ρ contained in $V-U$ is linearly equivalent to a divisor of the form $\alpha M+\beta S_\infty$, where α and β are non-negative integers such that $\beta\geqq\alpha a$. Since $U\cong \boldsymbol{A}_k^2$ and $V-U=\Gamma\cup S_\infty$, we know that Γ and S_∞ also generate $\mathrm{Pic}(V)$. Then $\alpha=1$, and Γ is a cross-section. This implies that $S\cap U$ is isomorphic to $\boldsymbol{A}_k^1$ for every fiber S of ρ other than S_∞. Hence the curve F_0 defined by $f=0$ is isomorphic to $\boldsymbol{A}_k^1$. Now, the assertion (4) follows from Abhyankar-Moh's theorem (cf. Abhyankar-Moh [1], Miyanishi [4]). The implication (4)$\Rightarrow$(1) is clear. Q.E.D.

1.8. DEFINITION. *Let f be a generically rational polynomial in $k[x, y]$. f is said to be of simple type if the equality $p=n+1$ holds.*

The second equality of Lemma 1.6 shows that f is of simple type if and only if $\Gamma_1, \cdots, \Gamma_p$ are all cross-sections of ρ.

Lemma. *Let f be a generically rational polynomial in $k[x, y]$ such that a general member of $\Lambda_0(f)$ has two places at infinity,* i.e., $n=1$. *Then f is of simple type. Moreover, $\Lambda_0(f)$ has only one reducible fiber, which has two irreducible components.*

Proof. By Lemma 1.7, we have $p\geqq 2$. Then, by the second equality of Lemma 1.6, we must have $p=2$, whence f is of simple type. Then the first equality of Lemma 1.6 implies that $r=1$ and $m_1=2$. Q.E.D.

1.9. Let f be a generically rational polynomial in $k[x, y]$. We assume in this paragraph that f is of simple type, i.e., $p=n+1$. Then every $\Gamma_j (1\leqq j\leqq p)$ is a nonsingular rational curve. We assume, furthermore, that $p\geqq 2$, i.e., $n\geqq 1$. Since U is isomorphic to $\boldsymbol{A}_k^2$, two distinct Γ_j and Γ_l do not meet each other on any fiber S of ρ other than S_∞; indeed, if otherwise, $\tilde{V}-\tilde{U}$ would contain a circular chain of irreducible components. Note also that $S_1, \cdots, S_r$ exhaust all singular fibers of ρ.

Let Δ be an irreducible component of $S_i - S_i\cap U$ (if it exists at all) for some $i (1\leqq i\leqq r)$. If there is a sequence of irreducible components $\{\Delta_1, \cdots, \Delta_t\}$ of $S_i-S_i\cap U$ such that $\Delta_1=\Delta$, $\Delta_l\cap\Delta_{l+1}\neq\phi$ for $1\leqq l<t$ and $\Delta_t\cap\Gamma_j\neq\phi$ for some j $(1\leqq j\leqq n+1)$, we say that Δ is connected to Γ_j. Since $V-U$ is a connected curve, every irreducible component Δ of $S_i-S_i\cap U$ is connected to some Γ_j $(1\leqq j\leqq n+1)$, while Δ is not connected to two distinct Γ_j and $\Gamma_{j'}$; indeed, if

otherwise, $\tilde{V}-\tilde{U}$ would contain a circular chain of irreducible components. For each pair (i,j) with $1\leqq i\leqq r$ and $1\leqq j\leqq n+1$, let E_{ij} be the union of all irreducible components Δ of $S_i-S_i\cap U$, which are connected to Γ_j. Then E_{ij} is a connected curve if it is not empty.

Set $P_\infty:=\rho(S_\infty)$ and $P_i:=\rho(S_i)$ for $1\leqq i\leqq r$. Then the points P_∞ and P_i's are determined by $u=\infty$ and $u=c_i(1\leqq i\leqq r)$, respectively, with respect to the inhomogeneous coordinate u of $\boldsymbol{P}_k^1$ (cf. the paragraph 1.5). Set $Z:=\boldsymbol{P}_k^1-\{P_\infty, P_1, \cdots, P_r\}$. Then $\rho^{-1}(Z)$ with the morphism $\rho: \rho^{-1}(Z)\rightarrow Z$ is a trivial $\boldsymbol{P}^1$-bundle over Z, i.e., there exists a Z-isomorphism $\eta: \rho^{-1}(Z)\xrightarrow{\sim} Z\times\boldsymbol{P}_k^1$. We may assume that there exist points $Q_1, \cdots, Q_{n+1}$ on $\boldsymbol{P}_k^1$ such that $\eta^{-1}(Z\times\{Q_j\})=\Gamma_j\cap\rho^{-1}(Z)$ for $1\leqq j\leqq n+1$. Now, choose an inhomogeneous coordinate v on $\boldsymbol{P}_k^1$ (=the fiber of ρ) such that Q_{n+1} is defined by $v=\infty$ and Q_j is defined by $v=d_j$ for $1\leqq j\leqq n$; in the sequel, we set $Q_\infty:=Q_{n+1}$ and $\Gamma_\infty:=\Gamma_{n+1}$. Let $p_2: Z\times\boldsymbol{P}_k^1\rightarrow\boldsymbol{P}_k^1$ be the second projection. Then the morphism $p_2\cdot\eta: \rho^{-1}(Z)\rightarrow\boldsymbol{P}_k^1$ (or equivalently saying, the inclusion of the subfield $k(v)$ into $k(x,y)$) defines a linear pencil L on V without fixed components such that $\Gamma_1, \cdots, \Gamma_{n+1}$ are contained in distinct members $\Xi_1, \cdots, \Xi_n$, $\Xi_\infty:=\Xi_{n+1}$ of L, respectively.

For each $i\,(1\leqq i\leqq r)$, let $S_{i1}, \cdots, S_{im_i}$ exhaust all irreducible components of S_i such that $S_{ia}\cap U\neq\phi$ for $1\leqq a\leqq m_i$. Set $F_{ia}:=S_{ia}\cap U$ and let f_{ia} be an irreducible polynomial in $k[x,y]$ such that F_{ia} is defined by $f_{ia}=0$, where $1\leqq a\leqq m_i$. Then, for each $i\,(1\leqq i\leqq r)$, we have

$$u-c_i=f-c_i=\lambda_i\cdot\prod_{a=1}^{m_i}(f_{ia})^{\alpha_{ia}},$$

where $\lambda_i\in k^*$ and α_{ia} is a positive integer. Then we have:

1.9.1. $k[x, y, (\prod_{i=1}^{r}\prod_{a=1}^{m_i}f_{ia})^{-1}]=k[u, v, \prod_{i=1}^{r}(u-c_i)^{-1}, \prod_{j=1}^{n}(v-d_j)^{-1}]$.

1.9.2. *If $n\geqq 2$, the linear pencil L has no base points which lie outside S_∞.*

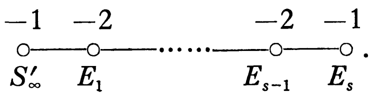
Proof. Suppose that Q is a base point of L with $Q\notin S_\infty$. Then it is clear that $Q\in S_i$ for some $i\,(1\leqq i\leqq r)$. Note that, for every $j\,(1\leqq j\leqq n+1)$, the underlying curve of Ξ_j is contained in the union of Γ_j and the underlying curves of S_i's $(1\leqq i\leqq r)$. Since $n+1\geqq 3$, this implies that either there are at least two components of S_i intersecting one of the cross-sections Γ_j's, or there are at least three components of S_i passing through the point Q, which contradicts the assertion (4) of Lemma 1.1.

1.9.3. *After a suitable modification*(*) *of V with centers at points of S_∞, we may*

(*) a succession of quadratic transformations with centers at points and contractions of exceptional curves of the first kind.

assume that L has no base points on S_∞.

