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<td>Author(s)</td>
<td>Ohno, Koji</td>
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<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 33(1) P.235-P.305</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1996</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/4927">https://doi.org/10.18910/4927</a></td>
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TOWARD DETERMINATION OF THE SINGULAR FIBERS OF MINIMAL DEGENERATION OF SURFACES WITH $k = 0$

KOJI OHNO

(Received December 21, 1994)

1. Introduction

Let $f : X \to \mathcal{D}$ be a projective surjective morphism from a complex normal 3-fold $X$ to a disk $\mathcal{D} : = \{ z \in \mathbb{C} ; |z| < 1 \}$. Assume that $f$ is a minimal degeneration of surfaces, i.e., $X$ has only $Q$-factorial terminal singularities with nef canonical divisor $K_X$, and that general fibers are smooth minimal surfaces with $x = 0$. The standard way for studying this degeneration is to use the so called semistable reduction, but it is impracticable in general. Another way was suggested by Y. Kawamata in [7], which may be called a log minimal reduction and explained as follows. Put $\Theta : = f^*(0)$, take a log resolution for the log pair $(X, \Theta)$, $\mu : (Y, \Theta_Y) \to (X, \Theta)$ and apply the log minimal model program for $(Y, \Theta_Y)$. Then after shrinking $\mathcal{D}$ with a projective surjective morphism $\tilde{f} : \tilde{X} \to \mathcal{D}$, where $\tilde{X}$ is normal $Q$-factorial 3-fold, $(\tilde{X}, \tilde{\Theta})$ is strictly log terminal in the sense of [20] and $K_X + \Theta$ is $\tilde{f}$-nef. We note here that $\tilde{X} \setminus \text{Supp } \tilde{\Theta}$ is smooth, and $\text{Supp } \tilde{\Theta} = \text{Supp } \tilde{f}^*(0)$. We call this new degeneration a log minimal degeneration. Log minimal degenerations can be studied in the same way as usual semistable degeneration, for example, irreducible components of the special fiber are normal and cross normally (see [20], Corollary 3.8). We should note that the theory of the log minimal degeneration was predicted in [18], (8.9). The aim of this paper is to determine (up to flops) the singular fiber of a minimal degeneration of surfaces with $x = 0$ of type II (see Definition 4.1) in the special case as explained in the statement of Theorem 4.3 and of type I (see Definition 5.1) under the condition that an associated log minimal degeneration has an irreducible component which is a $\nu_0$-log surface of abelian type (see Definition 5.3) by the above method. In the section 2, we firstly review degenerations of elliptic curves as warming-up. We classify $\nu_0$-log surfaces of type II in the section 3 and apply these results to degenerations of type II in the section 4. In the section 5, we classify $\nu_0$-log surfaces of abelian type which is an ideal generalization of a log Enriques surface whose log canonical cover is an

*This work is partially supported by the Fujukai Foundation.
abelian surface in the sense of D.-Q. Zhang [25] and apply these results to a
classification of degenerations of type I associated with ν₀-log surfaces of abelian
type in the section 5. So far our list in this section does not cover Itaka-Ueno’s
work on the first kind of degenerations of principally polarized abelian surfaces
[23], [24], but our statement is made under weaker assumptions on the general
fibie, which is important for applications to 3-folds. Our method is simple but
powerful, so we expect that this method would work in any characteristic.

**Notations and Conventions**

In what follows we shall use the following notations.

$A_{n,q}$: A surface singularity which is defined by the automorphism of $C^2$, $\sigma: (x, y) \mapsto (\xi x, \xi^q y)$ were $n, q \in \mathbb{N}$ and $\xi$ is the primitive $n$-th root of unity is called
the quotient singularity of type $A_{n,q}$.

$(1/n)(w_1, w_2, w_3)$: A 3-dimensional singularity which is defined by the
automorphism of $C^3$, $\sigma: (x, y, z) \mapsto (\xi^w_1 x, \xi^w_2 y, \xi^w_3 z)$ where $n, w_i \in \mathbb{N}$ for $i = 1, 2, 3$ and $\xi$ is the primitive $n$-th root of unity is called the quotient singularity of
type $(1/n)(w_1, w_2, w_3)$.

By $(C, \{xy = 0\})/\mathbb{Z}_2(1, 1, 1)$ for instance, we mean a pair;

$$(C^3/\langle \sigma \rangle, \{(x, y, z) \in C^3; \ xy = 0\}/\langle \sigma \rangle),$$

where $\sigma$ acts on $C^3$ such that $\sigma^*(x, y, z) = (-x, -y, -z)$.

$\Sigma_d$: Hirzebruch surface of degree $d$. $\infty$-section: A section on $\Sigma_d$ with
self-intersection number $d$.

$n$-section: An irreducible curve on a ruled surface whose intersection number
with a fibre of the ruling is $n$.

$(-n)$-curve: A smooth connected rational curve on a surface with self in-
tersection number $(-n)$, where $n \in \mathbb{N}$.

$\sim$: linear equivalence.

$\sim_{\text{num}}$: numerical equivalence.

$[\Delta]$ : reduced part of the boundary $\Delta$.

$\{\Delta\}$ : fractional part of the boundary $\Delta$.

$\chi_{\text{top}}$: topological Euler characteristic.

$\nu$: $X^\nu \to X$: The normalization of a scheme $X$.

We use terminology such as strictly log terminal, purely log terminal and so on
freely. For definition of these terminology, we refer the reader to [20] or [10].

Acknowledgment. The author would like to express his gratitude to Prof.
Y. Kawamata for initiating him into the “Philosophy of log”, to Prof. N. Na-
kayama for pointing out several gaps and giving him useful advice, to Profs. S.
Tsunoda, A. Fujiki, K. Oguiso, M. Kobayashi, R. Goto for valuable discussions, to
Prof. M. Miyanishi for correcting inaccuracies of the first version of this article and
for warm encouragement, to Prof. S. Mori for informing him about [1], Lemma 6.
2 and to the referee for correcting typographical errors and simplifying the part of the proof of Theorem 5.1 in the cases that Cartier indices are 3 and 5.

2. Degeneration of elliptic curves

Firstly, let us work log minimal degeneration of elliptic curves as warming-up. The log minimal reductions of minimal degeneration of elliptic curves are well known. Conversely, we can classify log minimal degeneration of elliptic curves by using the adjunction theory, the classification of surface log canonical singularities and [5], Lemma 6.1 as follows. We note that this gives another easy proof of [5], Theorem 6.1.

Proposition 2.1. Let \( \tilde{f} : \tilde{S} \to \mathcal{D} \) be a proper surjective morphism from a normal surface \( \tilde{S} \) onto a disk \( \mathcal{D} \). Assume that general fibers of \( \tilde{f} \) are smooth elliptic curves and \((\tilde{S}, \tilde{\Theta})\) is weak Kawamata log terminal and \( K_{\tilde{S}} + \tilde{\Theta} \) is \( \tilde{f} \)-nef, where \( \tilde{\Theta} : \tilde{f}^* (0) \to \tilde{S} \). Then the special fiber \( S_0 = \tilde{f}^* (0) \) is classified as follows. We note that they all exist.

- **\( m_{0, \log} \):** \( S_0 = m \tilde{\Theta} \), where \( m \in \mathbb{N} \) and \( \tilde{\Theta} \) is a smooth elliptic curve.
- **\( m_{b, \log} \):** \( S_0 \) is the singular fiber of a degeneration obtained by blowing up successively some singular loci of the support of the singular fiber of a minimal degeneration of type \( m_{b} (b \geq 2) \).
- **\( I_{0, \log} \):** \( S_0 = 2 \tilde{\Theta} \), where \( \tilde{\Theta} \) is an irreducible smooth rational curve on which lie four singular points of type \( A_{2,1} \).
- **\( I_{b, \log} \):** \( S_0 \) is the singular fiber obtained by blowing up singular points of the support of the singular fiber of a log minimal degeneration \( g : (\tilde{S}, \tilde{\Theta}) \to \mathcal{D} \). \( S_0 = g^* (0) = \sum_{i=0}^{b} \tilde{\Theta}_i (b \geq 1) \), where \( \tilde{\Theta}_i \) are irreducible smooth rational curves and \( \tilde{\Theta}_i \cdot \tilde{\Theta}_{i+1} = 1 \) for \( 0 \leq i \leq b-1 \), \( \tilde{\Theta}_i \cdot \tilde{\Theta}_j = 0 \) otherwise. And on each \( \tilde{\Theta}_0, \tilde{\Theta}_b \) lie two quotient singular points of \( S \) of type \( A_{2,1} \). Other singular points of \( \tilde{S} \) do not lie on \( \tilde{\Theta}_i, 1 \leq i \leq b-1 \).
- **\( II_{\log} \):** \( S_0 = 6 \tilde{\Theta} \), where \( \tilde{\Theta} \) is an irreducible smooth rational curve on which lie three quotient singular points of \( S \) of type \( A_{2,1}, A_{2,1}, A_{3,1} \) respectively.
- **\( II_{b, \log} \):** \( S_0 = 6 \tilde{\Theta} \), where \( \tilde{\Theta} \) is an irreducible smooth rational curve on which lie three quotient singular points of \( S \) of type \( A_{6,5}, A_{2,1}, A_{3,2} \) respectively.
- **\( III_{\log} \):** \( S_0 = 4 \tilde{\Theta} \), where \( \tilde{\Theta} \) is an irreducible smooth rational curve on which lie three quotient singular points of \( S \) of type \( A_{4,1}, A_{4,1}, A_{2,1} \) respectively.
- **\( III_{b, \log} \):** \( S_0 = 4 \tilde{\Theta} \), where \( \tilde{\Theta} \) is an irreducible smooth rational curve on which lie three quotient singular points of \( S \) of type \( A_{4,3}, A_{2,1}, A_{4,3} \) respectively.
IV\text{log}^*: S_0 = 3\bar{\Theta}, where \bar{\Theta} is an irreducible smooth rational curve on which lie three quotient singular points of \( S \) of type \( A_{3,1} \).

Proof. Take any irreducible component \( \bar{\Theta}_0 \) of \( \bar{\Theta} \). Let \( \{ P_i; i \in I \} \) be all singular points of \( S \) which lie on \( \bar{\Theta}_0 \). Put \( m_i := |\text{Weil}(0_s, \Lambda)| \) and \( n(\bar{\Theta}_0) := (\bar{\Theta} - \bar{\Theta}_0) \cdot \bar{\Theta}_0 \). Then

\[
0 = (K_S + \bar{\Theta}) \cdot \bar{\Theta}_0 = 2g(\bar{\Theta}_0) - 2 + \sum_{i \in I} \frac{m_i - 1}{m_i} + n(\bar{\Theta}_0),
\]

where \( g(\bar{\Theta}_0) \) is the genus of \( \bar{\Theta}_0 \) (see [20] or [10]). From the above formula, we can derive \( g(\bar{\Theta}_0) \leq 1 \). When \( g(\bar{\Theta}_0) = 1 \), \( S \) is smooth and singular fiber is of type \( m_{I_{0, \text{log}}} \).

So we may assume \( g(\bar{\Theta}_0) = 0 \) in what follows. Because we have \( n(\bar{\Theta}_0) \leq 2 \), we divide the proof into three cases \( n(\bar{\Theta}_0) = 0, 1, 2 \).

Case \( n(\bar{\Theta}_0) = 0 \). In this case we have \( \sum_{i \in I} (m_i - 1)/m_i = 2 \), hence \( (m_i; i \in I) = (2, 2, 2, 2, 2, 3, 3, 3) \).

Subcase \( (m_i; i \in I) = (2, 2, 2, 2) \). In this case, we can deduce that the singular fiber \( S_0 \) is of type \( I_{0, \text{log}} \).

Subcase \( (m_i; i \in I) = (2, 2, 3, 6) \). When the three singularities are of type \( A_{2,1}, A_{3,1}, A_{6,1} \) respectively, the strict transform \( \bar{\Theta}_0 \) of \( \bar{\Theta}_0 \) on the minimal resolution \( M \) is a \((-1\text{-})-\)curve. After blowing down \((-1\text{-})\)-curves, we get a singular fiber of type \( II_{\text{log}} \).

When the three singularities are of type \( A_{2,1}, A_{3,2}, A_{6,1} \) respectively, we have \( K_M \cdot \bar{\Theta}_0 = -(2/3) \), which is contradiction. When the three singularities are of type \( A_{2,1}, A_{3,1}, A_{6,5} \) respectively, we have \( K_M \cdot \bar{\Theta}_0 = -(1/3) \), which is contradiction.

When the three singularities are of type \( A_{2,1}, A_{3,2}, A_{6,5} \) respectively, the strict transform \( \bar{\Theta}_0 \) is a \((-2\text{-})\)-curve, of type \( A_{2,1}, A_{3,2}, A_{6,5} \) respectively, the strict transform \( \bar{\Theta}_0 \) is a \((-2\text{-})\)-curve, so we get a singular fiber of type \( III^* \). Hence multiplicity of \( \bar{\Theta}_0 \) in the singular fiber is \( 6 \) and we obtained a singular fiber of type \( II_{\text{log}}^* \).

Subcase \( (m_i; i \in I) = (2, 4, 4) \). When the three singularities are of type \( A_{2,1}, A_{4,1}, A_{4,1} \) respectively, the strict transform \( \bar{\Theta}_0 \) of \( \bar{\Theta}_0 \) on the minimal resolution is a \((-1\text{-})\)-curve and after blowing down \((-1\text{-})\)-curves we get a singular fiber of type \( III_{\text{log}} \).

When the three singularities are of type \( A_{2,1}, A_{4,1}, A_{4,3} \) respectively, we have \( K_M \cdot \bar{\Theta}_0 = -(1/2) \), which is a contradiction. When the three singularities are of type \( A_{2,1}, A_{4,3}, A_{4,3} \) respectively, the strict transform \( \bar{\Theta}_0 \) of \( \bar{\Theta}_0 \) on the minimal resolution is a \((-2\text{-})\)-curve and we get a singular fiber of type \( III \). Hence the multiplicity of \( \bar{\Theta}_0 \) in \( S_0 \) is 4 and we obtain a singular fiber of type \( III_{\text{log}}^* \).

Subcase \( (m_i; i \in I) = (3, 3, 3) \). When the three singularities are of type \( A_{3,1}, A_{3,1}, A_{3,1} \) respectively, the strict transform \( \bar{\Theta}_0 \) of \( \bar{\Theta}_0 \) on the minimal resolution is a \((-1\text{-})\)-curve and after blowing down the \((-1\text{-})\)-curve, we get a singular fiber of type
IV. Hence the multiplicity of $\Theta_0$ in $\tilde{S}_0$ is 3, so we obtained a singular fiber of type $IV_{\log}$. When the three singularities are of type $A_{3,1}$, $A_{3,1}$, $A_{3,2}$ respectively, we have $K_{\tilde{M}} \cdot \Theta_0 = -(2/3)$, which is a contradiction. When the three singularities are of type $A_{3,1}$, $A_{3,2}$, $A_{3,2}$ respectively, we have $K_{\tilde{M}} \cdot \Theta_0 = -(1/3)$, which is a contradiction. When the three singularities are of type $A_{3,2}$, the strict transform $\Theta_0$ is a $(−2)$-curve, so we get a singular fiber of type $IV^*$. Hence multiplicity of $\Theta_0$ in the singular fiber is 3 and we obtain a singular fiber of type $IV^*_{\log}$.

Case $n(\Theta_0)=1$. In this case we have $(m_i ; i \in I) = (2, 2)$, so two singularities of type $A_{2,1}$ lie on $\Theta_0$. $\bar{\Theta}$ has a unique component of $\bar{S}_0$, say $\bar{\Theta}_1$, which has non empty intersection with $\bar{\Theta}_0$. $\bar{\Theta}_1$ has the same type as $\bar{\Theta}_0$ or $n(\bar{\Theta}_1)=2$. Thus we get a chain of rational curves $\bar{\Theta}_0, \bar{\Theta}_1, \ldots, \bar{\Theta}_n$ and $\bar{S}$ has only four singularities of type $A_{2,1}$, each two of which lie on $\bar{\Theta}_0, \bar{\Theta}_n$ respectively. After taking the minimal resolution, this chain must be blown down to a singular fiber of type $I_n^*$. So we obtain a singular fiber of type $I_n^*_{\log}$.

Case $n(\Theta_0)=2$. Assume $\tilde{S}_0$ is not a singular fiber of type $I_2^*_{\log}$. Then we have a cycle of rational curves, which must be blown down to a singular fiber of type $mI_b$. Thus we obtained a singular fiber of type $mI_b,_{\log}$. 

3. Classification of $\nu_0$-log surfaces of type II

Let $\bar{f} : (\bar{X}, \bar{\Theta}) \to \bar{D}$ be a log minimal degeneration of surfaces with $x=0$ and let $\bar{\Theta}_i$ be any irreducible component of $\bar{\Theta}$. Then $(\bar{\Theta}_i, \text{Diff}_{\bar{\Theta}_i}(\bar{\Theta} - \bar{\Theta}_i))$ is a $\nu_0$-log surface in the following sense (see [20], (3.2.3)).

**Definition 3.1.** A normal surface with a boundary $(S, \Delta)$ is called a $\nu_0$-log surface, when the following conditions (1), (2), (3), (4) are satisfied.

1. $(S, \Delta)$ is weak Kawamata log terminal.
2. $K_S + \Delta \sim_{\text{num}} 0$, where $\sim_{\text{num}}$ is the numerical equivalence.
3. $\text{Supp}[\Delta] \cap \text{Supp}(\Delta) = \emptyset$, where $[\Delta]$ is the reduced part of $\Delta$ and $\{\Delta\}$ is the fractional part of $\Delta$.
4. All coefficients of $\Delta$ are elements of $\{(m-1)/m | m \in \mathbb{N} \cup \{\infty\}\}$

It is important to classify $\nu_0$-log surfaces and the following is a key lemma to study $\nu_0$-log surfaces which is proved essentially in the proof of Proposition 2.1.

**Lemma 3.1.** Let $(S, \Delta)$ be a $\nu_0$-log surface. Then a connected component $D$ of $[\Delta]$ and the singularities of $S$ in its neighborhood are one of the following 7 types.

$I_{0,\log}$: $D$ is a smooth elliptic curve and $S$ is smooth in its neighborhood.
\( I_{b, \text{log}} : D = \sum_{i=1}^{b} C_i (b \geq 2) \), where \( C_i \)'s form a cycle of rational curves forming a cycle and \( S \) is smooth in its neighborhood.

\( I_{0, \text{log}}^* : D \) is a smooth rational curve on which lie 4 quotient singularities of type \( A_2, 1 \).

\( I_{b, \text{log}}^* : D \) is a linear chain of rational curves, i.e., \( D = \sum_{i=0}^{b} C_i (b \geq 1) \), where \( C_i \)'s are irreducible smooth rational curves an \( C_i \cdot C_{i+1} = 1 (0 \leq i \leq b-1) \), \( C_i \cdot C_j = 0 \) otherwise, and each of edge curves \( C_0, C_b \) contains two singular points of \( S \) of type \( A_2, 1 \).

\( II_{\text{log}} : D \) is an irreducible smooth rational curve on which lie three quotient singular points of \( S \) of type \( A_{6,1} \) (or \( A_{6,2} \)), \( A_{2,1} \) (or \( A_{3,2} \)), respectively.

\( III_{\text{log}} : D \) is an irreducible smooth rational curve on which lie three quotient singular points of \( S \) of type \( A_{4,1} \) (or \( A_{4,2} \)), \( A_{4,1} \) (or \( A_{4,2} \)), \( A_{2,1} \), respectively.

\( IV_{\text{log}} : D \) is an irreducible smooth rational curve on which lie three quotient singular points of \( S \) of type \( A_{3,1} \) or \( A_{3,2} \).

In what follows we shall classify \( \nu_0 \)-log surfaces in certain cases.

**Lemma 3.2.** (cf. [2], Lemma (6.1)) Let \( (S, \Delta) \) be a \( \nu_0 \)-log surface.

(1) Assume that there is a connected reduced curve \( C_0 \subset \Delta \) of type \( I_{b, \text{log}} (b \leq 2) \) as in Lemma 3.1. Then \( \Delta = C_0, O_S(K_S + \Delta) \simeq O_S, S \) is rational and \( (S, 0) \) is canonical.

(2) Assume that there is a component of type \( I_{0, \text{log}} \) as in Lemma 3.1, i.e., a smooth elliptic curve \( C_0 \subset \Delta \). Then \( S \) is rational or birationally elliptic ruled and \( (S, \Delta) \) satisfies one of the following:

(a) \( \Delta = C_0 \) and \( (S, \Delta) \) is canonical.

(b) \( \Delta = C_0 + C_1 \), where \( C_1 \) is a smooth elliptic curve. \( O_S(K_S + \Delta) \simeq O_S, S \) is a birationally elliptic ruled surface and \( (S, \Delta) \) is canonical.

(c) \( \Delta = C_0 + (1/2)C_1 + (1/2)C_2 \), where \( C_1, C_2 \) are smooth elliptic curves, \( S \) is birationally elliptic ruled, \( (S, \Delta) \) is canonical and \( S \) is smooth in a neighborhood of \( \text{Supp} \Delta \).

(d) \( \Delta = C_0 + (1/2)C_1 \), where \( C_1 \) is an irreducible curve. \( S \) is birationally elliptic ruled, \( (S, \Delta) \) is canonical and \( S \) is smooth in a neighborhood of \( \text{Supp} \Delta \).

**Definition 3.2.** A \( \nu_0 \)-log surface \( (S, \Delta) \) is called a \( \nu_0 \)-log surface of type \( III, II, II_a, II_b, II_c, II_d \), when the conditions in (1), (2), (2-a), (2-b), (2-c), (2-d) of Lemma 3.2 are satisfied respectively.

**Proof.** First we note that \( S \) is rational or birationally ruled.

(1) Assume \( h^1(O_S) > 0 \). Let \( \mu : M \to S \) be the minimal resolution and \( \tau : 

\[ ... 

\]
$M \to N$ a birational morphism to a relatively minimal model $N$. $N$ has a $\mathbb{P}^1$-bundle structure $p: N \to \Gamma$ over a smooth curve of genus $h^1(\mathcal{O}_N)$. The assumption implies that there is a rational irreducible component of $\tau_*\mu^*C_0$ which dominates $\Gamma$ or $\tau_*\mu^*C_0$ turns out to be a fibre of $p$, though both cases are absurd. Hence $S$ is rational. From an exact sequence:

$$0 \to \mathcal{O}_S(-[\Delta]) \to \mathcal{O}_S \to \mathcal{O}_{[-\Delta]} \to 0,$$

we have the following exact sequence:

$$H^1(\mathcal{O}_S) \to H^1(\mathcal{O}_{[-\Delta]}) \to H^2(\mathcal{O}_S(-[\Delta])) \to 0 \quad (*)$$

Since $S$ is rational, we have an injection $H^1(\mathcal{O}_{[-\Delta]}) \to H^2(\mathcal{O}_S(-[\Delta]))$. On the other hand, we have $1 = h^1(\mathcal{O}_S) \leq h^1(\mathcal{O}_{[-\Delta]})$. Hence $h^2(\mathcal{O}_S(-[\Delta])) > 0$. By the Serre duality theorem, we get

$$h^2(\mathcal{O}(-[\Delta])) = h^0(\mathcal{H}om(\mathcal{O}_S(-[\Delta]), \omega_S)).$$

Since $\mathcal{H}om(\mathcal{O}_S(-[\Delta]), \omega_S)$ is torsion-free, we have an injection

$$H^0(\mathcal{H}om(\mathcal{O}_S(-[\Delta]), \omega_S)) \to H^0(\mathcal{O}_S(K_S + [\Delta])).$$

Hence $h^0(\mathcal{O}_S(K_S + [\Delta])) > 0$. Since $K_S + [\Delta] \sim_{num} 0$, for an ample divisor $H$ on $S$, we have

$$(K_S + [\Delta]) \cdot H = -[\Delta] \cdot H$$

and

$$(K_S + [\Delta]) \cdot H = [\Delta] \cdot H = 0.$$

Thus

$$[\Delta] = 0, \mathcal{O}_S(K_S + \Delta) \simeq \mathcal{O}_S.$$

From Lemma 3.1, we can deduce that any connected component of $[\Delta]$ other than $D$ is of type $I_{a,0} \log$ or $I_{b,0}$ $(b \geq 2)$, but since $h^1(\mathcal{O}_{[-\Delta]}) = 1$, we have $[\Delta] = C_0$, i.e., $\Delta = C_0$. $S$ has only quotient singularities which are all Gorenstein, so $(S, 0)$ is canonical.

(2) First we note that $h^1(\mathcal{O}_S) \leq 1$. For if $h^1(\mathcal{O}_S) > 0$, $C_0$ dominates $\Gamma$ for which we used the same notation as in (1). From $(*)$, we have the following exact sequence:

$$0 \to \text{Im } \alpha \to H^1(\mathcal{O}_{[-\Delta]}) \to H^2(\mathcal{O}_S(-[\Delta])) \to 0.$$

We note that dim Im $\alpha \leq 1$. First assume that dim Im $\alpha = 0$. In this case we have an injection $H^1(\mathcal{O}_{[-\Delta]}) \to H^2(\mathcal{O}_S(-[\Delta]))$. In the same way as in the argument in (1), we can deduce that $\Delta = C_0$, $\mathcal{O}_S(K_S + \Delta) \simeq \mathcal{O}_S$ and $(S, 0)$ is canonical. So we are in the case (2-a). In what follows we assume that dim Im $\alpha = 1$. Then, $S$ is
birationally elliptic ruled and we have $h^s(\mathcal{O}_{\mathbb{P}^1}) \leq 2$.

Case $h^s(\mathcal{O}_{\mathbb{P}^1}) = 2$; In this case, we have $\mathcal{A} = 0$, $\mathcal{O}_s(K_3 + \Delta) = \mathcal{O}_s$. Each connected component of $|\mathcal{A}|$ is of type $I_{0, \text{log}}$ or $I_{b, \text{log}} (b \geq 2)$, but from (1), there are no components of type $I_{b, \text{log}} (b \geq 2)$. Hence $\mathcal{A} = C_0 + C_1$, where $C_1$ is a smooth elliptic curve and we are in the case (2-b).

