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QUANTUM DEFORMATIONS OF CERTAIN PREHOMOGENEOUS VECTOR SPACES. II

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Introduction

Let G be a reductive algebraic group over the complex number field \mathbb{C} and let g be its Lie algebra. The quantized coordinate algebra $A_q(G)$ of G is constructed as a certain dual Hopf algebra of the quantized enveloping algebra $U_q(\mathfrak{g})$ of g. The Hopf algebras $U_q(\mathfrak{g})$ and $A_q(G)$ over $\mathbb{C}(q)$ tend to the ordinary enveloping algebra $U(\mathfrak{g})$ and the coordinate algebra A(G) respectively when the parameter q tends to 1 in a certain sense (Drinfeld [1], Jimbo [3]).

Let us consider what object we should regard as a quantum deformation of an affine variety X with G-action.

An affine variety X is endowed with an action of G if and only if its coordinate algebra A(X) is equipped with a right A(G)-comodule structure

$$\tau: A(X) \to A(X) \otimes A(G)$$

which is simultaneously an algebra homomorphism. By the duality between U(g) and A(G) we obtain a locally finite left U(g)-module structure

(*)
$$\gamma: U(\mathfrak{g}) \otimes A(X) \to A(X)$$

given by

(**)
$$\tau(n) = \sum_{i} n_{i} \otimes f_{i} \Rightarrow \gamma(u \otimes n) = \sum_{i} \langle u, f_{i} \rangle n_{i},$$

where $\langle , \rangle : U(\mathfrak{g}) \times A(G) \to \mathbb{C}$ is the dual pairing. Since τ is an algebra homomorphism, we have

$$(***) \quad u \in U(\mathfrak{g}), \ m, n \in A(X), \ \Delta(u) = \sum_{i} u_i \otimes v_i \ \Rightarrow \ u(mn) = \sum_{i} (u_i m)(v_i n),$$

where $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is the coproduct. Then the action of G on X is uniquely determined by the infinitesimal action γ . Moreover, for a locally finite left

 $U(\mathfrak{g})$ -module structure (*) on A(X) satisfying (* * *) and a certain condition on irreducible $U(\mathfrak{g})$ -modules appearing as submodules of A(X), there exists a unique action of G on X whose infinitesimal action is given by γ .

Now we define the notion of a quantum deformation of an affine variety X with G-action as follows. A (not necessarily commutative) $\mathbb{C}(q)$ -algebra $A_q(X)$ endowed with a locally finite left $U_q(\mathfrak{g})$ -module structure

$$\gamma_q: U_q(\mathfrak{g}) \otimes A_q(X) \to A_q(X)$$

is called a quantum deformation of X if $A_q(X)$ and γ_q tend to A(X) and $\gamma : U(\mathfrak{g}) \otimes A(X) \to A(X)$ respectively when q tends to 1 and if it satisfies

$$u \in U_q(\mathfrak{g}), \quad m, n \in A_q(X), \quad \Delta(u) = \sum_i u_i \otimes v_i \Rightarrow u(mn) = \sum_i (u_i m)(v_i n).$$

It seems to be an interesting problem to determine in which case X admits a quantum deformation. In this paper we consider the problem when X is a prehomogeneous vector space, that is, when X is a vector space with a linear G-action containing an open G-orbit. Such a quantum deformation was intensively studied in the case where $G = GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ and $X = M_{mn}(\mathbb{C})$ (see Taft-Towber [10], Hashimoto-Hayashi [2] and Noumi-Yamada-Mimachi [7]), and also in the case where $G = GL_n(\mathbb{C})$ and X is the set of skew symmetric matrices of degree n (see Strickland [8]).

In our previous paper [4] we gave a general method to construct quantum deformations of prehomogeneous vector spaces of parabolic type. Moreover, for each nonopen G-orbit C on X, we have shown that the defining ideal of the closure \overline{C} and its canonical generators admit quantum deformations inside $A_q(X)$. It includes the existence of the quantum deformation of the irreducible relative invariant when X is a regular prehomogeneous vector space. Indeed, the canonical generator of the defining ideal of the closure of the one-codimensional orbit is nothing but the irreducible relative invariant.

Quantum deformations of prehomogeneous vector spaces of commutative parabolic type associated to classical simple Lie algebras are intensively studied in Kamita [5]. In this paper we shall deal with the remaining two cases

- (I) $G = \mathbb{C}^{\times} \times \text{Spin}(10, \mathbb{C}), X = \mathbb{C}^{16}$, the scalar multiplication and the half-spin representation,
- (II) $G = \mathbb{C}^{\times} \times E_6$, $X = \mathbb{C}^{27}$, the scalar multiplication and the 27-dimensional irreducible representation of E_6 ,

which naturally arise from the exceptional simple Lie algebras of type E_6 and E_7 respectively using the method in our previous paper [4]. In Introduction we shall only state the results in case (II).

Let \mathfrak{g}_{E_7} be a simple Lie algebra of type E_7 over \mathbb{C} and let \mathfrak{h} be its Cartan subalgebra. We shall use the labelling of the vertices of the Dynkin diagram 1.



Dynkin diagram 1.

Set $I_0 = \{1, 2, ..., 7\}$, $I = I_0 \setminus \{1\}$. Let $\Delta \subset \mathfrak{h}^*$ be the root system of type E_7 . We denote the set of simple roots by $\{\alpha_i\}_{i \in I_0}$ and the set of positive roots by Δ^+ . Let $(,): \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ be a standard symmetric bilinear form. Set $D = \Delta^+ \setminus \sum_{i \in I} \mathbb{Z}\alpha_i$. Then we have $\sharp D = 27$. Set $\Lambda = \{1, 2, ..., 27\}$, and fix a bijection $\Lambda \ni j \mapsto \beta_j \in D$ such that $\beta_k - \beta_j \in \sum_{i \in I_0} \mathbb{Z}_{\geq 0}\alpha_i$ implies $j \leq k$, where $\mathbb{Z}_{\geq 0} = \{n \in \mathbb{Z} \mid n \geq 0\}$. Set $\delta = 3\alpha_1 + 4\alpha_2 + 5\alpha_3 + 6\alpha_4 + 3\alpha_5 + 4\alpha_6 + 2\alpha_7$. For each $n \in \Lambda$ there exist exactly five pairs $(i, j) \in \Lambda^2$ such that $\beta_i + \beta_j = \delta - \beta_n$, i < j. We denote them by $(i_1^n, j_1^n), (i_2^n, j_2^n), (i_3^n, j_3^n), (i_5^n, j_5^n) \in \Lambda^2$ where $i_5^n < i_4^n < i_3^n < i_2^n < i_1^n$. Let $K_1^{\pm 1}, E_i, F_i$ $(i \in I_0)$ be the canonical generators of $U_q(\mathfrak{g}_{E_7})$, and set $U_q(\mathfrak{g}) = \langle K_1^{\pm 3}, K_j^{\pm 1}, E_j, F_j \mid j \in I \rangle \subset U_q(\mathfrak{g}_{E_7})$. Then $U_q(\mathfrak{g})$ is isomorphic to the tensor product of $\mathbb{C}(q)[K, K^{-1}]$ and the quantized enveloping algebra of type E_6 , where $K = K_1^3 K_2^4 K_3^5 K_6^4 K_5^3 K_6^4 K_7^2$.

Theorem 0.1. A quantum deformation of the 27-dimensional irreducible prehomogeneous vector space X of $G = \mathbb{C}^{\times} \times E_6$ is given by the following.