Proof. If S_∞ is not contained in any member of L then L has no base points on S_∞. Suppose that L has a base point Q on S_∞. Let $s:=i(\Xi, \Xi'; Q)$ for distinct general members Ξ and Ξ' of L. Set $Q_0:=Q$, and let $Q_1, \cdots, Q_{s-1}$ be infinitely near points of Q_0, which lie on (the proper transforms of) Ξ, such that Q_i is an infinitely near point of Q_{i-1} of order one for $1 \leqq i < s$. Let σ_i: $V_i \rightarrow V_{i-1}$ be the quadratic transformation of V_{i-1} with center at Q_{i-1}, where $V_0:=V$, and let $\sigma=\sigma_1 \cdot \cdots \cdot \sigma_s$. Let $E_i:=(\sigma_{i+1} \cdot \cdots \cdot \sigma_s)'(\sigma_i^{-1}(Q_{i-1}))$ for $1 \leqq i < s$, and let $E_s:=\sigma_s^{-1}(Q_{s-1})$. Let $S'_\infty:=\sigma'(S_\infty)$, let $\Gamma'_j:=\sigma'(\Gamma_j)$ for $1 \leqq j \leqq n+1$, and let $L':=\sigma'(L)$. Then we have the following dual graph of $\sigma^{-1}(S_\infty)$:



Since $n+1 \geqq 2$, some member Ξ_j $(1 \leqq j \leqq n+1)$ does not contain S_∞. Moreover, since $(\Gamma_j \cdot S_\infty)=1$, we know that $Q_0, Q_1, \cdots, Q_{s-1}$ exhaust all base points (including infinitely near base points) of L centered on S_∞. Therefore, L' has no base points centered on S_∞. Thus E_s is a cross-section of L', and $S'_\infty \cup E_1 \cup \cdots \cup E_{s-1}$ is contained in some member of L'. Let $\bar{\sigma}$: $V_s \rightarrow \bar{V}$ be the contraction of $S'_\infty, E_1, \cdots, E_{s-1}$. Now, replace V by $\bar{V}$. Then the above modification $\bar{\sigma} \cdot \sigma^{-1}$: $V \rightarrow \bar{V}$ changes only S_∞. Q.E.D.

1.9.4. *Assume that $r \geqq 2$. Then, for every j $(1 \leqq j \leqq n+1)$, at most one of $E_{1j}, \cdots, E_{rj}$ is a nonempty set.*

Proof. Suppose that (at least) two of $E_{1j}, \cdots, E_{rj}$, say E_{1j} and E_{2j}, are nonempty sets. By 1.9.3, we may assume that L has no base points on S_∞. Then V is a normal compactification of U. Note that $(S_\infty^2)=0$ and no exceptional components are contained in $\bigcup_{i=1}^{r} E_{ij}$. By contracting (possible) exceptional components in $V-U$, we would obtain a minimal normal compactification of $\boldsymbol{A}_k^2$, for which the dual graph of the boundary curve is not linear. This contradicts Lemma 1.4. Q.E.D.

1.9.5. *For distinct pairs (i,j) and (i',j'), we have $E_{ij} \cap E_{i'j'}=\phi$. Furthermore, we have $S_i - S_i \cap U = \bigcup_{j=1}^{n+1} E_{ij}$ for every i $(1 \leqq i \leqq r)$.*

Proof. Clear from the construction and the above arguments.

1,9.6. Assume that $n \geqq 2$. With V modified so that L has no base points (cf. 1.9.3), L defines a $\boldsymbol{P}^1$-fibration τ: $V \rightarrow \boldsymbol{P}_k^1$ such that S_∞ is a cross-section of τ.

1.9.7. Assume that $n \geqq 1$. After the modification of V so that L has no base points on S_∞, we call V *a standard compactification of* $\boldsymbol{A}_k^2$ *with respect to* a generically rational polynomial f of simple type. Then we have the following

Lemma. *Let f be a generically rational polynomial of simple type in $k[x, y]$ with $n \geqq 1$. Let V and $\bar{V}$ be standard compactifications of $\boldsymbol{A}_k^2 := \mathrm{Spec}(k[x, y])$ with respect to f. Let $\rho: V \to \boldsymbol{P}_k^1$ (or $\bar{\rho}: \bar{V} \to \boldsymbol{P}_k^1$, resp.) be the $\boldsymbol{P}^1$-fibration of V (or $\bar{V}$, resp.) defined by f. Then there exists an isomorphism $\theta: V \to \bar{V}$ such that $\rho = \bar{\rho} \cdot \theta$.*

Proof. V (or $\bar{V}$, resp.) contains an open subset U (or $\bar{U}$, resp.) isomorphic to $\boldsymbol{A}_k^2$. The identity morphism $id.: U \xrightarrow{\sim} \bar{U}$ extends to a birational mapping $\theta: V \to \bar{V}$ such that $\rho = \bar{\rho} \cdot \theta$. We shall show that θ is an isomorphism. Let $\mathfrak{V}$ and $\bar{\mathfrak{V}}$ be the generic fibers of ρ and $\bar{\rho}$, respectively. Then both $\mathfrak{V}$ and $\bar{\mathfrak{V}}$ are complete normal models of the algebraic function field $k(x, y)$ of one variable over $\mathfrak{k} = k(f)$. Hence $\mathfrak{V}$ and $\bar{\mathfrak{V}}$ are isomorphic to $\boldsymbol{P}_{\mathfrak{k}}^1$. Let $\Gamma_1, \cdots, \Gamma_{n+1}$ (or $\bar{\Gamma}_1, \cdots, \bar{\Gamma}_{n+1}$, resp.) be the cross-sections of ρ (or $\bar{\rho}$, resp.) contained in $V - U$ (or $\bar{V} - \bar{U}$, resp.). Since $\mathfrak{V} - \{\Gamma_1 \cap \mathfrak{V}, \cdots, \Gamma_{n+1} \cap \mathfrak{V}\}$ is identified to $\bar{\mathfrak{V}} - \{\bar{\Gamma}_1 \cap \bar{\mathfrak{V}}, \cdots, \bar{\Gamma}_{n+1} \cap \bar{\mathfrak{V}}\}$ under θ, $\mathfrak{V}$ is isomorphic to $\bar{\mathfrak{V}}$ under θ, and we may assume that $\theta(\Gamma_j \cap \mathfrak{V}) = \bar{\Gamma}_j \cap \bar{\mathfrak{V}}$ for $1 \leqq j \leqq n+1$. Then $\theta: \Gamma_j \to \bar{\Gamma}_j$ is an isomorphism for $1 \leqq j \leqq n+1$. Let $S_1, \cdots, S_r$ (or $\bar{S}_1, \cdots, \bar{S}_r$, resp.) be reducible fibers of ρ (or $\bar{\rho}$, resp.). Define $\bar{E}_{ij}$ $(1 \leqq i \leqq r$ and $1 \leqq j \leqq n+1)$ for $\bar{V}$ in the same fashion as for V; note that $\bigcup_{i=1}^{r} \bigcup_{j=1}^{n+1} \bar{E}_{ij}$ contains no exceptional curves of the first kind. Let S_∞ (or $\bar{S}_\infty$, resp.) be the fiber of ρ (or $\bar{\rho}$, resp.) contained in $V - U$ (or $\bar{V} - \bar{U}$, resp.). If θ is not biregular, $V - U$ contains an irreducible curve T which becomes an exceptional curve of the first kind after a succession of quadratic transformations with centers at points in $V - U$. Since every component in $\bigcup_{i=1}^{r} \bigcup_{j=1}^{n+1} E_{ij}$ has self-intersection multiplicity $\leqq -2$, T must be S_∞. However, if S_∞ is contracted after a sequence of quadratic transformations, $\bar{\Gamma}_1, \cdots, \bar{\Gamma}_{n+1}$ meet each other at one point on $\bar{S}_\infty$, which is a contradiction. Hence θ is biregular. Similarly, θ^{-1} is biregular. Thus θ is an isomorphism such that $\rho = \bar{\rho} \cdot \theta$.

Q.E.D.

1.10. In the second section we use the following

Lemma (cf. Miyanishi [3,5]). *Let X be a nonsingular affine surface defined by an affine k-domain A. Assume that the following conditions are satisfied:*

(1) *A is a unique factorization domain and $A^* = k^*$.*

(2) *There exist nonsingular irreducible curves C_1 and C_2 on X such that $C_1 \cap C_2 = \{P\}$, and C_1 and C_2 intersect each other transversally at P.*

(3) *C_1 (resp. C_2) has only one place at infinity.*

(4) *Let a_2 be a prime element of A defining the curve C_2. Then $a_2-\alpha$ is a prime element of A for all $\alpha \in k$.*

(5) *There is a nonsingular projective surface V containing X as an open subset such that the closure $\bar{C}_2$ of C_2 in V is nonsingular and $(a_2)_0$ (=the zero part of the divisor of a_2)$=\bar{C}_2$. Then X is isomorphic to $\boldsymbol{A}_k^2$, and the curves C_1 and C_2 are sent to the axes of a suitable coordinate system of $\boldsymbol{A}_k^2$.*

1.11. Here, we recall a general result due to Russell [9; Cor. 3.7] on a generically rational polynomial.

Proposition. *Let f be a generically rational polynomial in $k[x, y]$. Then there are at most two points (including infinitely near points) of the curve $f=0$ on the line at infinity l_∞. In particular, the degree form of f has at most two distinct irreducible factors.*

2. Generically rational polynomials with $n=1$

In this section, f is a generically rational, irreducible polynomial in $k[x, y]$ with $n=1$. Then f is of simple type and $\Lambda_0(f)$ has only one reducible member consisting of two irreducible components (cf. Lemma 1.8). Let $f=c_1$ be the reducible member of $\Lambda_0(f)$. We use the notations of the previous section with due modifications.