Case $h^s(\mathcal{O}_{\mathbb{P}^1}) = 1$; Let $\mu : M \to S$, $\tau : M \to N$, $p : N \to \Gamma$ be as in the proof of (1). Since $S$ has only rational singularities and $\Gamma$ is an elliptic curve, there is a morphism $\pi : S \to \Gamma$ such that $p \circ \tau = \pi \circ \mu$. Let

$$\Delta = C_0 + \tilde{C} + \sum_{i \in I} \frac{m_i - 1}{m_i} C^{(i)}_i + \sum_{j \in J} \frac{n_j - 1}{n_j} C^{(j)}_j$$

be the decomposition of $\Delta$, where $\tilde{C}$ is a reduced curve and $C^{(i)}_i$ ($i \in I$) (resp. $C^{(j)}_j$ ($j \in J$)) are irreducible curves which are vertical (resp. horizontal) with respect to $\pi$. Let $l$ be a general fibre of $\pi$. Then we have

$$0 = (K_3 + \Delta) \cdot l = -2 + C_0 \cdot l + \tilde{C} \cdot l + \sum_{i \in I} \frac{n_i - 1}{n_i} C^{(i)}_i \cdot l,$$

from which we can deduce that $(A)(n_j; j \in J) = (2, 2)$, $\tilde{C} \cdot l = 0$, $C_0 \cdot l = 1$, $C_j \cdot l = 1(j = 1, 2)$, where $C_i := C^{(i)}_i$ or $(B)(n_j; j \in J) = (2)$, $\tilde{C} \cdot l = 0$, $C_0 \cdot l = 1$, $C_1 \cdot l = 2$, where $C_1 := C^{(j)}_j$ or $(C) J = \emptyset$, $\tilde{C} \cdot l = 0$, $C_0 \cdot l = 2$ since $\tilde{C}$ does not contain any reduced curve of type $I_{0, \text{log}}$ or $I_{b, \text{log}} (b \geq 2)$ in Lemma 3.1. Since $\mu$ is a minimal resolution, there is an effective $\mathbb{Q}$-divisor $D$ such that $K_3 + D = \mu^*(K_3 + \Delta)$. Put $\tilde{D} = \tau_* D$. Then we can write

$$\tilde{D} = \begin{cases} C_0 + (1/2) C_1 + (1/2) C_2 + F, & \text{Case (A)} \\ C_0 + (1/2) C_1 + F, & \text{Case (B)} \\ C_0 + F, & \text{Case (C)} \end{cases}$$

where $C_0$, $C_1$, $C_2$ are the strict transform of $C_0$, $C_1$, $C_2$ respectively and $F$ is an effective $\mathbb{Q}$-divisor composed of fibres of $p$. Note that and $K_N + D = \tau_* (K_3 + D) \sim_{\text{num}} 0$. Therefore, we have

$$0 = (K_N + \tilde{D}) \cdot C_0$$

$$= \begin{cases} \frac{(K_N + C_0)^2}{2} \cdot C_0 + (1/2) (C_1 + C_2) \cdot C_0 + C_0 \cdot F, & \text{Case (A)} \\ (K_N + C_0) \cdot C_0 + (1/2) C_1 \cdot C_0 + C_0 \cdot F, & \text{Case (B)} \\ (K_N + C_0) \cdot C_0 + C_0 \cdot F. & \text{Case (C)} \end{cases}$$

In the case $(A)$ (resp. $(B)$), since $C_0$ is a section of $p$, we have $C_i \cdot C_0 = 0 (i = 1, 2)$ (resp. $C_i \cdot C_0 = 0$) and $F = 0$. In the case $(C)$, let $\nu : C_0^\nu \to C_0$ be the normalization of $C_0$. Since

$$\frac{(K_N + C_0)^2}{2} \cdot C_0 + 1 = g(C_0^\nu) + \dim \nu_* \mathcal{O}_{C_0^\nu} / \mathcal{O}_{C_0},$$
we can deduce that \( F=0 \) and \( C_0 \) is a smooth elliptic curve. Since \( C'_1, \ C'_2 \) are smooth in the case (A) and \( C'_3 \) is a 2-section (i.e., intersection number with a fibre of \( p \) is 2) in the case (B), we can see that \((N, \overline{D})\) is canonical. So we can write

\[
K_M + \tau^{-1}\overline{D} = \tau^*(K_N + \overline{D}) + E,
\]

where \( E \) is an effective \( \mathbb{Q} \)-divisor which is \( \tau \)-exceptional. Since

\[
K_S + \Delta_h = \mu_*(K_M + \tau^{-1}\overline{D}) = \mu_*\tau^*(K_N + \overline{D}) + \mu_*E,
\]

where \( \Delta_h \) is the horizontal component of \( \Delta \), and

\[
0 \sim_{\text{num}} K_S + \Delta \sim_{\text{num}} \mu_*E + \Delta_v,
\]

where \( \Delta_v \) is the vertical component of \( \Delta \), we have \( \mu_*E = 0, \ \Delta_v = 0 \). Hence

\[
\Delta = \begin{cases} 
C_0 + (1/2)C_1 + (1/2)C_2, & \text{Case (A)} \\
C_0 + (1/2)C_1, & \text{Case (B)} \\
C_0, & \text{Case (C)}
\end{cases}
\]

and

\[
K_S + \Delta = \mu_*\tau^*(K_N + \overline{D}). \quad (**) \]

Write

\[
D = \mu^{-1}_*\Delta + \sum_{i\in I} a_iE_i,
\]

where \( \{E_i; \ i\in I\} \) are all \( \mu \)-exceptional divisors and \( a_i \)'s are non-negative rational numbers for \( i\in I \). Since

\[
0 \sim_{\text{num}} K_N + \overline{D} = \tau_*(K_M + \mu^{-1}_*\Delta) \sim_{\text{num}} \sum_{i\in I} a_i\tau_*E_i,
\]

we have \( \sum_{i\in I} a_i\tau_*E_i = 0 \). This implies that if \( \mu(E_i)\cap \text{Supp}\Delta \neq \emptyset \) or \( \mu(E_i) \) is a singular point of \( S \) other than a rational double point, then \( a_i > 0 \) and \( E_i \) is \( \tau \)-exceptional. So there is an open subset \( U \subset S \) such that the rational map \( \sigma: \tau\mu^{-1} \) is a morphism on \( U \) and \( S\setminus U \) consists of rational double points of \( S \) which do not lie in \( \text{Supp} \Delta \). Put \( V := \mu^{-1}(U) \). From \((**\)), we have \( K_S + \Delta|_V = \sigma^*(K_N + \overline{D}) \) and

\[
K_M + \tau^{-1}\overline{D}|_V = K_M + \mu^{-1}_*\Delta|_V = (\tau|_V)^*(K_N + \overline{D}) - \sum_{i\in I} a_iE_i|_V.
\]

This implies \( a_i \leq 0 \) if \( \mu(E_i) \subset U \). Thus we get \( a_i = 0 \) for all \( i\in I \), hence \( (S, \Delta) \) is canonical and \( S \) is smooth in a neighborhood of \( \text{Supp}\Delta \). In the case (A), \( C_1, \ C_2 \) are smooth elliptic curves and we are in the case (2-c). In the case (B)(resp. (C)), we know that we are in the case (2-d)(resp. (2-a)).
Definition 3.3. Let \((S, \Delta)\) be a \(\nu_0\)-log surface of type \(\Pi_c\) (resp. type \(\Pi_d\)). If \((S, \Delta)\) is terminal in a neighborhood of Supp \{\Delta\}, we call \((S, \Delta)\), a special \(\nu_0\)-log surface of type \(\Pi_c\) (resp. of type \(\Pi_d\)).

Definition 3.4.

(a) A log surface \((S, \Delta)\) is called an elliptic singular \(\nu_0\)-log surface of type \(\Pi_b\) (resp. \(\Pi_c\), resp. \(\Pi_d\)) if \(S\) has only one simple elliptic singular point \(P \in S\) and \((\tilde{S}, \Delta_\delta)\) is a \(\nu_0\)-log surface of type \(\Pi_b\) (resp. \(\Pi_c\), resp. \(\Pi_d\)), where \(\mu : \tilde{S} \to S\) is the minimal resolution of \(P \subset S\).

(b) Let \(S\) be a reduced irreducible surface which is Cohen-Macaulay and let \(\Delta\) be a boundary on \(S\). We call \((S, \Delta)\) a degenerate \(\nu_0\)-log surface of type \(\Pi_b\) (resp. \(\Pi_c\), resp. \(\Pi_d\)), if \((S, \Delta)\) is semi-canonical and \((S^\nu, \Theta)\) is a \(\nu_0\)-log surface of type \(\Pi_b\) (resp. \(\Pi_c\), resp. \(\Pi_d\)), where \(\nu : S^\nu \to S\) is the normalization of \(S\) and \(\Theta\) is defined by \(K_S + \Theta = \nu^*(K_{S^\nu} + \Delta)(\text{For the definition of "semi-canonical", see [10] or [9].})\).

(c) A log surface \((S, \Delta)\) is called a quasi \(K3\) surface if \(S\) has only two simple elliptic singular points \(P_1, P_2 \in S\) and \((\tilde{S}, \Delta_\delta)\) is a \(\nu_0\)-log surface of type \(\Pi_b\), where \(\mu : \tilde{S} \to S\) is the minimal resolution of \(P_1, P_2 \in S\).

(d) Let \(S\) be a reduced irreducible surface which is Cohen-Macaulay and let \(\Delta\) be a boundary on \(S\). We call \((S, \Delta)\) an elliptic singular degenerate \(\nu_0\)-log surface if \(S\) has only one simple elliptic singular point \(P \in S\), \((S \setminus \{P\}, \Delta)\) is semi-canonical and \((\tilde{S}, \Theta_\delta)\) is of type \(\Pi_b\), where \(\Theta\) is defined as above and \(\mu : \tilde{S} \to S^\nu\) is the minimal resolution of \(P\).

4. Degeneration of type \(\Pi\)

Definition 4.1. A minimal degeneration of surfaces with \(x=0 \ f : X \to \mathcal{D}\) is said to be of type \(\Pi\) if \(f\) has a log minimal reduction \(\tilde{f} : (\tilde{X}, \tilde{\Theta}) \to \mathcal{D}\) such that there is at least one irreducible component \(\tilde{\Theta}_i\) of \(\tilde{\Theta}\) such that \([\text{Diff}_{\tilde{\Theta}}(\tilde{\Theta} - \tilde{\Theta}_i)]\) contains a connected component of type \(I_{0,\log}\) as in Lemma 3.1.

From the results in the previous section, we obtain the following theorem.

Theorem 4.1. Let \((\tilde{X}, \tilde{\Theta})\) be a normal log 3-fold such that \((\tilde{X}, \tilde{\Theta})\) is strictly log terminal, \([\tilde{\Theta}] = \tilde{\Theta}\), Supp \(\tilde{\Theta}\) is connected and Sing \(\tilde{X} \subset \text{Supp} \tilde{\Theta}\). Assume that \((K_{\tilde{X}} + \tilde{\Theta})|_{\tilde{\Theta}} \sim_{\text{num}} 0\) and that there is at least one irreducible component \(\tilde{\Theta}_i\) of \(\tilde{\Theta}\) such that \([\text{Diff}_{\tilde{\Theta}}(\tilde{\Theta} - \tilde{\Theta}_i)]\) contains a connected component of type \(I_{0,\log}\) as in Lemma 3.1. Let \(\Theta = \sum_{i=1}^{r} \Theta_i\) be the irreducible decomposition. Then one of the following holds.

1) \(\tilde{X}\) has only terminal singularities. \((\tilde{\Theta}_i, \text{Diff}_{\tilde{\Theta}_i}(\tilde{\Theta} - \tilde{\Theta}_i))\) is a \(\nu_0\)-log surface
of type $II_b$ for all $i$. $\hat{\Theta}_i \cap \hat{\Theta}_j \neq \emptyset$ if $|i-j|=1$ or $(i, j)=(1, b)$ and $\hat{\Theta}_i \cap \hat{\Theta}_j = \emptyset$ if $|i-j|>1$ and $(i, j)=(1, b)$.

(2) $\hat{X}$ has only canonical singularities and $\hat{X} \setminus \text{Supp } (\text{Diff}_S(0))$ has only terminal singularities. $(\hat{\Theta}_i, \text{Diff}_{S}(\hat{\Theta} - \hat{\Theta}_i))$ is a $\nu_0$-log surface of type $II_b$ for $2 \leq i \leq b-1$ and of type $II_a$, $II_c$ or $II_d$ for $i=1, b$. $\hat{\Theta}_i \cap \hat{\Theta}_j \neq \emptyset$ if $|i-j|=1$ and $\hat{\Theta}_i \cap \hat{\Theta}_j = \emptyset$ if $|i-j|>1$.

Proof. From the assumption and Lemma 3.2, there is an irreducible component $\hat{\Theta}_i$ of $\hat{\Theta}$ such that $(\hat{\Theta}_i, \text{Diff}_{\hat{\Theta}_i}(\hat{\Theta} - \hat{\Theta}_i))$ is a $\nu_0$-log surface of type II. Since $\hat{\Theta}$ is connected, for any component $\hat{\Theta}_i$ of $\hat{\Theta}$, $(\hat{\Theta}_i, \text{Diff}_{\hat{\Theta}_i}(\hat{\Theta} - \hat{\Theta}_i))$ is a $\nu_0$-log surface of type II. We note that in a neighborhood of $\hat{\Theta}_i \setminus \text{Supp } (\text{Diff}_S(0))$, $\hat{X}$ is smooth. For a component $\hat{\Theta}_i$ of $\hat{\Theta}$, if $(\hat{\Theta}_i, \text{Diff}_{\hat{\Theta}_i}(\hat{\Theta} - \hat{\Theta}_i))$ is not a $\nu_0$-log surface of type $II_c$, we have $\{\text{Diff}_{\hat{\Theta}_i}(\hat{\Theta} - \hat{\Theta}_i)\}=0$, so $(\hat{X}, \hat{\Theta})$ is canonical in a neighborhood of $\hat{\Theta}_i \setminus \text{Supp } (\text{Diff}_{\hat{\Theta}_i}(\hat{\Theta} - \hat{\Theta}_i))$ by the following theorem.

**Theorem 4.2.** ([10], Corollary 17.2) Let $(X, S+B)$ be a normal log 3-fold with only $Q$-factorial singularities. Assume that $S$ is reduced, $(X, S+B)$ is log canonical in codimension 2. Then we have

$$\text{totaldiscrep } (S^\nu, \text{Diff}_S(B)) = \text{discrep } (\text{Center } \Pi S \Phi 0, X, S+B),$$

where $S^\nu$ is the normalization of $S$. For the definition of "totaldiscrep" and "discrep", see [10].

Since $\hat{X} \subseteq \text{Supp } \hat{\Theta}$, we can deduce that $(\hat{X}, 0)$ is terminal in a neighborhood of $\hat{\Theta}_i \setminus \text{Supp } (\text{Diff}_{\hat{\Theta}_i}(\hat{\Theta} - \hat{\Theta}_i))$. From the following lemma, we get the desired result.

**Lemma 4.1.** Let $X$ be a normal $Q$-factorial complex 3-fold and let $S$ be a reduced irreducible surface on $X$. Assume that $(X, S)$ is purely log terminal, $\text{Sing } X \subseteq S$, $(S, \text{Diff}_S(0))$ is canonical and all coefficients of the components of $\text{Diff}_S(0)$ are $1/2$. Then $X$ has only canonical singularities.

Proof. Let $\mu_1 : X^c \rightarrow X$ be the canonical blowing up, i.e., $\mu_1$ is a projective birational morphism from a normal 3-fold $X^c$ with only canonical singularities to $X$ and $K_{X^c}$ is $\mu_1$-ample. Let $\mu_2 : Y \rightarrow X^c$ be a $Q$-factorization of $X^c$, i.e., $\mu_2$ is a projective birational morphism from a normal $Q$-factorial 3-fold $Y$ with only canonical singularities to $X^c$ and $\mu_2$ is isomorphic in codimension 1. Put $\mu = \mu_1^c \mu_2$. Then we can write

$$K_Y + \sum_{j \in J} a_j E_j = \mu^* K_{X^c}, \mu^* S = \tilde{S} + \sum_{j \in J} r_j E_j$$

where $\tilde{S} : = \mu_1^* S$, the $E_j (j \in J)$ are all $\mu$-exceptional divisors and the $a_j$ and the
\( r_j \) \((j \in J)\) are positive rational numbers. From the above, we have
\[
K_Y + \bar{S} + \sum_{j \in J} (a_j + r_j)E_j = \mu^*(K_X + S)
\]
and we can show that \((Y, \bar{S})\) is purely log terminal, in particular, \(\bar{S}\) is normal. Taking the adjunction of the above equality, we get
\[
K_S + \text{Diff}_S(\sum_{j \in J} (a_j + r_j)E_j) = \mu^*(K_S + \text{Diff}_S(0)).
\]
Since \((S, \text{Diff}_S(0))\) is canonical, we have \(\sum_{j \in J} (a_j + r_j)E_j = 0\). By the \(\mathbb{Q}\)-factoriality of \(Y\), we can deduce that \((\bigcup_{j \in J} E_j) \cap \bar{S} = \emptyset\) which implies \(I = \emptyset\) and \(X\) has only canonical singularities.

Starting with Theorem 4.1, we can have insight into the minimal degenerations of type II.

**Theorem 4.3.** Let \(f : X \to \mathcal{D}\) be a minimal projective degeneration of surfaces with \(x = 0\) and let \(\tilde{f} : (\tilde{X}, \tilde{\Theta}) \to \mathcal{D}\) be a log minimal reduction of \(f\) with \(\mathcal{D}\) shrunk if necessary. Assume that there is at least one irreducible component \(\tilde{\Theta}_i\) of \(\tilde{\Theta}\) such that \([- \text{Diff}_\mathcal{D}(\tilde{\Theta} + \Theta)|_{\Theta}]\) contains a connected component of type \(I_{0,\log}\) as in Lemma 3.1 and that \(\tilde{\Theta}\) does not contain non-special \(\nu_0\)-log surfaces. Then after flopping \(X\) over \(\mathcal{D}\) if necessary, the singular fibre \(f^*(0)\) has one of the following types.

\[I : f^*(0) = m\Theta,\text{ where } m \in \mathbb{N}\text{ and } \Theta\text{ is an irreducible reduced surface such that the non-normal locus of } S\text{ is a smooth elliptic curve } C\text{ and } (\Theta^*, \nu^{-1}(C))\text{ is a } \nu_0\text{-log surface of type } II_b,\text{ where } \nu : \Theta^* \to \Theta\text{ is the normalization of } \Theta.\]

\[II : f^*(0) = \sum_{i=1}^{b} m\Theta_i,\text{ where } m \in \mathbb{N}\text{, } (\Theta_i, \sum_{j \neq i} \Theta_j|_{\Theta_i})\text{ is a } \nu_0\text{-log surface of type } II_b\text{ for all } i, \Theta_i \cap \Theta_j = \emptyset \text{ if } |i-j| > 1 \text{ and } (i, j) \neq (1, b).\]

\[III : f^*(0) = \sum_{i=1}^{b} m\Theta_i,\text{ where } m \in \mathbb{N}\text{. If } b \geq 2, (\Theta_i, \sum_{j \neq i} \Theta_j|_{\Theta_i})\text{ is a } \nu_0\text{-log surface of type } II_b\text{ for } 2 \leq i \leq b-1 \text{ and } (\Theta_i, \sum_{j \neq i} \Theta_j|_{\Theta_i})(i=1, b)\text{ is either a } \nu_0\text{-log surface of type } II_a\text{ or an elliptic singular or degenerate } \nu_0\text{-log surface of type } II_b. \Theta_i \cap \Theta_j = \emptyset \text{ if } |i-j| = 1. \Theta_i \cap \Theta_j = \emptyset \text{ if } |i-j| > 1. \text{ If } b=1, \Theta_1\text{ is either a quasi } K3\text{ surface or an elliptic singular or degenerate } \nu_0\text{-log surface.}\]

\[IV : \]
\[
f^*(0) = \begin{cases} 
\sum_{i=1}^{b} 2m\Theta_{1,i} + \sum_{j=1}^{b} m\Theta_{2,j}, & \text{Case (a)},
\sum_{i=1}^{b} 2m\Theta_{1,i} + \sum_{j=1}^{b} m\Theta_{2,j}, & \text{Case (b)},
\end{cases}
\]
where \(m \in \mathbb{N}\) and the \(\Theta_{2,j}\) are elliptic ruled surfaces. In the case (a), if \(b \leq 2, (\Theta_{1,i}, \sum_{i \neq j} \Theta_{1,i} + (1/2)\sum_{j=1}^{b} \Theta_{2,j}|_{\Theta_{1,i}})\) is a special \(\nu_0\)-log surface of type \(II_c\) for \(i=1\), of type \(II_b\) for \(2 \leq i \leq b-1\) and for \(i=b\), a \(\nu_0\)-log surface of type \(II_a\) or an elliptic singular or degenerate \(\nu_0\)-log surface of type \(II_b\). \(\Theta_{1,i} \cap \Theta_{1,i}\)
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$\Phi\psi$ if $i-j=1$. Moreover, if $b=1$, $(\Theta_{1,i}, (1/2)\sum_{i=1}^{2} \Theta_{i}\mid_{\Theta_{1,i}})$ is an elliptic singular or degenerate $\nu_0$-log surface of type $\Pi_c$. In the case $(\beta)$, $b \geq 2$ and $(\Theta_{1,i}, \sum_{i=1}^{2} \Theta_{1,i}+(1/2)\sum_{i=2}^{3} \Theta_{2,i}\mid_{\Theta_{1,i}})$ is a $\nu_0$-log surface of type $\Pi_b$ for $2 \leq i \leq b-1$, a special $\nu_0$-log surface of type $\Pi_c$ for $i=1,b$. $S_i \cap S_j \not= \emptyset$ if $|i-j|=1$ and $\Theta_{1,i} \cap \Theta_{1,j} = \emptyset$ if $|i-j|>1$.

\[ V: \quad f^*(\emptyset) = \sum_{i=1}^{b} 2m\Theta_{1,i} + m\Theta_2, \quad \text{Case (a)}, \]
\[ \sum_{i=1}^{b} 2m\Theta_{1,i} + \sum_{j=1}^{3} m\Theta_{2,j}, \quad \text{Case (\beta)}, \]

where $m \in \mathbb{N}$, the $\Theta_i$ and $\Theta_{2,j}$ are ruled surfaces. In the case (a), if $b \geq 2$, $(\Theta_{1,i}, \sum_{i=1}^{2} \Theta_{1,i}+(1/2)\sum_{i=1}^{2} \Theta_{2,i}\mid_{\Theta_{1,i}})$ is a special $\nu_0$-log surface of type $\Pi_d$ for $i=1$, of type $\Pi_b$ for $2 \leq i \leq b-1$ and for $i=b$, a $\nu_0$-log surface of type $\Pi_a$ or an elliptic singular or degenerate $\nu_0$-log surface of type $\Pi_b$. $\Theta_{1,i} \cap \Theta_{1,j} = \emptyset$ if $|i-j|=1$ and $\Theta_{1,i} \cap \Theta_{1,j} = \emptyset$ if $|i-j|>1$. Moreover, if $b=1$, $(\Theta_{1,i}, (1/2)\sum_{i=1}^{2} \Theta_{2,i}\mid_{\Theta_{1,i}})$ is an elliptic singular or degenerate $\nu_0$-log surface of type $\Pi_a$. In the case $(\beta)$, $b \geq 2$ and $(\Theta_{1,i}, \sum_{i=1}^{2} \Theta_{1,i}+(1/2)\sum_{i=1}^{2} \Theta_{2,i}\mid_{\Theta_{1,i}})$ is a $\nu_0$-log surface of type $\Pi_b$ for $2 \leq i \leq b-1$, a special $\nu_0$-log surface of type $\Pi_d$ for $i=1,b$. $\Theta_{1,i} \cap \Theta_{2,j} = \emptyset$ if $|i-j|=1$ and $\Theta_{1,i} \cap \Theta_{1,j} = \emptyset$ if $|i-j|>1$.

\[ VI: \quad f^*(\emptyset) = \sum_{i=1}^{b} 2m\Theta_{1,i} + \sum_{j=1}^{3} m\Theta_{2,j}, \]

where $m \in \mathbb{N}$, $\Theta_{2,1}, \Theta_{2,2}$ are elliptic ruled surfaces, $\Theta_{2,3}$ is a ruled surface and $(\Theta_{1,i}, \sum_{i=1}^{2} \Theta_{1,i}+(1/2)\sum_{i=1}^{2} \Theta_{2,i}\mid_{\Theta_{1,i}})$ is a $\nu_0$-log surface of type $\Pi_b$ for $2 \leq i \leq b-1$, a $\nu_0$-log surface of type $\Pi_c$ for $i=1$, of type $\Pi_d$ for $i=b$. $\Theta_{1,i} \cap \Theta_{2,j} = \emptyset$ if $|i-j|=1$ and $\Theta_{1,i} \cap \Theta_{1,j} = \emptyset$ if $|i-j|>1$.

EXAMPLE. There is a minimal degeneration whose special fibre has simple elliptic singularities even if the total space is smooth. For example, let $X$ be a hypersurface in $\mathbb{P}^3$ which is defined by the equation $X^4+Y^4+Z^2W^2+t(Z^4+W^4)$, where $X, Y, Z, W$ are homogeneous coordinates of $\mathbb{P}^3$ and $\mathcal{D} = \{ t \in \mathbb{C} \mid |t| < 1/2 \}$. Let $f : \mathbb{X} \rightarrow \mathcal{D}$ be the morphism induced by the natural projection $p : \mathbb{P}^3 \times \mathcal{D} \rightarrow \mathcal{D}$. Then, it is easy to verify that $X$ is smooth and has trivial canonical bundle, $X_t = f^*(t)$ is a smooth quartic surface for $t \neq 0$ and that $X_0$ is normal and has only two simple elliptic singularities of type $E_7$ as its singularities (see [19]).

Proof of Theorem 4.3. If $(\Theta_t, \text{Diff}_{\Theta_t}(\Theta_t - \Theta_t))$ is not a $\nu_0$-log surface of type $\Pi_c$ or $\Pi_d$ for any irreducible component $\Theta_t$ of $\Theta_t$, $f : X \rightarrow \mathcal{D}$ can be obtained from $f : \tilde{X} \rightarrow \mathcal{D}$ by running the minimal model program ($\mathcal{D}$shrunken if necessary). Let $\Theta$ be the strict transform of $\Theta_t$ on $X$, then, there is a positive integer $m$ such that $f_*(0) = m\Theta$ since some multiples $K_X$ and $K_X + \Theta$ are multiples of $f^*(0)$. If $I : =
\{i : \tilde{\Theta} \subset \tilde{\Theta}, (\tilde{\Theta}_i, \text{Diff}_s(\tilde{\Theta} - \tilde{\Theta}_i)) \text{ is a special } \nu_0\text{-log surface of type } II_c \text{ or } II_d \} \text{ is not empty, then } |I| = 1 \text{ or } 2 \text{ and } (\tilde{X}, 0) \text{ is terminal outside of } Z := U_{i \in I} \text{Supp}(\text{Diff}_s(\tilde{\Theta} - \tilde{\Theta}_i)). \text{ We can show that the singularities of } \tilde{X} \text{ in a neighborhood of } Z \text{ is } A_{2,1} \times Z \text{ by the following lemma.}

**Lemma 4.2.** Let \((p \in X, S)\) be a germ of 3-dimensional purely log terminal singularity, where \(S\) is a \(Q\)-Cartier prime divisor. Assume that \(p \in S, X \setminus S\) is smooth and that \((p \in S, \text{Diff}_s(0))\) is terminal. Then \(p \in X\) is a smooth point of \(X\) or there is a positive integer \(m\) such that \(X\) is isomorphic to \(C^3/\mathbb{Z}_m(1, q, 0)\) near \(p \in X\) and \(0 \in C^3/\mathbb{Z}_m(1, q, 0)\), where \(q\) is a positive integer such that \((m, q) = 1\).

**Proof of Lemma 4.2.** Assume first that \(p \in X\) does not lie on the support of \(\text{Diff}_s(0)\). From [10], Corollary 17.12, \((X, S)\) is canonical outside the support of \(\text{Diff}_s(0)\). Since \(\text{Sing } X \subset S\), \((p \in X, 0)\) is terminal but by the proof of Lemma 5.3 in [10], \(p \in X\) is in fact smooth because \(p \in S\) is smooth. Assume that \(p \in \text{Supp } \text{Diff}_s(0)\). Let \(\text{Diff}_s(0) = \sum_i (m_i - 1)/m_i \Gamma_i\) be the irreducible decomposition, where the \(m_i\) are integers which are equal to or larger than 2. Since \((p \in S, \text{Diff}_s(0))\) is terminal, \(p \in X\), we have \(\sum (m_i - 1)/m_i < 1\), whence \(l = 1\). So we may write \(\text{Diff}_s(0) = (m - 1)/m \Gamma\), where \(m \geq 2\). Take the log canonical cover \(\pi : \tilde{X} \to X\) with respect to \(K_X + S\) and put \(\tilde{S} := \pi^{-1}(S)\). Let \(mr\) be the local Cartier index of \(K_X + S\) at \(p\), where \(r\) is a positive integer, and assume that \(r < 1\). From [10], Lemma 16.13, \((\tilde{X}, \tilde{S})\) is purely log terminal, hence canonical. So \(\tilde{S}\) is a disjoint union of \(r\)-irreducible components, but this is absurd. Thus, we get \(r = 1\) and \(\tilde{S}\) is irreducible. Since \(\text{Sing } \tilde{X} \subset \tilde{S}, (\tilde{X}, 0)\) is terminal, we have

\[K_{\tilde{S}} = (\pi|_S)^*(K_S + m - 1/m \Gamma)\]

and the proof of Corollary 2.2 in [20] shows that \(\tilde{S}\) is smooth. We can also check this directly. Hence \(\tilde{X}\) is smooth and \(X\) must be a cyclic quotient singularity. Thus the lemma follows from [6], Lemma 9.9.

We blow-up these singularities to obtain \(\sigma : \tilde{X} \to X\), where \((\tilde{X}, 0)\) is terminal. Let \(\tilde{\Theta}_2, (j \in J)\) be exceptional divisors of \(\sigma\) and put \(\tilde{\Theta} : = \sigma^{-1}_* \tilde{\Theta} + \sum_{j \in J} (1/2) \tilde{\Theta}_2, j\). Then we have \(K_{\tilde{X}} + \tilde{\Theta} = \sigma^*(K_X + \Theta)\) and \(f : X \to \tilde{X}\) can be obtained from \(f \circ \sigma : (\tilde{X}, \tilde{\Theta}) \to \tilde{X}\) by running the minimal model program with shrunk if necessary. Let \(\Theta\) be the strict transform of \(\tilde{\Theta}\) on \(X\). Then some multiples of \(K_X + \Theta\) and \(K_X\) are multiples of \(f^*(0)\) and there is a positive integer \(m\) such that \(f^*(0) = 2m\Theta\). In the course of applying the minimal model program, we have to care about divisorial contractions which might produce bad degenerations. We know that irrational surfaces are not contracted to points by the contraction associated with an extremal ray.