(a) $A_q(X)$ is an associative $\mathbb{C}(q)$ -algebra defined by the following generators and fundamental relations:

Generators: Y_i with i = 1, ..., 27. Fundamental relations: For i < j

 $Y_i Y_j = \begin{cases} q Y_j Y_i & \text{if } \beta_i + \beta_j \text{ does not have another decomposition } \beta + \beta', \ \beta, \ \beta' \in D, \\ Y_j Y_i + q Y_b Y_a - q^{-1} Y_a Y_b \\ & \text{if there exist } k \in I, \ a, b \in \Lambda \text{ such that } \beta_a = \beta_i + \alpha_k, \ \beta_b = \beta_j - \alpha_k, \\ Y_j Y_i & \text{otherwise.} \end{cases}$

(b) The action $\gamma_q : U_q(\mathfrak{g}) \otimes A_q(X) \to A_q(X)$ is given by the following. For $2 \le k \le 7$, $1 \le m \le 7$

$$\gamma_q(F_k \otimes Y_i) = \begin{cases} Y_j & \text{if there exists } j \text{ such that } \beta_j = \beta_i + \alpha_k, \\ 0 & \text{otherwise,} \end{cases}$$

$$\gamma_q(E_k \otimes Y_i) = \begin{cases} Y_j & \text{if there exists } j \text{ such that } \beta_j = \beta_i - \alpha_k, \\ 0 & \text{otherwise,} \end{cases}$$

$$\gamma_q(K_m \otimes Y_i) = q^{-(\alpha_m, \beta_i)}Y_i.$$

(c) The quantum deformation of the irreducible relative invariant of X is given by

$$\varphi = \sum_{n \in \Lambda} (-q)^{|\beta_n| - 1} Y_n \psi_n,$$

where $|\beta| = \sum_{i \in I_0} m_i$ $(\beta = \sum_{i \in I_0} m_i \alpha_i)$, $\psi_n = Y_{i_5^n} Y_{j_5^n} - qY_{i_4^n} Y_{j_4^n} + q^2 Y_{i_3^n} Y_{j_3^n} - q^3 Y_{i_2^n} Y_{j_2^n} + q^4 Y_{i_1^n} Y_{j_1^n}$.

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1. Preliminaries

Let g be a simple Lie algebra of type E_6 or E_7 over the complex number field \mathbb{C} , and let \mathfrak{h} be a Cartan subalgebra of g. Let $\Delta \subset \mathfrak{h}^*$ be the root system, and let $W \subset GL(\mathfrak{h})$ be the Weyl group. We denote the set of positive roots by Δ^+ and the set of simple roots by $\{\alpha_i\}_{i \in I_0}$, where I_0 is an index set. For $i \in I_0$ we denote the simple reflection corresponding to α_i by $s_i \in W$. Let $(,): \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be the invariant symmetric bilinear form such that $(\alpha, \alpha) = 2$ for any $\alpha \in \Delta$. Set $a_{ij} = (\alpha_i, \alpha_j)$. The matrix $(a_{ij})_{i,j\in I_0}$ is called the Cartan matrix of type E_6 or E_7 . For $\alpha \in \Delta$ we denote the corresponding root space by \mathfrak{g}_{α} . Set $\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{\alpha}$, $\mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$. For a subset $I \subset I_0$ we define

$$\Delta_I = \Delta \cap \sum_{i \in I} \mathbb{Z} \alpha_i, \quad W_I = \langle s_i \mid i \in I \rangle.$$

We set

$$\mathfrak{l}_I = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_I} \mathfrak{g}_{\alpha}\right), \quad \mathfrak{n}_I^+ = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_I} \mathfrak{g}_{\alpha}, \quad \mathfrak{n}_I^- = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_I} \mathfrak{g}_{-\alpha}.$$

Let G be a connected algebraic group with Lie algebra g. We denote by L_I the subgroup of G corresponding to l_I . Then L_I acts on n_I^{\pm} via the adjoint action.

The quantized enveloping algebra $U_q(\mathfrak{g})$ (Drinfel'd [1], Jimbo [3]) is an associative algebra over the rational function field $\mathbb{C}(q)$ generated by the elements E_i , F_i , K_i , K_i^{-1} ($i \in I_0$) satisfying the following fundamental relations:

$$\begin{split} K_i K_j &= K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \\ K_i E_j &= q^{a_{ij}} E_j K_i, \quad K_i F_j = q^{-a_{ij}} F_j K_i, \\ E_i F_j &- F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\ E_i E_j &= E_j E_i \quad (i \neq j, \ a_{ij} = 0), \\ E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \quad (i \neq j, \ a_{ij} = -1), \\ F_i F_j &= F_j F_i \quad (i \neq j, \ a_{ij} = 0), \\ F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 = 0 \quad (i \neq j, \ a_{ij} = -1). \end{split}$$

A Hopf algebra structure on $U_q(\mathfrak{g})$ is defined as follows. The comultiplication $\Delta: U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ is the algebra homomorphism satisfying

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i.$$

The counit $\epsilon: U_q(\mathfrak{g}) \to \mathbb{C}(q)$ is the algebra homomorphism satisfying

$$\epsilon(K_i) = 1, \quad \epsilon(E_i) = \epsilon(F_i) = 0.$$

The antipode $S: U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ is the algebra antiautomorphism satisfying

$$S(K_i) = K_i^{-1}, \quad S(E_i) = -E_i K_i, \quad S(F_i) = -K_i^{-1} F_i.$$

Using the Hopf algebra structure, we define the adjoint action of $U_q(\mathfrak{g})$ on $U_q(\mathfrak{g})$ as follows. For $x, y \in U_q(\mathfrak{g})$ write $\Delta(x) = \sum_k x_k^1 \otimes x_k^2$ and set $\operatorname{ad}(x)y = \sum_k x_k^1 y S(x_k^2)$. Then $\operatorname{ad} : U_q(\mathfrak{g}) \to \operatorname{End}_{\mathbb{C}(q)}(U_q(\mathfrak{g}))$ is an algebra homomorphism. For $x, y, z \in U_q(\mathfrak{g})$ we have $\operatorname{ad}(x)(yz) = \sum_k (\operatorname{ad}(x_k^1)y)(\operatorname{ad}(x_k^2)z)$, where $\Delta(x) = \sum_k x_k^1 \otimes x_k^2$.

We define subalgebras $U_q(\mathfrak{n}^-)$ and $U_q(\mathfrak{l}_I)$ for $I \subset I_0$ by

$$U_q(\mathfrak{n}^-) = \langle F_i \mid i \in I_0 \rangle, \quad U_q(\mathfrak{l}_I) = \langle E_i, F_i, K_j, K_j^{-1} \mid i \in I, \ j \in I_0 \rangle.$$

For $i \in I_0$ we define an algebra automorphism T_i of $U_q(\mathfrak{g})$ by

$$T_{i}(K_{j}) = K_{j}K_{i}^{-a_{ij}},$$

$$T_{i}(E_{j}) = \begin{cases} -F_{i}K_{i} & (i = j) \\ E_{j} & (i \neq j, a_{ij} = 0) \\ E_{i}E_{j} - q^{-1}E_{j}E_{i} & (i \neq j, a_{ij} = -1), \end{cases}$$

$$T_{i}(F_{j}) = \begin{cases} -K_{i}^{-1}E_{i} & (i = j) \\ F_{j} & (i \neq j, a_{ij} = 0) \\ F_{j}F_{i} - qF_{i}F_{j} & (i \neq j, a_{ij} = -1). \end{cases}$$

(see Lusztig [6]). For $w \in W$ choose a reduced expression $w = s_{i_1} \cdots s_{i_r}$ and set $T_w = T_{i_1} \cdots T_{i_r}$. It is known that T_w does not depend on the choice of a reduced expression.

We shall use the following later (see Lusztig [6]).

Lemma 1.1. If $w(\alpha_i) = \alpha_j$ for $w \in W$ and $i, j \in I_0$, then we have $T_w(F_i) = F_j$.