2.1. Lemma. *With the notations of* 1.9, *we have one of the following cases*:

(1) *Case $F_{11} \cap F_{12} \neq \phi$. Then $(S_{11} \cdot S_{12})=1$ and $F_{1a} \cong \boldsymbol{A}_k^1$ for $a=1, 2$.*

(2) *Case $F_{11} \cap F_{12}=\phi$. Then $(S_{11} \cdot S_{12})=0$, and one of F_{11} and F_{12}, say F_{11}, is isomorphic to $\boldsymbol{A}_k^1$ and the other, say F_{12}, is isomorphic to $\boldsymbol{A}_*^1$ $(=\boldsymbol{A}_k^1-$(one point)).*

Proof. (1) Suppose $F_{11} \cap F_{12} \neq \phi$. Then $(S_{11} \cdot S_{12})=1$ (cf. Lemma 1.1, (3)). Since $E_{11} \cap E_{12}=\phi$, we may assume that $(\Gamma_a \cup E_{1a}) \cap S_{1a} \neq \phi$ for $a=1, 2$. Since the dual graph of S_1 is a tree, it is easy to see that $F_{1a} \cong \boldsymbol{A}_k^1$ for $a=1, 2$.
(2) Suppose $F_{11} \cap F_{12}=\phi$. Then one of S_{11} and S_{12}, say S_{12}, intersects both $\Gamma_1 \cup E_{11}$ and $\Gamma_2 \cup E_{12}$, and the other one, say S_{11}, intersects only one of $\Gamma_1 \cup E_{11}$ and $\Gamma_2 \cup E_{12}$. Then $(S_{11} \cdot S_{12})=0$, $F_{11} \cong \boldsymbol{A}_k^1$ and $F_{12} \cong \boldsymbol{A}_*^1$. Q.E.D.

2.2. Lemma. *With the notations of* 1.9 *and after a suitable change of coordinates of $k[x, y]$, we have one of the following cases*:

(1) *$f_{11}=x$ and $f_{12}=y$.*

(2) *$f_{11}=x$ and $f_{12}=x^l y+P(x)$, where $l>0$ and $P(x) \in k[x]$ with* deg $P(x)<l$ *and $P(0) \neq 0$.*

Proof. (1) Suppose $F_{11} \cap F_{12} \neq \phi$. Then, by virtue of Lemmas 1.7 and 1.10, we may change coordinates of $k[x, y]$ so that $f_{11}=x$ and $f_{12}=y$.
(2) Suppose $F_{11} \cap F_{12}=\phi$. Then, by virtue of Abhyankar-Moh's Theorem

(cf. Lemma 1.7), we may assume that $f_{11}=x$. Taking account of 1.9.1, we have only to prove the following assertion:

Let $g\in k[x,y]$ be an irreducible polynomial such that $g\notin k[x]$ and $g(0,y)\neq 0$. Assume that we have the following identification of rings,

$$k[x, y, x^{-1}, g^{-1}] = k[u, v, u^{-1}, v^{-1}]\,.$$

Then g is of the form:

$$g = \lambda(x^l y+P(x)), \text{ where } \lambda\in k^*,\ l>0 \text{ and } P(x)\in k[x]$$
$$\text{with } \deg P(x)<l \text{ and } P(0)\neq 0\,.$$

Indeed, by comparison of the unit groups of both rings, we have: $u\sim x^{\alpha}g^{\beta}$ and $v\sim x^{\gamma}g^{\delta}$, where $\alpha, \beta, \gamma, \delta\in \boldsymbol{Z}$ such that $\alpha\delta-\beta\gamma=\pm 1$. This implies that g satisfies

$$x^a g^b y = G(x, g), \quad \text{where} \quad G(x, g)\in k[x, g]\,.$$

If $b>0$ then we have a relation: $gh(x, y)\in k[x]$ with $h(x, y)\in k[x, y]$, which implies $g\in k[x]$, a contradiction. Hence $b=0$. Write

$$g = A_0(x)y^M+\cdots+A_M(x) \quad \text{and} \quad G(x, g) = B_0(x)g^N+\cdots+B_N(x)\,,$$

where $A_i(x)$, $B_j(x)\in k[x]$ ($0\leqq i\leqq M$ and $0\leqq j\leqq N$) and $A_0(x)B_0(x)\neq 0$. Then we have

$$x^a y = B_0(x)A_0(x)^N y^{NM}+\cdots\cdots\,,$$

whence $M=1$ and $A_0(x)\sim x^l$. Therefore, g is written in the stated form after replacing y by $y+C(x)$. Q.E.D.

2.3. Theorem (cf. Saito [12; p. 332], Sugie [14]). *Let f be a generically rational, irreducible polynomial in $k[x, y]$ with $n=1$. Then, after a suitable change of coordinates, f is reduced to either one of the following two forms:*

(1) $f\sim x^{\alpha}y^{\beta}+1$, *where* $\alpha>0$, $\beta>0$ *and* $(\alpha, \beta)=1$.

(2) $f\sim x^{\alpha}(x^l y+P(x))^{\beta}+1$, *where* $\alpha, \beta, l>0$, $(\alpha, \beta)=1$ *and* $P(x)\in k[x]$ *with* $\deg P(x)<l$ *and* $P(0)\neq 0$.

Proof. Clear by Lemma 2.2.

2.4. Proposition. *Assume that the curve C_0 on $\boldsymbol{P}_k^2$ defined by $f=0$ intersects l_∞ in only one point P. Let $d_1:=\mathrm{mult}_P C_0$. Then $d_1<d=$the degree of f. Moreover, there exists a birational automorphism ρ of $\boldsymbol{P}_k^2$ such that ρ induces a biregular automorphism on $\boldsymbol{A}_k^2:=\boldsymbol{P}_k^2-l_\infty$ and that the proper transform C_0' of C_0 by ρ intersects l_∞ in two points with $(C_0'\cdot l_\infty)\leqq d_1$.*

The existence of an automorphism ρ such that $\rho|_{\boldsymbol{A}_k^2}$ is a biregular automorphism and that the proper transform $\rho'(C_0)$ intersects l_∞ in two points follows from the above theorm. However, the above proposition is proved by a constructive method depending on Lemma 1.2, which allows us to determine more explicitly an automorphism ρ with the stated properties. For more details, the readers will be referred to Miyanishi [5; Chap. II, §6].

2.5. It is clear that the integers α and β in Theorem 2.3 are the multiplicities of components S_{11} and S_{12} of the fiber S_1, respectively, in the standard compactification of $\boldsymbol{A}_k^2$ with respect to f. On the other hand, Lemma 1.9.7 implies that the standard compactification of $\boldsymbol{A}_k^2$ with respect to f is obtained by assuming that f is written in the form stated in Theorem 2.3. By a straightforward computation, we know that the dual graphs of E_{11} and E_{12} are linear. For details, see [5; ibid.].

3. Generically rational polynomials of simple type with $n>1$

In this section, f is a generically rational, irreducible polynomial of simple type with $n>1$. In the paragraphs 3.1~3.4, we consider the case $r=1$; in the paragraphs 3.5~3.8, we consider the case where $r\geqq 2$ and $m_i<n$ for every $i\,(1\leqq i\leqq r)$; in the paragraphs 3.9~3.12, we consider the case where $r=2$, $m_1=n$ and $m_2=2$.