**Claim 1.** If \((\tilde{\Theta}_i, (\tilde{\Theta} - \tilde{\Theta}_i)|_{\tilde{\Theta}})\) is a special \(\nu_0\)-log surface of type \(II_c\) or \(II_d\), \(\tilde{\Theta}_i\) is
not contracted in the course of running the minimal model program.

Proof of Claim 1. Let \( \tilde{f}^{(i)} : \tilde{X}^{(i)} \to \mathcal{D} \) be a 3-fold over \( \mathcal{D} \) which is obtained from \( X \) by divisorial contractions and flips. Let \( \rho : \tilde{X}^{(i)} \to \tilde{X}^{(i+1)} \) be a divisorial contraction which contracts \( \tilde{\Theta}^{(i)} \), where \( \tilde{\Theta}^{(i)} \) is the strict transform of \( \Theta_1 \) on \( \tilde{X}^{(i)} \). First suppose that there is an irreducible component \( \tilde{\Theta}^{(i)} \) of \( \lfloor \tilde{\Theta}^{(i)} \rfloor \) other than \( \tilde{\Theta}^{(i)} \) which has non-empty intersection with \( \Theta^0 \), where \( \Theta^0 \) is the strict transform of \( \Theta \) on \( X \). Let \( n_1, n_2 \) be multiplicities of \( \tilde{\Theta}^{(i)} \) and \( \Theta^0 \) respectively, and let \( l \) be a general fibre of the ruling of \( \Theta^0 \). Since \( (K_{X^{(0)}} + \Theta^0) \cdot l = 0 \), we have \( \Theta^0 \cdot l = -K_{X^{(0)}} \cdot l - 2 \). Since \( H^1(\omega_{X^{(0)}}\otimes \mathcal{O}_l) = 0 \) and \( l \subset \text{Reg } X^{(i)} \), we have \( -K_{X^{(0)}} \cdot l = 1 \) and \( \Theta^0 \cdot l = -1 \). From this, we get
\[
0 = \tilde{f}^{(i)*}(0) \cdot l = n_1 \Theta^0 \cdot l + n_2 = n_1 = n_2,
\]
which is a contradiction. If there is no irreducible component of \( \lfloor \tilde{\Theta}^{(i)} \rfloor \), then \( K_{X^{(0)}} \) is numerically trivial over \( \mathcal{D} \) and this leads to a contradiction.

Claim 2. The divisorial contraction of a \( \nu_0 \)-log surface of type \( \Pi_b \) does not change singularities locally on neighbouring surfaces.

Proof of Claim 2. Let \( \rho : \tilde{X}^{(i)} \to \tilde{X}^{(i+1)} \) be a divisorial contraction associated with an extremal ray which contracts \( \tilde{\Theta}^{(i)} \) to a curve, where \( (\tilde{\Theta}^{(i)}, (\tilde{\Theta}^{(i)} - \tilde{\Theta}^{(1)}))_{\mathcal{D}^0} \) is a \( \nu_0 \)-log surface of type \( \Pi_b \), and let \( l \) be a general fibre of the ruling of \( \tilde{\Theta}^{(i)} \). Let \( \tilde{\Theta}^{(i+1)} \) be one of the neighbouring surfaces. Since we have \( (- \tilde{\Theta}^{(i)} - K_{X^{(0)}}) \cdot l = 0 \), \( -\tilde{\Theta}^{(i)} - K_{X^{(0)}} \) is \( \rho \)-trivial, hence \( R^1 \rho_* \mathcal{O}_{X^{(0)}} (-\tilde{\Theta}^{(i)} - \tilde{\Theta}^{(i+1)}) = 0 \) and \( \mathcal{O}_{\mathcal{D}^0} \simeq \rho_* \mathcal{O}_{\mathcal{D}^0} \), where \( \tilde{\Theta}^{(i+1)} = \rho_* \tilde{\Theta}^{(i)} \). So \( \rho \) induces an isomorphism \( \tilde{\Theta}^{(i)} \simeq \tilde{\Theta}^{(i+1)} \).

Claim 3. When a \( \nu_0 \)-log surface of type \( \Pi_a \) is contracted to a point by a divisorial contraction, this contraction produces a simple elliptic singularity on a neighbouring surface.

Proof of Claim 3. Let \( \rho : \tilde{X}^{(i)} \to \tilde{X}^{(i+1)} \) be a divisorial contraction of an extremal ray which contracts \( \tilde{\Theta}^{(i)} \) to a point, where \( (\tilde{\Theta}^{(i)}, (\tilde{\Theta}^{(i)} - \tilde{\Theta}^{(1)}))_{\mathcal{D}^0} \) is a \( \nu_0 \)-log surface of type \( \Pi_a \). Let \( \tilde{\Theta}^{(i+1)} \) be a neighbouring surface. Since \( -\tilde{\Theta}^{(i)} - \tilde{\Theta}^{(i+1)} - K_{X^{(0)}} \) is \( \rho \)-trivial, \( R^1 \rho_* \mathcal{O}_{X^{(0)}} (-\tilde{\Theta}^{(i)} - \tilde{\Theta}^{(i+1)}) = 0 \). So we have a surjection
\[
\mathcal{O}_{\mathcal{D}^0} \twoheadrightarrow \rho_* \mathcal{O}_{\mathcal{D}^0} \oplus \mathcal{O}_{\mathcal{D}^0}.
\]
From an exact sequence;
\[
0 \to \mathcal{O}_{\mathcal{D}^0}(-\tilde{\Theta}^{(i)}) \to \mathcal{O}_{\mathcal{D}^0} \oplus \mathcal{O}_{\mathcal{D}^0} \to \mathcal{O}_{\mathcal{D}^0} \to 0,
\]
we have the following exact sequence
\[
\rho_* \mathcal{O}_{\mathcal{D}^0} \oplus \mathcal{O}_{\mathcal{D}^0} \twoheadrightarrow \rho_* \mathcal{O}_{\mathcal{D}^0} \to R^1 \rho_* \mathcal{O}_{\mathcal{D}^0}(-\tilde{\Theta}^{(i)}).
\]
Put \( \Gamma = \tilde{\Theta}^{(i)}|_{\mathcal{D}^0} \). Since \( \Gamma \) is irreducible, we have an injection \( H^1(\mathcal{O}_{\mathcal{D}^0}(-\Gamma)) \to H^1(\mathcal{O}_{\mathcal{D}^0}) \). But since \( \tilde{\Theta}^{(i)} \) is rational, we deduce that \( R^1 \rho_* \mathcal{O}_{\mathcal{D}^0}(-\tilde{\Theta}^{(i)}) = H^1(\mathcal{O}_{\mathcal{D}^0}) \).
\((-\Gamma')=0\). So we have a surjection
\[ \rho_* \mathcal{O}_{\tilde{\Theta}^{(i+1)}} \to \rho_* \mathcal{O}_{\tilde{\Theta}^{(i)}}. \] (4.2)

From (4.1) and (4.2), we deduce that \( \mathcal{O}_{\tilde{\Theta}^{(i+1)}} \cong \rho_* \mathcal{O}_{\tilde{\Theta}^{(i)}} \) and \( \Gamma' \) is contracted to a simple elliptic singularity on \( \tilde{\Theta}^{(i+1)} \).

Claim 4. When a \( \nu_0 \)-log surface of type \( II_a \) is contracted to a curve by a
divisorial contraction, this contraction produces non-normal singularities on the
neighbouring surfaces, but these singularities are Cohen-Macaulay and semi-
canonical.

Proof of Claim 4. Let \( \rho : X^{(i)} \to X^{(i+1)} \) be a divisorial contraction of an
extremal ray which contracts \( \tilde{\Theta}^{(i)} \) to a curve, where \((\tilde{\Theta}^{(i)}, (\tilde{\Theta}^{(i)} - \tilde{\Theta}^{(i)})|_{\tilde{\Theta}^{(i)}})\) is a \( \nu_0 \)-log
surface of type \( II_a \). Let \( \tilde{\Theta}^{(i+1)} \) be a neighbouring surface. As in the above argument,
we have a surjection
\[ \mathcal{O}_{\tilde{\Theta}^{(i+1)}} \to \rho_* \mathcal{O}_{\tilde{\Theta}^{(i)}}. \] (4.3)

Let \( \pi : S_2(\tilde{\Theta}^{(i+1)}) \to \tilde{\Theta}^{(i+1)} \) be the \( S_2 \)-infication of \( \tilde{\Theta}^{(i+1)} \) and put \( \tilde{\Theta}' = \tilde{\Theta} \times_{\tilde{\Theta}^{(i)}} \)
\( S_2(\tilde{\Theta}^{(i+1)}) \), where \( \tilde{\Theta} : = \tilde{\Theta}^{(i)} \cup \tilde{\Theta}^{(i+1)} \). Let \( \pi' : \tilde{\Theta}' \to \tilde{\Theta} \) be the first projection and let \( \rho' : \tilde{\Theta}' \to S_2(\tilde{\Theta}^{(i+1)}) \) be the second projection,
\[ \begin{array}{ccc}
\tilde{\Theta}' & \xrightarrow{\pi'} & \tilde{\Theta} \\
\rho' \downarrow & & \rho \\
S_2(\tilde{\Theta}^{(i+1)}) & \xrightarrow{\pi} & \tilde{\Theta}^{(i+1)}
\end{array} \]

Since \( \pi \) is finite and isomorphic in codimension 1, \( \pi' \) is finite, birational on
each component and isomorphic on the generic point of the double locus. So \( \pi' \)
is also isomorphism in codimension 1 and \( \mathcal{O}_{\tilde{\Theta}'} \cong \pi'_* \mathcal{O}_{\tilde{\Theta}} \) since \( \tilde{\Theta} \) is Cohen-
Macaulay. Thus \( \pi' \) is an isomorphism. From (4.3), the natural inclusions
\[ \mathcal{O}_{\tilde{\Theta}^{(i+1)}} \to \pi'_* \mathcal{O}_{\tilde{\Theta}^{(i+1)}} \] are surjective. Therefore \( \tilde{\Theta}^{(i+1)} \) satisfies Serre’s condition \( S_2 \) i.e., \( \tilde{\Theta}^{(i+1)} \) is Cohen-
Macaulay. Since the normalization of the new singularity of \( \tilde{\Theta}^{(i+1)} \) coincides with
\( \tilde{\Theta}^{(i)} \), it is easy to see that \( \tilde{\Theta}^{(i+1)} \) is semi-canonical.

Claim 5. Any degenerate \( \nu_0 \)-log surface of type \( II_b \) is not contracted by a
divisorial contraction.

Proof of Claim 5. Let \( \rho : X^{(i)} \to X^{(i+1)} \) be a divisorial contraction associated
with an extremal ray which contracts \( \tilde{\Theta}^{(i)} \), where \((\tilde{\Theta}^{(i)}, (\tilde{\Theta}^{(i)} - \tilde{\Theta}^{(i)})|_{\tilde{\Theta}^{(i)}})\) is a degenerate
\( \nu_0 \)-log surface of type \( II_b \). Since \( h^1(\mathcal{O}_{\tilde{\Theta}^{(i)}}) \neq 0 \), \( \rho(\tilde{\Theta}^{(i)}) \) is not a point, but a curve.
Let \( l \) be a general fibre of \( \rho|_{\tilde{\Theta}^{(i)}} : \tilde{\Theta}^{(i)} \to \rho(\tilde{\Theta}^{(i)}) \) and let \( \tilde{\Theta}^{(i+1)} \) be a neighbouring
surface. Then since \( K_{\varphi^*} \cdot l = -1 \) and \( -\varphi(l) \cdot l = 1 \), we have \( \varphi(l) \cdot l = (K_{\varphi^*} + \varphi(l)) \cdot l = 1 \), which is a contradiction.

By the following lemma, we can see that essentially new singularities do not appear after flips and flops.

**Lemma 4.3.** Let \( \varphi \) (resp. \( \varphi^+ \)): \( X \) (resp. \( X^+ \)) \( \to Z \) be a projective birational morphism from a normal complex 3-fold \( X \) (resp. \( X^+ \)) to a normal 3-fold \( Z \), such that \( \varphi \) (resp. \( \varphi^+ \)) is an isomorphism in codimension 1, \((X, 0)(\text{resp. } (X^+, 0))\) is Kawamata log terminal and \(-K_X \) (resp. \( K_{X^+} \)) is \( \varphi \)-nef (resp. \( \varphi^+ \)-nef). Furthermore, let \( S \) be a reduced surface on \( X \) such that \((X, S)\) is log canonical and \(-K_X + s\) is \( \varphi \)-numerically trivial. Assume that any irreducible component of the exceptional locus of \( \varphi \) is not contained in the non-normal locus of \( S \). Let \( S^+ \) be the strict transform of \( S \) on \( X^+ \). In the above situation, if \( S \) is Cohen-Macaulay, \( S^+ \) is Cohen-Macaulay, too.

**Proof.** We have \( O_S \cong \varphi_* O_S \) by the vanishing theorem, where \( S := \varphi^* S \). Let \( \pi: S_2(S) \to S \) be the \( S_2 \)-ification of \( S \) and let \( S' := S \times_S S_2(S) \). Let \( \pi': S' \to S \) be the first projection and let \( \varphi': S' \to S_2(S) \) be the second projection,

\[
\begin{array}{ccc}
S' & \xrightarrow{\pi'} & S \\
\downarrow{\pi} & & \downarrow{\varphi} \\
S_2(S) & \xrightarrow{\pi} & S
\end{array}
\]

Since \( \pi \) is finite and isomorphic in codimension 1, \( \pi' \) is finite and birational on each component. By the assumption, \( \pi' \) is also an isomorphism in codimension 1. In the same way as in the proof of Claim 4, we can see that \( \pi' \) is an isomorphism and \( S \) is Cohen-Macaulay. Let \( \pi^+: S_2(S) \to S^+ \) be the \( S_2 \)-ification of \( S^+ \). From an exact sequence

\[0 \to O_S \to \pi^* O_{\partial(S)} \to N \to 0,\]

where \( N \) is a sheaf such that \( \dim \text{Supp } N = 0 \), we have the following exact sequence

\[0 \to \varphi^+_S O_{S} \to \varphi^+_S \pi^* O_{\partial(S)} \to \varphi^+_N \to R^1 \varphi^+_S O_S.\]

We note that the last term of the above exact sequence is 0, because \( R^1 \varphi^+_S O_X = 0 \) and \( R^2 \varphi^+_S O_X(-S^+) = 0 \). Since \( S \) is Cohen-Macaulay, we have \( O_S \cong \varphi^+_S \pi^+_S O_{\partial(S)} \). So we have the inclusions

\[O_S \to \varphi^+_S O_S \to \varphi^+_S \pi^+_S O_{\partial(S)},\]

hence \( N = 0 \), which implies that \( S^+ \) is Cohen-Macaulay.
Claim 6. The non-normal locus of a degenerate $\nu_0$-log surface is not contracted by a flipping (or flopping) contraction.

Proof of Claim 6. Let $\varphi : \tilde{X}^{(i)} \to \tilde{X}^{(i+1)}$ be a flipping (or flopping) contraction of an extremal ray which contracts the non-normal locus, say $C$, of $\tilde{S}^{(i)}$, where $(\tilde{S}^{(i)}, (\tilde{\Theta}^{(i)} - \tilde{S}^{(i)})|_{\tilde{S}^{(i)}})$ is a degenerate $\nu_0$-log surface. From an exact sequence:

$$0 \to \mathcal{O}_{\tilde{S}^{(i)}} \to \nu_0^* \mathcal{O}_{\tilde{S}^{(i)}} \to \mathcal{O}_C \to 0,$$

we have an exact sequence:

$$R^1 \varphi_* \mathcal{O}_{\tilde{S}^{(i)}} \to R^1 \varphi_* (\nu_0^* \mathcal{O}_{\tilde{S}^{(i)}}) \to R^1 \varphi_* \mathcal{O}_C,$$

where the last term is 0 since $C$ is a rational curve. We can show that the first term is also 0 since $R^1 \varphi_* \mathcal{O}_{\tilde{S}^{(i)}} = R^2 \varphi_* \mathcal{O}_{\tilde{S}^{(i-1)}} = 0$. Hence $R^1 \varphi_*(\nu_0^* \mathcal{O}_{\tilde{S}^{(i)}}) = 0$. Let $\varphi' : \tilde{S}^{(i)} \to \tilde{S}^{(i+1)}$ be the morphism induced by $\varphi$. Since

$$0 = R^1 \varphi_* (\nu_0^* \mathcal{O}_{\tilde{S}^{(i)}}) = R^1 (\varphi \circ \nu)_* \mathcal{O}_{\tilde{S}^{(i-1)}} = R^1 (\nu_0 \varphi')_* \mathcal{O}_{\tilde{S}^{(i-1)}} = \nu_0^* R^1 \varphi'_* \mathcal{O}_{\tilde{S}^{(i-1)}},$$

we have $R^1 \varphi'_* \mathcal{O}_{\tilde{S}^{(i-1)}} = 0$, which is a contradiction because $\varphi'$ contracts an elliptic curve.

Flips and flops may produce new non-normal singularities, but if the non-normal locus contains a curve, we can show that this assumption leads to a contradiction by the classification of $\nu_0$-log surfaces. Recalling that Serre's conditions $R_2$ and $R_1$ are equivalent to the normality, we can deduce that new non-normal points do not appear. We note by easy observation that the speciality of $\nu_0$-log surfaces is preserved under flips but not under flops. Thus we have proved Theorem 4.3.

5. Classification of $\nu_0$-log surfaces of abelian type

Let $\tilde{f} : (\tilde{X}, \tilde{\Theta}) \to \tilde{X}$ be a log minimal degeneration of surfaces with $\kappa = 0$ and assume that $\tilde{\Theta}$ is irreducible. Then $(\tilde{\Theta}, \text{Diff}(0))$ is a $\nu_0$-log surface of type I in the following sense.

Definition 5.1. Let $(S, \Delta)$ be a $\nu_0$-log surface. $(S, \Delta)$ is called a $\nu_0$-log surface of type I, if $[\Delta] = 0$.

We note that a $\nu_0$-log surface of type I is a Log Enriques surface in the sense of De-Qi Zhang [25], if $\Delta = 0$ and $q(S) = 0$.

Definition 5.2. Let $(S, \Delta)$ be a $\nu_0$-log surface of type I. A number defined by
\[ \text{CI}(S, \Delta) = \min \{ n \in \mathbb{N} : n(K_S + \Delta) \text{ is Cartier} \} \]

is called the Cartier index of \((S, \Delta)\).

Let \((S, \Delta)\) be as above and let \(r\) the minimum value such that \(r(K_S + \Delta) \sim 0\). We define the log canonical cover of \((S, \Delta)\) as

\[ \pi : \tilde{S} = \text{Spec} \bigoplus_{i=0}^{\infty} \mathcal{O}_S[-i(K_S + \Delta)] \to S, \]

where the \(\mathcal{O}_S\)-algebra structure of \(\bigoplus_{i=0}^{\infty} \mathcal{O}_S[-i(K_S + \Delta)]\) is given by a nowhere vanishing section of \(\mathcal{O}_S(r(K_S + \Delta))\). This definition does not depend on the choice of the nowhere vanishing sections up to isomorphisms. By the definition and [20], Corollary 2.2, \(S\) is a normal surface with only rational double points and has trivial canonical bundle. So \(\tilde{S}\) is a \(K3\) surface with only rational double points or an abelian surface by the classification theory of surfaces.

**Definition 5.3.** Let \((S, \Delta)\) be a \(\nu_0\)-log surface of type I, and \(\pi : \tilde{S} \to S\) be the log canonical cover. When \(\tilde{S}\) is \(K3\) surface with only rational double points (resp. abelian surface), \((S, \Delta)\) is called \(\nu_0\)-log surface of type \(K3\) (resp. \(\nu_0\)-log surface of abelian type).

The next lemma gives us a hope of classifying \(\nu_0\)-log surfaces. We refer the reader to [16] Theorem 3.1 or [25] Lemma 2.3.

**Lemma 5.1.** Let \(\bar{\rho}\) be the Picard number of the minimal resolution of the log canonical cover \(\tilde{S}\) and let \(\varphi\) be the Euler function. Then \(\varphi(\text{CI}(S, \Delta))(22 - \bar{\rho})\) if \((S, \Delta)\) is a \(\nu_0\)-log surface of type \(K3\) and \(\varphi(\text{CI}(S, \Delta))(6 - \bar{\rho})\) under the assumption that \((S, \Delta)\) is a \(\nu_0\)-log surface of abelian type and that \(\text{CI}(S, \Delta)(K_S + \Delta) \sim 0\).

In what follows, we mean by writing \(\text{Sing } S = \sum_{n,q} m_{n,q} A_{n,q}\), that the singular locus of \(S\) is composed of \(m_{n,q}\) singular points of type \(A_{n,q}\).

**Theorem 5.1.** \(\nu_0\)-log surfaces of abelian type \((S, \Delta)\) can be classified as follows. In the list below, we mean by writing \(C\), the strict transform of a curve \(C \subset S\) on the minimal resolution of \(S\).

**I:** \(S\) is an abelian surface or a hyperelliptic surface and \(\Delta = 0\).

**II:** \(S = P_E(\mathcal{O}_E \oplus \mathcal{L})\), where \(E\) is a smooth elliptic curve and \(\mathcal{L} \in \text{Pic}^0 E\). Moreover, \(\text{Supp } \Delta\) is smooth and \(\Delta\) has one of the following types.

**II_a:** \(\Delta = \sum \frac{1}{2} C_i\), where \(C_i\) is a section and \(C_i^2 = 0\) for every \(i\).

**II_b:** \(\Delta = \sum \frac{1}{2} C_i\), where \(C_1\) is a 3-section and \(C_2\) is a section. \(C_i\) is a smooth elliptic curve and \(C_i^2 = 0\) for every \(i\).

**II_c:** \(\Delta = \sum \frac{1}{2} C_i\), where \(C_i\) is a 2-section which is a smooth elliptic curve...
and $C_i^2 = 0$ for every $i$.

$II_1: \Delta = \sum_{i=1}^{3}(1/2)C_i$, where $C_1$ is a 2-section which is a smooth elliptic curve, $C_i$ is a section for $i = 2, 3$ and $C_i^2 = 0$ for every $i$.

$II_2: \Delta = (1/2)C$, where $C$ is a 4-section which is a smooth elliptic curve and $C^2 = 0$.

$III_1: S \cong P_2(\mathcal{O}_E \oplus \mathcal{L})$, where $E$ is a smooth elliptic curve and $\mathcal{L} \in \text{Pic}^0 E$. Moreover, $\Delta = \sum_{i=3}^{5}(2/3)C_i$, where $C_i(i = 1, 2, 3)$ are sections with self-intersection number 0 and they are disjoint from each other.

$III_2: S \cong P_2(\mathcal{O}_E \oplus \mathcal{L})$, where $E$ is a smooth elliptic curve and $\mathcal{L} \in \text{Pic}^0 E$. Moreover, $\Delta = \sum_{i=3}^{5}(2/3)C_i$, where $C_1$ is a 2-section which is a smooth elliptic curve and $C_2$ is a section. Moreover, $C_i(i = 1, 2)$ are disjoint from each other and have self-intersection number 0.

$III_3: S \cong P_2(\mathcal{O}_E \oplus \mathcal{L})$, where $E$ is a smooth elliptic curve and $\mathcal{L} \in \text{Pic}^0 E$, and $\Delta = (2/3)C$, where $C$ is a 3-section which is a smooth elliptic curve and $C^2 = 0$.

$IV_1: S$ is a normal rational surface with $\rho(S) = 4$, $\text{Sing} S = 9A_{3,1}$ and $\Delta = 0$. The minimal resolution $M$ of $S$ is obtained by blowing up $\sum d$ ($d \leq 3$).

$IV_2: S \cong P_2(\mathcal{O}_E \oplus \mathcal{L})$, where $E$ is a smooth elliptic curve and $\mathcal{L} \in \text{Pic}^0 E$, and $\Delta = \sum_{i=1}^{3}(3/4)C_{1,i} + (1/2)C_2$, where $C_{1,i}, C_2$ are sections with self-intersection number 0.

$IV_3: S \cong P_2(\mathcal{O}_E \oplus \mathcal{L})$, where $E$ is a smooth elliptic curve and $\mathcal{L} \in \text{Pic}^0 E$, and $\Delta = (3/4)C_1 + (1/2)C_2$, where $C_1$ is a 2-section which is a smooth elliptic curve, $C_2$ is a section and $C_i^2 = 0$ for $i = 1, 2$.

$IV_4: S$ is a normal rational surface with $\rho(S) = 2$ and $\text{Sing} S = 8A_{2,1}$. The minimal resolution $M$ of $S$ is obtained by blowing up $\sum d$ ($d \leq 4$). Moreover, $\Delta = \sum_{i=4}^{9}(1/2)C_{2,i}$, where $C_{2,i}$ is a smooth elliptic curve with $C_{2,i}^2 = 0$, $C_{2,i} \approx P^1$ with $C_{2,i}^2 = -2$ for $i = 2, 3$, and $C_{2,i} \cap \text{Sing} S = 4A_{2,1}$ for $i = 1, 2$.

$IV_5: S$ is a normal rational surface with $\rho(S) = 2$ and $\text{Sing} S = 8A_{2,1}$. The minimal resolution $M$ of $S$ is obtained by blowing up $\sum d$ ($d \leq 4$). Moreover, $\Delta = \sum_{i=4}^{9}(1/2)C_{2,i}$, where $C_{2,i} \approx P^1$ with $C_{2,i}^2 = -2$ for $i = 1, 2$, and $C_{2,i} \cap \text{Sing} S = 4A_{2,1}$ for $i = 1, 2$.

$V: S$ is a rational surface with $\rho(S) = 2$, $\text{Sing} S = 5A_{5,2}$ and $\Delta = 0$. The minimal resolution $M$ of $S$ is obtained by blowing up $\sum d$ ($d \leq 3$).

$VI_1: S \cong P_2(\mathcal{O}_E \oplus \mathcal{L})$, where $E$ is a smooth elliptic curve and $\mathcal{L} \in \text{Pic}^0 E$, and $\Delta = (5/6)C_1 + (2/3)C_2 + (1/2)C_3$, where $C_i(i = 1, 2, 3)$ are sections with
self-intersection number 0 and disjoint from each other.

VI*: \( S \cong \mathbb{P}^1 \times \mathbb{P}^1 \) and \( \Delta = \sum_{i=1}^{3}(2/3)C_{2,i} + \sum_{j=1}^{3}(1/2)C_{3,j} \), where \( C_{2,i} \) (\( i = 1, 2, 3 \)) are fibres of the first projection \( S \to \mathbb{P}^1 \) and \( C_{3,j} \) (\( j = 1, 2, 3, 4 \)) are fibres of the second projection \( S \to \mathbb{P}^1 \).

VI+: \( S \) is a rational surface with \( \rho(S) = 2 \), and \( \text{Sing } S = 3A_{3,1} + 3A_{3,2} \). The minimal resolution \( M \) of \( S \) is obtained by blowing up \( \sum_{d}(d \leq 4) \).

Furthermore, \( \Delta = \sum_{i=1}^{2}(1/2)C_{3,i} \), where \( C_{3,1} \) is a smooth elliptic curve with self-intersection number 0, \( C_{3,2} \cong \mathbb{P}^1 \) with \( C_{3,2}^2 = -2 \), \( C_{3,1} \cap \text{Sing } S = \emptyset \), \( C_{3,2} \cap \text{Sing } S = 3A_{3,2} \) and \( C_{3,1} \cap C_{3,2} = \emptyset \).

VI*: \( S \) is a rational surface with \( \rho(S) = 2 \), and \( \text{Sing } S = 3A_{3,1} + 3A_{3,2} \). The minimal resolution \( M \) of \( S \) is obtained by blowing up \( \sum_{d}(d \leq 4) \).

Furthermore, \( \Delta = (1/2)C_3 \), where \( C_3 \cong \mathbb{P}^1 \) with \( C_3^2 = -2 \), \( C_3 \cap \text{Sing } S = 3A_{3,2} \) and \( C_{3,1} \cap C_{3,2} = \emptyset \).