For $I \subset I_0$ let w_I be the longest element of W_I and let w_0 be the longest element of W. Choose a reduced expression $w_I w_0 = s_{i_1} \cdots s_{i_r}$ of $w_I w_0$ and set

$$\beta_j = s_{i_1} s_{i_2} \cdots s_{i_{j-1}} (\alpha_{i_j}), \quad Y_j = Y_{\beta_j} = T_{i_1} \cdots T_{i_{j-1}} (F_{i_j})$$

for $1 \le j \le r$. Then it is known that $\{\beta_j \mid 1 \le j \le r\} = \Delta^+ \setminus \Delta_I$. Set

$$U_q(\mathfrak{n}_l^-) = \sum_{d_j \ge 0} \mathbb{C}(q) Y_1^{d_1} \cdots Y_r^{d_r}$$

Then $\{Y_1^{d_1} \cdots Y_r^{d_r} \mid d_j \in \mathbb{Z}_{\geq 0}, 1 \leq j \leq r\}$ is a basis of $U_q(\mathfrak{n}_l^-)$ and $U_q(\mathfrak{n}_l^-)$ is a subalgebra of $U_q(\mathfrak{n}^-)$. we have

$$U_q(\mathfrak{n}_I^-) = U_q(\mathfrak{n}^-) \cap T_{w_I}^{-1} U_q(\mathfrak{n}^-)$$

and $U_q(\mathbf{n}_l)$ does not depend on the choice of a reduced expression of $w_l w_0$ (see Lusztig [6]).

If $\mathfrak{n}_I^+ \neq \{0\}$, $[\mathfrak{n}_I^+, \mathfrak{n}_I^+] = \{0\}$, then Y_β for $\beta \in \Delta^+ \setminus \Delta_I$ does not depend on the choice of a reduced expression of $w_I w_0$ (see [4]). In this case we denote the $\mathbb{C}(q)$ -algebra $U_q(\mathfrak{n}_I^-)$ by A_q . We can regard it as a quantum deformation of the coordinate algebra $A = \mathbb{C}[\mathfrak{n}_I^+]$ of \mathfrak{n}_I^+ as explained in [4].

2. Case of type E_6

Let g be a simple Lie algebra of type E_6 . We shall use the labelling of the vertices of the Dynkin diagram 2.



Dynkin diagram 2.

Hence we have $I_0 = \{1, 2, 3, 4, 5, 6\}$. Set $I = \{2, 3, 4, 5, 6\}$. In this case we have $n_I^* \neq \{0\}, [n_I^+, n_I^+] = \{0\}$. Then I_I is isomorphic to $\mathbb{C} \oplus \mathfrak{o}(10, \mathbb{C})$ and n_I^+ is a 16-dimensional irreducible prehomogeneous vector space. There are three L_I -orbits $\{0\}, C_0, O$ on n_I^+ satisfying $\{0\} \subset \overline{C_0} \subset \overline{O}$. Let $J_{C_0} \subset \mathbb{C}[n_I^+]$ be the defining ideal of the closure of C_0 , and let $J_{C_0}^0$ denote the subspace of J_{C_0} consisting of the polynomials in J_{C_0} with homogeneous degree 2. Then $J_{C_0}^0$ is a ten-dimensional irreducible I_I -module, and it generates the ideal J_{C_0} .

We fix a reduced expression

$$w_1 w_0 = s_1 s_2 s_3 s_4 s_5 s_3 s_2 s_1 s_6 s_5 s_3 s_2 s_4 s_3 s_5 s_6$$

of $w_1 w_0$ and define the elements Y_i $(i \in \Lambda = \{1, 2, ..., 16\})$ as in Section 1.

Set $I'_0 = \{1, 2, 3, 4, 5\}$, $I' = \{2, 3, 4, 5\}$, $\Lambda' = \{1, 2, ..., 8\}$. Then $\{\alpha_i\}_{i \in I'_0}$ is a set of simple roots of type D_5 . Let g' be the simple subalgebra of g corresponding to I'_0 . We choose a reduced expression $w_{I'}w_{I'_0} = s_1s_2s_3s_4s_5s_3s_2s_1$ of $w_{I'}w_{I'_0}$. The elements Y_i $(i \in \Lambda')$ can be computed inside $U_q(g')$.

Let $\beta_j = \sum_{i \in I_0} m_i^j \alpha_i$ and set $\mathbf{m}^j = (m_1^j, \dots, m_6^j)$ for $j \in \Lambda$. Then we have

$$\begin{split} \mathbf{m}^1 &= (1,\,0,\,0,\,0,\,0,\,0), \quad \mathbf{m}^2 &= (1,\,1,\,0,\,0,\,0,\,0), \quad \mathbf{m}^3 &= (1,\,1,\,1,\,0,\,0,\,0), \\ \mathbf{m}^4 &= (1,\,1,\,1,\,1,\,0,\,0), \quad \mathbf{m}^5 &= (1,\,1,\,1,\,0,\,1,\,0), \quad \mathbf{m}^6 &= (1,\,1,\,1,\,1,\,1,\,0), \\ \mathbf{m}^7 &= (1,\,1,\,2,\,1,\,1,\,0), \quad \mathbf{m}^8 &= (1,\,2,\,2,\,1,\,1,\,0), \quad \mathbf{m}^9 &= (1,\,1,\,1,\,0,\,1,\,1), \\ \mathbf{m}^{10} &= (1,\,1,\,1,\,1,\,1,\,1), \quad \mathbf{m}^{11} &= (1,\,1,\,2,\,1,\,1,\,1), \quad \mathbf{m}^{12} &= (1,\,2,\,2,\,1,\,1,\,1), \\ \mathbf{m}^{13} &= (1,\,1,\,2,\,1,\,2,\,1), \quad \mathbf{m}^{14} &= (1,\,2,\,2,\,1,\,2,\,1), \quad \mathbf{m}^{15} &= (1,\,2,\,3,\,1,\,2,\,1), \\ \mathbf{m}^{16} &= (1,\,2,\,3,\,2,\,2,\,1). \end{split}$$

If $(\beta_j, \alpha_k) = -1$ for $j \in \Lambda$ and $k \in I$, then $s_k(\beta_j) = \beta_j + \alpha_k \in \Delta^+$. Since $k \neq 1$ and $m_1^j = 1$, we have $\beta_j + \alpha_k \notin \Delta_I$. Therefore there exists $l \in \Lambda$ satisfying $\beta_j + \alpha_k = \beta_l$. Conversely if $\beta_j + \alpha_k = \beta_l$ $(j, l \in \Lambda, k \in I)$, then we have $(\beta_j, \alpha_k) = -1$, $s_k(\beta_j) = \beta_l$.

There exist 20 triplets $(k, j, l) \in I \times \Lambda \times \Lambda$ satisfying $\beta_j + \alpha_k = \beta_l$. The triplets are the following: (2, 1, 2), (3, 2, 3), (4, 3, 4), (5, 3, 5), (5, 4, 6), (4, 5, 6), (3, 6, 7), (2, 7, 8), (6, 5, 9), (4, 9, 10), (3, 10, 11), (2, 11, 12), (5, 11, 13), (5, 12, 14), (2, 13, 14), (3, 14, 15), (4, 15, 16), (6, 6, 10), (6, 7, 11), (6, 8, 12).

For $k \in I$, $j \in \Lambda$, we have $\beta_j - 2\alpha_k$, $\beta_j + 2\alpha_k \notin \Delta^+ \setminus \Delta_I$.