3.1. We shall consider the case $r=1$. For convenience's sake, we simplify the notations as follows: $\Sigma:=S_1$; $\Sigma_a:=S_{1a}$, $F_a:=F_{1a}$, $f_a:=f_{1a}$, $\alpha_a:=\alpha_{1a}$ for $1\leqq a\leqq m_1$, where $m_1=p=n+1$; $E_j:=E_{1j}$ for $1\leqq j\leqq n+1$; we may assume that $c_1=1$ (cf. 1.9.) Note that the pencil L on the standard compactification V of $\boldsymbol{A}_k^2$ with respect to f has no base points; hence L defines a $\boldsymbol{P}^1$-fibration $\tau\colon V\to\boldsymbol{P}_k^1$. Let Ξ be a general fiber of τ. Since $(S_\infty\cdot\Xi)=1$, we have $(\Sigma\cdot\Xi)=1$. This implies that there exists an irreducible component Δ of Σ such that $(\Delta\cdot\Xi)=1$, i.e., Δ is a cross-section of τ, and the other components of Σ are contained in the fibers $\Xi_1,\cdots,\Xi_{n+1}$ and possibly one other fiber of L, as will be seen below. First, we shall prove the following:

Lemma. *With the assumptions and the notations as above, we have one of the following cases:*

(1) $\Delta\cap U\neq\phi$. *We may assume that* $\Delta=\Sigma_{n+1}$. *Then* $\Xi\cap U\cong\boldsymbol{A}_k^1$ *and* $F_a\cong\boldsymbol{A}_k^1$ *for* $1\leqq a\leqq n$; $\Sigma_a\,(1\leqq a\leqq n)$ *belongs to one and only one of* $\Xi_1,\cdots,\Xi_{n+1}$, *while none of* $\Xi_1,\cdots,\Xi_{n+1}$ *contains two of* Σ_a*'s; thus, after a suitable change of inhomogeneous coordinate* v *of* $\boldsymbol{P}_k^1$ (*=the fiber of* ρ) *and a suitable change of indices, we may assume that* Σ_a *belongs to* Ξ_a *for* $1\leqq a\leqq n$; $F_{n+1}=\Delta-(\Delta\cap\Xi_{n+1})\cup\bigcup_{j=1}^{n}((\Gamma_j\cup E_j)\cap\Delta)$, *where* $\Delta\cong\boldsymbol{P}_k^1$; $\alpha_{n+1}=1$.

(2) $\Delta\cap U=\phi$. *We may assume that* Δ *is a component of* E_{n+1}. *Then* $\Xi\cap U\cong\boldsymbol{A}_*^1$; $\Sigma_a\,(1\leqq a\leqq n+1)$ *belongs to one and only one of* $\Xi_1,\cdots,\Xi_n$; *none of*

Σ_a's belongs to Ξ_{n+1}; thus, after a suitable change of indices, we may sasume that Σ_a belongs to Ξ_a for $1 \leqq a \leqq n$ and Σ_{n+1} belongs to either one of Ξ_j's $(1 \leqq j \leqq n)$, say Ξ_1, or none of them; $F_1 \cong A^1_*$ for $2 \leqq a \leqq n$ and $F_{n+1} \cong A^1_k$. Moreover, (i) *$F_1 \cong A^1_*$ either if Σ_{n+1} is a component of Ξ_1 and $\Sigma_1 \cap \Sigma_{n+1} = \phi$ or if Σ_{n+1} belongs to none of Ξ_j's $(1 \leqq j \leqq n)$, and* (ii) *$F_1 \cong A^1_k$ if Σ_{n+1} is a component of Ξ_1 and $\Sigma_1 \cap \Sigma_{n+1} \neq \phi$.*

Proof. (1) Suppose$\Delta \cap U \neq \phi$. As in the statement, we may assume that $\Delta = \Sigma_{n+1}$. Then $\Xi \cap U = \Xi - \Xi \cap S_\infty$, whence $\Sigma \cap U \cong A^1_k$. Since $(\Xi \cdot \Sigma_a) = 0$ for $1 \leqq a \leqq n$, Σ_a belongs to the fibers of L. Hence we may assume that one of $\Xi_1, \cdots, \Xi_{n+1}$, say Ξ_{n+1}, contains none of Σ_a's $(1 \leqq a \leqq n)$. Then the inhomogeneous coordinate v of P^1_k (=the fiber of ρ) is an element of $k[x, y]$; indeed, the polar divisor of v on V has no irreducible components meeting the open set $U = \mathrm{Spec}(k[x, y])$. Since $\Xi \cap U \cong A^1_k$, we may assume, after a suitable change of coordinates in $k[x, y]$, that $v = x$ (cf. Abhyankar-Moh's Theorem). Then the curve $v = \alpha$ on U is isomorphic to A^1_k for every $\alpha \in k$. This implies that Σ_a $(1 \leqq a \leqq n)$ belongs to one and only one of $\Xi_1, \cdots, \Xi_n$, and that none of Ξ_j's $(1 \leqq j \leqq n)$ contains two of Σ_a's. Note that $\Gamma_j \cup E_j$ is contained in Ξ_j for $1 \leqq j \leqq n+1$. It is now clear that $F_{n+1} = \Delta - (\Delta \cap \Xi_{n+1}) \cup \bigcup_{j=1}^{n} ((\Gamma_j \cup E_j) \cap \Delta)$, where $(\Gamma_j \cup E_j) \cap \Delta$ is a single point if it is nonempty and $\Delta \cong P^1_k$. Since $(\Xi \cdot \Sigma) = 1$, the multiplicity α_{n+1} of Δ in Σ equals 1.

(2) Suppose $\Delta \cap U = \phi$. After a suitable change of the inhomogeneous coordinate v, we may assume that Δ is a component of E_{n+1}. We have $(\Xi \cdot \Sigma_a) = 0$ for $1 \leqq a \leqq n+1$ and $(\Xi_j \cdot \Delta) = 1$ for $1 \leqq j \leqq n$. Hence $\Gamma_j \cup E_j$, which is contained in Ξ_j, is connected to Δ by means of one or more of Σ_a's for $1 \leqq j \leqq n$. This implies that Ξ_{n+1} contains at most one of Σ_a's and Ξ_j $(1 \leqq j \leqq n)$ contains at most two of Σ_a's. Suppose that Ξ_{n+1} contains one of Σ_a's, say Σ_{n+1}. Then Ξ_j $(1 \leqq j \leqq n)$ contains one and only one of Σ_a's $(1 \leqq a \leqq n)$. After a suitable change of indices, we may assume that Ξ_j contains Σ_j for $1 \leqq j \leqq n$. Then it is easy to see that $v - d_j \sim f_j^{\beta_j} f_{n+1}^{-\gamma}$ for $1 \leqq j \leqq n$, where $\beta_j > 0$ $(1 \leqq j \leqq n)$ and $\gamma > 0$; γ is independent of j. Since $u - 1 \sim f_1^{\alpha_1} \cdots f_{n+1}^{\alpha_{n+1}}$, the identification of rings in 1.9.1 implies (by computing the unit groups of the rings on both hand sides) that the following matrix A is unimodular,

$$A = \begin{pmatrix} \alpha_1 \cdots\cdots \alpha_n & \alpha_{n+1} \\ \beta_1 & -\gamma \\ \ddots \quad 0 & \vdots \\ \quad \ddots & \vdots \\ 0 \qquad \beta_n & -\gamma \end{pmatrix},$$

However, $\det A = (-1)^n \{\beta_1 \cdots \beta_n \alpha_{n+1} + \gamma \sum_{i=1}^{n} \beta_1 \cdots \beta_{i-1} \alpha_i \beta_{i+1} \cdots \beta_n\}$, whence A is not unimodular because $n \geqq 2$. Therefore, Ξ_{n+1} contains none of Σ_a's $(1 \leqq a \leqq n+1)$. Then, after a suitable change of indices, we may assume that Σ_j is used to connect $\Gamma_j \cup E_j$ to Δ for $1 \leqq j \leqq n$. The remaining component Σ_{n+1} belongs to one of Ξ_j's $(1 \leqq j \leqq n)$, say Ξ_1, or none of them. It is then clear that $F_a \cong A^1_*$ for $2 \leqq a \leqq n$. If either Σ_{n+1} belongs to none of Ξ_j's $(1 \leqq j \leqq n)$ or Σ_{n+1} is a component of Ξ_1 and $\Sigma_1 \cap \Sigma_{n+1} = \phi$, then $F_1 \cong A^1_*$. If Σ_{n+1} is a component of Ξ_1 and $\Sigma_1 \cap \Sigma_{n+1} \neq \phi$, then $F_1 \cong A^1_k$. In any case, $F_{n+1} \cong A^1_k$. Since $\Xi \cap U = \Xi - (\Xi \cap S_\infty) \cup (\Xi \cap \Delta)$, it is isomorphic to A^1_*. Q.E.D.

3.2. Lemma. *With the assumptions and the notations of Lemma* 3.1, *after a suitable change of coordinates of* $k[x, y]$, *we have one of the following four cases*:

(1) *Case* $\Delta \cap U \neq \phi$:

$$f_j \sim x - d_j \quad (1 \leqq j \leqq n) \quad \textit{and}$$

$$f_{n+1} \sim y \cdot \prod_{j=1}^{n} (x - d_j)^{\varepsilon_j} + P(x), \quad \varepsilon_j \geqq 0, \quad P(x) \in k[x],$$

where $\varepsilon_j = 0$ *whenever* $(\Gamma_j \cup E_j) \cap \Delta = \phi$, *and where* $\varepsilon_j > 0$ *and* $P(d_j) \neq 0$ *whenever* $(\Gamma_j \cup E_j) \cap \Delta \neq \phi$. *Moreover*, $\alpha_{n+1} = 1$.