XII*: \( S \cong \mathbb{P}^1 \times \mathbb{P}^1 \) and \( \Delta = \sum_{i=1}^{4}(3/4)C_{1,i} + \sum_{j=1}^{3}(2/3)C_{2,j} + (1/2)C_3 \), where \( C_{1,i} \) (\( i = 1, 2, 3 \)) and \( C_3 \) are fibres of the first projection \( S \to \mathbb{P}^1 \) and \( C_{2,j} \) (\( j = 1, 2, 3 \)) are fibres of the second projection \( S \to \mathbb{P}^1 \).

XII*: \( S \) is a normal rational surface with \( \rho(S) = 2 \) and \( \text{Sing } S = 4A_{2,1} \). The minimal resolution \( M \) of \( S \) is obtained by blowing up \( \sum_{d}(d \leq 6) \).

Furthermore, \( \Delta = \sum_{i=1}^{4}(2/3)C_{2,i} + \sum_{j=1}^{3}(1/2)C_{3,j} \), where \( C_{2,i} \), \( C_{3,j} \approx \mathbb{P}^1 \), \( C_{3,1} = C_{3,2} = C_{2,1}^2 = -1 \) and \( C_{3,2}^2 = C_{2,2}^2 = 0 \). The configuration of \( \text{Supp } \Delta \) and the singular loci of \( S \) are given as follows.

Proof. Let \( M \to S \) be the minimal resolution and let \( \pi : \tilde{S} \to S \) be the global log canonical cover with respect to the pair \((S, \Delta)\). By \( \sigma \), we signify a generator of the covering transformation group \( \text{Gal } (\tilde{S}/S) \). Put \( \rho = \rho(S) \). We fix these notations in what follows. From Lemma 5.1, the possible value of \( \text{Cl } (S, \Delta) \) is 1, 2, 3, 4, 5, 6, 8, 10 or 12. We may assume that \( \text{Cl } (S, \Delta) \geq 2 \).

Case \( \text{Cl } (S, \Delta) = 2 \). After taking an étale cover of \( S \), we may assume that \( 2(K_S + \Delta) \sim 0 \) since the étale quotient of an elliptic ruled surface is also an elliptic ruled
Let \( p \in \tilde{S} \) be any fixed point of under the action of \( \text{Gal}(\tilde{S}/S) \). The generator \( \sigma \) of \( \text{Gal}(\tilde{S}/S) \) acts on \( m_p/m^2_p \) in such a way that \( \sigma^*(x, y) = (x, -y) \) for a suitable basis \( x, y \). Therefore \( S \) is a smooth surface with \( q = 1 \) and \( \Delta = (1/2)C \), where \( C \) is a not necessarily connected smooth curve. Thus \( S \approx P_k(\mathcal{O}_k \oplus \mathcal{E}) \), where \( E \) is a smooth elliptic curve and \( \mathcal{E} \in \text{Pic}^0E \) (see [17], 4.2.1 or [1], Lemma 6.2).

Case CI \((S, \Delta) = 3\). If \( S \) is not rational, we are in one of the cases \( III_a, III_b \) or \( III_c \). Assume that \( S \) is rational.

Take \( p \in \tilde{S} \) as above. Then \((a)\sigma^*(x, y) = (\xi x, y)\), \((b)(\xi x, \xi^2 y)\) or \((c)(\xi x, \xi y)\) for a suitable basis \( x, y \) of \( m_p/m^2_p \), where \( \xi \) is a primitive cubic root of unity. But case \((a)\) is excluded from the assumption that \( S \) is rational and \((b) \) is also excluded since \( K_S \) is Cartier at \( \pi(p) \). Therefore we are in the case \((c)\). Put \( \tilde{S} = V/L \), where \( V = T_{\tilde{s}, p} \) and \( L \) is a rank 4 free \( \mathbb{Z} \)-module. Since the action of \( \langle \sigma \rangle \) on \( L \) is faithfull and torsion free and \( \mathbb{Z}[\langle \sigma \rangle] \approx \mathbb{Z}[\xi] \) is a principal ideal domain, we have \( L \approx \mathbb{Z}[\xi]^{\oplus 2} \) as \( \mathbb{Z}[\xi] \)-module. From the assumption that \( \sigma^*(x, y) = (\xi x, \xi y) \), \( V \) is unique as the 2-dimensional eigenspace of \( T_{\tilde{s}, p} \otimes T_{\tilde{s}, p} \approx \mathbb{Z}[\xi]^{\oplus 2} \otimes C \) associated with the eigen value \( \xi \) under the action of \( \xi \otimes \text{id} \). Therefore \( V/L \) and the action of \( \langle \sigma \rangle \) is unique up to isomorphism. Hence \( \tilde{S} \approx E_{\xi} \times E_{\xi} \), where \( E_{\xi} = C^2/Z + \xi Z \) and \( \sigma([z, w]) = [\xi z, \xi w] \) for \([z, w] \in E_{\xi} \times E_{\xi} \). Thus \( S \) has 9 singular points of type \( A_{3,1} \) and \( \Delta = 0 \). Let \( Z = \text{Sing } S \). Since \( \pi|_{\tilde{S}\setminus \{Z\}} : \tilde{S}\setminus \{Z\} \to S\setminus Z \) is \( \text{étale} \), we have

\[
\chi_{\text{top}}(\tilde{S}) - 9 = 3(\chi_{\text{top}}(M) - 18). \tag{5.1}
\]

Nothing that \( \chi_{\text{top}}(\tilde{S}) = 0 \) and that \( \chi_{\text{top}}(M) = 2 + \rho + s \), we obtain that \( \rho = 4 \). Thus we are in the case \( III_a \).

Case CI \((S, \Delta) = 4\). If \( S \) is not rational, we are in the case \( IV_a \) or \( IV_b \). Assume \( S \) is rational. Let \( p \in \tilde{S} \) be as above. Then \((a)\sigma^*(x, y) = (-1x, y)\), \((b)(-1x, -1y)\) or \((c)(-x, -1y)\) or \((d)(-1x, \sqrt{-1} y)\) for a suitable basis \( x, y \) of \( m_p/m^2_p \). But case \((a)\) is excluded by the assumption that \( S \) is rational and cases \((b) \) and \((d) \) are also excluded since \( 2K_S \) and \( K_S \) are Cartier at \( \pi(p) \) respectively. Therefore all singular points of \( S \) are of type \( A_{2,1} \) and \( \Delta \) can be written as \( \Delta = (1/2)C \), where \( C \) is a smooth reduced curve such that \( \text{Sing } S \subset \text{Supp } C \). Let \( \mu : M \to S \) be the minimal resolution and put \( C' = \mu^*_kC \). Since \( \pi|_{\tilde{S}\setminus \{\text{Sing } S\}} : \tilde{S}\setminus \{\text{Sing } S\} \to S\setminus \text{Supp } \Delta \) is \( \text{étale} \), we have

\[
\chi_{\text{top}}(\tilde{S}) - \chi_{\text{top}}(\pi^{-1}(C\setminus Z)) - s = 4(\chi_{\text{top}}(C) - \chi_{\text{top}}(C') - 2s + s), \tag{5.2}
\]

where \( Z = \text{Sing } S \) and \( s \) is the number of the singular points of \( S \). Since \( \chi_{\text{top}}(\tilde{S}) = 0 \), \( \chi_{\text{top}}(\pi^{-1}(C\setminus Z)) = 2(\chi_{\text{top}}(C\setminus Z) = -2(4K_M \cdot C' + C^2) - 2s \) and \( \chi_{\text{top}}(M) = 2 + \rho + s \), we obtain that

\[
K_M \cdot C' + C^2 = (1/2)s - 2\rho - 4. \tag{5.3}
\]
On the other hand, we have
\[ K_M + (1/2)C' + (1/4)E \sim_{\text{num}} 0, \]  
(5.4)
where \( E = \sum_{i=1}^s E_i \) and \( E_i (1 \leq i \leq s) \) are \((-2)\)-curves. From (5.4), we get \( K_M^2 + (1/2)K_M \cdot C' = 0 \) and \( K_M \cdot C' + (1/2)C'^2 + (1/4)s = 0 \). Hence \( K_M \cdot C' = 2\rho + 2s - 20 \) and \( C'^2 = 40 - 4\rho - (9/2)s \), since \( K_M^2 = 10 - \rho - s \). These two equations plugged into (5.3) yield \( s = 8 \). Let \( C_i \) be any irreducible component of \( \text{Supp} \Delta \). Since \( \pi^{-1}(C_i) \) is a disjoint union of elliptic curves, \( C_i \) is (1) an elliptic curve or (2) isomorphic to \( P^1 \), and the number of singular points of \( S \) which is contained in \( C_i \) is 4. In the case (1), we have \( C_i^2 = 0 \) and in the case (2), \( C_i^2 = -2 \). Assume that there are two or more elliptic components of \( C \). Let \( \tau : M \to N \) be a birational morphism from \( M \) to a relatively minimal model \( N \) and let \( \overline{C} = \tau_*C \) and \( \overline{E} = \tau_*E \). If \( N \cong P^2 \), then \( \overline{C} \) is a union of two smooth cubic curves and \( \overline{E} = 0 \). If \( N \cong \Sigma_d \), then \( \overline{C} \) is a union of two smooth elliptic curves and \( \overline{E} = 0 \). Hence \( E = 0 \) and this is a contradiction. So we are in the cases \( IV_r \) or \( IV_s \).

Case \( CI(S, \Delta) = 5 \). If \( S \) is not rational, then \( S \) is an elliptic ruled surface. Let \( f \) be a fibre of the ruling. Then we have \( (K_S + (4/5)C, f) = 0 \), hence \( C \cdot f = 5/2 \), which is absurd. Assume \( S \) is rational. Let \( p \in S \) be as above. Then \( (a) \sigma^*(x, y) = (\xi x, y), (b)(\xi^3 x, \xi^2 y), (c)(\xi^2 x, \xi^3 y) \) or \( (d)(\xi x, \xi^4 y) \) for a suitable basis \( x, y \) of \( m_p/m_p^2 \), where \( \xi \) is a primitive fifth root of unity. But the case (a) is excluded by the assumption that \( S \) is rational and the case (d) is also excluded since \( K_S \) is Cartier at \( p \). Put \( \bar{S} = V/L \), where \( V = TS_p \) and \( L \) is a rank 4 free \( \mathbb{Z} \)-module. Since the action of \( \langle \sigma \rangle \) on \( L \) is faithful and torsion free and \( Z[\langle \sigma \rangle] \cong Z[\xi] \) is a principal ideal domain, we have \( L \cong Z[\xi] \) as \( Z[\xi] \)-module. Assume we are in the case (b). Then the eigen vector space of \( T_{S_p} \cong Z[\xi] \otimes C \) associated with the eigen value \( \xi \) under the action of \( \xi \otimes id \) has dimension 1, which is absurd. Hence we are in the case (c). From the assumption that \( \sigma^*(x, y) = (\xi^2 x, \xi^3 y) \), \( V \) is unique as the direct summand of the two eigen vector spaces of \( T_{S_p} \otimes T_{S_p} \cong Z[\xi] \otimes C \) associated with the eigen values \( \xi \) and \( \xi^2 \) under the action of \( \xi \otimes id \). Therefore \( V/L \) and the action of \( \langle \sigma \rangle \) is unique up to isomorphism. Hence \( S \cong C^2/L \), where \( L = \langle(n_1 + \xi n_3 + (\xi + \xi^3)n_4, n_2 + (\xi + \xi^3)n_3 - \xi^4 n_4) | n_i \in \mathbb{Z} (i = 1, 2, 3, 4) \rangle \) and \( \sigma([z, w]) = [(\xi^3 + \xi)z + w, -\xi^4 z] \) for \( [z, w] \in C^2/L \) (see [23]). Thus \( S \) has 5 singular points of type \( A_5 \) and \( \Delta = 0 \). Let \( Z = \text{Sing} S \). Since \( \pi|_{\Sigma_{\nu-\Delta}} : \bar{S} \setminus \pi^{-1}(Z) \to S \setminus Z \) is étale, we have
\[ \chi_{\text{top}}(\bar{S}) - 5 = 5(\chi_{\text{top}}(M) - 15). \]  
(5.5)
Since \( \chi_{\text{top}}(\bar{S}) = 0 \) and \( \chi_{\text{top}}(M) = \rho + 12 \), we obtain that \( \rho = 2 \). Thus we are in the case \( V \).

Case \( CI(S, \Delta) = 6 \). If \( S \) is not rational, we are in the case \( VI_s \). Assume that \( S \) is rational. Let \( p \in \bar{S} \) be as above. Then \( (a) \sigma^*(x, y) = (\xi x, y), (b)(\xi^3 x, \xi^2 y), (c)(\xi^2 x, \xi^3 y) \) for a suitable basis \( x, y \) of \( m_p/m_p^2 \). In the same way as in the argument in the case \( CI(S, \Delta) = 4 \), we can exclude the case (a). Therefore \( \Delta \) can
be written as \( \mathcal{A} = (2/3)C_1 + (1/2)C_2 \), where \( C_i (i = 1, 2) \) is smooth reduced curve such that \( C_1 \) and \( C_2 \) meet transversely and singular points of \( S \) are of type \( A_{3,1} \). Moreover, if \( p \in S \) is a singular point of type \( A_{3,1} \) (resp. \( A_{3,2} \)), then \( p \notin \text{Supp} \ \mathcal{A} \) (resp. \( p \notin C_1 \setminus C_1) \). Put \( C_i = \mu_\delta^\ast C_i (i = 1, 2) \), \( Z = \text{Sing} S \cup \text{Sing} \text{Supp} \mathcal{A} \). Let \( s \) (resp. \( s_2 \)) be the number of the singular point of \( S \) of type \( A_{3,2} \) (resp. \( A_{3,1} \)) and \( s_3 \) be the intersection number of \( C_1 \) and \( C_2 \). Since \( \pi\mid_{S\times\pi^{-1}(Z)}: S\setminus\pi^{-1}(Z) \rightarrow S \setminus Z \) is étale, we have

\[
\chi_{\text{top}}(\mathcal{S}) - \chi_{\text{top}}(\pi^{-1}(C_1 \setminus Z)) - \chi_{\text{top}}(\pi^{-1}(C_2 \setminus Z)) - s_1 - 2s_2 - s_3
\]

\[
= 6(\chi_{\text{top}}(M) - \chi_{\text{top}}(C_1 \setminus \mu^{-1}Z) - \chi_{\text{top}}(C_2 \setminus \mu^{-1}Z))
\]

\[- 3s_1 - 2s_2 - s_3, \tag{5.6}\]

Since \( \chi_{\text{top}}(\mathcal{S}) = 0 \), \( \chi_{\text{top}}(\pi^{-1}(C_1 \setminus Z)) = 2\chi_{\text{top}}(C_1 \setminus Z) = -2(K_M \cdot C_1 + C_1^2) - 2s_3 \), \( \chi_{\text{top}}(\pi^{-1}(C_2 \setminus Z)) = 3\chi_{\text{top}}(C_2 \setminus Z) = -3(K_M \cdot C_1 + C_1^2) - 3s_1 - 3s_3 \) and \( \chi_{\text{top}}(M) = 2 + \rho + 2s_1 + s_2 \), we obtain that

\[
4(K_M \cdot C_1 + C_1^2) + 3(K_M \cdot C_2 + C_2^2) = 2s_1 + 4s_2 - 2s_3 - 6\rho - 12. \tag{5.7}\]

On the other hand, we have

\[
K_M + (2/3)C_1 + (1/2)C_2 + (1/3)E_1 + (1/6)E_2 + (1/3)E_3 \sim 0, \tag{5.8}\]

where \( E_k = \sum_{i=1}^{s_1} E_{k,i} (k = 1, 2) \), \( E_{k,i} (1 \leq j \leq s_1) \) are \((-2)\)-curves, \( E_3 = \sum_{i=1}^{s_2} E_{3,i} (1 \leq j \leq s_2) \) are \((-3)\)-curves. Form (5.8), we get

\[
K_M + (2/3)K_M \cdot C_1 + (1/2)K_M \cdot C_2 + (1/3)s_2 = 0 ,
\]

hence

\[
4K_M \cdot C_1 + 3K_M \cdot C_2 = 12s_1 + 4s_2 + 6\rho - 60 , \tag{5.9}\]

since \( K_M^2 = 10 - 2s_1 - s_2 - \rho \). And we have

\[
K_M \cdot C_1 + (2/3)C_1^2 = -(1/2)s_3 , \tag{5.10}\]

\[
K_M \cdot C_2 + (1/2)C_2^2 = -(1/3)s_1(2/3)s_3 . \tag{5.11}\]

Let \( H_1 \subseteq \text{Gal}(\mathcal{S}/S) \) be the subgroup of order 3 and put \( \mathcal{S}_1 = \mathcal{S}/H_1 \). Let \( \pi_1 : \mathcal{S}_1 \rightarrow S \) be the induced morphism. Define the boundary \( \mathcal{A}_1 \) on \( S_1 \) such that \( K_{S_1} + \mathcal{A}_1 = \pi_1^\ast (K_S + \mathcal{A}) \). Then \((S_1, \mathcal{A}_1)\) is a \( \log \)-surface of abelian type with \( \text{Cl}(S_1, \mathcal{A}_1) = 3 \). Since \( \pi_1^{-1}(C_1) \) is a disjoint union of smooth elliptic curves, we have

\[
K_M \cdot C_1 + C_1^2 + (1/2)s_3 = 0 , \tag{5.12}\]

by the Hurwitz formula. By the same argument as above, we have

\[
K_M \cdot C_2 + C_2^2 + (2/3)(s_1 + s_3) = 0 . \tag{5.13}\]

From (5.10), (5.11), (5.12) and (5.13), we have

\[
K_M \cdot C_1 = -(1/2)s_3 , C_1^2 = 0 , K_M \cdot C_2 = -(2/3)s_3 , C_2^2 = -(2/3)s_1 . \tag{5.14}\]

From (5.9) and (5.14), we obtain
2\(3s_1 + s_2 + s_3 = 3(10 - \rho)\), \quad (5.15)

hence \(2|\rho\). Since \(\rho \leq \rho(\tilde{S})\) and \(\rho(\tilde{S}) = 2\) or 4, we have \(\rho = 2\) or 4. Noting that \(s_1 \equiv 0(\text{mod } 3)\) and \(s_3 \equiv 0(\text{mod } 12)\) from (5.14) and the fact that \(K_M \cdot C_1 + C_1^2 = 0(\text{mod } 2)\), we have the following possibilities:

1. \(\rho = 2\) and \(s_1 = s_2 = 3, s_3 = 0\),
2. \(\rho = 2\) and \(s_1 = 0, s_2 = 12, s_3 = 0\),
3. \(\rho = 2\) and \(s_1 = s_2 = 0, s_2 = 12\),
4. \(\rho = 4\) and \(s_1 = 3, s_2 = s_3 = 0\),
5. \(\rho = 4\) and \(s_1 = 0, s_2 = 9, s_3 = 0\).

On the other hand, from (5.7) and (5.14), we have

\[25s_1 + 2s_2 + s_3 = 3p + 6, \quad (5.16)\]

hence the cases (2) and (4) are excluded. Let \(\tau : M \to N\) be a birational morphism from \(M\) to \(N \simeq \Sigma_d\). For \(i = 1, 2\), Put \(\overline{C}_i : = \tau_* \overline{C}_i\), \(\overline{E}_i : = \tau_* \overline{E}_i\) and \(\overline{C}_i \sim n_i \theta + l_i f\), where \(\theta\) is a section such that \(\theta^2 \leq 0\) and \(f\) is a fibre of \(N\). We note that \(4n_1 + 3n_2 \leq 12\) and \(4l_1 + 3l_2 \leq 6d + 12\) from \(K_N + (2/3) \overline{C}_1 + (1/2) \overline{C}_2 \sim \text{num} 0\). Assume that we are in the case (1). Let \(H_1 \subset \text{Gal}(\tilde{S}/S)\) be the subgroup of order 3 and put \(\tilde{S}_1 : = \tilde{S}/H_1\). We note that \((\tilde{S}_1, (2/3) \pi_1^{-1} C_1)\) is a \(\nu_0\)-log surface of abelian type with \(\text{Cl}(\tilde{S}, (2/3) \pi_1^{-1} C_1) = 3\), where \(\pi_1 : \tilde{S}_1 \to S\) is the induced morphism. If \(C_1 \neq 0\), then \(\tilde{S}_1\) is an elliptic ruled surface, which contradicts \(s_2 = 3\). Hence \(C_1 = 0\). Assume that \(C_2\) contains at least two elliptic components. Since \(n_2 \leq 4\), we have \(\overline{C}_2 = \overline{C}_{2,1} + \overline{C}_{2,2}\), where \(\overline{C}_{2,i}, (i = 1, 2)\) is a 2-section. Put \(\overline{C}_{2,i} \sim 2\theta + l_{2,i} f\) for \(i = 1, 2\). Since \(\pi(\overline{C}_{2,i}) = l_{2,i} - i - 1 \geq 1\) for \(i = 1, 2\), and \(3l_2 = 3(l_{2,1} + l_{2,2}) \leq 6d + 12\), we have \(E_j = 0\) for \(j = 1, 2, 3\) and \(C_{2,i}\) is an elliptic curve for \(i = 1, 2\). Hence \(E_j = 0\) for \(j = 1, 2, 3\), which is absurd.

Nothing that \(K_M \cdot C_2 + C_2^2 = -2\), we conclude that we are in the cases \(VI_1\) or \(VI_2\). Assume that we are in the case (3). Since \(4n_1 + 3n_2 = 12\), we have \((3\cdot a)n_1 = 3, n_2 = 0\) or \((3\cdot b)n_1 = 0, n_2 = 4\). Consider the case (3-a). From the equation \(K_S \cdot C_2 + C_2^2 = -2\), we have \(l_2 = 4\) and \(l_1 = (3/2)d\). On the other hand, we have \(l_1 \geq 2d\) by the assumption, hence \(l_1 = d = 0\) and \(C_1 \cdot \theta = 0\). Thus we are in the case \(VI_2\). Under the assumption in the case (3-b), we conclude that we are also in the case \(VI_2\) by the same way as above. Assume that we are in the case (5). If \(C_2 = 0\), then \(3(K_S + \Delta)\) is Cartier, which contradicts the assumption. Let \(H_2 \subset \text{Gal}(\tilde{S}/S)\) be the subgroup of order 2 and put \(\tilde{S}_2 : = \tilde{S}/H_2\). We note that \((\tilde{S}_2, (1/2) \pi_2^{-1} C_2)\) is a \(\nu_0\)-log surface of abelian type with \(\text{Cl}(\tilde{S}_2, (1/2) \pi_2^{-1} C_2) = 2\), where \(\pi_2 : \tilde{S}_2 \to S\) is the induced morphism. Let \(p \in S\) be a singular point of \(S\) and put \(\tilde{p} : = \pi_2^{-1}(p)\). Let \(L\) be the fibre of the ruling on \(\tilde{S}_2\) which goes through the point \(\tilde{p}\). By construction, we have a faithful and fixed point free group action on the set \(\pi_2^{-1}(C_2) \cap L\) but this set is composed of exactly four points, which is absurd.

Case \(\text{Cl}(S, \Delta) = 8\). We claim that this case does not occur. Decompose \(\Delta\) as \(\Delta = (7/8)C_1 + (3/4)C_2 + (1/2)C_3\), where \(C_i (i = 1, 2, 3)\) is a reduced curves. If \(S\) is not rational, then \(S\) is an elliptic ruled surface. Let \(f\) be a fibre of the ruling of \(S\) and put \(n_i : = (C_i, f)\), then we have \(7n_1 + 6n_2 + 4n_3 = 16\), hence \(n_1 = 0\) and \(4(K_S + \Delta)\) is Cartier, which is absurd. Assume that \(S\) is rational. Let \(p \in \tilde{S}\) be as above. Then (a) \(\sigma^*(x, y) = (\xi x, y)\), (b) \((\xi x, \xi^2 y)\), (c) \((\xi^2 x, \xi^3 y)\), or (d) \((\xi x, \xi^4 y)\), where \(\xi\) is a primitive eighth root of unity, for a suitable basis \(x, \circ\) of \(m_p/m_p^2\). Therefore
Supp $\Delta$ is smooth and all singular points of $S$ are of type $A_{4,1}$, $A_{4,3}$ or $A_{2,1}$ and if $p \in S$ is a singular point of $S$, then $p \in C_2$ and $p$ is of type $A_{2,1}$ or $p \in C_3$ and $p$ is type $A_{4,1}$, $A_{4,3}$ or $A_{2,1}$. We can get $C_1 = 0$ by the same way as in the argument in the case $\text{Cl}(S, \Delta) = 4$. Let $H_1 \subset \text{Gal}(\tilde{S}/S)$ be the subgroup of order 2 and put $\tilde{S}_1 := S/H_1$. Let $\pi_1: S_1 \to S$ be the induced morphism and define the boundary $\tilde{\Delta}_1$ on $S_1$ such that $K_S + \tilde{\Delta}_1 = \pi_1^*(K_S + \Delta)$. Then $(\tilde{S}_1, \tilde{\Delta}_1)$ is a $\nu_0$-log surface of abelian type with $\text{Cl}(S_1, \tilde{\Delta}_1) = 2$. Let $p \in S_1$ be a fixed point of the action of $\sigma^2$ on $S_1$ and $L$ be the fibre of the ruling on $\tilde{S}_1$ which passes through $p$. Since the cyclic group of order four acts on the set $L \cap \text{Supp} \tilde{\Delta}_1$ which is composed of exactly four points, this set decomposes to a disjoint union of orbits whose cardinality is $1, 1, 1, 1$ or $1, 1, 2$ respectively. But in the first case, $\sigma^2$ acts trivially on $L$ and in the second case, $\sigma^2$ acts trivially on $L$, which is absurd.

Case $\text{Cl}(S, \Delta) = 10$. We claim that this case does not occur. We may assume that $10(K_S + \Delta) \sim 0$. Let $H_1 \subset \text{Gal}(\tilde{S}/S)$ be the subgroup of order 2 and put $\tilde{S}_1 := S/H_1$. Let $\pi_1: S_1 \to S$ be the induced morphism and define the boundary $\tilde{\Delta}_1$ on $S_1$ such that $K_S + \tilde{\Delta}_1 = \pi_1^*(K_S + \Delta)$. Then $(\tilde{S}_1, \tilde{\Delta}_1)$ is a $\nu_0$-log surface of abelian type with $\text{Cl}(S_1, \tilde{\Delta}_1) = 2$. $\sigma^2$ acts on $S_1$, hence on $\text{Alb} S_1$, but since it is well known that group action of order 5 on an elliptic curve is trivial or fixed point free, the action of $\sigma^2$ on $S_1$ is fixed point free, which contradicts the assumption.