Lemma 2.1. Let $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ satisfying $\beta + \alpha_k = \beta'$ ($k \in I$). Then we can choose a reduced expression $w_I w_0 = s_{i_1} s_{i_2} \cdots s_{i_{16}}$ and $p \in \Lambda$ satisfying

$$\beta = s_{i_1} s_{i_2} \cdots s_{i_{p-1}} (\alpha_{i_p}), \ \beta' = s_{i_1} s_{i_2} \cdots s_{i_{p-1}} s_{i_p} (\alpha_{i_{p+1}}), \ (\alpha_{i_p}, \alpha_{i_{p+1}}) = -1,$$

$$\alpha_k = s_{i_1} s_{i_2} \cdots s_{i_{p-1}} (\alpha_{i_{p+1}}).$$

Proof. Among the 20 triplets (k, j, l) satisfying $\beta_j + \alpha_k = \beta_l$ $(k \in I, j, k \in \Lambda)$, the 12 triplets satisfy l = j + 1, $(\alpha_{i_j}, \alpha_{i_{j+1}}) = -1$. Therefore it is sufficient to deal with the remaining 8 cases. In the cases (k, j, l) = (5, 3, 5), (5, 4, 6), (5, 11, 13), (5, 12, 14), the reduced expression

$$w_1 w_0 = s_1 s_2 s_3 s_5 s_4 s_3 s_2 s_1 s_6 s_5 s_3 s_4 s_2 s_3 s_5 s_6$$

of $w_1 w_0$ with p = 3, 5, 11, 13 respectively satisfies the required properties. In the cases (k, j, l) = (6, 5, 9), (6, 6, 10), (6, 7, 11), (6, 8, 12), the reduced expression

$$w_I w_0 = s_1 s_2 s_3 s_4 s_5 s_6 s_3 s_5 s_2 s_3 s_1 s_2 s_4 s_3 s_5 s_6$$

of $w_1 w_0$ with p = 5, 7, 9, 11 respectively satisfies the required properties.

It is known that $U_q(\mathfrak{n}_I^+)^1 = \bigoplus_{\beta \in \Delta^+ \setminus \Delta_I} \mathbb{C}(q) Y_\beta$ is an irreducible $U_q(\mathfrak{l}_I)$ -module. (see [4])

Lemma 2.2. For $k \in I$, $j \in \Lambda$, we have

$$ad(F_k)Y_j = \begin{cases} Y_l \text{ if there exists } l \in \Lambda \text{ such that } \beta_l = \beta_j + \alpha_k, \\ 0 \text{ otherwise,} \end{cases}$$

$$ad(E_k)Y_j = \begin{cases} Y_l \text{ if there exists } l \in \Lambda \text{ such that } \beta_l = \beta_j - \alpha_k, \\ 0 \text{ otherwise.} \end{cases}$$

Proof. Since $\bigoplus_{j \in \Lambda} \mathbb{C}(q)Y_j$ is a $U_q(\mathfrak{l}_I)$ -module, we have $\operatorname{ad}(F_k)Y_j = 0$ if $\beta_j + \alpha_k \notin \Delta^+ \setminus \Delta_I$, and we have $\operatorname{ad}(E_k)Y_j = 0$ if $\beta_j - \alpha_k \notin \Delta^+ \setminus \Delta_I$.

We shall show $\operatorname{ad}(F_k)Y_{\beta} = Y_{\beta'}$ for $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ and $k \in I$ satisfying $\beta' = \beta + \alpha_k$. By Lemma 2.1 we can choose a reduced expression of $w_I w_0 = s_{i_1} s_{i_2} \cdots s_{i_{16}}$ satisfying $\beta = s_{i_1} s_{i_2} \cdots s_{i_{p-1}} (\alpha_{i_p}), \beta' = s_{i_1} s_{i_2} \cdots s_{i_{p-1}} s_{i_p} (\alpha_{i_{p+1}}), (\alpha_{i_p}, \alpha_{i_{p+1}}) = -1$. Then we can write $Y_{\beta} = T_{i_1} T_{i_2} \cdots T_{i_{p-1}} (F_{i_p}), Y_{\beta'} = T_{i_1} T_{i_2} \cdots T_{i_{p-1}} T_{i_p} (F_{i_{p+1}})$. Since $(\alpha_{i_p}, \alpha_{i_{p+1}}) = -1$, we have $T_{i_p}(F_{i_{p+1}}) = F_{i_{p+1}} F_{i_p} - q F_{i_p} F_{i_{p+1}}$. Moreover, since $\alpha_k = s_{i_1} s_{i_2} \cdots s_{i_{p-1}} (\alpha_{i_{p+1}})$, we have $T_{i_1} T_{i_2} \cdots T_{i_{p-1}} (F_{i_{p+1}}) = F_k$ by Lemma 1.1, and hence

$$Y_{\beta'} = T_{i_1}T_{i_2}\cdots T_{i_{p-1}}T_{i_p}(F_{i_{p+1}})$$

= $T_{i_1}T_{i_2}\cdots T_{i_{p-1}}(F_{i_{p+1}}F_{i_p}-qF_{i_p}F_{i_{p+1}}) = F_kY_\beta - qY_\beta F_k.$

Since $(\beta, \alpha_k) = -1$, we have $ad(F_k)Y_\beta = F_kY_\beta - qY_\beta F_k$. Hence we have $ad(F_k)Y_\beta = Y_{\beta'}$.

Let us show $\operatorname{ad}(E_k)Y_{\beta} = Y_{\beta'}$ for β , $\beta' \in \Delta^+ \setminus \Delta_I$ and $k \in I$ satisfying $\beta' = \beta - \alpha_k$. By the above argument we have $Y_{\beta} = \operatorname{ad}(F_k)Y_{\beta'} = F_kY_{\beta'} - qY_{\beta'}F_k$. Since $\beta' - \alpha_k = \beta - 2\alpha_k \notin \Delta^+ \setminus \Delta_I$, we have $\operatorname{ad}(E_k)Y_{\beta'} = 0$, and hence $E_kY_{\beta'} = Y_{\beta'}E_k$. Since $(\beta', \alpha_k) = -1$, we have $K_kY_{\beta'} = qY_{\beta'}K_k$. Hence we have

$$ad(E_{k})Y_{\beta} = (E_{k}Y_{\beta} - Y_{\beta}E_{k})K_{k} = (E_{k}(F_{k}Y_{\beta'} - qY_{\beta'}F_{k}) - (F_{k}Y_{\beta'} - qY_{\beta'}F_{k})E_{k})K_{k}$$
$$= \left(\frac{K_{k} - K_{k}^{-1}}{q - q^{-1}}Y_{\beta'} - qY_{\beta'}\frac{K_{k} - K_{k}^{-1}}{q - q^{-1}}\right)K_{k} = (Y_{\beta'}K_{k}^{-1})K_{k} = Y_{\beta'}.$$

Next we shall consider quadratic fundamental relations among the elements Y_i . Since we have

$$\sum_{i,j\in\Lambda}\mathbb{C}(q)Y_iY_j=\bigoplus_{s\leq t}\mathbb{C}(q)Y_sY_t,$$

we can write

$$Y_i Y_j = \sum_{\substack{s \leq i \\ \beta_i + \beta_j = \beta_s + \beta_t}} a_{s,t}^{i,j} Y_s Y_t \quad (a_{s,t}^{i,j} \in \mathbb{C}(q))$$

for i > j (see [4]). Hence if $\beta_i + \beta_j$ does not have another decomposition $\beta + \beta' (\beta, \beta' \in \Delta^+ \setminus \Delta_I, \beta_i + \beta_j = \beta + \beta')$ then we have $Y_i Y_j = a_{i,j} Y_j Y_i$ for some $a_{i,j} \in \mathbb{C}(q)$. We denote the set of weights of the ten-dimensional irreducible highest weight l_I -module $J_{C_0}^0$ with highest weight $-\beta_1 - \beta_8$ by Γ . For $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ a weight $\beta + \beta'$ has another decomposition if and only if we have $-(\beta + \beta') \in \Gamma$. We fix a bijection

 $\{1, 2, \ldots, 10\} \ni n \mapsto -\delta_n \in \Gamma$ such that if $\delta_m - \delta_n \in \sum_{i \in I_0} \mathbb{Z}_{\geq 0} \alpha_i$, then $n \leq m$. For each *n* there exist exactly four pairs $(i, j) \in \Lambda^2$ such that $i < j, \beta_i + \beta_j = \delta_n$. We denote them by $(i_1^n, j_1^n), (i_2^n, j_2^n), (i_3^n, j_3^n), (i_4^n, j_4^n) \in \Lambda^2$ where $i_4^n < i_3^n < i_2^n < i_1^n$. Set $\mathbf{A}(n) = (i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n) \in \Lambda^8$ $(1 \leq n \leq 10)$. Then we have

We denote the set $\{i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n\}$ by $|\mathbf{A}(n)|$ for $1 \le n \le 10$. For any $i, j \in \Lambda$ there exists *n* satisfying $i, j \in |\mathbf{A}(n)|$.