(2) *Case* $\Delta \cap U = \phi$ *and* $\Sigma_{n+1} \not\subset |\Xi_j|$[(*)] *for* $1 \leqq j \leqq n$:

$$f_{n+1} \sim x \quad \textit{and} \quad f_j \sim x^l(x^t y + P(x)) - d_j \quad \text{for} \quad 1 \leqq j \leqq n,$$

where $l > 0$, $t \geqq 0$ *and* $P(x) \in k[x]$; $\deg P(x) < t$ *and* $P(0) \neq 0$ *if* $t > 0$, *and* $P(x) = 0$ *if* $t = 0$. *Moreover*, $\alpha_{n+1} = 1$.

(3) *Case* $\Delta \cap U = \phi$, $\Sigma_{n+1} \subset |\Xi_1|$ *and* $\Sigma_1 \cap \Sigma_{n+1} \neq \phi$:

$$f_{n+1} \sim x, \quad f_1 \sim y \quad \textit{and} \quad f_j \sim x^l y + d_1 - d_j \quad \textit{for} \quad 2 \leqq j \leqq n,$$

where $l > 0$. *Moreover*, $\alpha_{n+1} - \alpha_1 l = \pm 1$.

(4) *Case* $\Delta \cap U = \phi$, $\Sigma_{n+1} \subset |\Xi_1|$ *and* $\Sigma_1 \cap \Sigma_{n+1} = \phi$:

$$f_{n+1} \sim x, \quad f_1 \sim x^t y + P(x) \quad \textit{and} \quad f_j \sim x^l(x^t y + P(x)) + d_1 - d_j \quad \textit{for} \quad 2 \leqq j \leqq n,$$

where l, $t > 0$ *and* $P(x) \in k[x]$ *with* $\deg P(x) < t$ *and* $P(0) \neq 0$. *Moreover*, $\alpha_{n+1} - l\alpha_1 = \pm 1$.

Proof. (1) Since none of Σ_a's $(1 \leqq a \leqq n+1)$ is contained in Ξ_{n+1} we have $v \in k[x, y]$, and since $\Xi \cap U \cong A^1_k$, we may assume that $v = x$ (cf. Abhyankar-Moh's Theorem). Then, each fiber Ξ_j $(1 \leqq j \leqq n)$ of L (or τ) has only one component Σ_j which intersects U, and $\Sigma_j \cap U = F_j$. Since Ξ_j corresponds to the value $v = d_j$, we have $f_j \sim x - d_j$. Now, the curve F_{n+1}, which is defined by $f_{n+1} = 0$, is written in the form:

(*) For an effective divisor D on V, we denote by $|D|$ the underlying reduced curve of D.

$$f_{n+1}=P_0(x)y^N+\cdots P_N(x)\,,$$

where $P_0(x), \cdots, P_N(x)\in k[x]$ and $P_0(x)\neq 0$. Since F_{n+1} meets the curve $x=\alpha$ transversally in a single point for $\alpha\in k$ such that $\alpha\neq d_j$ $(1\leqq j\leqq n)$ or $\alpha=d_j$ with $F_j\cap F_{n+1}\neq\phi$. Hence we know that $N=1$ and $P_0(x)\sim\prod_{j=1}^{n}(x-d_j)^{\varepsilon_j}$, where $\varepsilon_j=0$ if $F_j\cap F_{n+1}\neq\phi$, i.e., $(\Gamma_j\cup E_j)\cap\Delta=\phi$. If $F_j\cap F_{n+1}=\phi$, i.e., $(\Gamma_j\cup E_j)\cap\Delta\neq\phi$, we have $\varepsilon_j>0$ and $P_1(d_j)\neq 0$. Thus, we obtain the stated form for f_{n+1}. On the other hand, we have:

$$u-1\sim f_1^{\alpha_1}\cdots f_{n+1}^{\alpha_{n+1}} \quad\text{and}\quad v-d_j\sim f_j^{\beta_j} \quad\text{with}\quad \beta_j>0 \quad\text{for}\quad 1\leqq j\leqq n\,.$$

The identification of rings in 1.9.1 implies that the following matrix A is unimodular,

$$A=\begin{pmatrix} \alpha_1 \cdots \alpha_n & \alpha_{n+1} \\ \beta_1 & 0 \\ \ddots \quad 0 & \vdots \\ 0 \quad \ddots & \vdots \\ \beta_n & 0 \end{pmatrix}.$$

Since $\det A=(-1)^n\alpha_{n+1}\beta_1\cdots\beta_n$, we have $\beta_1=\cdots=\beta_n=\alpha_{n+1}=1$.

(2) Since $\Sigma_{n+1}\not\subset|\Xi_j|$ for $1\leqq j\leqq n$, there is one more reducible fiber Ξ_0, other than Ξ_j's $(1\leqq j\leqq n+1)$, such that

$$\Xi_0=T+l\Sigma_{n+1}+Z\,,$$

where $(T\cdot S_\infty)=1$, $(T\cdot\Sigma_{n+1})\leqq 1$, $|Z|\subset E_{n+1}$ if $Z>0$, and $l>0$. Then T_0 $(:=T\cap U)\cong \boldsymbol{A}_k^1$ if $(T\cdot\Sigma_{n+1})=1$ and $T_0\cong\boldsymbol{A}_*^1$ if $(T\cdot\Sigma_{n+1})=0$. Since $\Xi\cap U\cong\boldsymbol{A}_*^1$, we may assume, by virtue of Theorem 2.3 that $f_{n+1}=x$ and T_0 is defined by $y=0$ (if $T\cong\boldsymbol{A}_k^1$) or $x^ty+P(x)=0$ (if $T\cong\boldsymbol{A}_*^1$), where $t>0$ and $P(x)\in k[x]$ with $\deg P(x)<t$ and $P(0)\neq 0$. Moreover, we may assume that the fiber Ξ_0 corresponds to the value $v=0$, where $v\in k[x,y]$ because $|\Xi_{n+1}|\cap U=\phi$. Since Ξ_j corresponds to the value $v=d_j$ and $F_j=|\Xi_j|\cap U$ for $1\leqq j\leqq n$, we have $f_j\sim x^l(x^ty+P(x))-d_j$. We obtain $\alpha_{n+1}=1$ by the same argument as in the case (1).

(3) The fiber Ξ_1 is, then, written in the form:

$$\Xi_1=\Sigma_1+l\Sigma_{n+1}+Z\,,$$

where $(\Sigma_1\cdot\Delta)=1$, $(\Sigma_1\cdot\Sigma_{n+1})=1$, $|Z|=E_1\cup\Gamma_1$, and $l>0$. Since $F_1\cong F_{n+1}\cong\boldsymbol{A}_k^1$, we may assume that $f_{n+1}=x$ and $f_1=y$ (cf. Lemma 1.10). Since $v\in k[x,y]$ and the fiber Ξ_1 corresponds to the value $v=d_1$, we may assume that $v=x^ly+d_1$. Then we have $f_j\sim v-d_j=x^ly+d_1-d_j$ for $2\leqq j\leqq n$, because $\Sigma_j\subset|\Xi_j|$ and $(\Sigma_j\cdot\Delta)=1$. On the other hand, we have:

$$u-1\sim f_1^{\alpha_1}\cdots f_{n+1}^{\alpha_{n+1}}\,,\quad v-d_1\sim f_1f_{n+1}^l$$
$$\text{and}\quad v-d_j\sim f_j \quad\text{for}\quad 2\leqq j\leqq n\,.$$

By the same reasoning as in the case (1), we have the following unimodular matrix

$$A=\begin{pmatrix} \alpha_1\cdots\alpha_n & \alpha_{n+1} \\ 1 & l \\ 1 \quad 0 & \\ \ddots & 0 \\ 0 \quad \ddots & \\ & 1 \end{pmatrix}.$$

Since $\det A=(-1)^n(\alpha_{n+1}-\alpha_1 l)$, we have $\alpha_{n+1}-\alpha_1 l=\pm 1$.