Case $\text{Cl}(S, \Delta) = 12$. If $S$ is not rational, then $S$ is an elliptic ruled surface and $\text{Supp} \Delta$ is a disjoint union of smooth elliptic curves. Let $\Delta = (11/12)C_1 + (5/6)C_2 + (3/4)C_3 + (2/3)C_4 + (1/2)C_5$ be the decomposition of $\Delta$ and $f$ be a fibre of the ruling of $S$. Put $n_i := (C_i, f)$. We have $11n_1 + 10n_2 + 9n_3 + 8n_4 + 6n_5 = 24$ from the assumption, hence $(n_1; 1 \leq i \leq 5) = (0, 1, 0, 1, 1), (0, 0, 2, 0, 1), (0, 0, 0, 0, 4)$ or $(0, 0, 0, 3, 0)$ and $4(K_S + \Delta)$ is Cartier, which is absurd. Therefore $S$ is rational. Let $\pi: S_1 \to S$ and $p \in S$ be as above. We have $(a_1) \sigma^4(x, y) = (\xi x, y), (b_1)(\xi^2 x, \xi^3 y), (c_1)(\xi^2 x, \xi^3 y), (d_1)(\xi x, \xi^4 y), (e_1)(\xi^3 x, \xi^4 y), (f_1)(\xi x, \xi^4 y)$, where $\xi$ is a primitive twelfth root of unity, for a suitable basis $x, y$ of $m_p/m_p^2$. Therefore, $C_i$ is smooth for $1 \leq i \leq 5$. $\text{Supp} C_i \cap \text{Supp} C_j = \emptyset$ for $i < j$ except for $(i, j) = (3, 4), (4, 5)$ and each components of $C_3$ and $C_4, C_4$ and $C_5$ intersect transversely. If $p$ is any singular point of $S$, then $p$ is of type $A_{3,1}$ and $p \in S \setminus \text{Supp} \Delta$ or $p \in C_3$, of type $A_{3,2}$ and $p \in C_5$, of type $A_{6,5}$ and $p \in C_5$, of type $A_{2,1}$ and $p \in C_5$ or $p \in C_5$. Let $s_1$ be the number of the singular points $p \in S$ of type $A_{2,1}$ such that $p \in C_2$, $s_2$ be the number of the singular points $p \in S$ of type $A_{3,1}$ such that $p \in C_3$, $s_3$ be the number of singular points $p \in S$ of type $A_{2,1}$ such that $p \in C_5$, $s_4$ be the number of the singular points $p \in S$ of type $A_{6,5}$ such that $p \in C_5$, $s_5$ be the number of the singular points $p \in S$ of type $A_{3,2}$ such that $p \in C_5$, $s_6$ be the number of the singular points $p \in S$ of type $A_{2,1}$ such that $p \in C_4 \setminus C_1$, $s_7$ be the number of the point $p \in S$ such that $p \in C_3 \setminus C_4$, $s_8$ be the number of the point $p \in S$ such that $p \in C_4 \setminus C_4$ and $S$ is smooth at $p$ and $s_9$ be the number of the singular points $p \in S$ of type $A_{3,1}$ such that $p \in \text{Supp} \Delta$. Let $H_1 \subset \text{Gal}(\tilde{S}/S)$ be the subgroup of order 6 and put $S_1 := S/H_1$ and let $\pi_1: S_1 \to S$ be the induced morphism. Assume that $C_1 \neq 0$ or $C_2 \neq 0$. If $\Delta$ is smooth and all singular points of $S$ are of type $A_{4,1}, A_{4,3}$ or $A_{2,1}$ and if $p \in S$ is a singular point of $S$, then $p \in C_2$ and $p$ is of type $A_{2,1}$ or $p \in C_3$ and $p$ is type $A_{4,1}, A_{4,3}$ or $A_{2,1}$. We can get $C_1 = 0$ by the same way as in the argument in the case $\text{Cl}(S, \Delta) = 4$. Let $H_1 \subset \text{Gal}(\tilde{S}/S)$ be the subgroup of order 2 and put $S_1 := S/H_1$. Let $\pi_1: S_1 \to S$ be the induced morphism and define the boundary $\tilde{\Delta}_1$ on $S_1$ such that $K_S + \tilde{\Delta}_1 = \pi_1^*(K_S + \Delta)$. Then $(\tilde{S}_1, \tilde{\Delta}_1)$ is a $\nu_0$-log surface of abelian type with $\text{Cl}(S_1, \tilde{\Delta}_1) = 2$. Let $p \in S_1$ be a fixed point of the action of $\sigma^2$ on $S_1$ and $L$ be the fibre of the ruling on $\tilde{S}_1$ which passes through $p$. Since the cyclic group of order four acts on the set $L \cap \text{Supp} \tilde{\Delta}_1$ which is composed of exactly four points, this set decomposes to a disjoint union of orbits whose cardinality is $1, 1, 1, 1$ or $1, 1, 2$ respectively. But in the first case, $\sigma^2$ acts trivially on $L$ and in the second case, $\sigma^2$ acts trivially on $L$, which is absurd.
0. Define the boundary $\tilde{A}_1$ as

$$\tilde{A}_1 := (5/6)\pi^{-1}(C_1 \cup C_2) + (2/3)\pi^{-1}(C_4) + (1/2)\pi^{-1}(C_3 \cup C_5).$$

We note that $K_s + \tilde{A}_1 = \pi^*(K_\Sigma + \Delta)$ and $(\tilde{S}_1, \tilde{A})$ is a $\nu$-log surface of type $VI_\delta$ by construction. The induced group action on $\tilde{S}_1$ has fixed point $p \in \tilde{S}_1$ by assumption. Let $L$ be the fibre of the ruling on $\tilde{S}_1$ which passes through $p$. Since the sets $L \cap \pi^{-1}(C_1 \cup C_2)$, $L \cap \pi^{-1}(C_4)$ and $L \cap \pi^{-1}(C_3 \cup C_5)$ are Gal ($\tilde{S}_1/S$)-invariant, Gal ($\tilde{S}_1/S$) acts on $L$ trivially, which is absurd. Thus we get $C_1 = 0$, $C_2 = 0$ and $s_1 = 0$. Assume that $s_2 \neq 0$. Then there is a singular point $p \in S$ of type $A_{3,1}$ such that $p \in C_3$. Let $H_2 \subset \text{Gal}(\tilde{S}/S)$ be the subgroup of order 4 and put $\tilde{S}_2 := \tilde{S}/H_2$ and let $\pi_2 : \tilde{S}_2 \to S$ be the induced morphism. Define the boundary $\tilde{A}_2$ as $\tilde{A}_2 := (3/4)\pi^{-1}(C_3) + (1/2)\pi^{-1}(C_5)$. We note that $K_5 + \tilde{A}_2 = \pi^*_2(K_5 + \Delta)$ and $(\tilde{S}_2, \tilde{A}_2)$ is a $\nu$-log surface of type $IV_\alpha$ or $IV_\beta$ by construction. The induced group action on $\tilde{S}_2$ has a fixed point $p \in \pi^{-1}(p)$. Let $L$ be the fibre of the ruling on $\tilde{S}_2$ which passes through $p$. Since the sets $L \cap \pi^{-1}(C_3)$ and $L \cap \pi^{-1}(C_5)$ is Gal ($\tilde{S}_2/S$)-invariant, the action of Gal ($\tilde{S}_2/S$) on $L$ has three fixed points, hence trivial, which is absurd. Therefore we obtain $s_2 = 0$. Put $Z := \text{Sing Supp } \mathcal{A} \cup \text{Sing } S$, $C_i := \mu^i C_i, (3 \leq i \leq 5)$. Since $\pi|_{\tilde{S} \setminus (\text{Supp } \mathcal{A} \cup \text{Sing } S)} : \tilde{S} \setminus (\text{Supp } \mathcal{A} \cup \text{Sing } S) \to S \setminus (\text{Supp } \mathcal{A} \cup \text{Sing } S)$ is étale, we have

$$\chi_{\text{top}}(\tilde{S}) - \frac{5}{3} \chi_{\text{top}}(\pi^{-1}(C_i \setminus Z)) = s_3 - s_4 - 2s_5 - 3s_6 - 2s_8 - 4s_9 = 12(\chi_{\text{top}}(M) - \frac{5}{3} \chi_{\text{top}}(C_i \setminus \mu^{-1}Z)) - 2s_3 - 6s_4 - 3s_5 - 2s_8 - s_7 - s_8 - 2s_9).$$

This gives us $\chi_{\text{top}}(\tilde{S}) = 0$,

$$\chi_{\text{top}}(\pi^{-1}(C_i \setminus Z)) = 3\chi_{\text{top}}(C_i \setminus Z) = 3(\chi_{\text{top}}(C_3) - s_7),$$

$$\chi_{\text{top}}(\pi^{-1}(C_4 \setminus Z)) = 4\chi_{\text{top}}(C_4 \setminus Z) = 4(\chi_{\text{top}}(C_4) - s_3 - s_7 - s_8),$$

$$\chi_{\text{top}}(\pi^{-1}(C_5 \setminus Z)) = 6\chi_{\text{top}}(C_5 \setminus Z) = 6(\chi_{\text{top}}(C_5) - s_3 - s_4 - s_5 - s_6 - s_8),$$

and

$$\chi_{\text{top}}(M) = 2 + \rho + s_3 + 5s_4 + 2s_5 + s_6 + s_9,$$

we obtain

$$9(K_M \cdot C_3 + C_3^2) + 8(K_M \cdot C_4 + C_4^2) + 6(K_M \cdot C_5 + C_5^2) = -3s_3 + 5s_4 + 4s_5 + 3s_6 - 6s_7 - 4s_8 + 8s_9 - 12\rho - 24.$$ (5.18)

On the other hand, we have

$$K_5 + (3/3)C_3 + (2/3)C_4 + (1/2)C_5$$

$$+ (7/12)E_1 + (5/12)E_2 + (1/3)E_2 + (1/4)E_3 + (1/6)E_4$$

$$+ (1/12)E_5 + (1/3)E_5, + (1/12)E_3 + (1/6)E_3 + (1/4)E_4 + (1/3)E_5$$

$$= \mu^*(K_5 + \Delta) \sim \text{num } 0.$$ (5.19)
where \( E_1 := \sum_{i=1}^{5} E_i(i) \), \( E_{2,j} := \sum_{i=1}^{5} E_{2,j}(i) \) (\( 1 \leq j \leq 5 \)), \( E_{3,j} := \sum_{i=1}^{5} E_{3,j}(i) \) (\( j = 1, 2 \)), \( E_4 := \sum_{i=1}^{5} E_4(i) \), \( E_5 := \sum_{i=1}^{5} E_5(i) \), \( E_6(i) \) (\( 1 \leq i \leq s_0 \)), \( E_{6,j}(i) \) (\( 1 \leq i \leq s_0, 1 \leq j \leq 5 \)), \( E_{5,j}(i) \) (\( 1 \leq i \leq s_0, j = 1, 2 \)) and \( E_4(i) \) (\( 1 \leq i \leq s_0 \)) are \((-2)\)-curves and \( E_5(i) \) (\( 1 \leq i \leq s_0 \)) are \((-3)\)-curves. From (5.18), we have

\[
K_3 + (3/4)K_M \cdot C_4 + (2/3)K_M \cdot C_5 + (1/2)K_M \cdot C_6 + (7/12)s_2 + (1/3)s_9 = 0,
\]

(5.20)

\[
K_M \cdot C_4 + (3/4)C_2^2 + (2/3)s_7 = 0,
\]

(5.21)

\[
K_M \cdot C_4 + (1/3)C_2^2 + (7/12)s_3 + (3/4)s_7 + (1/2)s_8 = 0,
\]

(5.22)

and

\[
K_M \cdot C_5 + (2/3)C_2^2 + (7/12)s_3 + (5/12)s_4 + (1/3)s_5 + (1/4)s_6 + (2/3)s_8 = 0.
\]

(5.23)

Since we have

\[
K_4 = 12 - \chi_{top}(M) - 10 - \rho - 5s_4 - 2s_5 - s_6 - s_9,
\]

(5.24)

we get

\[
9K_M \cdot C_5 + 8K_M \cdot C_4 + 6K_M \cdot C_6
= 12\rho - 120 + 12s_8 + 60s_4 + 24s_5 + 12s_6 + 8s_9.
\]

(5.25)

Let \( \pi_2 : \tilde{S} \to S \) as above. Since \( \pi_2^{-1}(C_3) \) is 0 or a disjoint union of elliptic curves, we have

\[
K_M \cdot C_4 + C_2^2 + (2/3)s_7 = 0.
\]

(5.26)

Let \( H_3 \subset \text{Gal}(\tilde{S}/S) \) be the subgroup of order 3 and put \( \tilde{S}_3 := \tilde{S}/H_3 \) and let \( \pi_3 : \tilde{S}_3 \to S \) be the induced morphism. Define the boundary \( \tilde{A}_3 \) as \( \tilde{A}_3 := (2/3)\pi_3^{-1}(C_4) \). We note that \( K_3 + \tilde{A}_3 = \pi_3^*(K_3 + \Delta) \) and \( (\tilde{S}_3, \tilde{A}_3) \) is a \( \nu_0 \)-log surface with \( \text{Cl}(\tilde{S}_3, \tilde{A}_3) = 3 \). Since \( \pi_3^{-1}(C_4) \) is 0 or a disjoint union of elliptic curves, we have

\[
K_M \cdot C_4 + C_2^2 + (3/4)s_3 + (3/4)s_7 + (1/2)s_8 = 0.
\]

(5.27)

Let \( H_4 \subset \text{Gal}(\tilde{S}/S) \) be the subgroup of order 2 and put \( \tilde{S}_4 := \tilde{S}/H_4 \) and let \( \pi_4 : \tilde{S}_4 \to S \) be the induced morphism. Define the boundary \( \tilde{A}_4 \) as \( \tilde{A}_4 := (1/2)\pi_4^{-1}(C_3 \cup C_5) \). We note that \( K_3 + \tilde{A}_4 = \pi_4^*(K_3 + \Delta) \) and \( (\tilde{S}_4, \tilde{A}_4) \) is a \( \nu_0 \)-log surface with \( \text{Cl}(\tilde{S}_4, \tilde{A}_4) = 2 \). Since \( \pi_4^{-1}(C_4) \) is 0 or a disjoint union of elliptic curves, we have

\[
K_M \cdot C_5 + C_2^2 + (5/6)s_3 + (5/6)s_4 + (2/3)s_5 + (1/2)s_6 + (2/3)s_8 = 0.
\]

(5.28)

From (5.18), (5.26), (5.27) and (5.28), we obtain

\[
4s_3 + 5s_4 + 4s_5 + 3s_6 + 3s_7 + 2s_8 + 4s_9 = 6(\rho + 2).
\]

(5.29)

From (5.21) and (5.26), we get

\[
K_M \cdot C_3 = -(2/3)s_7, \ C_3^2 = 0.
\]

(5.30)

From (5.22) and (5.27), we get
\[ K_M \cdot C_4 = -(1/4)s_3 - (3/4)s_7 - (1/2)s_8, \quad C_4^2 = -(1/2)s_3. \]  
(5.31)

From (5.23) and (5.28), we get
\[ K_M \cdot C_5 = -(1/3)s_3 - (2/3)s_8, \quad C_5^2 = -(1/2)s_3 - (5/6)s_4 - (2/3)s_5 - (1/2)s_6. \]  
(5.32)

From (5.25), (5.30), (5.31) and (5.32), we obtain
\[ 4s_3 + 15s_4 + 6s_5 + 3s_6 + 3s_7 + 2s_8 + 2s_9 = 3(10 - \rho). \]  
(5.33)

Since \( 4|6 - \rho(S) \), we have \( \rho(S) = 2 \), hence \( \rho \leq \rho(S) = 2 \). From (5.29) and (5.33), we get \( \rho \equiv 0 \pmod{2} \), hence \( \rho = 2 \) and
\[ 2(2s_3 + 4s_5 + s_9) + 3(s_6 + s_7) = 24, \quad s_4 = 0, \quad s_9 = s_5. \]  
(5.34)

We note that since \( K_M \cdot C_4, C_4^2, K_M \cdot C_5 \in \mathbb{Z} \) and \( K_M \cdot C_4 + C_4^2 \equiv 0 \pmod{2} \), we have
\[ s_7 = 0 \pmod{3}, \]  
(5.35)
\[ s_3 = 0 \pmod{2}, \]  
(5.36)
\[ 3s_3 + 3s_7 + 2s_8 \equiv 0 \pmod{8} \]  
(5.37)

and
\[ s_3 + 2s_8 \equiv 0 \pmod{3} \]  
(5.38)

from (5.30), (5.31) and (5.32) and that in fact, we have
\[ s_7 \equiv 0 \pmod{6}, \]  
(5.39)

from (5.35), (5.36) and (5.37). From (5.34), (5.36), (5.37), (5.38) and (5.39), we obtain that (1) \( s_5 = s_9 = 3, \ s_i = 0 \) for \( i = 3, 6, 7, 8 \) (2) \( s_8 = 12, \ s_i = 0 \) for \( i = 3, 5, 6, 7, 9 \) (3) \( s_3 = s_7 = 2, \ s_5 = s_1 = 0 \) for \( i = 5, 7, 9 \) (4) \( s_7 = 6, \ s_3 = 3, \ s_i = 0 \) for \( i = 3, 5, 6, 9 \) or (5) \( s_3 = 8, \ s_i = 0 \) for \( i = 3, 5, 7, 8, 9 \).

Case (1). If \( C_3 = 0 \), then \( 6(K_5 + \Delta) \) is Cartier, which is absurd. Therefore, we have \( C_3 \neq 0 \) and \( (\tilde{S}_2, \tilde{A}_2) \) is a \( \nu_1 \)-log surface of type \( IV_\alpha \) or \( IV_\beta \). Let \( p \in S \) be a singular point of \( S \) such that \( p \in C_3 \) and \( L \) be a fibre of the ruling on \( \tilde{S}_1 \) which passes through \( \tilde{p} \in \pi^{-1}(p) \). By construction, \( L \cap \pi^{-1}(C_3) \) admits fixed point free action of \( \text{Gal}(\tilde{S}_2/S) \), which is absurd since \( L \cap \pi^{-1}(C_3) \) is composed of exactly two points and the order of \( \text{Gal}(\tilde{S}_2/S) \) is three.

Case (2). If \( C_3 = 0 \), then \( C_3 \) is a disjoint union of elliptic curves. But since \( (\tilde{S}_1, \tilde{A}_1) \) is a \( \nu_1 \)-log surface of type \( VI_\beta \), each component of \( \pi^{-1}(C_3) \) is a rational curve, which is absurd. Therefore \( C_3 = 0 \) and \( 6(K_5 + \Delta) \) is Cartier. Thus we get a contradiction.

Case (3). From the assumption, \( (\tilde{S}_1, \tilde{A}_1) \) is a \( \nu_1 \)-log surface of type \( VI_\beta \). Since each component of \( \text{Supp } \tilde{A}_1 = \pi^{-1}(\text{Supp } \Delta) \) is a rational curve, we have \( C_3 = 0 \) and each components of \( C_4 \) and \( C_5 \) is a rational curve. Since \( K_M \cdot C_4 + C_4^2 = -4 \) and \( K_M \cdot C_5 + C_5^2 = -6 \), we have irreducible decompositions of \( C_4 \) and \( C_5 \), \( C_4 = C_{4,1} + C_{4,2} \) and \( C_5 = C_{5,1} + C_{5,2} + C_{5,3} \), where \( C_{4,i}, C_{5,j} \approx P^1 \) for \( i = 1, 2, j = 1, 2, 3 \). Let
be the number of the singular points \( p \in S \) of type \( A_{2,1} \) such that \( p \in C_{4,i} \cap C_5 \),

be the number of the singular points \( p \in S \) of type \( A_{2,1} \) such that \( p \in C_4 \cap C_{5,j} \),

be the number of the points \( p \in C_{4,i} \cap C_5 \) such that \( S \) is smooth at \( p \),

be the number of the points \( p \in C_{4,i} \cap C_{5,j} \) such that \( S \) is smooth at \( p \).

In the same way as above, we have

\[ K_M \cdot C_{i,i} = -(1/4) s_{4,i} - (1/2) s_{6,i}, \quad C_{i,i}^2 = -(1/2) s_{6,i} \]

and

\[ K_M \cdot C_{5,j} = -(1/3) s_{5,j} - (2/3) s_{6,j}, \quad C_{6,j}^2 = -(1/2) s_{5,j} - (1/2) s_{6,j}, \]

where \( C_{4,i} := \mu_* C_{4,i} \) and \( C_{6,j} := \mu_* C_{5,j} \). From (5.40), we have

\( (s_{4,i}^{(5)}, s_{6,i}^{(5)}, C_{6,i}^2) = (0, 4, 0) \) or \( (2, 1, -1) \).

Since we have \( C_{4,i}^2 = -1 \), we obtain

\( (s_{4,i}^{(4)}, s_{6,i}^{(4)}, C_{6,i}^2) = (0, 4, 0) \),

and

\( (s_{4,i}^{(4)}, s_{6,i}^{(4)}, C_{6,i}^2) = (2, 1, -1) \).

From (5.41), we have

\( (s_{5,j}^{(5)}, s_{6,j}^{(5)}, C_{6,j}^2) = (1, 1, 1, 1, -1), (0, 4, 0, -2) \) or \( (0, 0, 3, 0) \).

Since we have \( K_M \cdot C_6 = -4 \) and \( C_6^2 = -2 \), we obtain

\( (s_{5,j}^{(5)}, s_{6,j}^{(5)}, C_{6,j}^2) = (1, 1, 1, 1, -1) \) for \( j = 1, 2 \)

and

\( (s_{5,j}^{(5)}, s_{6,j}^{(5)}, C_{6,j}^2) = (0, 0, 3, 0) \).

For any \( i, j \), we have \( (\pi_C^* C_{4,i}, \pi_C^* C_{5,j}) = 1, 2 \) or \( 4 \), hence \( (C_{4,i}, C_{5,j}) = 1/2, 1 \) or \( 2 \).

Thus we conclude that we are in the case \( XII_\beta \).

Case (4). Since \( S \) is nonsingular and \( \rho = 2 \), we have \( S \cong \Sigma_d \) for some \( d \geq 0 \).

We note that \( K_5 \cdot C_3 = -4, C_3^2 = 0, K_S \cdot C_4 = -6, C_4^2 = 0, K_5 \cdot C_5 = -2, C_5^2 = 0 \) by assumption. Assume that \( C_i \sim n_i \theta + l_i f \) for \( i = 3, 4, 5 \), where \( \theta \) is a section such that \( \theta^2 \leq 0 \) and \( f \) is a fibre of the ruling on \( S \). Since we have \( 9 n_3 + 8 n_4 + 6 n_5 = 24 \),

we have \( (n_3, n_4, n_5) = (2, 0, 1), (0, 3, 0) \) or \( (0, 0, 4) \). If \( (n_3, n_4, n_5) = (2, 0, 1) \), then we have \( (l_3, l_4, l_5) = (d, 3, (1/2)d) \). Since \( (C_3, \theta) \geq 0 \) or \( (C_5, \theta) \geq 0 \), we have \( l_3 = l_5 = d = 3 \). Thus we are in the case \( XII_\beta \). If \( (n_3, n_4, n_5) = (0, 3, 0) \), then we have \( (l_3, l_4, l_5) = (2, (3/2)d, 1) \). Since \( l_4 \geq 2d \), we have \( l_4 = d = 0 \). Thus we are in the case \( XII_\beta \) again.

If \( (n_3, n_4, n_5) = (0, 0, 4) \), then we have \( (l_3, l_4, l_5) = (2, 3, 2d - 3) \) but we have \( l_5 \geq 3d \), which is absurd.

Case (5). Since we have \( C_4 \neq 0 \) by assumption, \( (\tilde{S}_i, \tilde{A}_i) \) is a \( \nu_0 \)-log surface of
EXAMPLES. (1) Put $E := C/\mathbb{Z} + \zeta \mathbb{Z}$, where $\zeta$ is a primitive third root of unity and $A := E \times E$. Consider the action $\sigma$ on $A$ defined as $\sigma([z_1], [z_2]) = ([\zeta^2 z_2], [\zeta z_1])$ for $([z_1], [z_2]) \in A$. Put $S := A/\langle \sigma \rangle$ and $D := (1/2)\{(z, [\zeta z]) | z \in C\}$. This log surface $(S, D)$ gives an example of $\nu_0$-log surface of type $II_e$.

(2) Let $\zeta$ and $E$ be as above. Consider the action $\sigma$ on $E \times P^1$ such that $\sigma([z], [w_1 : w_2]) = ([\zeta z], [\zeta w_1 : w_2])$. Put $S := E \times P^1/\langle \sigma \rangle$ and $D := (1/2)E_1 \times ([1 : 0], [\zeta^2 : 1], [\zeta : 1], [1 : 1])/\langle \sigma \rangle$. This log surface $(S, D)$ gives an example of $\nu_0$-log surface of type $VI_f$.

(3) Examples of $\nu_0$-log surface of type $IV_7$, $VI_8$ and $XII_9$ are known. We refer the reader to [23].

6. Degeneration of type I associated with $\nu_0$-log surface of abelian type

DEFINITION 6.1 A minimal degeneration of surfaces $f : X \to D$ with $x = 0$ is said to be of type I if $f$ has a log minimal reduction $\tilde{f} : (\tilde{X}, \tilde{D}) \to D$ such that $(\tilde{D}, \text{Diff}_D(0))$ is a $\nu_0$-log surface of type I.

In this section, we study the singular fibres by using the results in the previous section.

Theorem 6.1. Let $\tilde{f} : (\tilde{X}, \tilde{D}) \to D$ be a projective log minimal degeneration of surfaces with $x = 0$ and assume that $(\tilde{D}, \text{Diff}_D(0))$ is a $\nu_0$-log surface of abelian type then the generic fibre is an abelian or hyperelliptic surface and there is a projective degeneration $f : X \to D$ which is bimeromorphically equivalent to $\tilde{f} : \tilde{X} \to D$ (we shrink $D$ if necessary) such that $X$ is a normal $Q$-factorial 3-fold with only terminal singularities and one of the following holds.

I : $X$ is smooth and $f^*(0) = m\Theta$, $\Theta$ is an abelian surface or a hyperelliptic surface.

II : $X$ is smooth and $f^*(0) = 2m\Theta_0 + \sum_{i=1}^4 m\Theta_{1,i}$, where $m \in N$, $\Theta_{1,i}$ is an elliptic ruled surface for any $i$. $\Theta_{1,i} \cap \Theta_0$ is a section whose self-intersection number is 0 on each of the two components for $i \geq 1$. $\Theta_{1,i} \cap \Theta_{1,j} = 0$ for $i \neq j \geq 1$.

III : $X$ is smooth and $f^*(0) = 2m\Theta_0 + \sum_{i=1}^2 m\Theta_{1,i}$, where $m \in N$, $\Theta_0$ and $\Theta_{1,i}$ are elliptic ruled surfaces for $i = 1,2$. $\Theta_{1,2} \cdot \Theta_0$ is a 3-section on $\Theta_0$ which is a smooth elliptic curve with the self-intersection number 0 and is an section on $\Theta_{1,2}$. $\Theta_{1,2} \cap \Theta_0$ is a section whose self-intersection number 0 on each of the two components. $\Theta_{1,i} \cap \Theta_{1,j} = 0$ for $i \neq j \geq 1$.

IV : $X$ is smooth and $f^*(0) = 2m\Theta_0 + \sum_{i=1}^3 m\Theta_{1,i}$, where $m \in N$, $\Theta_{1,i}$ is an elliptic ruled surface for $i = 0,1,2$. $\Theta_{1,i} \cap \Theta_0$ is a 2-section on $\Theta_0$ which is a smooth
elliptic curve with the self-intersection number 0 and is a section on \( \Theta_{1,i} \) for \( i = 1, 2 \). 
\( \Theta_{i,j} \cap \Theta_{i,j} = \emptyset \) for \( i > j \geq 1 \).

**IIa**: \( X \) is smooth and \( f^*(0) = 2m\Theta_0 + \sum_{i=1}^3 m\Theta_{i,i} \), where \( m \in \mathbb{N} \), \( \Theta_0 \) and \( \Theta_{i,i} \) are elliptic ruled surfaces for any \( i \). \( \Theta_{i,i} \cdot \Theta_0 \) is a 2-section on \( \Theta_0 \) which is a smooth elliptic curve with the self-intersection number 0 and is a section on \( \Theta_{1,2} \cdot \Theta_{1,2} \cdot \Theta_0 \) is a section whose self-intersection number 0 on each of the two components. 

**IIb**: \( X \) is smooth and \( f^*(0) = 2m\Theta_0 + m\Theta_i \), where \( m \in \mathbb{N} \), \( \Theta_i \) is an elliptic ruled surface for \( i = 0, 1, 2 \). \( \Theta_i \cdot \Theta_0 \) is a 4-section on \( \Theta_0 \) which is a smooth elliptic curve with the self-intersection number 0 and is a section on \( \Theta_{i,2} \cdot \Theta_{i,2} \cdot \Theta_0 \) is a section whose self-intersection number 0 on each of the two components.

**IIIa-1**: \( X \) is smooth and \( f^*(0) = 3m\Theta_0 + \sum_{i=1}^3 (2m\Theta_{i,1} + m\Theta_{i,2}) \), where \( m \in \mathbb{N} \), \( \Theta_0 \) and \( \Theta_{i,i,j} \) are elliptic ruled surfaces for any \( i, j \). \( \Theta_{i,i,j} \cdot \Theta_0 \) and \( \Theta_{i,i} \cdot \Theta_{i,i,j} \) are sections with the self-intersection number 0 on each of the two components for \( i = 1, 2, 3 \). \( \Theta_{i,j} \cap \Theta_{k,i} = \emptyset \) if \( i \neq k \) and \( \Theta_{i,j} \cap \Theta_0 = \emptyset \) for \( i = 1, 2, 3 \) (see Figure IIIa-1).

**IIIa-2**: \( X \) is smooth and \( f^*(0) = \sum_{i=1}^3 m\Theta_{i,i} \), where \( m \in \mathbb{N} \), \( \Theta_{i,i} \) is an elliptic ruled surface for \( i = 1, 2, 3 \). \( \Theta_{i,1} \cdot \Theta_{i,2} = \Theta_{i,2} \cdot \Theta_{i,3} = \Theta_{i,3} \cdot \Theta_{i,1} \) is a smooth elliptic curve which is a section on each \( \Theta_{i,i} \) (see Figure IIIa-2).