Set

$$\mathcal{A} = \{ (k, n, n') \in I \times \Lambda \times \Lambda \mid \delta_n + \alpha_k = \delta_{n'} \}.$$

Then

$$\mathcal{A} = \{ (6, 1, 2), (5, 2, 3), (3, 3, 4), (2, 4, 5), (4, 4, 6), \\ (2, 6, 7), (4, 5, 7), (3, 7, 8), (5, 8, 9), (6, 9, 10) \}.$$

For any $n \in \{2, 3, ..., 10\}$ we can take a sequence $((k_1, n_1, n'_1), ..., (k_s, n_s, n'_s))$ of \mathcal{A} satisfying $n_1 = 1, n'_s = n, n'_j = n_{j+1}$ $(1 \le j \le s - 1)$.

For $(k, n, n') \in \mathcal{A}$ and $m \in \{1, 2, 3, 4\}$, we have either (\mathbf{P}_m^+) $(\beta_{i_m^n}, \alpha_k) = 0, i_m^{n'} = i_m^n, (\beta_{j_m^n}, \alpha_k) = -1, \beta_{j_m^{n'}} = \beta_{j_m^n} + \alpha_k$ or (\mathbf{P}_m^-) $(\beta_{i_m^n}, \alpha_k) = -1, \beta_{i_m^{n'}} = \beta_{i_m^n} + \alpha_k, (\beta_{j_m^n}, \alpha_k) = 0, j_m^{n'} = j_m^n.$

Proposition 2.3. For any $i, j \in \Lambda$ satisfying i < j, we have

$$(Q6) Y_i Y_j = \begin{cases} Y_j Y_i & \text{if there exists } n \text{ such that } i = i_1^n, j = j_1^n, \\ Y_{j_2^n} Y_{i_2^n} + (q - q^{-1}) Y_{i_1^n} Y_{j_1^n} & \text{if there exists } n \text{ such that } i = i_2^n, j = j_2^n, \\ Y_{j_m^n} Y_{i_m^n} + q Y_{j_{m-1}^n} Y_{i_{m-1}^n} - q^{-1} Y_{i_{m-1}^n} Y_{j_{m-1}^n} \\ & \text{if there exist } n, m \in \{3, 4\} \text{ such that } i = i_m^n, j = j_m^n, \\ q Y_j Y_i & \text{otherwise.} \end{cases}$$

Proof. Since there exists some *n* satisfying $i, j \in |\mathbf{A}(n)|$ for any $i, j \in \Lambda$, it is sufficient to show that for any $1 \le n \le 10$ the elements $Y_{i_m^n}, Y_{j_m^n}$ $(1 \le m \le 4)$ satisfy

the following relations.

$$Y_{i_1^n} Y_{j_1^n} = Y_{j_1^n} Y_{i_1^n}$$
(Rn, 1)

$$(\mathbf{R}n) \begin{cases} Y_{l_m^n} Y_{j_m^n} = Y_{j_m^n} Y_{l_m^n} + q Y_{j_{m-1}^n} Y_{l_{m-1}^n} - q^{-1} Y_{l_{m-1}^n} Y_{j_{m-1}^n} (2 \le m \le 4) & (\mathbf{R}n, 2) \\ Y_{l_1} Y_{l_2} = q Y_{l_2} Y_{l_1} & (l_1, l_2 \in |\mathbf{A}(n)|, l_1 < l_2, (l_1, l_2) \neq (l_m^n, j_m^n) (1 \le m \le 4)) & (\mathbf{R}n, 3) \end{cases}$$

When n = 1, the elements Y_i $(1 \le i \le 8)$ satisfy the same relations as those for type D_5 , hence the relations (R1) hold.

For any m > 1 there exists a sequence $((k_1, n_1, n'_1), \dots, (k_s, n_s, n'_s))$ of \mathcal{A} satisfying $n_1 = 1, n'_s = m, n'_j = n_{j+1}$ $(1 \le j \le s - 1)$, and hence it is sufficient to show the relations $(\mathbb{R}n')$ for $(k, n, n') \in \mathcal{A}$ assuming the relations $(\mathbb{R}n)$.

Let $(k, n, n') \in \mathcal{A}$. Assume that the relations $(\mathbb{R}n)$ hold.

We first show that the relation $(\mathbb{R}n',1)$ holds. If the condition (\mathbb{P}_1^+) is satisfied, then we have $Y_{i_1^{n'}} = Y_{i_1^n}$, $F_k Y_{i_1^n} = Y_{i_1^n} F_k$, $Y_{j_1^{n'}} = \operatorname{ad}(F_k) Y_{j_1^n} = F_k Y_{j_1^n} - q Y_{j_1^n} F_k$. Since $Y_{i_1^n} Y_{j_1^n} = Y_{j_1^n} Y_{i_1^n}$, we have

$$Y_{i_1^{n'}}Y_{j_1^{n'}} = Y_{i_1^n} \operatorname{ad}(F_k)Y_{j_1^n} = Y_{i_1^n}(F_kY_{j_1^n} - qY_{j_1^n}F_k)$$

= $(F_kY_{j_1^n} - qY_{j_1^n}F_k)Y_{i_1^n} = Y_{j_1^{n'}}Y_{j_1^{n'}}.$

If the condition (P_1^-) is satisfied, then we can prove the formula (Rn',1) similarly.

Next we prove the formula (Rn',2). Assume the condition (P_m^+) is satisfied, then we have

$$\begin{aligned} Y_{i_m^{n'}}Y_{j_m^{n'}} &= Y_{i_m^{n}}(F_kY_{j_m^{n}} - qY_{j_m^{n}}F_k) \\ &= F_kY_{j_m^{n}}Y_{i_m^{n}} - qY_{j_m^{n}}F_kY_{i_m^{n}} \\ &+ q(F_kY_{j_{m-1}}Y_{i_{m-1}} - qY_{j_{m-1}}Y_{i_{m-1}}F_k) \\ &- q^{-1}(F_kY_{i_{m-1}}Y_{j_{m-1}} - qY_{i_{m-1}}Y_{j_{m-1}}F_k). \end{aligned}$$

If the condition (P_{m-1}^+) is satisfied, then we have

$$F_{k}Y_{j_{m-1}^{n}}Y_{j_{m-1}^{n}} - qY_{j_{m-1}^{n}}Y_{j_{m-1}^{n}}F_{k} = Y_{j_{m-1}^{n}}(F_{k}Y_{i_{m-1}^{n}} - qY_{i_{m-1}^{n}}F_{k}) = Y_{j_{m-1}^{n'}}Y_{i_{m-1}^{n'}},$$

$$F_{k}Y_{i_{m-1}^{n}}Y_{j_{m-1}^{n}} - qY_{i_{m-1}^{n}}Y_{j_{m-1}^{n}}F_{k} = (F_{k}Y_{i_{m-1}^{n}} - qY_{i_{m-1}^{n}}F_{k})Y_{j_{m-1}^{n}} = Y_{i_{m-1}^{n'}}Y_{j_{m-1}^{n'}},$$

and if the condition (P_{m-1}^{-}) is satisfied, then we have

$$F_k Y_{j_{m-1}^n} Y_{i_{m-1}^n} - q Y_{j_{m-1}^n} Y_{i_{m-1}^n} F_k = (F_k Y_{j_{m-1}^n} - q Y_{j_{m-1}^n} F_k) Y_{i_{m-1}^n} = Y_{j_{m-1}^n'} Y_{i_{m-1}^n'},$$

$$F_k Y_{i_{m-1}^n} Y_{j_{m-1}^n} - q Y_{i_{m-1}^n} Y_{j_{m-1}^n} F_k = Y_{i_{m-1}^n} (F_k Y_{j_{m-1}^n} - q Y_{j_{m-1}^n} F_k) = Y_{i_{m-1}^n'} Y_{j_{m-1}^n'}.$$

Hence we have $Y_{i_m'}Y_{j_m'} = Y_{j_m'}Y_{i_m'} + qY_{j_{m-1}'}Y_{i_{m-1}'} - q^{-1}Y_{i_{m-1}'}Y_{j_{m-1}'}$. The formula (Rn',2) is proved. When the condition (P_m) is satisfied, we can prove it similarly.