(4) The fiber Ξ_1 is, then, written in the form:

$$\Xi_1=\Sigma_1+l\Sigma_{n+1}+Z,$$

where $(\Xi_1\cdot\Delta)=1$, $(\Sigma_1\cdot\Sigma_{n+1})=0$, $\Sigma_{n+1}\cap|Z|\neq\phi$, $|Z|=E_1\cup\Gamma_1$ and $l>0$. Since $F_{n+1}\cong \boldsymbol{A}_k^1$, we may assume that $f_{n+1}=x$. Since $v\in k[x,y]$ and $\Xi\cap U\cong\boldsymbol{A}_*^1$, v is a generically rational polynomial with $n=1$, and the curve $v=d_1$ is a reducible member of $\Lambda_0(v)$ with two mutually non-intersecting components. By virtue of Lemma 2.2, we may set

$$f_1=x^t y+P(x), \quad t>0, \quad P(x)\in k[x]$$
$$\text{with} \quad \deg P(x)<t \quad \text{and} \quad P(0)\neq 0.$$

Then we may assume that $v-d_1=f_1 f_{n+1}^l=x^l(x^t y+P(x))$. Hence $f_j\sim v-d_j=x^l(x^t y+P(x))+d_1-d_j$ for $2\leqq j\leqq n$. We obtain $\alpha_{n+1}-l\alpha_1=\pm 1$ by the same argument as in the case (3). Q.E.D.

3.3. Theorem. *Let f be a generically rational, irreducible polynomial of simple type in $k[x,y]$ with $n>1$ and only one reducible member in $\Lambda_0(f)$. Then, after a suitable change of coordinates in $k[x,y]$, f is reduced to one of the following four polynomials:*

(1) $f\sim\left(\prod_{j=1}^n (x-d_j)^{\alpha_j}\right)\cdot\left(y\cdot\prod_{j=1}^n (x-d_j)^{\varepsilon_j}+P(x)\right)+1$,

where $d_1,\cdots,d_n$ are mutually distinct elements in k and $P(x)\in k[x]$; $\alpha_j>0$ and $\varepsilon_j\geqq 0$ for $1\leqq j\leqq n$; $P(d_j)\neq 0$ if $\varepsilon_j>0$.

(2) $f\sim x\cdot\prod_{j=1}^n (x^l(x^t y+P(x))-d_j)^{\alpha_j}+1$,

where $l>0$, $t\geqq 0$ and $P(x)\in k[x]$; $\deg P(x)<t$ and $P(0)\neq 0$ if $t>0$ and $P(x)=0$ if $t=0$; α_j's and d_j's are as in the case (1).

(3) $f\sim x^\beta y^{\alpha_1}\cdot\prod_{j=2}^n (x^l y-d_j)^{\alpha_j}+1$,

where $d_2,\cdots,d_n$ are mutually distinct elements in k^*; $\beta>0$, $l>0$ and $\alpha_j>0$ for

$1\leqq j\leqq n$; $\beta-\alpha_1 l=\pm 1$.

(4) $f\sim x^\beta\cdot(x^t y+P(x))^{\alpha_1}\cdot\prod_{j=2}^{n}(x^l(x^t y+P(x))-d_j)^{\alpha_j}+1$,

where $t>0$ *and* $P(x)\in k[x]$ *with* $\deg P(x)<t$ *and* $P(0)\neq 0$; β, l, α_j*'s and* d_j*'s are as in the case* (3).

Proof. Follows easily from Lemma 3.2.

3.4. By comparison of the unit groups of two rings, which are connected to each other by the identification of rings as in 1.9.1, we can prove the following:

Proposition. *Let* $f_1,\cdots,f_{n+1}$ $(n\geqq 2)$ *be mutually distinct irreducible polynomials in* $k[x, y]$. *Assume that we have the identification of rings,*

$$k[x, y, (f_1\cdots f_{n+1})^{-1}]=k[u, v, u^{-1}, (\prod_{j=1}^{n}(v-d_j))^{-1}],$$

where $u=f_1^{\alpha_1}\cdots f_{n+1}^{\alpha_{n+1}}$ *with* $\alpha_i>0$ $(1\leqq i\leqq n+1)$, *and* $d_1,\cdots,d_n$ *are mutually distinct elements of* k. *Then, after a suitable change of indices and a suitable change of variables u and v, we are reduced to the following case*:

$$v-d_1\sim f_1^{\beta_1}f_2^{\beta_2}\quad and\quad v-d_j\sim f_{j+1}\quad for\quad 2\leqq j\leqq n,$$

where β_1, $\beta_2\geqq 0$ *and* $\alpha_1\beta_2-\alpha_2\beta_1=\pm 1$.

3.5. Next, we shall consider the case where $r\geqq 2$ and $m_i<n$ for every i $(1\leqq i\leqq r)$. Here, we retain the notations in 1.9. Since $m_i\geqq 2$, we have $n\geqq 3$.

Lemma. *With the assumptions and the notations as above, we have*:

(1) *For every* i $(1\leqq i\leqq r)$, $E_{ij}\cap\Xi=\phi$ *for* $1\leqq j\leqq n+1$, *where* Ξ *is a general fiber of the* $\boldsymbol{P}^1$*-fibration* $\tau: V\rightarrow\boldsymbol{P}_k^1$ (cf. 1.9.6), *and there exists an irreducible component* Δ_i *among* S_{ia}*'s* $(1\leqq a\leqq m_i)$ *such that* $(\Delta_i\cdot\Xi)=1$, i.e., Δ_i *is a cross-section of* τ; *moreover*, $E_{ij}\cup\Gamma_j\subset|\Xi_j|$ *for* $1\leqq i\leqq r$ *and* $1\leqq j\leqq n+1$.

(2) $\Xi\cap U\cong\boldsymbol{A}_k^1$.

(3) *For every* i $(1\leqq i\leqq r)$, $F_{ia}\cong\boldsymbol{A}_k^1$ *if* $S_{ia}\neq\Delta_i$ $(1\leqq a\leqq m_i)$ *and* $\Delta_i\cap U=\Delta_i-\bigcup_{j=1}^{n+1}((\Gamma_j\cup E_{ij})\cap\Delta_i)$.

(4) *One of* Ξ_j*'s* $(1\leqq j\leqq n+1)$, *say* Ξ_{n+1}, *contains none of* S_{ia}*'s* $(1\leqq i\leqq r$ *and* $1\leqq a\leqq m_i)$. *Then, every* Ξ_j $(1\leqq j\leqq n)$ *contains one and only one of* S_{ia}*'s* $(1\leqq i\leqq r$ *and* $1\leqq a\leqq m_i)$ *with multiplicity* 1. *After a suitable change of indices, we may assume that, for every* i $(1\leqq i\leqq r)$, *we have*:

$$\Delta_i=S_{i1}\quad and\quad S_{ia}\subset|\Xi_b|,$$

where $b:=\gamma(i, a)=(m_1-1)+\cdots+(m_{i-1}-1)+(a-1)$ *for* $2\leqq a\leqq m_i$.

Proof. (1) Suppose that, for some i $(1\leqq i\leqq r)$, S_i contains an irreducible component Δ such that $\Delta\cap U=\phi$ and $(\Delta\cdot\Xi)=1$, i.e., Δ is a cross-section of τ. Suppose, for convenience's sake, that $\Delta\subset E_{i,n+1}$. Then $(\Gamma_j\cup E_{ij})\cap\Xi=\phi$ for $1\leqq j\leqq n$, whence $\Gamma_j\cup E_{ij}\subset|\Xi_j|$. Since Δ is a cross-section of τ, every $\Gamma_j\cup E_{ij}$ $(1\leqq j\leqq n)$ is connected to Δ by one or more components of S_{ia}'s $(1\leqq a\leqq m_i)$. However, since $m_i<n$, this is impossible. This implies that there exists a component Δ_i among S_{ia}'s $(1\leqq a\leqq m_i)$ such that $(\Delta_i\cdot\Xi)=1$, i.e., Δ_i is a cross-section. Then $(\Gamma_j\cup E_{ij})\cap\Xi=\phi$ for $1\leqq j\leqq n+1$, whence $(\Gamma_j\cup E_{ij})\subset|\Xi_j|$.

(2) It is now clear that $\Xi\cap U=\Xi-(\Xi\cap S_\infty)\cong \boldsymbol{A}_k^1$.

(3) Suppose that $S_{ia}\neq\Delta_i$ $(1\leqq a\leqq m_i)$. Since $S_{ia}\cap\Xi=\phi$, S_{ia} is a component of some Ξ_j $(1\leqq j\leqq n+1)$. If $S_{ia}\cap\Delta_i\neq\phi$, then $S_{ia}\cap(\Gamma_j\cup E_{ij})\neq\phi$; indeed, if otherwise, $S_{ia}\subset U$, which is a contradiction. Hence $F_{ia}\cong\boldsymbol{A}_k^1$. If $S_{ia}\cap\Delta_i=\phi$ then $E_{ij}\neq\phi$ and $E_{ij}\cap S_{ia}\neq\phi$, whence $F_{ia}\cong\boldsymbol{A}_k^1$. It is now clear that $\Delta_i\cap U=\Delta_i-\bigcup_{j=1}^{n+1}((\Gamma_j\cup E_{ij})\cap\Delta_i)$.