**IIIb-1**: \( X \) is smooth and \( f^*(0) = 3m\Theta_0 + \sum_{i=1}^3 (2m\Theta_{i,1} + m\Theta_{i,2}) \), where \( m \in \mathbb{N} \), \( \Theta_0 \) and \( \Theta_{i,i,j} \) are elliptic ruled surfaces for any \( i, j \). \( \Theta_{i,i,j} \cdot \Theta_0 \) is a 2-section on \( \Theta_0 \) which is a smooth elliptic curve with the self-intersection number 0 and is a section on \( \Theta_{i,2,3} \cdot \Theta_{1,2,3} \cdot \Theta_0 \) are sections with the self-intersection number 0 on each of the two components for \( i = 1, 2 \). \( \Theta_{i,j} \cap \Theta_{k,i} = \emptyset \) if \( i \neq k \) and \( \Theta_{i,j} \cap \Theta_0 = \emptyset \) for \( i = 1, 2 \) (see Figure IIIb-1).

**IIIb-2**: \( X \) is smooth and \( f^*(0) = \sum_{i=1}^3 m\Theta_{i,i} \), where \( m \in \mathbb{N} \). There is a projective birational morphism \( \mu : Y \rightarrow X \) from a smooth 3-fold \( Y \) such that \( \tilde{f}^*(0) = 3m\Theta_0 + m\Theta_{1,1} + m\Theta_{1,2} \), where \( \tilde{f} := f \circ \mu, \Theta_i := \mu^* \Theta_i \) for \( i = 1, 2 \). \( \Theta_{1,1} \) is an elliptic ruled surface for \( i = 0, 1, 2 \). \( \Theta_{1,1} \cdot \Theta_0 \) is a 2-section on \( \Theta_0 \) which is a smooth elliptic curve with the self-intersection number 0 and is a section on \( \Theta_{1,2} \cdot \Theta_0 \) is a section whose self-intersection number 0 on each of the two components. \( \Theta_{1,1} \cap \Theta_{1,2} = \emptyset \) (see Figure IIIb-2).

**IIIc-1**: \( X \) is smooth and \( f^*(0) = 3m\Theta_0 + 2m\Theta_{1,1} + m\Theta_{1,2} \), where \( m \in \mathbb{N} \), \( \Theta_{1,1} \) is an elliptic ruled surface for any \( i \). \( \Theta_{1,1} \cdot \Theta_0 \) is a 3-section on \( \Theta_0 \) which is a smooth elliptic curve with the self-intersection number 0 and is a section on \( \Theta_{1,2} \cdot \Theta_{1,3} \cdot \Theta_0 \) is a section with the self-intersection number 0 on each of the two components. \( \Theta_{1,1} \cap \Theta_{1,2} = \emptyset \) (see Figure IIIc-1).

**IIIc-2**: \( X \) is smooth and \( f^*(0) = m\Theta_0 \), where \( m \in \mathbb{N} \). There is a projective birational morphism \( \mu : Y \rightarrow X \) from a smooth 3-fold \( Y \) such that \( \tilde{f}^*(0) = 3m\Theta_0 \).
+ m\tilde{\Theta}_i$, where \( \tilde{f} := f \circ \mu \), \( \tilde{\Theta}_1 := \mu^{-1} \Theta_1 \). \( \tilde{\Theta}_i \) is an elliptic ruled surface for \( i = 0, 1 \). \( \tilde{\Theta}_1 \cdot \tilde{\Theta}_0 \) is a 3-section on \( \tilde{\Theta}_0 \) with the self-intersection number 0 which is a smooth elliptic curve (see Figure III-b).

**IIIb**: \( f^*(0) = 3\Theta_0 + \sum_{i=1}^s \Theta_{1,i} \), where \( \Theta_0 \) is a normal rational surface with \( \rho(\Theta_0) = 4 + t \) and \( \Theta_{1,i} \equiv \mathbb{P}^2 \) for \( i \geq 1 \). Sing \( \Theta_0 = \left\{ p_i ; 1 \leq i \leq s \right\} \), where \( p_i \in \Theta_0 (1 \leq i \leq s) \) are singular points of type \( A_{1,1} \) and \( s := 9 - t \). \( \Theta_{1,i} \cdot \Theta_0 \) is a \((-3)\)-curve on \( \Theta_0 \) and is a line on \( \Theta_{1,i} \) for \( i \geq 1 \). If \( \left\{ p_i ; 1 \leq i \leq s \right\} = \text{Sing } \Theta_0 \), then \( \text{Sing } X = \left\{ p_i ; 1 \leq i \leq s \right\} \) and analytic locally around \( p_i \). \( (p_i \in X, \Theta) \) is isomorphic to \((0 \subset \mathbb{C}^3, \{ z = 0 \})/\mathbb{Z}(1, 1, 2) \). Moreover, if \( X_t \) is an abelian surface for \( t \in \mathbb{D}^* \), then \( t = 0 \) or 9 (see Figure IIIb).

**IVa-1**: \( X \) is smooth and \( f^*(0) = 4 \Theta_0 + \sum_{i=1}^s (3 \Theta_{1,i,1} + 2 \Theta_{1,i,2} + m \Theta_{1,i,3}) + 2 \Theta_2 \), where \( \Theta_0, \Theta_{1,i,1}, \Theta_2 \) are elliptic ruled surfaces. \( \Theta_{1,i,1} \cdot \Theta_0 \) \((i = 1, 2)\), \( \Theta_2 \cdot \Theta_0 \), \( \Theta_{1,i,2} \cdot \Theta_{1,i,1} \) and \( \Theta_{1,i,2} \cdot \Theta_{1,i,1} \) are sections of \( \Theta_0 \) with the self-intersection number 0 on each of the two components. \( \Theta_{1,i,1} \cap \Theta_{1,j,1} = \emptyset \) if \( i \neq k \), \( \Theta_{1,i,3} \cap \Theta_{1,i,1} = \emptyset \) for \( i = 1, 2 \) and \( \Theta_2 \cap \Theta_{1,i,1} = \emptyset \) for any \( i, j \) (see Figure IVa-1).

**IVa-2**: \( X \) is smooth and \( f^*(0) = m \Theta_{1,1} + m \Theta_{1,2} \), where \( m \in \mathbb{N}, \Theta_i \) is an elliptic ruled surface for \( i = 1, 2 \). \( \Theta_{1,1} \cdot \Theta_{1,2} = 2 \Gamma \) where \( \Gamma \) is a section with the self-intersection number 0 on each of the two components (see Figure IVa-2).

**IVb-1**: \( X \) is smooth and \( f^*(0) = 4 \Theta_0 + 3 \Theta_{1,1} + 2 \Theta_{1,2} + m \Theta_{1,3} + 2 \Theta_2 \), where \( m \in \mathbb{N}, \Theta_{1,i} \) and \( \Theta_2 \) are elliptic ruled surfaces for any \( i \). \( \Theta_{1,1} \cdot \Theta_0 \) is a 2-section with the self-intersection number 0 which is a smooth elliptic curve and is a section on \( \Theta_{1,1} \). \( \Theta_2 \cdot \Theta_0 \), \( \Theta_{1,2} \cdot \Theta_{1,1} \) and \( \Theta_3 \cdot \Theta_2 \) are sections with the self-intersection number 0 on each of the two components. \( \Theta_{1,1} \cap \Theta_{1,3} = \emptyset \), \( \Theta_0 \cap \Theta_{1,i} = \emptyset \) for \( i = 2, 3 \) and \( \Theta_3 \cap \Theta_{1,i} = \emptyset \) for \( i = 1, 2, 3 \) (see Figure IVb-1).

**IVb-2**: \( X \) is smooth and \( f^*(0) = m \Theta_0 \), where \( m \in \mathbb{N} \) and \( \Theta_0 \) is irreducible. there is a projective birational morphism \( \mu : Y \to X \) from a smooth 3-fold \( Y \) such that \( \tilde{f}^*(0) = 4 m \tilde{\Theta}_0 + m \tilde{\Theta}_{1,1} + m \tilde{\Theta}_{1,2} \), where \( \tilde{f} := f \circ \mu \), \( \tilde{\Theta}_{1,1} \) is an elliptic ruled surface, \( \tilde{\Theta}_{1,1} \cdot \tilde{\Theta}_0 = m^{-1} \Theta_0 \). \( \tilde{\Theta}_{1,1} \cdot \tilde{\Theta}_0 \) is a 2-section with the self-intersection number 0 on \( \tilde{\Theta}_0 \) which is a smooth elliptic curve and is a section on \( \tilde{\Theta}_{1,1} \cdot \tilde{\Theta}_{1,2} \cdot \tilde{\Theta}_0 \) is a section with the self-intersection number 0 on each of the two components. \( \tilde{\Theta}_{1,1} \cap \tilde{\Theta}_{1,2} = \emptyset \) (see Figure IVb-2).

**IVc**: \( f^*(0) = 4 \Theta_0 + \sum_{i=1}^2 \Theta_{1,i} + \sum_{i=1}^2 \sum_{j=1}^t \Theta_{1,(i,j)} \), where \( \Theta_0 \) and \( \Theta_{1,i} \) are normal rational surfaces with \( \rho(\Theta_0) = 2 + t_1 + t_2 \) and \( \rho(\Theta_{1,i}) = 2 \) for \( i = 1, 2 \). \( \Theta_{1,3} \) is an elliptic ruled surface and \( \Theta_{1,(i,j)} \equiv \mathbb{P}^2 \). \( t_i = 0 \) or 2 or 4 for \( i = 2, 3 \) and \( s := 8 - \sum_{i=1}^2 t_i \). \( \Theta_{1,3} \cdot \Theta_0 \) is a smooth elliptic curve whose self-intersection number 0 on each of the two components. The strict transform of \( \Theta_{1,i} \cdot \Theta_0 \) is a \((-2)\)-curve on the minimal resolution of \( \Theta_0 \) and is a \(((1/2)t_i - 2)\)-curve on the minimal resolution of \( \Theta_{1,i} \) for \( i = 1, 2 \). \( \Theta_{1,(i,j)} \cdot \Theta_0 \) is a \((-2)\)-curve on \( \Theta_0 \) and is a fibre of the ruling on \( \Theta_{1,(i,j)} \) for any \((i, j)\). \( \Theta_{1,(i,j)} \cdot \Theta_{1,i} \) is a 0-curves on \( \Theta_{1,i} \).
and is a $(-2)$-curve on $\Theta_{t,j,k}$ for $i=1, 2, 1 \leq j', k \leq t_i$. $\Theta_{t,i,j} \cap \Theta_{t,i} = \emptyset$ if $i \neq j$, $\Theta_{t,i,j} \cap \Theta_{t,i,k} = \emptyset$ if $i \neq k$ and $\Theta_{t,i,j,k}'s$ are disjoint from each other. Putting Sing $\Theta_0 = \{p_{i,j}^0 \in \Theta_{t,i}; 0 \leq j \leq 8 - t_i \ (i=1, 2)\}$ and Sing $\Theta_{t,i} = \{p_{i,j}^t, p_{i,0}^t; 0 \leq j \leq 8 - t_i \ (i=1, 2)\}$, we have Sing $X = \{p_{i,j}^t, p_{i,0}^t; 0 \leq j \leq 8 - t_i \ (i=1, 2)\}$ and analytic locally around each $p_{i,j}^0$, $(p_{i,j}^t \in X, \Theta)$ is isomorphic to $(0 \in C^3, \{x, y = 0\})/\mathbb{Z}_2(1, 1, 1)$, around each $p_{i,j}^t$, $(p_{i,j}^t \in X, \Theta)$ is isomorphic to $(0 \in C^3, \{x = 0\})/\mathbb{Z}_2(1, 1, 1)$. Moreover, if $X_t$ is an abelian surface for $t \in \mathbb{D}^*$, then $(t_1, t_2) = (0, 0)$, or $(4, 4)$ (see Figure IV.7).

$IV_5: f^*(0) = 4\Theta_0 + \sum_{i=1}^2 2\Theta_{t,i} + \sum_{i=1}^t \sum_{k=1} s \Theta_{i,k}$, where $\Theta_0$ and $\Theta_{t,i}$ are normal rational surfaces with $\rho(\Theta_0) = 2 + t_1 + t_2$ and $\rho(\Theta_{t,i}) = 2$ for $i = 1, 2$ and $\Theta_{t,i,j}$ is a $(-2)$-curve on the minimal resolution of $\Theta_0$ and is $a((1/2)t - 2)$-curve on the minimal resolution of $\Theta_{t,i}$ for $i = 1, 2$. $\Theta_{t,i,j}$ is a $(-2)$-curve on $\Theta_0$ and is a fibre of the ruling on $\Theta_{t,i}$ for any $(i, j)$. $\Theta_{t,i,j} \cap \Theta_{t,i} = \emptyset$ if $i \neq j$, $\Theta_{t,i,j} \cap \Theta_{t,i,k} = \emptyset$ if $i \neq k$ and $\Theta_{t,i,j,k}'s$ are disjoint from each other. Putting Sing $\Theta_0 = \{p_{i,j}^0 \in \Theta_{t,i}; 0 \leq j \leq 8 - t_i \ (i=1, 2)\}$ and Sing $\Theta_{t,i} = \{p_{i,j}^t, p_{i,0}^t; 0 \leq j \leq 8 - t_i \ (i=1, 2)\}$, we have Sing $X = \{p_{i,j}^t, p_{i,0}^t; 0 \leq j \leq 8 - t_i \ (i=1, 2)\}$ and analytic locally around each $p_{i,j}^0$, $(p_{i,j}^t \in X, \Theta)$ is isomorphic to $(0 \in C^3, \{x, y = 0\})/\mathbb{Z}_2(1, 1, 1)$, around each $p_{i,j}^t$, $(p_{i,j}^t \in X, \Theta)$ is isomorphic to $(0 \in C^3, \{x = 0\})/\mathbb{Z}_2(1, 1, 1)$. Moreover, if $X_t$ is an abelian surface for $t \in \mathbb{D}^*$, then $(t_1, t_2) = (0, 0)$, or $(4, 4)$ (see Figure IV.5).

$V-1: X$ is smooth and $f^*(0) = 5\Theta_0 + \sum_{i=1}^5 (2\Theta_{t,i} + 2\Theta_{2,i})$, where $\Theta_0$ is a smooth rational surface with $\rho(\Theta_0) = 12$, $\Theta_{t,i} \approx \mathbb{P}^2$ for $1 \leq i \leq 5$. $\Theta_{t,i,j} \cap \Theta_{t,i} = \emptyset$ if $i \neq j$. $\Theta_{t,i,j} \cap \Theta_{t,i,k} = \emptyset$ if $i \neq k$ and $\Theta_{t,i,j,k}'s$ are disjoint from each other. Putting Sing $\Theta_0 = \{p_{i,j,k}^0 \in \Theta_{t,i}; 0 \leq j \leq 8 - t_i \ (i=1, 2)\}$ and Sing $\Theta_{t,i} = \{p_{i,j,k}^t, p_{i,0}^t; 0 \leq j \leq 8 - t_i \ (i=1, 2)\}$, we have Sing $X = \{p_{i,j,k}^t, p_{i,0}^t; 0 \leq j \leq 8 - t_i \ (i=1, 2)\}$ and analytic locally around each $p_{i,j,k}^0$, $(p_{i,j,k}^t \in X, \Theta)$ is isomorphic to $(0 \in C^3, \{x, y = 0\})/\mathbb{Z}_2(1, 1, 1)$, around each $p_{i,j,k}^t$, $(p_{i,j,k}^t \in X, \Theta)$ is isomorphic to $(0 \in C^3, \{x = 0\})/\mathbb{Z}_2(1, 1, 1)$. Moreover, if $X_t$ is an abelian surface for $t \in \mathbb{D}^*$, then $(t_1, t_2) = (0, 0)$, or $(4, 4)$ (see Figure IV.5).

$V-2: f^*(0) = 5\Theta$, where $\Theta$ is a normal rational surface with $\rho(\Theta) = 2$ and has five quotient singularities $\{p_j; 1 \leq j \leq 5\}$ of type $A_{1,2}$. Sing $X = \{p_j; 1 \leq j \leq 5\}$ and around each $p_j$, $(p_j \in X, \Theta)$ is isomorphic to $(0 \in C^3, \{x = 0\})/\mathbb{Z}_2(1, 2, 3)$.

$V-3: X$ is smooth and $f^*(0) = \sum_{i=1}^5 \Theta_{t,i}$, where $\Theta_{t,i}$ is a smooth rational surface for $1 \leq i \leq 5$ and $\sum_{i=1}^5 \rho(\Theta_{t,i}) = 20$. There is projective birational morphism $\mu: Y \to X$ from a smooth 3-fold $Y$ to $X$ such that $g^*(0) = 5\Theta_0 + \sum_{i=1}^5 \Theta_{t,i}$, where $g$ is the induced morphism from $Y$ to $\mathbb{D}$ and $\Theta_0 \approx \sum_{d} (d \leq 3)$, $\Theta_{t,i,j} : \mu^* \Theta_{t,i} = \sum_{i=1} \sum_{j=1} \sum_{k=1} s \Theta_{i,j,k}$, is a smooth rational surface which is obtained by blowing up $\sum_{i=1}^{20}$ for $1 \leq i \leq 5$. $\Theta_{t,i,j} = \Theta_{t,i} = \Theta_{t,i,j} = \Theta_{t,i} = \Theta_{t,i,j} = \Theta_{t,i}$ is a section on $\Theta_0$ and is the strict transform of a fibre of the ruling on $\Theta_{t,i,j}$ for $1 \leq i \leq 5$. For $i > j$, $\Theta_{t,i,j} = \sum_{i=1} \sum_{j=1} \sum_{k=1} s \Theta_{i,j,k}$, where $\{\Gamma_k; 1 \leq k \leq 5\}$ are rational curves which are disjoint from each other such that $\Theta_{t,i,j} = \sum_{i=1} \sum_{j=1} \sum_{k=1} s \Theta_{i,j,k}$, $\Gamma_k = \sum_{i=1} \sum_{j=1} \sum_{k=1} s \Theta_{i,j,k}$, and either $(1) \sum_{i=1} \sum_{j=1} \sum_{k=1} s \Theta_{i,j,k} = 1$, $\Gamma_k$ is a $(1)$-curve on $\Theta_{t,i,j}$ (or on $\Theta_{t,i,j}$) and $(2)$-curve on $\Theta_{t,i,j}$ (or on
\( \tilde{\Theta}_{1,i} \) or (2) \( m(i, j; k) = 2 \), \( \Gamma_k \) is a \((-1)\)-curve on \( \tilde{\Theta}_{1,i} \) and \( \tilde{\Theta}_{1,j} \) (see Figure V-3).

**VI\(_{a}\)-1:** \( X \) is smooth and \( f^* (0) = 6 m \Theta_0 + 5 m \Theta_{1,1} + 4 m \Theta_{1,2} + 3 m \Theta_{1,3} + 2 m \Theta_{1,4} + m \Theta_{1,5} + 4 m \Theta_{2,1} + 2 m \Theta_{2,2} + 3 m \Theta_{3,0} \), where \( m \in \mathbb{N} \), \( \Theta_0 \), \( \Theta_{1,i} \) and \( \Theta_{3} \) are elliptic ruled surfaces for any \( i, j \). \( \Theta_0 \cdot \Theta_{1,i} (i = 1, 2) \), \( \Theta_{1,i} \cdot \Theta_{1,i+1} (1 \leq i \leq 4) \), \( \Theta_{2,1} \cdot \Theta_{2,2} \) and \( \Theta_0 \cdot \Theta_3 \) are sections with the self-intersection number 0 on each component. \( \Theta_{i,j} \cap \Theta_{k,i} = \emptyset \) if \( i \neq k \) or \( i = k = 1 \) and \( |j - l| > 1 \), \( \Theta_i \cap \Theta_{i,j} = \emptyset \) for any \( i, j \) and \( \Theta_0 \cap \Theta_{i,j} \) if \( i = j = 2 \) or \( i = 1, j \geq 2 \) (see Figure VI\(_{a}\)-1).

**VI\(_{a}\)-2:** \( X \) is smooth and \( f^* (0) = m \Theta \), where \( \Theta \) is an elliptic ruled surface. The non-normal locus of \( \Theta \) is an smooth elliptic curve (say \( \Gamma \)) and around any point \( p \in \text{Sing} \Theta \), \( \Theta \) is defined by the equation \( y^2 - x^3 = 0 \) analytic locally. \( \nu^{-1} \Gamma \) is a section of \( \Theta^* \) with the self-intersection number 0 (see Figure VI\(_{a}\)-2).

**VI\(_{b}\)-1:** \( X \) is smooth and

\[
f^* (0) = 6 \Theta_0 + \sum_{i=1}^{3} (4 \Theta_{1,i,1} + 2 \Theta_{1,i,2}) + \sum_{j=1}^{3} 3 \Theta_{2,j} + 1 : \sum_{k=3} \Theta_{(k,i)},
\]

where \( \Theta_0 \simeq P^1 \times P^1 \), \( \Theta_{1,i} \simeq \sum_{2} \), \( \Theta_{i,1} \simeq \sum_{4} \), \( \Theta_{2,j} \) is a smooth rational surface with \( \rho(\Theta_{2,j}) = 11 \) for \( j = 1, 2, 3 \) and \( \Theta_{(k,i),0} \simeq P^2 \) for \( 1 \leq k \leq 3, 1 \leq i \leq 3 \). \( \Theta_{1,1} \cdot \Theta_0 \) is a fibre of the first projection \( \Theta_0 \to P^1 \) for \( i = 1, 2, 3 \) and \( \Theta_{2,j} \cdot \Theta_0 \) is a fibre of the second projection \( \Theta_0 \to P^1 \) for \( j = 1, 2, 3 \). \( \Theta_{1,1} \cdot \Theta_{1,2} \) is a section on \( \Theta_{1,1,1} \) which is disjoint from the negative section and is a \((-4)\)-curve on \( \Theta_{1,1,2} \) for \( i = 1, 2, 3, \Theta_{1,i,j} \cdot \Theta_{2,k} \) is a fibre of the ruling on \( \Theta_{1,i,j} \) and is a \((-2)\)-curve on \( \Theta_{2,k} \) for any \( i, j, k \) and \( \Theta_{2,i} \cdot \Theta_{(k,i)} \) is a \((-3)\)-curve on \( \Theta_{2,i} \) and is a line on \( \Theta_{(k,i)} \cdot \Theta_{1,i,j} \cap \Theta_{1,i,j} \cap \Theta_{(k,i)} = \emptyset \) if \( i \neq j \), \( i, j, k, l \), \( \Theta_{2,j} \cdot \Theta_{(k,i)} = \emptyset \) if \( j \neq l \), \( \Theta_{(k,i)} \cdot \Theta_{(k,i)} = \emptyset \) are disjoint from each other (see Figure VI\(_{b}\)-1).

**VI\(_{b}\)-2:** \( f^* (0) = 2 \Theta_{1,1} + 2 \Theta_{1,2} + 2 \Theta_{1,3} \), where \( \Theta_{1,i} \) is a normal rational surface with \( \rho(\Theta_{1,i}) = 2 \) such that \( \text{Sing} \Theta_{1,i} = 6 A_{2,1} \). \( \Theta_{1,i} \cdot \Theta_{1,2} = \Theta_{1,2} \cdot \Theta_{3,1} = \Theta_{1,3} \cdot \Theta_{1,1} = : \Gamma \) and the strict transform of \( \Gamma \) on the minimal resolution of each component is a \((-2)\) curve for any \( i, j \). The singular locus of \( X \) consists of the points \( p_i \in \Gamma (i = 1, 2, 3) \) and the points \( p_j^{(k)} \in \Theta_{1,k} \setminus \Gamma (j = 1, 2, 3, k = 1, 2, 3) \) and analytic locally around \( p_i \), \( (p_i \in X, \Theta) \simeq (0 \in \mathbb{C}^3, \{xy(x-y) = 0\})/\mathbb{Z}_2(1, 1, 1) \) for \( i = 1, 2, 3, \) around \( p_j^{(k)}, (p_j^{(k)} \in X, \Theta) \simeq (0 \in \mathbb{C}^3, \{z = 0\})/\mathbb{Z}_2(1, 1, 1) \) for any \( j, k \) (see Figure VI\(_{b}\)-2).

**VI\(_{a}\)-1:** \( X \) is smooth and

\[
f^* (0) = 6 \Theta_0 + 3 \Theta_1 + 3 \Theta_2 + \sum_{i=1}^{3} (2 \Theta_{1,i,1} + \Theta_{1,i,2} + 2 \Theta_{3,i}),
\]

where \( \Theta_0 \) is a smooth rational surface with \( \rho(\Theta_0) = 11 \), \( \Theta_1 \simeq P^1 \times P^1 \), \( \Theta_2 \) is an elliptic ruled surface, \( \Theta_{1,1,1} \simeq \sum_2 \), \( \Theta_{1,2,2} \simeq \sum_4 \) for \( i = 1, 2, 3 \) and \( \Theta_{3,i} \simeq P^3 \) for \( i \)
$\Theta_1 \cdot \Theta_0$ is a $(-2)$-curve on $\Theta_0$ and is a fibre of the first projection $\Theta_1 \to P^1$, $\Theta_2 \cdot \Theta_0$ is an elliptic curve with the self-intersection number 0 on each component, $\Theta_{i,j} \cdot \Theta_0$ is a $(-2)$-curve on $\Theta_0$ and is a fibre of the ruling on $\Theta_{i,j}$ for $i = 1, 2, 3$, $j = 1, 2, 3$. $\Theta_{i,1} \cap \Theta_2 = \emptyset$, $\Theta_1 \cap \Theta_{i,12} = \emptyset$, $\Theta_1 \cap \Theta_{3,j} = \emptyset$, $\Theta_2 \cap \Theta_{i,1j} = \emptyset$ and $\Theta_2 \cap \Theta_{3,j} = \emptyset$ for any $i, j$. $\Theta_{i,1j} \cap \Theta_{3,i'} = \emptyset$ for any $i, j, i'$. (see Figure VI.1).

**VI.1**: $f^*(0) = 6\Theta_0 + 3\Theta_1 + 3\Theta_2$, where $\Theta_i$ is a normal rational surface with $\rho(\Theta_i) = 2$ such that $\mathrm{Sing} \, \Theta_i = 3A_3 + 3A_3$ for $i = 0, 1$ and $\Theta_2$ is an elliptic ruled surface. The strict transform of $\Theta_i \cdot \Theta_0$ is a $(-2)$-curve on the minimal resolution of $\Theta_0$, (1,-1) curve on the minimal resolution of $\Theta_i$, $\Theta_2 \cdot \Theta_0$ is an elliptic curve with the self-intersection number 0 on each component. $\Theta_1 \cap \Theta_2 = \emptyset$. The singular locus of $X$ consists of the points $p_i \in \Theta_1 \cap \Theta_0$ ($i = 1, 2, 3$) and the points $p_{i,j} \in \Theta_i \setminus (\Theta_1 \cap \Theta_0)$ ($i = 1, 2, 3, j = 0, 1$) and analytic locally around $p_i, (p_i \in X, \Theta) \simeq (0 \in C^3, \{x = 0\})/Z_3(1, 1, 2)$ for $i = 1, 2, 3$, where $\{x = 0\}$ corresponds to $\Theta_0$ and $\{z = 0\}$ corresponds to $\Theta_i$, around $p_{i,0}^0, (p_{i,0}^0 \in X, \Theta) \simeq (0 \in C^3, \{z = 0\})/Z_3(1, 1, 2)$, around $p_{1,1}^0, (p_{1,1}^0 \in X, \Theta) \simeq (0 \in C^3, \{x = 0\})/Z_3(1, 1, 2)$ for $i = 1, 2, 3$. $\Theta_{i,1} \cap \Theta_{i,2} = \emptyset$, $\Theta_1 \cap \Theta_{3,j} = \emptyset$, $\Theta_1 \cap \Theta_{2,j} = \emptyset, \Theta_1 \cap \Theta_{i,j} = \emptyset$ for any $i, j, i'$ (see Figure VI.2).