Finally we prove the formula $(\mathbb{R}n',3)$. Let $l'_1, l'_2 \in |\mathbf{A}(n')|$ satisfying $l'_1 < l'_2$ and $(l'_1, l'_2) \neq (i_m^{n'}, j_m^{n'})$ for $1 \leq m \leq 4$. When $l'_p = i_m^{n'} \in |\mathbf{A}(n')|$ (resp. $l'_p = j_m^{n'}$), we denote $i_m^n \in |\mathbf{A}(n)|$ (resp. j_m^n) by l_p for p = 1, 2. Since $l_1 < l_2$ and $(l_1, l_2) \neq (i_m^n, j_m^n)$ for $1 \leq m \leq 4$, we have $Y_{l_1}Y_{l_2} = qY_{l_2}Y_{l_1}$. We have the following possibilities:

- (1) $l'_1 = l_1, l'_2 = l_2, (\beta_{l_1}, \alpha_k) = (\beta_{l_2}, \alpha_k) = 0,$
- (2) $l'_1 = l_1, \ (\beta_{l_1}, \alpha_k) = 0, \ \beta_{l'_2} = \beta_{l_2} + \alpha_k, \ (\beta_{l_2}, \alpha_k) = -1,$
- (3) $\beta_{l'_1} = \beta_{l_1} + \alpha_k, \ (\beta_{l_1}, \alpha_k) = -1, \ l'_2 = l_2, \ (\beta_{l_2}, \alpha_k) = 0,$
- (4) $\beta_{l'_1} = \beta_{l_1} + \alpha_k, \ \beta_{l'_2} = \beta_{l_2} + \alpha_k, \ (\beta_{l_1}, \alpha_k) = (\beta_{l_2}, \alpha_k) = -1.$

In the case (1) the formula (Rn',3) is obvious.

In the case (2) we have $F_k Y_{l_1} = Y_{l_1}F_k$, $Y_{l_2'} = \operatorname{ad}(F_k)Y_{l_2} = F_k Y_{l_2} - qY_{l_2}F_k$. Hence we have

$$Y_{l_1'}Y_{l_2'} = Y_{l_1}(F_kY_{l_2} - qY_{l_2}F_k) = q(F_kY_{l_2} - qY_{l_2}F_k)Y_{l_1} = qY_{l_2'}Y_{l_1'}$$

In the case (3) we can prove it similarly to the case (2).

In the case (4) we have $Y_{l'_p} = F_k Y_{l_p} - q Y_{l_p} F_k$ for p = 1, 2. Since $\beta_{l'_p} + \alpha_k = \beta_{l_p} + 2\alpha_k \notin \Delta^+ \setminus \Delta_I$ and $(\beta_{l'_p}, \alpha_k) = 1$, we have $\operatorname{ad}(F_k)Y_{l'_p} = F_k Y_{l'_p} - q^{-1}Y_{l'_p} F_k = 0$ for p = 1, 2. Hence we have $F_k F_k Y_{l_p} - (q + q^{-1})F_k Y_{l_p} F_k + Y_{l_p} F_k F_k = 0$, $F_k Y_{l_p} F_k = (q + q^{-1})^{-1}(F_k F_k Y_{l_p} + Y_{l_p} F_k F_k)$ for p = 1, 2. By these formulas we have

$$\begin{aligned} Y_{l_1'}Y_{l_2'} &= (F_kY_{l_1} - qY_{l_1}F_k)(F_kY_{l_2} - qY_{l_2}F_k) \\ &= F_kY_{l_1}F_kY_{l_2} - qF_kY_{l_1}Y_{l_2}F_k - qY_{l_1}F_kF_kY_{l_2} + q^2Y_{l_1}F_kY_{l_2}F_k \\ &= \frac{1}{q+q^{-1}}F_kF_kY_{l_1}Y_{l_2} + \frac{1}{q+q^{-1}}Y_{l_1}F_kF_kY_{l_2} - qF_kY_{l_1}Y_{l_2}F_k - qY_{l_1}F_kF_kY_{l_2} \\ &+ \frac{q^2}{q+q^{-1}}Y_{l_1}F_kF_kY_{l_2} + \frac{q^2}{q+q^{-1}}Y_{l_1}Y_{l_2}F_kF_k \\ &= \frac{1}{q+q^{-1}}F_kF_kY_{l_1}Y_{l_2} - qF_kY_{l_1}Y_{l_2}F_k + \frac{q^2}{q+q^{-1}}Y_{l_1}Y_{l_2}F_kF_k. \end{aligned}$$

Similarly we have

$$Y_{l'_2}Y_{l'_1} = \frac{1}{q+q^{-1}}F_kF_kY_{l_2}Y_{l_1} - qF_kY_{l_2}Y_{l_1}F_k + \frac{q^2}{q+q^{-1}}Y_{l_2}Y_{l_1}F_kF_k.$$

Since $Y_{l_1}Y_{l_2} = qY_{l_2}Y_{l_1}$, we have $Y_{l'_1}Y_{l'_2} = qY_{l'_2}Y_{l'_1}$.

By [4] and Proposition 2.3 we obtain the following:

Theorem 2.4. The formulas (Q6) give fundamental relations for the generator system $\{Y_i\}_{i \in \Lambda}$ of the algebra $A_q = U_q(\mathfrak{n}_l^-)$.

We shall construct a quantum deformation of the lowest degree part $J_{C_0}^0$ of the defining ideal J_{C_0} and we shall give canonical generators of a quantum analogue of

 J_{C_0} .

Set

$$\psi_n = Y_{i_4^n} Y_{j_4^n} - q Y_{i_3^n} Y_{j_3^n} + q^2 Y_{i_2^n} Y_{j_2^n} - q^3 Y_{i_1^n} Y_{j_1^n},$$

for $1 \le n \le 10$. Recall that $\mathbf{A}(n) = (i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n)$. Using the formulas (Rn,1), (Rn,2), we can write $\psi_n = Y_{j_4^n} Y_{i_4^n} - q^{-1} Y_{j_3^n} Y_{i_3^n} + q^{-2} Y_{j_2^n} Y_{i_2^n} - q^{-3} Y_{j_1^n} Y_{i_1^n}$.

Lemma 2.5. We have

$$ad(F_k)\psi_n = \begin{cases} \psi_{n'} & \text{if there exists } n' \text{ such that } \delta_n + \alpha_k = \delta_{n'}, \\ 0 & \text{otherwise}, \end{cases}$$
$$ad(E_k)\psi_n = \begin{cases} \psi_{n'} & \text{if there exists } n' \text{ such that } \delta_n - \alpha_k = \delta_{n'}, \\ 0 & \text{otherwise} \end{cases}$$

for $k \in I$, and

$$\mathrm{ad}(K_k)\psi_n=q^{-(\delta_n,\alpha_k)}\psi_n$$

for $k \in I_0$.