(4) Since Δ_i $(1\leqq i\leqq r)$ is a cross-section of τ, Δ_i is not contained in any one of Ξ_j's $(1\leqq j\leqq n)$. The remaining components of S_{ia}'s $(1\leqq i\leqq r$ and $1\leqq a\leqq m_i)$ are contained in the union of $\Xi_1,\cdots,\Xi_{n+1}$. Since $\sum_{i=1}^r(m_i-1)=n$ (cf. Lemma 1.6), one of Ξ_j's, say Ξ_{n+1}, contains none of S_{ia}'s. Then the inhomogeneous coordinate v is an element of $k[x,y]$. Since $\Xi\cap U\cong\boldsymbol{A}_k^1$, we may assume that $v=x$ (cf. Abhyankar-Moh's Theorem). Then $|\Xi_j|\cap U$, which corresponds to the value $v=d_j$, is irreducible for $1\leqq j\leqq n$. Hence, every Ξ_j $(1\leqq j\leqq n)$ contains one and only one of S_{ia}'s. The remaining assertion is easy to prove.

Q.E.D

3.6. Lemma. *With the assumptions and the notations as in Lemma* 3.5, *we have*:

(1) $f_{ia}\sim x-d_b$ *with* $b=\gamma(i,a)$, *for* $1\leqq i\leqq r$ *and* $2\leqq a\leqq m_i$.

(2) $f_{i1}\sim y\cdot\prod_{j=1}^n(x-d_j)^{\varepsilon_j}+P_i(x)$ *with* $\varepsilon_j\geqq 0$ *and* $P_i(x)\in k[x]$, *where* $\varepsilon_j=0$ *if* $(\Gamma_j\cup E_{ij})\cap\Delta_i=\phi$, *and* $\varepsilon_j>0$ *and* $P_i(d_j)\neq 0$ *if* $(\Gamma_j\cup E_{ij})\cap\Delta_i\neq\phi$.

(3) $\alpha_{i1}=1$ *for every* i $(1\leqq i\leqq r)$.

Proof. (1) As in the proof of Lemma 3.5, we may assume that $v=x$. Then, since $S_{ia}\subset|\Xi_b|$ with $b=\gamma(i,a)$ and $S_{ia}=|\Xi_b|\cap U$, we have $f_{ia}\sim x-d_b$.

(2) f_{i1} can be determined by the same argument as used in the proof of Lemma 3.2, the assertion (1). We only note that Δ_i meets (or does not meet, resp.) the unique component among S_{ia}'s contained in Ξ_j if and only if $(\Gamma_j\cup E_{ij})\cap\Delta_i=\phi$ (or $(\Gamma_j\cup E_{ij})\cap\Delta_i\neq\phi$).

(3) We have:

$$u-c_i\sim\prod_{a=1}^{m_i}f_{ia}^{\alpha_{ia}}\quad\text{for}\quad 1\leqq i\leqq r$$

and $v-d_j\sim f_{ia}$ with $j=\gamma(i,a)$, for $1\leqq j\leqq n$.

Let A_i ($1\leqq i\leqq r$) and A be the following square matrices of size m_i and $n+r$, respectively;

$$A_i=\begin{pmatrix} \alpha_{i1}\cdots\alpha_{im_i} & & \\ 1 & & \\ & \ddots & 0 \\ 0 & \ddots & \\ & & 1\end{pmatrix} \quad\text{and}\quad A=\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_r\end{pmatrix}.$$

The identification of rings in 1.9.1 implies that A is a unimodular matrix. Hence $\alpha_{i1}=1$ for $1\leqq i\leqq r$. Q.E.D.

3.7. Theorem. *Let f be a generically rational, irreducible polynomial of simple type in $k[x, y]$ with $n>1$, $r\geqq 2$ and $n>m_i$ for every i ($1\leqq i\leqq r$); indeed, $n\geqq 3$. Then, after a suitable change of coordinates in $k[x, y]$, f has one of the following presentations*:

$$f\sim\Big(\prod_{a=2}^{m_i}(x-d_{e(a)})^{\alpha_{ia}}\Big)\cdot\Big(y\cdot\prod_{j=1}^{n}(x-d_j)^{\varepsilon_j}+P_i(x)\Big)+c_i \quad \textit{for} \quad 1\leqq i\leqq r\,,$$

where $d_1, \cdots, d_n$ are mutually distinct elements of k, $c_i\in k^*$; $\alpha_{ia}>0$, $\varepsilon_j\geqq 0$ and $e(a):=\gamma(i, a)$ (cf. *Lemma* 3.5); *$P_i(d_j)\neq 0$ if $\varepsilon_j>0$; $\varepsilon_j>0$ if $j\neq e(a)$ for all* a *($1\leqq a\leqq m_i$).*

Proof. Follows easily from Lemmas 3.5 and 3.6.

3.8. Finally, we shall consider the case where $r\geqq 2$ and $m_i=n$ for some i ($1\leqq i\leqq r$). We may assume that $m_1=n$. Lemma 1.6 then implies $r=2$, $m_1=n$ and $m_2=2$. We consider first the case $n\geqq 3$; the case $n=2$ will be treated in the paragraphs 3.11 and 3.12. The $\boldsymbol{P}^1$-fibration $\rho\colon V\to\boldsymbol{P}_k^1$ has two singular fibers S_1 and S_2, where $S_1\cap U$ has n irreducible components and $S_2\cap U$ has 2 irreducible components. Here we retain the notations in 1.9. Let Δ_i ($i=1, 2$) be the irreducible component in S_i such that $(\Delta_i\cdot\Xi)=1$, where Δ_2 must be one of S_{21} and S_{22}, say S_{21}, (cf. the proof of the assertion (1) in Lemma 3.5).

Lemma. *Assume that $n\geqq 3$. With the assumptions and the notations as above, we have one of the following two cases*:

(I) *Case $\Delta_1\cap U\neq\phi$. Then we have the same situation as stated in Lemma* 3.5 *with $r=2$.*

(II) *Case $\Delta_1\cap U=\phi$. We may assume that $\Delta_1\subset E_{1,n+1}$. Then the following assertions hold true*:

(1) *For every j ($1\leqq j\leqq n$), $\Gamma_j\cup E_{1j}\subset|\Xi_j|$; S_{1a} ($1\leqq a\leqq n$) belongs to one and only one of Ξ_j's ($1\leqq j\leqq n$); after a change of indices, we may assume that S_{1a} is contained in Ξ_a with multiplicity* 1 *for $1\leqq a\leqq n$; $F_{1a}\cong\boldsymbol{A}_*^1$ for $1\leqq a\leqq n$.*

(2) *$\Xi\cap U\cong\boldsymbol{A}_*^1$, where Ξ is a general fiber of the $\boldsymbol{P}^1$-fibration $\tau\colon V\to\boldsymbol{P}_k^1$.*

(3) *S_{22} belongs to one of Ξ_j's ($1\leqq j\leqq n+1$), and $S_{22}\not\subset|\Xi_{n+1}|$; we may*

assume that $S_{22}\subset|\Xi_1|$; $F_{22}\cong A_k^1$ *and* $F_{21}=S_{21}-\bigcup_{j=1}^{n+1}(\Gamma_j\cup E_{2j})\cap S_{21}$.

Proof. (I) If $\Delta_1\cap U\neq\phi$, we have only to follow the arguments in Lemma 3.5.

(II) Assume that $\Delta_1\cap U=\phi$. We may assume that $\Delta_1\subset E_{1,n+1}$. Since $E_{1j}\cap\Xi=\phi$, we have $\Gamma_j\cup E_{1j}\subset|\Xi_j|$ for $1\leqq j\leqq n$. Since $(\Gamma_j\cup E_{1j})\cap\Delta_1=\phi$, $\Gamma_j\cup E_{1j}$ is connected to Δ_1 by one and only one of S_{1a}'s for $1\leqq j\leqq n$. After a suitable change of indices a, we may assume that $S_{1a}\subset|\Xi_a|$. Then it is clear that S_{1a} is a component of Ξ_a with multiplicity 1 and that $F_{1a}\cong A_*^1$ for $1\leqq a\leqq n$. This proves the assertion (1).

(2) Since $\Xi\cap U=\Xi-(\Xi\cap S_\infty)\cup(\Xi\cap\Delta_1)$, $\Xi\cap U\cong A_*^1$.