**VI.2**: $f^*(0) = 6\Theta_0 + 3\Theta_1 + 3\Theta_2$, where $\Theta_i$ is a normal rational surface with $\rho(\Theta_i) = 2$ such that $\mathrm{Sing} \, \Theta_i = 3A_3 + 3A_3$ for $i = 0, 1$ The strict transform of $\Theta_i \cdot \Theta_0$ is a $(-2)$-curve on the minimal resolution of $\Theta_0$ and is a fibre of the first projection $\Theta_i \to P^1$. $\Theta_{i,j} \cdot \Theta_0$ is a $(−3)$-curve on $\Theta_0$ and is a fibre of the ruling on $\Theta_{i,j}$ for $i = 1, 2, 3$, $j = 1, 2, 3$. $\Theta_{i,1} \cap \Theta_2 = \emptyset$, $\Theta_1 \cap \Theta_{i,2} = \emptyset$, $\Theta_1 \cap \Theta_{3,j} = \emptyset$, $\Theta_2 \cap \Theta_{i,1j} = \emptyset$ and $\Theta_2 \cap \Theta_{3,j} = \emptyset$ for any $i, j$. $\Theta_{i,1j} \cap \Theta_{3,i'} = \emptyset$ for any $i, j, i'$. (see Figure VI.1).

**XII.1**: $X$ is smooth and

$$f^*(0) = 12\Theta_0 + \sum_{i=1}^{2}(9\Theta_{1,i,1} + 6\Theta_{1,i,2} + 3\Theta_{1,i,3}) + \sum_{j=1}^{3}(8\Theta_{2,j,1} + 4\Theta_{2,j,2})$$
$$+ 6\Theta_3 + \sum_{1 \leq i \leq 2 \leq j \leq 3} (5\Theta_{(i,j,1)} + 2\Theta_{(i,j,2)} + \Theta_{(i,j,3)}) + \sum_{j=1}^{3} \Theta_{(3,j)},$$

where $\Theta_0 \simeq P^1 \times P^1$, $\Theta_{1,i,1} \simeq \Sigma_2$, $\Theta_{1,i,2} \simeq \Sigma_4$, $\Theta_{1,i,3} \simeq \Sigma_6$ for $i = 1, 2, 3$; $\Theta_{2,j,1}$ is a $(-2)$-curve on $\Theta_0$ and is a fibre of the first projection $\Theta_1 \to P^1$, $\Theta_{2,j,2}$ is an elliptic ruled surface. $\Theta_{2,j,3}$ is an elliptic curve with the self-intersection number 0 on each component. $\Theta_{3,j}$ is a $(−3)$-curve on $\Theta_0$ and is a fibre of the ruling on $\Theta_{3,j}$ for $i = 1, 2, 3$. $\Theta_{1} \cap \Theta_2 = \emptyset$, $\Theta_1 \cap \Theta_{12} = \emptyset$, $\Theta_1 \cap \Theta_{3,j} = \emptyset$, $\Theta_2 \cap \Theta_{11j} = \emptyset$ and $\Theta_2 \cap \Theta_{3,j} = \emptyset$ for any $i, j$. $\Theta_{1,1j} \cap \Theta_{3,i'} = \emptyset$ for any $i, j, i'$ (see Figure VI.2).
smooth rational surface with $x=0$, $\Theta_{2,j,2}$ is a smooth rational surface with $\rho(\Theta_{2,j,2})=6$ for $j=1, 2, 3$, $\Theta_{3}\simeq \Sigma_2$, $\Theta_{(i,j,1)}\simeq \Sigma_2$, $\Theta_{(i,j,2)}\simeq \Sigma_3$, $\Theta_{(i,j,3)}\simeq \Sigma_2$ and $\Theta_{(i,j,2)}\simeq \Sigma_2$ for $i=1, 2, j=1, 2, 3$. $\Theta_{i,j,1} \cdot \Theta_0 (i=1, 2)$ and $\Theta_3 \cdot \Theta_0$ are fibres of the first projection $\Theta_0\to \mathbb{P}^1$ and is a $(-2)$-curve on $\Theta_{i,j,1}$ for $i=1, 2, 3$. $\Theta_{i,j,1} \cdot \Theta_{i,j,3} \simeq \infty$-section on $\Theta_{i,j,1}$ and is a $(-4)$-curve on $\Theta_{i,j,2}, \Theta_{i,j,2} \cdot \Theta_{i,j,3}$ is an $\infty$-section of $\Theta_{i,j,2}$ and is a $(-6)$-curve on $\Theta_{i,j,3}$ for $i=1, 2$. $\Theta_{2,j,1} \cdot \Theta_{2,j,2}$ is a $(-1)$-curve on each component for $j=1, 2, 3$. $\Theta_{i,j,k} \cdot \Theta_{2,j,1}$ (resp. $\Theta_{3} \cdot \Theta_{2,j,1}$) is a $(-2)$-curve on $\Theta_{2,j,1}$ and is a fibre of the ruling on $\Theta_{i,j,k}$ (resp. $\Theta_3$) for $i=1, 2$, $j=1, 2, 3$, $k=1, 2, 3$. $\Theta_{(i,j,2)} \cdot \Theta_{(i,j,3)}$ is a $(-2)$-curve on $\Theta_{(i,j,1)}$ and is a fibre of the ruling on $\Theta_{(i,j,2)}$, $\Theta_{(i,j,1)} \cdot \Theta_{(i,j,3)}$ is a $(-4)$-curve on $\Theta_{(i,j,1)}$ and is an $\infty$-section on $\Theta_{(i,j,1)}$. $\Theta_{(i,j,1)} \cdot \Theta_{(i,j,2)}$ is a $(-2)$-curve on $\Theta_{(i,j,2)}$ and is a fibre of the ruling on $\Theta_{(i,j,1)}$, $\Theta_{(i,j,2)} \cdot \Theta_{(i,j,3)}$ is a $(-1)$-curve on $\Theta_{(i,j,2)}$ and is an $(-1)$-curve on $\Theta_{(i,j,3)}$, $\Theta_{(i,j,3)} \cdot \Theta_{(i,j,2)}$ is a $(-2)$-curve on $\Theta_{(i,j,2)}$ and is a fibre of the ruling on $\Theta_{(i,j,3)}$ for $i=1, 2, j=1, 2, 3$. $\Theta_{i,j} \cdot \Theta_{2,j,1}$ is a $(-2)$-curve on $\Theta_{2,j,1}$ and is a fibre of the ruling on $\Theta_{3}$. $\Theta_{2,j,1} \cdot \Theta_{(3,j)}$ is a $(-2)$-curve on $\Theta_{2,j,1}$ and is a fibre of the ruling on $\Theta_{(3,j)} \cdot \Theta_{j,2} \cdot \Theta_{(3,j)}$ is $(-2)$-curve on $\Theta_{(3,j)}$ and is a 0-curve on $\Theta_{2,j,2}$ for $j=1, 2, 3$. $\Theta_0 \cap \Theta_{i,j,k}=\emptyset$ for $i=1, 2, k=1, 2, 3$. $\Theta_0 \cap \Theta_{2,j,2}=\emptyset$ for $j=1, 2, 3$. $\Theta_0 \cap \Theta_{(i,j,2)}=\emptyset$ for $i=1, 2, k=1, 2, 3$. $\Theta_0 \cap \Theta_{(i,j,3)}=\emptyset$ for $j=1, 2, 3$. $\Theta_{1} \cdot \Theta_{(i,j,k)}=\emptyset$ for $i=1, 2, j=1, 2, 3$. $\Theta_0 \cap \Theta_{(3,j)}=\emptyset$ for $j=1, 2, 3$. $\Theta_{i,j,k} \cap \Theta_{1,2,k'}=\emptyset$ if $k=k'$. $\Theta_{i,1} \cap \Theta_{1,2,3}=\emptyset$ for $i=1, 2$. $\Theta_{2,j,2} \cap \Theta_{1,2,3}=\emptyset$ for $i=1, 2, j=1, 2, 3$. $\Theta_{3} \cap \Theta_{2,j,2}=\emptyset$ for $j=1, 2, 3$. $\Theta_{i,j,k} \cap \Theta_{(i',j',k')}=\emptyset$ if $i \neq i'$, $j \neq j'$, $k \neq k'$. $\Theta_{i,j,k} \cap \Theta_{(3,j')}=\emptyset$ if $j \neq j'$. $\Theta_{i,j,k} \cap \Theta_{(i,j,k')}=\emptyset$ if $(i,j) \neq (i',j')$ or $(k,k')=(1,3)$. $\Theta_{(i,j,k)} \cap \Theta_{(3,j')}=\emptyset$ for any $i, j, j'$. $\Theta_{3} \cap \Theta_{(3,j')}=\emptyset$ for $j=1, 2, 3$ (see Figure XII-1).

\[ f^*(0) = 12 \Theta_0 + \sum_{i=1}^{3} (9 \Theta_{i,1,1} + 6 \Theta_{i,1,2} + 3 \Theta_{i,1,3}) + \sum_{j=1}^{3} 4 \Theta_{2,j} + 6 \Theta_3 + \sum_{1 \leq i \leq 2 \leq j \leq 3} \Theta_{(i,j)} \]

where $\Theta_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$, $\Theta_{i,1} \simeq \Sigma_1, \Theta_{i,2} \simeq \Sigma_2, \Theta_{i,3} \simeq \Sigma_3$ for $i=1, 2, \Theta_{2,j}$ is a normal rational surface with $\rho(\Theta_{2,j})=10$ which has only one singular point $p_j$ of type $A_{2,1}$ for $j=1, 2, 3$, $\Theta_3 \simeq \Sigma_1, \Theta_{(i,j)} \simeq \mathbb{P}^2$. Sing $X=\{p_j; 1 \leq j \leq 3\}$ and analytic locally around $p_j$, $(p_j \in X, \Theta) \simeq (0 \in C^3, \Theta=0)\mathbb{Z}_2(1, 1, 1)$ for $i=1, 2, 3$. $\Theta_{i,j} \cdot \Theta_3 (i=1, 2)$ and $\Theta_3 \cdot \Theta_0$ is a fibre of the first projection $\Theta_0 \to \mathbb{P}^1$, $\Theta_{i,j} \cdot \Theta_0$ is a fibre of the second projection $\Theta_0 \to \mathbb{P}^1$ for $j=1, 2, 3$. $\Theta_{i,j,1} \cdot \Theta_{i,j,2}$ is an $\infty$-section on $\Theta_{i,j,1}$ and is a $(-2)$-curve on $\Theta_{i,j,2}, \Theta_{i,j,2} \cdot \Theta_{i,j,3}$ is an $\infty$-section on $\Theta_{i,j,2}$ and is a $(-3)$-curve on $\Theta_{i,j,3}$ for $i=1, 2, \Theta_{i,j,2} \cdot (\Theta_{3} \cdot \Theta_{2,j})$ is a $(-2)$-curve on $\Theta_{2,j}$ and is a fibre of the ruling on $\Theta_{i,j,k}$ (resp. $\Theta_{3}$) for $i=1, 2$, $j=1, 2, 3$, $k=1, 2, 3$. $\Theta_{i,j} \cdot \Theta_{2,j}$ is a $(-4)$-curve on $\Theta_{2,j}$ and
is a line on \( \Theta_{(i,j)} \) for \( i = 1, 2, j = 1, 2, 3 \). \( \Theta_{0} \cap \Theta_{(i,j,k)} = \emptyset \) for \( i = 1, 2, j = 1, 2, 3 \). \( \Theta_{0} \cap \Theta_{(i,i,k)} = \emptyset \) for \( i = 1, 2, k = 1, 2, 3 \). \( \Theta_{1,i,k} \cap \Theta_{1,2,k} = 0 \) if \( k \neq k' \). \( \Theta_{i,1,i} \cap \Theta_{1,i,i} = 0 \) for \( i = 1, 2 \). \( \Theta_{3} \cap \Theta_{1,i,k} = 0 \) for \( i = 1, 2, k = 1, 2, 3 \). \( \Theta_{i,1,i} \cap \Theta_{1,i,j} = 0 \) if \( j \neq j' \). \( \Theta_{i,j,i} \cap \Theta_{(i,j),j'} = 0 \) if \( i, j \neq (i', j') \). \( \Theta_{3} \cap \Theta_{(i,j)} = 0 \) for \( i = 1, 2, j = 1, 2, 3 \) (see Figure XIIa-2).

**XIIa-3:**

\[
f^*(0) = 12 \Theta_{0} + \sum_{i=1}^{2} 3 \Theta_{i,1} + \sum_{j=1}^{3} (8 \Theta_{2,i,1} + 4 \Theta_{2,i,2}) + 6 \Theta_{3} + \sum_{j=1}^{3} 2 \Theta_{3,j},
\]

where \( \Theta_{0} \cong P^1 \times P^1 \), \( \Theta_{i,i} \) is a normal rational surface with \( \rho(\Theta_{1,i}) = 8 \) which has three singular points \( \{p_{j}^{(i)} ; 1 \leq j \leq 3 \} \) of type \( A_{3,1} \) for \( i = 1, 2 \), \( \Theta_{2,i,1} \cong \Sigma_1 \). \( \Theta_{2,i,2} \cong \sum_{j=1}^{3} \Theta_{2,i,j} \) for \( j = 1, 2, 3 \), \( \Theta_{3} \) is a smooth rational surface with \( \rho(\Theta_{3}) = 11 \), \( \Theta_{3,j} \cong P^2 \) for \( j = 1, 2, 3 \). \( \text{Sing} X = \{p_{j}^{(i)} ; 1 \leq i \leq 2, 1 \leq j \leq 3 \} \) and analytically \( p_{j}^{(i)}(X, \Theta) \approx (0 \in C^3, \{z = 0\})/Z_2(1, 1, 2) \) for \( i = 1, 2, j = 1, 2, 3 \). \( \Theta_{1,i} \cdot \Theta_{0}(i = 1, 2) \) (resp. \( \Theta_{3} \cdot \Theta_{0} \)) is a fibre of the first projection \( \Theta_{0} \to P^1 \) and is a \((-2)\)-curve on \( \Theta_{1,i} \) (resp. \( \Theta_{3} \)). \( \Theta_{3,j} \cdot \Theta_{0} \) is a fibre of the second projection \( \Theta_{0} \to P^1 \) and is a \((-1)\)-curve on \( \Theta_{2,i,j} \) for \( j = 1, 2, 3 \). \( \Theta_{2,i,1} \cdot \Theta_{2,i,2} \) is an \((-3)\)-section on \( \Theta_{2,i,1} \) and is a \((-2)\)-curve on \( \Theta_{2,i,2} \) for \( j = 1, 2, 3 \). \( \Theta_{2,i} \cdot \Theta_{2,i,k} \) is a \((-2)\)-curve on \( \Theta_{2,i} \) (resp. \( \Theta_{3} \)) and is a fibre of the ruling on \( \Theta_{2,i,k} \) for \( i = 1, 2, j = 1, 2, 3, k = 1, 2 \). \( \Theta_{3,j} \cdot \Theta_{3} \) is a \((-3)\)-curve on \( \Theta_{3} \) and is a line on \( \Theta_{3,j} \) for \( j = 1, 2, 3 \). \( \Theta_{0} \cap \Theta_{2,i,j} = 0 \) for \( j = 1, 2, 3, \Theta_{0} \cap \Theta_{3,j} = 0 \) for \( J = 1, 2, 3 \). \( \Theta_{1,i} \cap \Theta_{1,i} = 0 \) for \( i = 1, 2 \). \( \Theta_{1,i} \cap \Theta_{3,j} = 0 \) for \( j = 1, 2, 3 \). \( \Theta_{3,i} \cap \Theta_{3,j} = 0 \) if \( i \neq j \). (see Figure XIIa-3).

**XIIa-4:**

\[
f^*(0) = 12 \Theta_{0} + \sum_{i=1}^{2} 3 \Theta_{i,1} + \sum_{j=1}^{3} 4 \Theta_{2,i,j} + 6 \Theta_{3},
\]

where \( \Theta_{0} \cong P^1 \times P^1 \), \( \Theta_{i,i} = \Sigma_1 \) for \( i = 1, 2 \), \( \Theta_{2,i} \) is a normal rational surface with \( \rho(\Theta_{2,i}) = 5 \) which has two singular points \( \{p_{j}^{(i)} ; 1 \leq i \leq 2\} \) of type \( A_{4,3} \) and one singular point \( p_{j}^{(i)} \) of type \( A_{2,1} \) for \( j = 1, 2, 3 \). \( \Theta_{0} \cong \Sigma_1 \). \( \text{Sing} X = \{p_{j}^{(i)} ; 1 \leq i \leq 3, 1 \leq j \leq 3 \} \) and analytically \( (p_{j}^{(i)}(X, \Theta) \approx (0 \in C^3, \{z = 0\})/Z_4(1, 1, 1, 1) \) for \( i = 1, 2, j = 1, 2, 3 \) and \( (p_{j}^{(i)}(X, \Theta) \approx (0 \in C^3, \{z = 0\})/Z_4(1, 1, 1, 1) \) for \( j = 1, 2, 3 \). \( \Theta_{1,i} \cdot \Theta_{0}(i = 1, 2, 3) \) (resp. \( \Theta_{3} \cdot \Theta_{0} \)) is a fibre of the first projection \( \Theta_{0} \to P^1 \) and is a \((-1)\)-curve on \( \Theta_{1,i} \) (resp. \( \Theta_{3} \)). \( \Theta_{2,i,1} \cdot \Theta_{0} \) is a fibre of the second projection \( \Theta_{0} \to P^1 \) and is a \((-1)\)-curve on \( \Theta_{2,i} \) for \( j = 1, 2, 3 \). \( \Theta_{2,i} \cdot \Theta_{2,i} \) is a \((-4)\)-curve on \( \Theta_{2,i} \) and is a fibre of the ruling on \( \Theta_{1,i} \) for \( i = 1, 2 \). \( \Theta_{2,i} \cdot \Theta_{2,i} \) is a \((-2)\)-curve on \( \Theta_{2,i} \) and is a fibre of the ruling on \( \Theta_{3} \) for \( j = 1, 2, 3 \). \( \Theta_{1,1} \cap \Theta_{1,2} = \emptyset \). \( \Theta_{3} \cap \Theta_{1,1} = \emptyset \) for \( i = 1, 2 \). (see Figure XIIa-4).
where \( \mathcal{D} := \{(2, 1), (2, 2), (1, 3), (2, 3), (3, 3)\}, \Theta_0 \) is a normal rational surface with \( \rho(\Theta_0) = 2 \) which has four singular points \( \{p^j, q^j; j = 1, 2\} \) of type \( A_{2,1} \).
\(\Theta_{1,1,1} \simeq \Sigma_2, \Theta_{1,1,2} \simeq \Sigma_4, \Theta_{1,2,1} \) is a normal rational surface with \(\rho(\Theta_{1,1,1}) = 2\) which has four singular points \(\{q^{(i)}; l = 1, 2, j = 1, 2\}\) of type \(A_{2,1}\), \(\Theta_{1,2,2}\) is a normal rational surface with \(\rho(\Theta_{1,2,2}) = 2\) which has four singular points \(\{q^{(j)}; l = 2, 3, j = 1, 2\}\) of type \(A_{2,1}\), \(\Theta_{2,1}\) is a normal rational surface with \(\rho(\Theta_{2,1}) = 8\) which has five singular points \(\{p^{(i)}_k, q^{(j)}_l; 1 \leq k \leq 2, 1 \leq l \leq 3\}\) of type \(A_{2,2}\). \(\Theta_{2,2}\) is a smooth rational surface with \(\rho(\Theta_{2,2}) = 11\), \(\Theta_{1,2}\) is a normal rational surface with \(\rho(\Theta_{1,2,1}) = 8\) which has five singular points \(\{p^{(i)}_k, q^{(j)}_l; 1 \leq k \leq 2, 1 \leq l \leq 3\}\) of type \(A_{2,2}\). \(\Theta_{2,3}\) is a smooth rational surface with \(\rho(\Theta_{2,3}) = 2\) which has four singular points \(\{q^{(i)}; l = 1, 2, j = 1, 2\}\) of type \(A_{2,2}\). \(\Theta_{2,4}\) of type \(A_{2,2}\) is a normal rational surface with \(\rho(\Theta_{2,4}) = 5\) which has three singular points \(\{r^{(i)}; l = 1, 2, 3\}\), \(r \in \Theta_{1,2}\) is of type \(A_{4,3}\) for \(l = 1, 2\), and \(r \in \Theta_{1,2}\) is of type \(A_{2,1}\). \(\Theta_{2,5}\) is a normal rational surface with \(\rho(\Theta_{2,5}) = 2\) which has two singular points \(\{p^{(i)}; 1 \leq k \leq 2\}\) of type \(A_{2,1}\) for \(j = 1, 2\), \(\Theta_{2,6} \simeq \Sigma_1\) and \(\Theta_{2,7} \simeq \Sigma_2\) for \(j = 1, 2\). Sing \(X = \{p^{(i)}_k, 1 \leq k \leq 2, 1 \leq l \leq 3\}\) and analytic locally, \(\rho(\Theta_{1,2}, \Theta) = (C^3, \{xy = 0\})/\mathbb{Z}(1, 1, 1)\) and analytic locally around \(p^{(j)}_l(1 \leq j \leq 2), q^{(j)}_l(1 \leq j \leq 4), r_3, (X, \Theta)\) is isomorphic.

**XII.3**:

\[f^*(0) = 12 \theta_0 + \sum_{l=1}^{2} 4 \theta_{1,1} + \sum_{j=1}^{3} 6 \theta_{2,2} + \sum_{j=1}^{3} 5 \theta_{2,j}\]

where \(\Theta_0\) is a normal rational surface with \(\rho(\Theta_0) = 4\) which has two singular points \(\{p^{(i)}; j = 1, 2\}\) of type \(A_{2,1}\), \(\Theta_{1,1}\) is a normal rational surface with \(\rho(\Theta_{1,1}) = 6\) which has four singular points \(\{q^{(j)}; 1 \leq j \leq 4\}\) of type \(A_{2,1}\), \(\Theta_{2,1}\) is a normal rational surface with \(\rho(\Theta_{2,1}) = 5\) which has three singular points \(\{r^{(i)}; l = 1, 2, 3\}\), \(r \in \Theta_{1,2}\) is of type \(A_{4,3}\) for \(l = 1, 2\), and \(r \in \Theta_{1,2}\) is of type \(A_{2,1}\). \(\Theta_{2,2}\) is a normal rational surface with \(\rho(\Theta_{2,2}) = 2\) which has two singular points \(\{p^{(i)}; 1 \leq k \leq 2\}\) of type \(A_{2,1}\) for \(j = 1, 2\), \(\Theta_{2,3} \simeq \Sigma_1\) and \(\Theta_{2,4} \simeq \Sigma_2\) for \(j = 1, 2\). Sing \(X = \{p^{(i)}_k, 1 \leq j \leq 2, 1 \leq l \leq 3\}\) and analytic locally, \(\rho(\Theta_{1,2}, \Theta) = (C^3, \{xy = 0\})/\mathbb{Z}(1, 1, 1)\) and analytic locally around \(p^{(j)}_l(1 \leq j \leq 2), q^{(j)}_l(1 \leq j \leq 4), r_3, (X, \Theta)\) is isomorphic.
to the germ of the origin of \((\mathbb{C}^3, \{z=0\})/\mathbb{Z}_2(1, 1, 1)\) and \((r_t \in X, \Theta) = (0 \in \mathbb{C}^3, \{z=0\})/\mathbb{Z}_4(1, 1, 3)\) for \(l=1, 2\). \(\Theta_0 \cdot \Theta_{1,1}\) is a 0-curve on \(\Theta_0\) and is a \((-2)\)-curve on \(\Theta_{1,1}\). \(\Theta_0 \cdot \Theta_{1,2}\) is a \((-1)\)-curve on \(\Theta_0\) and \(\Theta_{1,2}\). The strict transform of \(\Theta_0 \cdot \Theta_{2,j}\) is a \((-1)\)-curve on the minimal resolution of \(\Theta_0\) and the minimal resolution of \(\Theta_{2,j}\) for \(j=1, 2\). \(\Theta_0 \cdot \Theta_{2,3}\) is a 0-curve on \(\Theta_0\) and is a \((-1)\)-curve on \(\Theta_{2,3}\). \(\Theta_{1,1} \cdot \Theta_{2,j}\) is a \((-2)\)-curve on \(\Theta_{1,1}\) and is a 0-curve on \(\Theta_{2,j}\) for \(j=1, 2\). \(\Theta_{1,1} \cdot \Theta_{2,3}\) are two disjoint \((-2)\)-curves on \(\Theta_{1,1}\) and are two 0-curves on \(\Theta_{2,3}\). \(\Theta_{1,2} \cdot \Theta_{2,j}\) is a \((-4)\)-curve on \(\Theta_{1,2}\) and is an \(\infty\)-section of \(\Theta_{2,j}\) for \(j=1, 2\). \(\Theta_{1,2} \cdot \Theta_{2,3}\) is a \((-2)\)-curve on \(\Theta_{1,2}\) and is a fibre of the ruling on \(\Theta_{2,3}\). \(\Theta_{2,j} \cdot \Theta_{2,j}\) is a 0-curve on \(\Theta_{2,j}\) and is a \((-2)\)-curve on \(\Theta_{2,j}\) for \(j=1, 2\). \(\Theta_{1,1} \cap \Theta_{2,j}=\emptyset\) if \(j \neq j'\). \(\Theta_{1,1} \cap \Theta_{2,j}\) is a \((-2)\)-curve on \(\Theta_{1,1}\) and is a fibre of the ruling on \(\Theta_{2,j}\) and is a \((-2)\)-curve on \(\Theta_{1,j}\) for \(j=1, 2\). 

**XII.4**

\[f^*(0) = 12 \Theta_0 + \sum_{j=1}^3 4 \Theta_{1,1} + \sum_{j=1}^3 6 \Theta_{2,j} + \sum_{j=1}^3 3 \Theta_{1,2} \]

where \(\Theta_0\) is a normal rational surface with \(\rho(\Theta_0)=4\) which has two singular points \(\{q_{1,1}^{(j)} ; j=1, 2\}\) of type \(A_{2,1}\), \(\Theta_{1,1}\) is a normal rational surface with \(\rho(\Theta_{1,1})=6\) which has four singular points \(\{q_{1,1}^{(j)} ; 1 \leq j \leq 4\}\) of type \(A_{2,1}\), \(\Theta_{1,2}\) is a normal rational surface with \(\rho(\Theta_{1,2})=5\) which has seven singular points \(q_{1,2}^{(j)}\) \((1 \leq j \leq 2, 1 \leq k \leq 3)\), \(q_{1,2}^{(3)}\) \((1 \leq j \leq 2, 1 \leq k \leq 2)\), \(q_{1,2}^{(4)}\) \((1 \leq j \leq 2, 1 \leq k \leq 3)\), \(q_{1,2}^{(5)}\) \((1 \leq j \leq 2) \subseteq \Theta_{1,2}\) are of type \(A_{2,1}\) and \(q_{1,2}^{(6)}\) \((1 \leq j \leq 2) \subseteq \Theta_{1,2}\) are of type \(A_{4,1}\), \(\Theta_{2,j}\) is a normal rational surface with \(\rho(\Theta_{2,j})=2\) which has two singular points \(\{q_{2,j}^{(k)} ; 1 \leq k \leq 2\}\) of type \(A_{2,1}\) for \(j=1, 2\). \(\Theta_{2,j} \cong \sum_1 \Theta_{1,1} + \Theta_{1,2}\) for \(j=1, 2\). \(\text{Sing } X = \{q_{1,1}^{(j)} ; 1 \leq j \leq 4\} \cup \{q_{1,2}^{(1, 1, 1)} ; 1 \leq j \leq 2, 1 \leq k \leq 3\}\) and analytic locally around \(\Theta_{1,1}\) is isomorphic to the germ of the origin of \((\mathbb{C}^3, \{z=0\})/\mathbb{Z}_2(1, 1, 1)\), \((q_{1,1}^{(k)} \in X, \Theta) = C_3, \{xyz=0\})/\mathbb{Z}_2(1, 1, 1)\), \((q_{1,2}^{(k)} \in X, \Theta) = (\mathbb{C}^3, \{xy=0\})/\mathbb{Z}_2(1, 1, 1)\) and \((q_{1,2}^{(k)} \in X, \Theta) = (\mathbb{C}^3, \{z=0\})/\mathbb{Z}_2(1, 1, 3)\) for \(j=1, 2\). \(\Theta_0 \cdot \Theta_{1,1}\) is a 0-curve on \(\Theta_0\) and is a \((-2)\)-curve on \(\Theta_{1,1}\). The strict transform of \(\Theta_0 \cdot \Theta_{1,2}\) is a \((-1)\)-curve on the minimal resolution of \(\Theta_0\) and is a \((-2)\)-curve on the minimal resolution of \(\Theta_{1,2}\). The strict transform of \(\Theta_0 \cdot \Theta_{2,j}\) is a \((-1)\)-curve on the minimal resolution of \(\Theta_0\) and on the minimal resolution of \(\Theta_{2,j}\) for \(j=1, 2\). \(\Theta_0 \cdot \Theta_{2,3}\) is a 0-curve on \(\Theta_0\) and is a \((-1)\)-curve on \(\Theta_{2,3}\). \(\Theta_0 \cdot \Theta_{1,2}\) is a \((-2)\)-curve on \(\Theta_0\) and is a fibre of the ruling on \(\Theta_{1,2}\) for \(j=1, 2\). \(\Theta_{1,1} \cdot \Theta_{2,j}\) is a \((-2)\)-curve on \(\Theta_{1,1}\) and is a 0-curve on \(\Theta_{2,j}\) for \(j=1, 2\). \(\Theta_{1,1} \cdot \Theta_{2,3}\) consists of two disjoint \((-2)\)-curves on \(\Theta_{1,1}\) and is two fibres of the ruling on \(\Theta_{2,3}\). The strict transform of \(\Theta_{1,2} \cdot \Theta_{2,j}\) is a \((-2)\)-curve on the minimal resolution of \(\Theta_{2,3}\) and is a \((-1)\)-curve on the minimal resolution of \(\Theta_{2,j}\) for \(j=1, 2\). \(\Theta_{1,2} \cdot \Theta_{2,3}\) is a \((-2)\)-curve on \(\Theta_{1,2}\) and is a fibre of the ruling on \(\Theta_{2,3}\). \(\Theta_{1,2} \cdot \Theta_{1,1}\) is a fibre of the ruling on \(\Theta_{2,j}\) and is a \((-2)\)-curve on \(\Theta_{1,j}\) for \(j=1, 2\). \(\Theta_{1,1} \cap \Theta_{1,2} = \emptyset\), \(\Theta_{2,j} \cap \Theta_{2,j} = \emptyset\) if \(j \neq j'\). \(\Theta_{1,j} \cap (\Theta \cup \Theta_{1,j}) = \emptyset\) for
\[ j=1, 2 \text{ (see Figure XII$\beta$-4)}. \]

**Remark 6.1.** The above degeneration \( f : X \to \mathcal{D} \) is minimal degeneration except the cases \( \text{XII}_2-2, 3, 4, \text{XII}_4-2, 3, 4 \).