Proof. Let $(k, n, n') \in A$. We shall show $\operatorname{ad}(F_k)\psi_n = \psi_{n'}$. If the condition (\mathbf{P}_m^+) is satisfied, then we have $\operatorname{ad}(F_k)Y_{i_m^n} = 0$, $Y_{i_m^{n'}} = Y_{i_m^n}$, $\operatorname{ad}(K_k)Y_{i_m^n} = Y_{i_m^n}$ ad $(F_k)Y_{j_m^n} = Y_{j_m^{n'}}$. Hence

$$\mathrm{ad}(F_k)(Y_{i_m^n}Y_{j_m^n}) = (\mathrm{ad}(F_k)Y_{i_m^n})Y_{j_m^n} + (\mathrm{ad}(K_k)Y_{i_m^n})(\mathrm{ad}(F_k)Y_{j_m^n}) = Y_{i_m^{n'}}Y_{j_m^{n'}}.$$

If the condition (P_m^-) is satisfied, then we have $ad(F_k)Y_{i_m^n} = Y_{i_m^{n'}}$, $ad(F_k)Y_{j_m^n} = 0$. Hence $ad(F_k)(Y_{i_m^n}Y_{j_m^n}) = Y_{i_m^{n'}}Y_{j_m^{n'}}$ similarly. Therefore we have $ad(F_k)\psi_n = \psi_{n'}$.

Next we prove $\operatorname{ad}(E_k)\psi_{n'} = \psi_n$. We have $\operatorname{ad}(E_k)Y_{i_m^{n'}} = 0$, $\operatorname{ad}(E_k)Y_{j_m^{n'}} = Y_{j_m^n}$ if the condition (P_m^+) is satisfied, and we have $\operatorname{ad}(E_k)Y_{i_m^{n'}} = Y_{i_m^n}$, $\operatorname{ad}(K_k^{-1})Y_{j_m^{n'}} = Y_{j_m^{n'}}$, $j_m^{n'} = j_m^n$, $\operatorname{ad}(E_k)Y_{i_m^{n'}} = 0$ if the condition (P_m^-) is satisfied. Hence we have

$$\mathrm{ad}(E_k)(Y_{i_m'}Y_{j_m'}) = (\mathrm{ad}(E_k)Y_{i_m'})(\mathrm{ad}(K_k^{-1})Y_{j_m'}) + Y_{i_m'}(\mathrm{ad}(E_k)Y_{j_m'}) = Y_{i_m'}Y_{j_m}$$

for $1 \le m \le 4$. Therefore we have $ad(E_k)\psi_{n'} = \psi_n$.

In other 50 cases, where $\delta_n + \alpha_k \notin {\delta_l \mid 1 \le l \le 10}$, we can check $ad(F_k)\psi_n = 0$ by a case-by-case consideration as follows.

In the 10 cases where there exists n' satisfying $ad(F_k)\psi_{n'} = \psi_n$, ((k, n) = (6, 2), (5, 3), (3, 4), (2, 5), (4, 6), (2, 7), (4, 7), (3, 8), (5, 9), (6, 10)), we have $ad(F_k)Y_{i_m^n} = ad(F_k)Y_{j_m^n} = 0$ for $1 \le m \le 4$, and hence the assertion is obvious.

In the 8 cases (k, n) = (5, 1), (6, 3), (6, 4), (6, 5), (6, 6), (6, 7), (6, 8), (5, 10), we have $ad(F_k)Y_{i_m^n} = ad(F_k)Y_{j_m^n} = 0$ for m = 3, 4, $ad(F_k)Y_{i_1^n} = Y_{j_1^n}$, $ad(F_k)Y_{j_2^n} = 0$,

 $ad(F_k)Y_{i_1^n} = Y_{j_2^n}$, $ad(F_k)Y_{j_1^n} = 0$, and hence $ad(F_k)(Y_{i_2^n}Y_{j_2^n}) = Y_{j_1^n}Y_{j_2^n}$, $ad(F_k)(Y_{i_1^n}Y_{j_1^n}) = Y_{j_2^n}Y_{j_1^n}$. Thus we have $ad(F_k)\psi_n = q^2(Y_{j_1^n}Y_{j_2^n} - qY_{j_2^n}Y_{j_1^n}) = 0$ by Proposition 2.3.

In the remaining 32 cases there exists $m' \in \{2, 3, 4\}$ such that $ad(F_k)Y_{i_m^n} = 0$ $(m \neq m')$, $ad(F_k)Y_{j_m^n} = 0$ $(m \neq m'-1)$, $ad(F_k)Y_{i_{m'}} = Y_{i_{m'-1}}^n$, $ad(F_k)Y_{j_{m'-1}}^n = Y_{j_{m'}}^n$, $ad(K_k)Y_{i_{m'-1}}^n = q^{-1}Y_{i_{m'-1}}^n$. Then we have $ad(F_k)(Y_{i_{m'}}^n Y_{j_{m'}}^n) = Y_{i_{m'-1}}^n Y_{j_{m'}}^n$, $ad(F_k)(Y_{i_{m'-1}}^n Y_{j_{m'-1}}^n) = q^{-1}Y_{i_{m'-1}}^n Y_{j_{m'}}^n$, $ad(F_k)\psi_n = q^{4-m'}(1-qq^{-1})Y_{i_{m'-1}}^n Y_{j_{m'}}^n = 0$.

The weight $\beta_{i_m^n} + \beta_{j_m^n}$ does not depend on *m*. Hence we have $ad(K_k)\psi_n = q^{-(\delta_n,\alpha_k)}\psi_n$ where $\delta_n = \beta_{i_m^n} + \beta_{j_m^n}$.

Finally we show $\operatorname{ad}(E_k)\psi_n = 0$ if $\delta_n - \alpha_k \notin \{\delta_l \mid 1 \leq l \leq 10\}$. We can check $\operatorname{ad}(E_k)\psi_1 = 0$ for any $k = 2, 3, \ldots, 6$ directly. It follows that $\sum_{n=1}^{10} \mathbb{C}(q)\psi_n = U_q(\mathfrak{l}_l)\psi_1$ and hence $\sum_{n=1}^{10} \mathbb{C}(q)\psi_n$ is an $\operatorname{ad} U_q(\mathfrak{l}_l)$ -stable subspace with weights in $\{-\delta_l \mid 1 \leq l \leq 10\}$. \Box

Proposition 2.6. $\sum_{n=1}^{10} \mathbb{C}(q)\psi_n$ is an irreducible highest weight $U_q(\mathfrak{l}_l)$ -module with highest weight vector ψ_1 .

Proof. By Lemma 2.5 $\sum_{n=1}^{10} \mathbb{C}(q)\psi_n$ is a finite dimensional $U_q(\mathfrak{l}_I)$ -submodule generated by a highest weight vector ψ_1 with highest weight $-\delta_1$. Thus it is irreducible.

By [4] and Proposition 2.6 we obtain the following:

Theorem 2.7. A quantum analogue of the defining ideal J_{C_0} of the closure of the non-trivial non-open orbit C_0 is given by the two-sided ideal of A_q generated by $\{\psi_n \mid 1 \le n \le 10\}$.

3. Case of type E_7

Let \mathfrak{g} be a simple Lie algebra of type E_7 . We shall use the labelling of the vertices of the Dynkin diagram 1. Hence we have $I_0 = \{1, 2, 3, 4, 5, 6, 7\}$. Set $I = \{2, 3, 4, 5, 6, 7\}$. In this case we have $\mathfrak{n}_I^+ \neq \{0\}$, $[\mathfrak{n}_I^+, \mathfrak{n}_I^+] = \{0\}$. Then \mathfrak{l}_I is isomorphic to $\mathbb{C} \oplus \mathfrak{g}_{E_6}$, where \mathfrak{g}_{E_6} is a Lie algebra of type E_6 over \mathbb{C} , and \mathfrak{n}_I^+ is a 27-dimensional irreducible prehomogeneous vector space. There are four L_I -orbits $\{0\}, C_1, C_2, O$ on \mathfrak{n}_I^+ satisfying $\{0\} \subset \overline{C_1} \subset \overline{C_2} \subset \overline{O}$. Let $J_{C_1} \subset \mathbb{C}[\mathfrak{n}_I^+]$ be the defining ideal of the closure of C_1 , and let $J_{C_1}^0$ denote the subspace of J_{C_1} consisting of the polynomials in J_{C_1} with homogeneous degree 2. Then $J_{C_2}^0 \subset \mathbb{C}[\mathfrak{n}_I^+]$ be the defining ideal of the closure of C_2 , and let $J_{C_2}^0$ denote the subspace of J_{C_2} consisting of the polynomials in J_{C_2} with homogeneous degree 3. Then $J_{C_2}^0$ is a one-dimensional irreducible \mathfrak{l}_I -module generated by the irreducible relative invariant, and it generates the ideal J_{C_2} .