(3) Suppose that S_{22} belongs to none of Ξ_j's $(1\leqq j\leqq n+1)$. Then $S_{22}\subset U$, which is a contradiction. Hence S_{22} belongs to one of Ξ_j's $(1\leqq j\leqq n+1)$. Suppose that $S_{22}\subset|\Xi_{n+1}|$. Then, in the identification of rings in 1.9.1, we have:

$$u-c_1\sim\prod_{a=1}^{n}f_{1a}^{\alpha_{1a}},\qquad u-c_1\sim f_{21}^{\alpha_{21}}f_{22}^{\alpha_{22}}$$
$$v-d_j\sim f_{1j}f_{22}^{-\beta}\quad\text{for}\quad 1\leqq j\leqq n\,,$$

where $\beta>0$, β is independent of j. By comparison of the unit groups of the rings of both hand sides in 1.9.1, we know that the following matrix A is unimodular,

$$A=\begin{pmatrix}\alpha_{11}\cdots\alpha_{1n} & 0 & 0\\ 0\ \cdots\ 0 & \alpha_{21} & \alpha_{22}\\ 1\qquad & 0 & -\beta\\ \ddots\ \ 0 & \vdots & \vdots\\ 0\ \ \ddots\quad & \vdots & \vdots\\ \qquad 1 & 0 & -\beta\end{pmatrix}.$$

Since $\det A=-\beta\alpha_{21}\cdot(\sum_{i=1}^{n}\alpha_{1i})$ and $n\geqq3$, A is not a unimodular matrix, which is a contradiction. Hence $S_{22}\not\subset|\Xi_{n+1}|$. Then we may assume that $S_{22}\subset|\Xi_1|$. Then $F_{22}\cong A_k^1$ (cf. the proof of the assertion (3) of Lemma 3.5), and $F_{21}=S_{21}-\bigcup_{j=1}^{n+1}(\Gamma_j\cup E_{2j})\cap S_{21}$. Q.E.D.

3.9. Lemma. *With the assumptions and the notations as in Lemma* 3.8, *assume that* $\Delta_1\cap U=\phi$. *Then, after a suitable change of coordinates in* $k[x,y]$, *we have*:

(1) $f_{22}\sim x$, $f_{11}\sim x^ly+P(x)$, *and* $f_{1j}\sim x(x^ly+P(x))+d_1-d_j$ *for* $2\leqq j\leqq n$, *where* $l>0$, $P(x)\in k[x]$, $\deg P(x)<l$ *and* $P(0)\neq0$.

(2) $\alpha_{11}=\alpha_{21}=1$.

Proof. Since $F_{22}\cong A_k^1$, we may assume that $f_{22}=x$ (cf. Abhyankar-Moh's Theorem). Since $|\Xi_{n+1}|\cap U=\phi$, the inhomogeneous coordinate v is an element of $k[x,y]$. Since $\Xi\cap U\cong A_*^1$, v is a generically rational polynomial with $n=1$, and the curve $v=d_1$ is the unique reducible member of $\Lambda_0(v)$ with two disjoint components F_{22} and F_{11}. Since $(S_{11}\cdot\Delta_1)=1$, the fiber Ξ_1 is of the form:

$$\Xi_1 = S_{11}+tS_{22}+Z \quad \text{with} \quad |Z| = \Gamma_1 \cup E_{11}\,.$$

On the other hand, since $F_{11}\cap F_{22}=\phi$, f_{11} is written in the form:

$$f_{11} = x^l y + P(x)\,,$$

where $l>0$ and $P(x)\in k[x]$ with $\deg P(x)<l$ and $P(0)\neq 0$. Then we have:

$$v-d_j \sim x^t(x^l y+P(x))+d_1-d_j \quad \text{for} \quad 1\leqq j\leqq n\,,$$
$$v-d_j \sim f_{1j} \quad \text{for} \quad 2\leqq j\leqq n\,.$$

Since $u-c_1\sim \prod_{a=1}^{n} f_{1a}^{\alpha_{1a}}$ and $u-c_2\sim f_{21}^{\alpha_{21}} f_{22}^{\alpha_{22}}$, we have the following unimodular matrix,

$$A = \begin{pmatrix} \alpha_{11} \cdots \alpha_{1n} & 0 & 0 \\ 0 \cdots\cdots 0 & \alpha_{21} & \alpha_{22} \\ \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & \begin{matrix} 0 \\ \\ \end{matrix} & \begin{matrix} t \\ 0 \\ \end{matrix} \end{pmatrix}.$$

Since $\det A=\alpha_{11}\alpha_{21}t=1$, we have $\alpha_{11}=\alpha_{21}=t=1$. Q.E.D.

3.10. Theorem. *Let f be a generically rational, irreducible polynomial of simple type in $k[x,y]$ with $n\geqq 3$, $r=2$, $m_1=n$ and $m_2=2$. Then, after a suitable change of coordinates in $k[x,y]$, f is reduced to a polynomial of one of the following types:*

(I) *f is a polynomial of the type given in Theorem 3.7, where $r=2$, $m_1=n$ and $m_2=2$;*

(II) $f\sim (x^l y+P(x))\cdot \prod_{j=2}^{n} (x(x^l y+P(x))-d_j)^{\alpha_j}+c,$

where $l>0$, $\alpha_j>0$ $(2\leqq j\leqq n)$, $P(x)\in k[x]$ with $\deg P(x)<l$ and $P(0)\neq 0$, $c\in k^*$, and d_j's $(2\leqq j\leqq n)$ are mutually distinct elements in k^*.

Proof. Follows from Lemmas 3.8 and 3.9.

3.11. We shall consider the remaining case: $n=2$ and $m_1=m_2=2$. As in 3.8, let Δ_i be an irreducible component of S_i such that $(\Delta_i\cdot\Xi)=1$, where $i=1, 2$. We shall prove the following:

Lemma. *With the assumptions and the notations as above, we have either $\Delta_1\cup U\neq\phi$ or $\Delta_2\cap U\neq\phi$.*

Proof. Suppose that $\Delta_i\cap U=\phi$ for $i=1, 2$. We may assume that $\Delta_1\subset E_{13}$ and $\Gamma_i\cup E_{1i}\cup S_{1i}\subset|\Xi_i|$ for $i=1, 2$, (cf. the proof of Lemma 3.8); the component S_{1i} has multiplicity 1 in Ξ_i for $i=1, 2$. Suppose $\Delta_2 \subset E_{23}$. Then $E_{13} \neq \phi$ and $E_{23} \neq \phi$, which contradicts the property 1.9.4; this is the only one place where we have to depend on Lemma 1.4. Therefore, $\Delta_2 \not\subset E_{23}$. We may assume that $\Delta_2 \subset E_{22}$, $\Gamma_1 \cup E_{21} \cup S_{21} \subset |\Xi_1|$ and $\Gamma_3 \cup E_{23} \cup S_{22} \subset |\Xi_3|$. Then we have: $u-c_i \sim f_{i1}^{\alpha_{i1}} f_{i2}^{\alpha_{i2}}$ for $i=1, 2$, $v-d_1 \sim f_{11} f_{21} f_{22}^{-\gamma}$ and $v-d_2 \sim f_{12} f_{22}^{-\gamma}$, where $\gamma > 0$. Hence we have the following unimodular matrix,

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 & 0 \\ 0 & 0 & \alpha_{21} & \alpha_{22} \\ 1 & 0 & 1 & -\gamma \\ 0 & 1 & 0 & -\gamma \end{pmatrix}.$$

Since $\det A = -\alpha_{11}\alpha_{22} - \gamma(\alpha_{11}\alpha_{21} + \alpha_{12}\alpha_{21}) < -1$, this is a contradiction. Q.E.D.

Therefore, we have two cases: $\Delta_i \cap U \neq \phi$ for $i=1, 2$; $\Delta_1 \cap U = \phi$ and $\Delta_2 \cap U \neq \phi$. The case $\Delta_1 \cap U \neq \phi$ and $\Delta_2 \cap U = \phi$ is reduced to the second case. In both cases, we are reduced to the same situation as in Lemma 3.8 with $r=2$ and $m_i=2$ ($i=1, 2$). Therefore, we have:

3.12. Theorem. *Theorem* 3.10 *is valid in the case* $n=2$.

3.13. According to Russell [10], a polynomial f in $k[x, y]$ is said to be a *good* field generator if there exists a polynomial g in $k[x, y]$ such that $k(x, y) = k(f, g)$; otherwise, f is said to be a *bad* field generator. The observations in §§2, 3 imply the following:

Theorem. *Let f be a generically rational polynomial of simple type in $k[x, y]$. Then there exists a generically rational polynomial g with at most two places at infinity such that $k(x, y) = k(f, g)$. In particular, f is a good field generator.*

We note that a field generator is not necessarily good. An example of a bad field generator was given in [10].

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