**Remark 6.2.** Let \((S, \mathcal{D})\) be a \( \nu_\mathcal{D} \)-log surface of abelian type and \( \pi : \tilde{S} \to S \) be the global log canonical cover. Let \( r \) be the order of \( \text{Gal}(\tilde{S}/S) \). Consider the action of \( \text{Gal}(\tilde{S}/S) \) to \( \tilde{S} \times \mathcal{D} \) such that \( \sigma((p, t)) = (\sigma(p), \xi^w t) \) for any \((p, t) \in \tilde{S} \times \mathcal{D} \), where \( \sigma \) is a generator of \( \text{Gal}(\tilde{S}/S) \) and \( \xi \) is a primitive \( r \)-th root of unity. For an appropriate \( w \in \mathbb{N} \), \( \tilde{f} : \tilde{S} \times \mathcal{D} \to \mathcal{D}/\langle \sigma \rangle \) is a log minimal degeneration. In this way, we can easily construct examples of degeneration except the cases \( \Pi_\beta, \Pi_7, \Pi_8, \Pi_9-1, 2, \Pi_7-1, 2, \Pi_8 \ (0 > 0, s > 0), \ IV_\beta-1, 2, IV_7(t_1, t_2) \neq (0, 0), (4, 4) \) and \( IV_8 \).

From the above theorem, we can calculate the Euler number of the special fibre in certain cases.

**Corollary 6.1.** The Euler number of the special fibre of the above degeneration is 0 in the cases I, II$\alpha$, II$\beta$, II$\gamma$, II$\delta$, III$\beta-1, 2$, III$\gamma-1, 2$, III$\delta$ \((t=9, s=0)\), IV$\beta-1, 2$, IV$\gamma(t_1, t_2) \neq (0, 0)$, \((4, 4)\) and \( IV_\delta \).

**Remark 6.3.** The above numbers do not depend on the choice of minimal models (see [8]).

To prove Theorem 6.1, we prepare the following lemma.

**Lemma 6.1.** Let \( \tilde{f} : (\tilde{X}, \tilde{\Theta}) \to \mathcal{D} \) be a log minimal degeneration such that \( \tilde{\Theta} \) is irreducible and suppose that \( \mathcal{O}_\mathcal{D}(r(K_\mathcal{D} + \text{Diff}_\mathcal{D}(0))) \simeq \mathcal{O}_\mathcal{D} \) for \( r \in \mathbb{N} \). Then we have \( \mathcal{O}_{\tilde{X}}(r(K_{\tilde{X}} + \tilde{\Theta})) \simeq \mathcal{O}_{\tilde{X}} \) for any \( i \in \mathbb{Z} \).

**Proof.** Put \( \mathcal{F} := \text{Coker}(\mathcal{O}_{\tilde{X}}(rK_{\tilde{X}}) \to \mathcal{O}_{\tilde{X}}(r(K_{\tilde{X}} + \tilde{\Theta}))) \) and \( U := \{ p \in X ; \mathcal{O}_{\tilde{X}}(r(K_{\tilde{X}} + \tilde{\Theta})) \text{ is Cartier in the neighborhood of } p \} \). We note that \( U \) is an open subset of \( \tilde{X} \) which has codimension 3. Let \( j : U \to \tilde{X} \) be the natural embedding.

Claim \( R^1j_*((j^{-1}\mathcal{O}_{\tilde{X}}(rK_{\tilde{X}})))=0 \).

Proof of the Claim. We may assume that \( \tilde{X} \) is affine. Let \( \pi : \tilde{X} \to X \) be the log canonical cover with respect to \( K_\tilde{X}, \pi_0 \) be the restriction of \( \pi \) to \( \tilde{U} := \pi^{-1}(U) \) and \( \tilde{f} : \tilde{U} \to \tilde{X} \) be the natural embedding. We note that \( \tilde{X} \) has only Gorenstein canonical singularities. Since \( \pi \) is finite and \( \tilde{X} \) is Cohen Macaulay, we have \( R^1j_*(j^{-1}\pi_*\mathcal{O}_{\tilde{X}}(iK_{\tilde{X}}))=R^1j_*(\pi_0^*\mathcal{O}_0)=R^1(j_0^*\pi_0)^*\mathcal{O}\tilde{U}=R^1(\pi_0^*\tilde{f})_*\mathcal{O}_0=\pi_*R^1\tilde{f}_*\mathcal{O}_0=0 \), hence \( R^1j_*(j^{-1}\mathcal{O}_{\tilde{X}}(iK_{\tilde{X}})))=0 \) for any \( i \in \mathbb{Z} \).
Proof of Lemma 6.1 continued. By the above claim, we get $\mathcal{F} = j_* j^{-1} \mathcal{F} = \mathcal{O}_\varnothing$ and the following exact sequence:

$$0 \to \mathcal{O}_X(r \mathcal{K}_X) \to \mathcal{O}_X(r (\mathcal{K}_X + \varnothing)) \to \mathcal{O}_\varnothing \to 0,$$

The above exact sequence induces the following exact sequence:

$$\tilde{f}_* \mathcal{O}_X(r (\mathcal{K}_X + \varnothing)) \to H^0(\mathcal{O}_\varnothing) \to R^1 \tilde{f}_* \mathcal{O}_X(r (\mathcal{K}_X + \varnothing)) \to R^1 \tilde{f}_* \mathcal{O}_X(r (\mathcal{K}_X + \varnothing)).$$

Since $\mathcal{K}_X$ is $\tilde{f}$-semi-ample, $R^1 \tilde{f}_* \mathcal{O}_X(r \mathcal{K}_X)$ is torsion free and $\gamma$ is an isomorphism on $\mathcal{D}^\ast$. Therefore we have $\text{Ker } \gamma = 0$, hence $\beta$ is a zero map and $\alpha$ is surjective. Let $\theta$ be a section of $H^0(\mathcal{O}_X(r (\mathcal{K}_X + \varnothing)))$ such that $\alpha(\theta) = 1$. By construction, we have $\dim(\text{Supp } \text{div } \theta \cap \text{Supp } \varnothing) = 0$, hence $\text{Ker } \text{div } \theta = 0$ since $\text{Supp } \text{div } \theta$ is $\mathbb{Q}$-Cartier. Thus we get the assertion.

Proof of Theorem 6.1. Firstly, we note that $\text{Sing } \tilde{X} \subset \text{Supp } \text{Diff}_\varnothing(0) \cup \text{Sing } \varnothing$ by [20], Corollary 3.7. Put $r := \text{Cl}(\varnothing, \text{Diff}_\varnothing(0))$ and Let $\pi : \tilde{X} \to \tilde{X}$ be the global log canonical cover with respect to $(\tilde{X}, \varnothing)$. Since $(\tilde{X}, \varnothing)$ is purely log terminal, $(\tilde{X}, \pi^{-1} \varnothing)$ is also purely log terminal by [20], Corollary 2.2 and in fact, canonical, since $\mathcal{K}_X + \pi^{-1} \varnothing$ is Cartier. Taking analytic Stein factorization, we have a surjective and connected projective morphism $\widetilde{f} : \tilde{X} \to \mathcal{D}$ from $\tilde{X}$ to a complex disk $\mathcal{D}$ such that $\widetilde{f} \circ \pi = \tau \circ \tilde{f}$, where $\tau : \mathcal{D} \to \mathbb{D}$ is the induced finite morphism. Put $\varnothing := \pi^{-1} \varnothing$. Taking adjunction from $\mathcal{K}_X + \varnothing = \pi^* (\mathcal{K}_X + \varnothing)$, we have $0 \sim \mathcal{K}_\varnothing = \pi^* (\mathcal{K}_\varnothing + \text{Diff}_\varnothing(0))$, hence $\pi : \varnothing \to \varnothing$ is the global log canonical cover with respect to $(\varnothing, \text{Diff}_\varnothing(0))$. Since $\varnothing$ is smooth by assumption, $\tilde{X}$ is smooth by [20], Corollary 3.7, hence $\tilde{X}$ has only quotient singularities. Since the support of singular fibre of $\widetilde{f}$ is an abelian surface, $\widetilde{f}^* (\mathcal{I})$ is also an abelian surface for $\mathcal{I} \in \mathcal{D}^\ast$, hence $\widetilde{f}^* (t)$ is an abelian surface or a hyperelliptic surface for $t \in \mathcal{D}^\ast$. Assume that $\varnothing$ is rational. From Lemma 6.1, $r \mathcal{K}_X \sim 0$ for $t \in \mathcal{D}^\ast$. In particular, if $r = 5$, then $\tilde{X}_t$ is an abelian surface for $t \in \mathcal{D}^\ast$ by the classification of surfaces. Let $m$ be the multiplicity of $\varnothing$. Since $\varnothing(0) = r \varnothing$ from [5], Lemma 6.1, we have $r \varnothing = \pi^* r \varnothing(0) = \varnothing^* r \varnothing(0) = \deg \tau m \varnothing$. Put $l := \text{Min } \{ n \in \mathbb{N} ; n \mathcal{K}_X \sim 0 (t \in \mathcal{D}^\ast) \}$. Then we have $\deg \tau = r/l$, hence $m = l$. Announce that $\tilde{X}_t$ is an abelian surface for $t \in \mathcal{D}^\ast$. By the assumption, $\tilde{f}$ is smooth. Let $\sigma$ be a generator of $\text{Gal}(\tilde{X}/\tilde{X})$. Choose the basis $\omega_1, \omega_2$ of $H^0(\varnothing, \mathcal{O}_\varnothing)$ such that $\sigma^* \omega_i = \zeta^{w_i} \omega_i$ ($i = 1, 2$), where $\zeta$ is a primitive $r$-th root of unity and $w_i$, $i = 1, 2$ is a non-negative integer. Since $\varnothing$ is $\sigma$-invariant, we can write $\sigma^* \tilde{f} = \zeta^{w_3} \tilde{f}$, where $w_3$ is a non-negative integer. Noting that $\varnothing$ is an abelian surface, for all fixed point $p \in \varnothing$ under the action of $\sigma$, $\pi(p) \in \tilde{X}$ is a quotient singularity of type $(1/r)(w_1, w_2, w_3)$, that is, all singularities of images of fixed points of $\sigma$ are of the same type if the generic fibre is an abelian surface.

Cases where $(\varnothing, \text{Diff}_\varnothing(0))$ is of type $I$, $II$, $III_\alpha$, $III_\beta$, $III_\gamma$, $IV_\alpha$, $IV_\beta$ or $IV_\alpha$ can be treated in the same way as degeneration of elliptic curves.

Assume that $(\varnothing, \text{Diff}_\varnothing(0))$ is of type $III_\beta$ in Theorem 5.1. Let $p \in \tilde{X}$ be a fixed
point of $\sigma$. From the above argument, analytic locally around $\pi(p)$, $(\tilde{X}, \tilde{\Theta})$ is isomorphic to the germ of the origin of $(1) (C^3, \{z=0\})/\mathbb{Z}_3(1, 1, 1)$ or $(2) (C^3, \{z=0\})/\mathbb{Z}_3(1, 2, 1)$. We blow up the singular points of type $(1)$ and we are in the case $III_\Theta$ in Theorem 6.1.

Assume that $(\tilde{\Theta}, \text{Diff}_\Theta)$ is of type $IV_7$ or $IV_{9\gamma}$ in Theorem 5.1. Let $p \in \tilde{X}$ be a fixed point of $\sigma$. Then analytic locally around $\pi(p)$, $(\tilde{X}, \tilde{\Theta})$ is isomorphic to the germ of the origin of $(1) (C^3, \{z=0\})/\mathbb{Z}_3(1, 2, 1)$ or $(2) (C^3, \{z=0\})/\mathbb{Z}_3(1, 2, 3)$. By taking a crepant blowing-up, we see that we are in the case $IV_7$ or $IV_{9\gamma}$ in Theorem 6.1.

Assume that $(\tilde{\Theta}, \text{Diff}_\Theta)$ is of type $V$ in Theorem 5.1. Let $p \in \tilde{X}$ be a fixed point of $\sigma$. Then analytic locally around $\pi(p)$ is isomorphic to the germ of the origin of $(1) (C^3, \{z=0\})/\mathbb{Z}_3(1, 2, 2)$ or $(2) (C^3, \{z=0\})/\mathbb{Z}_3(1, 2, 3)$. (3) $(C^3, \{z=0\})/\mathbb{Z}_3(1, 2, 1)$. From the above argument, all of the singularities of $\tilde{X}$ is of the same type. In case $(1)$, (resp. $(2)$), we are in the case $V-I$ (resp. $V-2$) in Theorem 6.1. Assume that we are in the case $(3)$. Let $Y \to \tilde{X}$ be the resolution as in the Figure $V-3.1$. Then we have

$$K_Y = \mu^* K_X - \sum_{i=1}^{5} (1/5) \tilde{\Theta}_{1,i} + \sum_{i=1}^{5} (2/5) \tilde{\Theta}_{2,i},$$

$$\mu^* \tilde{\Theta} = \tilde{\Theta}_0 + \sum_{i=1}^{5} (1/5) \tilde{\Theta}_{1,i} + \sum_{i=1}^{5} (3/5) \tilde{\Theta}_{2,i},$$

where $\tilde{\Theta}_0 := \mu^* \tilde{\Theta}_0$ and $\tilde{\Theta}_{1,i}, \tilde{\Theta}_{2,i}$ are $\mu$-exceptional divisors for $1 \leq i \leq 5$. Since $K_Y \cdot l_i = -1$, where $l_i \subset \tilde{\Theta}_{2,i}$ be a line, we can see that $\{l_i; 1 \leq i \leq 5\}$ generate extremal rays. Let $\varphi_1: Y_0 := Y \to Y_1$ be the blow down of all of these rays (see Figure $V-3.2$) and put $\tilde{\Theta}_0^{(1)} := \varphi_{1*} \tilde{\Theta}_0, \tilde{\Theta}_1^{(1)} := \varphi_{1*} \tilde{\Theta}_{1,i}$. We note that all of the support of extremal rays are contained in $\tilde{\Theta}^{(1)}$. Let $\varphi_2: Y_1 \to Y_2$ be the contraction of an extremal ray. Assume first that $\tilde{\Theta}_0^{(1)}$ is divisorially contracted. By the $Q$-factoriality of $Y_2$, $\varphi_2(\tilde{\Theta}_0^{(1)})$ is a curve. We use the same notation for the induced morphism $\varphi_2: \tilde{\Theta}_0^{(1)} \to \tilde{\Theta}_2^{(1)} \simeq P^1$. For any point $p \in \varphi_2(\tilde{\Theta}_0^{(1)})$, $\varphi_2^*(p)$ is written as $\varphi_2^*(p) = \sum_j m_j l_j$, where $l_j \simeq P^1$ and $\cup l_j$ is a tree of rational curves. Since we have $1 = (-K_{Y_1}, \varphi_2^*(p)) = \sum_j m_j (-K_{Y_1}, l_j)$ and $2 (-K_{Y_1}, l_j) \in \mathbb{N}$ for any $j$, $\varphi_2^*(p)$ is one of the following: (1) $\varphi_2^*(p) = l$, where $l \simeq P^1$ and $\tilde{\Theta}_0^{(1)}$ is smooth in the neighborhood of $l$, (2) $\varphi_2^*(p) = l_1 + l_2$, where $l_j \simeq P^1$ for $j = 1, 2$ and $(l_1, l_2) = 1$. $\tilde{\Theta}_0^{(1)}$ has two singular points $q_j(j = 1, 2)$ of type $A_{2j}$ on $\text{Supp} \varphi_2^*(p)$ such that $q_j \in l_j \cap \cap l_j(j = 1, 2)$, (3) $\varphi_2^*(p) = 2l$, where $l \simeq P^1$ and $\tilde{\Theta}_0^{(1)}$ has one singular point $q$ of type $A_{2j}$ on $\text{Supp} \varphi_2^*(p)$ or (4) $\varphi_2^*(p) = l_1 + l_2$, where $l_j \simeq P^1$ for $j = 1, 2$ and $l_1$ and $l_2$ intersect at one point $q$. $\tilde{\Theta}_0^{(1)}$ has one singular point $q$ of type $A_{2j}$ on $\text{Supp} \varphi_2^*(p)$. The cases (2) and (3) are excluded by trivial reason. Put $C_i^{(1)} := \tilde{\Theta}_0^{(1)}|_{q_i}$ and let $f$ be a general fibre of $\varphi_2: \tilde{\Theta}_0^{(1)} \to P^1$. Since we have $\sum_{i=1}^{5} (C_i^{(1)}, f) = 5$ and $(C_i^{(1)}, f) > 0$ for any $i$ by the $Q$-factoriality of $Y_2$, we get $(C_i^{(1)}, f) = 1$ for any $i$. Let $q \in \tilde{\Theta}_0^{(1)}$ be any point described as in (4). If $C_i^{(1)}$ does not pass through $q$, then we may assume that $(\tilde{\Theta}_0^{(1)}, l_1) = 1$ and $(\tilde{\Theta}_0^{(1)}, l_2) = 0$. Since $\varphi_2$ is an extremal contraction, we
get a contradiction. Therefore, for any \( i \), \( C_i^{(i)} \) passes through \( q \), but which is absurd. Thus we conclude that \( \varphi_2 \) is a small contraction. Let \( C \) be an irreducible curve which is contained in the exceptional locus of \( \varphi_2 \). From \([13]\), we can see that \( C \cong P^1 \) and \( C \) passes through only one singular point of \( Y_1 \). Put \( C_i^{(i)} := \bar{\Theta}_i^{(i)}|_{\Theta_i^{(i)}} \) and \( C_i^{(i)} := \bar{\Theta}_i^{(i)}|_{\Theta_i^{(i)}} \) for \( 1 \leq i \leq 5 \). Let \( C' \) be the strict transform of \( C \) on \( \Theta_0^{(0)} \). Since we have \((-K_Y, C) = 1/2 \), we have
\[
\frac{5}{2} = \left( \mu^* \left( \sum_{i=1}^{5} C_i^{(i)}, C' \right) \right) = \left( \sum_{i=1}^{5} C_i^{(i)}, C' \right) + \left( \frac{1}{2} \right) \left( \sum_{i=1}^{5} C_i^{(i)}, C' \right).
\]
Noting that \( K_Y \) and \( K_Y + \bar{\Theta}_0 + (1/5) \sum_{i=1}^{5} \bar{\Theta}_1^{(i)} + (3/5) \sum_{i=1}^{5} \bar{\Theta}_2^{(i)} \) is relatively numerical equivalent over \( \mathcal{D} \), we see that \( (Y_2, \bar{\Theta}_2^{(2)} + (1/5) \sum_{i=1}^{5} \bar{\Theta}_2^{(i)}) \) is divisorially log terminal, hence \( \bar{\Theta}_2^{(2)} \) is normal and \( (\bar{\Theta}_2^{(2)}, (1/5) \sum_{i=1}^{5} C_i^{(i)}) \) is also divisorially log terminal by \([20] \), (3.2.3), where \( \bar{\Theta}_2^{(2)} := \varphi_2^* \bar{\Theta}_2^{(2)}, \bar{\Theta}_2^{(i)} := \varphi_2^* \bar{\Theta}_2^{(i)} \) and \( C_i^{(i)} := \varphi_2^* C_i^{(i)} \). Thus we conclude that \( (\sum_{i=1}^{5} C_i^{(i)}, C') = 2 \) and \( (\sum_{i=1}^{5} C_i^{(i)}, C') = 1 \). Moreover, since we have \( K_{\bar{\Theta}_2^*=C'} = -(2/5) \sum_{i=1}^{5} C_i^{(i)} - (1/5) \sum_{i=1}^{5} C_i^{(i)}, C' = -1 \) and \( C'^2 < 0 \), we get \( C'^2 = -1 \). We can get the flip of \( C \) by blowing-up along \( C' \) and contract the exceptional divisor which is isomorphic to \( P^1 \times P^1 \) along the other ruling (see Figure V-3.2 and V-3.3). By the same way as above, we carry out flips four more times and we get the model as in Figure V-3.4. The strict transform of \( \bar{\Theta}_0 \) on this model is isomorphic to a Hirzebruch surface and after contracting this component along fibres of the ruling, we get a minimal model as described in Theorem 6.1 V-3.

Assume that \( (\bar{\Theta}, \text{Diff}_0(0)) \) is of type \( VI_{\phi} \) in Theorem 5.1. Let \( p \in \bar{X} \) be a fixed point of \( \sigma \). We see that analytic locally around \( \pi(p), (\bar{X}, \bar{\Theta}) \) is isomorphic to the germ of the origin of \((1) \ C^3, \{z=0\}/\mathbb{Z}_6(3, 2, 1) \) or \((2) \ C^3, \{z=0\}/\mathbb{Z}_6(3, 2, 5) \). We resolve these singularities and calculate the intersection number with the special fibre of the induced fibration and the strict transform of an irreducible component of \( \text{Diff}_0(0) \) whose coefficient is 1/2 to see that for all of the fixed points \( p \in \bar{X} \) of \( \sigma, \pi(p) \) has the same type described as above. Thus we are in the case \( VI_{\phi^1} \) or \( VI_{\phi^2} \) in Theorem 6.1.

Assume that \( (\bar{\Theta}, \text{Diff}_0(0)) \) is of type \( VI_{\tau} \) or \( VI_{\delta} \) in Theorem 5.1. Let \( p \in \bar{X} \) be a fixed point of \( \sigma \). We can see that analytic locally around \( \pi(p), (\bar{X}, \bar{\Theta}) \) is isomorphic to the germ of the origin of \((1) \ C^3, \{z=0\}/\mathbb{Z}_6(2, 5, 5) \) or \((2) \ C^3, \{z=0\}/\mathbb{Z}_6(2, 5, 1) \). We resolve these singularities and calculate the intersection number with the special fibre of the induced fibration and the strict transform of an irreducible component of \( \text{Diff}_0(0) \) whose coefficient is 1/2 to see that for all of the fixed point \( p \in \bar{X} \) of \( \sigma, \pi(p) \) has the same type described as above. From the proof of Theorem 5.1, \( \bar{\Theta} \) has a structure of \( P^1 \)-fibration all of whose fibres are irreducible. Take an irreducible reduced curve \( \Gamma \) which is contained in a fibre and passes through the singular points of \( \bar{\Theta} \). We resolve \( \bar{X} \) and calculate the intersection number with the special fibre of the induced fibration and the strict transform of \( \Gamma \) to see that analytic locally around all the other singular points of \( \bar{X}, (\bar{X}, \bar{\Theta}) \) is
isomorphic to the germ of the origin of \((C^3, \{z=0\})/\mathbb{Z}_3(1, 1, 1)\) in the case (1) and \((C^3, \{z=0\})/\mathbb{Z}_3(1, 1, 2)\) in the case (2). Thus we are in the case \(VI_7\cdot 1, VI_8\cdot 1, VI_9\cdot 2\) or \(VI_{10}\cdot 2\) in Theorem 6.1.

Assume that \((\tilde{\Theta}, \text{Diff}_{\Theta}(0))\) is of type \(XII_a\) in Theorem 5.1. Let \(\tilde{p} \in \tilde{X}\) be a fixed point of \(\sigma\). From the above argument, analytic locally around \(\pi(\tilde{p}), (\tilde{X}, \tilde{\Theta})\) is isomorphic to the germ of the origin of (1) \((C^3, \{z=0\})/\mathbb{Z}_{12}(4, 3, 7)\) or (2) \((C^3, \{z=0\})/\mathbb{Z}_{12}(2, 3, 7)\). (3) \((C^3, \{z=0\})/\mathbb{Z}_{12}(2, 3, 5)\), (4) \((C^3, \{z=0\})/\mathbb{Z}_{12}(2, 3, 11)\). In the same way as above, we see that for all of the fixed points \(p \in \tilde{X}\) of \(\sigma, \pi(p)\) has the same type described as above and we are in the cases \(XII_{a\cdot 1}, 2, 3, 4\) in Theorem 6.1.

Assume that \((\tilde{\Theta}, \text{Diff}_{\Theta}(0))\) is of type \(XII_b\) in Theorem 5.1. Let \(p \in \tilde{X}\) be a fixed point of \(\sigma\). From the above argument, analytic locally around \(\pi(p), (\tilde{X}, \tilde{\Theta})\) is isomorphic to the germ of the origin of (1) \((C^3, \{z=0\})/\mathbb{Z}_{12}(2, 3, 7)\) or (2) \((C^3, \{z=0\})/\mathbb{Z}_{12}(2, 3, 11)\). In the same way as above, we see that for all of the fixed point \(p \in \tilde{X}\) of \(\sigma, \pi(p)\) has the same type described as above and we are in the cases \(XII_{b\cdot 1}, 2, 3, 4\) in Theorem 6.1.
Figure IIIγ-1

Figure IIIγ-2

Figure IIIδ

$(t,s)=(3,6)$
MINIMAL DEGENERATION OF SURFACES WITH $x = 0$
Figure IVβ-1

Figure IVβ-2
Figure IV_1

Figure IV_2
Figure V-1

Figure V-2
Figure V–3.1

[Diagram showing a flipping curve in minimal degeneration of surfaces with $x=0$.]

Flipping curve
Figure V-3.2
Figure 3-3

MINIMAL DEGENERATION OF SURFACES WITH $x=0$
Case $\tilde{\Theta}_0 \equiv p^1 \times p^1$

Figure V-3.4
MINIMAL DEGENERATION OF SURFACES WITH $x=0$
Figure VIs-1
Figure VI \(\beta-2\)
Figure VI.1-1
Figure VI.2
Figure VI.5-1
Figure VI.2
Figure XII-1
Figure XIIa-2

\[ \Theta_{2,1} \]
\[ 1/2(1,1,1) \]
\[ p_1 \]
\[ A_{2,1} \]

\[ \Theta_{2,2} \]
\[ 1/2(1,1,1) \]
\[ p_2 \]
\[ A_{2,1} \]

\[ \Theta_{2,3} \]
\[ 1/2(1,1,1) \]
\[ p_3 \]
\[ A_{2,1} \]

\[ \Theta_{0,0} \]
\[ \Theta_{1,1} \]
\[ \Theta_{1,2} \]
\[ \Theta_{0,1} \]
\[ \Theta_{0,2} \]

\[ 1 \]
\[ 0 \]
\[ 0 \]
\[ 0 \]
\[ 0 \]

\[ 4 \]
\[ 3 \]
\[ 6 \]
\[ 9 \]
\[ 12 \]
Figure XIIa-1

Figure XIIb-2
Figure XII-3
Figure XIIb-4
References

MINIMAL DEGENERATION OF SURFACES WITH $x=0$

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