We fix a reduced expression

 $w_I w_0 = s_1 s_2 s_3 s_4 s_5 s_6 s_4 s_3 s_2 s_1 s_7 s_6 s_4 s_3 s_5 s_4 s_6 s_7 s_2 s_3 s_4 s_6 s_5 s_4 s_3 s_2 s_1$

of $w_1 w_0$ and define the elements Y_i $(i \in \Lambda = \{1, 2, ..., 27\})$ as in Section 1.

Set $I'_0 = \{1, 2, 3, 4, 5, 6\}$, $I' = \{2, 3, 4, 5, 6\}$, $\Lambda' = \{1, 2, ..., 10\}$. Then $\{\alpha_i\}_{i \in I'_0}$ is a set of simple roots of type D_6 . Let \mathfrak{g}' be the simple subalgebra of \mathfrak{g} corresponding to I'_0 . We choose a reduced expression $w_{I'}w_{I'_0} = s_1s_2s_3s_4s_5s_6s_4s_3s_2s_1$ of $w_{I'}w_{I'_0}$. The elements Y_i $(i \in \Lambda')$ can be computed inside $U_q(\mathfrak{g}')$.

Let $\beta_j = \sum_{i \in I_0} m_i^j \alpha_i$ and set $\mathbf{m}^j = (m_1^j, \dots, m_7^j)$ for $j \in \Lambda$. Then we have $\mathbf{m}^1 = (1, 0, 0, 0, 0, 0, 0), \quad \mathbf{m}^2 = (1, 1, 0, 0, 0, 0, 0), \quad \mathbf{m}^3 = (1, 1, 1, 0, 0, 0, 0, 0), \quad \mathbf{m}^4 = (1, 1, 1, 1, 0, 0, 0), \quad \mathbf{m}^5 = (1, 1, 1, 1, 1, 0, 0), \quad \mathbf{m}^6 = (1, 1, 1, 1, 0, 0, 0), \quad \mathbf{m}^7 = (1, 1, 1, 1, 1, 1, 0), \quad \mathbf{m}^8 = (1, 1, 1, 2, 1, 1, 0), \quad \mathbf{m}^9 = (1, 1, 2, 2, 1, 1, 0), \quad \mathbf{m}^{10} = (1, 2, 2, 2, 1, 1, 0), \quad \mathbf{m}^{11} = (1, 1, 1, 2, 2, 1, 1, 1), \quad \mathbf{m}^{12} = (1, 1, 1, 1, 1, 1, 1, 1), \quad \mathbf{m}^{13} = (1, 1, 1, 2, 1, 1, 1), \quad \mathbf{m}^{14} = (1, 1, 2, 2, 1, 1, 1), \quad \mathbf{m}^{15} = (1, 1, 1, 2, 1, 2, 1), \quad \mathbf{m}^{16} = (1, 1, 2, 2, 1, 2, 1), \quad \mathbf{m}^{17} = (1, 1, 2, 3, 1, 2, 1), \quad \mathbf{m}^{18} = (1, 1, 2, 3, 2, 2, 2), \quad \mathbf{m}^{18} = (1, 1, 2, 3, 2, 2, 2), \quad \mathbf{m}^{18} = (1, 1, 2, 3, 2, 2, 2), \quad \mathbf{m}^{18} = (1, 1, 2, 3, 2, 2, 2), \quad \mathbf{m}^{18} = (1, 1, 2, 3, 2, 2), \quad \mathbf{m}^{18} = (1, 1, 2, 3, 2, 2), \quad \mathbf{m}^{18} = (1, 1, 2, 3, 2, 2), \quad \mathbf{m}^{18} = (1, 1, 2, 3, 2), \quad \mathbf{m}^{18} = (1, 1$

 $\mathbf{m}^{19} = (1, 2, 2, 2, 1, 1, 1), \quad \mathbf{m}^{20} = (1, 2, 2, 2, 1, 2, 1), \quad \mathbf{m}^{21} = (1, 2, 2, 3, 1, 2, 1), \\ \mathbf{m}^{22} = (1, 2, 2, 3, 2, 2, 1), \quad \mathbf{m}^{23} = (1, 2, 3, 3, 1, 2, 1), \quad \mathbf{m}^{24} = (1, 2, 3, 3, 2, 2, 1), \\ \mathbf{m}^{25} = (1, 2, 3, 4, 2, 2, 1), \quad \mathbf{m}^{26} = (1, 2, 3, 4, 2, 3, 1), \quad \mathbf{m}^{27} = (1, 2, 3, 4, 2, 3, 2).$

If $(\beta_j, \alpha_k) = -1$ for $j \in \Lambda$ and $k \in I$, then $s_k(\beta_j) = \beta_j + \alpha_k \in \Delta^+ \setminus \Delta_I$ and there exists $l \in \Lambda$ satisfying $\beta_j + \alpha_k = \beta_l$. Conversely if $\beta_j, \beta_l \in \Delta^+ \setminus \Delta_I$ satisfying $\beta_l - \beta_j = \alpha_k$ ($k \in I$), then we have $(\beta_j, \alpha_k) = -1$, $s_k(\beta_j) = \beta_l$.

For $k \in I$, $j \in \Lambda$, we have $\beta_j - 2\alpha_k$, $\beta_j + 2\alpha_k \notin \Delta^+ \setminus \Delta_I$. Set

$$\mathcal{B} = \{(k, j, l) \in I \times \Lambda \times \Lambda \mid \beta_j + \alpha_k = \beta_l\}.$$

We have

 $\mathcal{B} = \{(2, 1, 2), (3, 2, 3), (4, 3, 4), (5, 4, 5), (6, 4, 6), (6, 5, 7), (5, 6, 7), (4, 7, 8), (3, 8, 9), \\ (2, 9, 10), (7, 6, 11), (7, 7, 12), (7, 8, 13), (7, 9, 14), (7, 10, 19), (5, 11, 12), \\ (4, 12, 13), (3, 13, 14), (6, 13, 15), (6, 14, 16), (3, 15, 16), (4, 16, 17), (5, 17, 18), \\ (2, 14, 19), (2, 16, 20), (2, 17, 21), (2, 18, 22), (6, 19, 20), (4, 20, 21), (5, 21, 22), \\ (3, 21, 23), (3, 22, 24), (5, 23, 24), (4, 24, 25), (6, 25, 26), (7, 26, 27)\}.$ In particular, we have $|\mathcal{B}| = 36.$

Lemma 3.1. Let β , $\beta' \in \Delta^+ \setminus \Delta_I$ satisfying $\beta + \alpha_k = \beta'$ ($k \in I$). Then we can choose a reduced expression $w_I w_0 = s_{i_1} s_{i_2} \cdots s_{i_{22}}$ and $p \in \Lambda$ satisfying

$$\beta = s_{i_1} s_{i_2} \cdots s_{i_{p-1}} (\alpha_{i_p}), \quad \beta' = s_{i_1} s_{i_2} \cdots s_{i_{p-1}} s_{i_p} (\alpha_{i_{p+1}}), \quad (\alpha_{i_p}, \alpha_{i_{p+1}}) = -1,$$

$$\alpha_k = s_{i_1} s_{i_2} \cdots s_{i_{p-1}} (\alpha_{i_{p+1}}).$$

Proof. The 21 triplets (k, j, l) in \mathcal{B} satisfy l = j + 1, $(\alpha_{i_j}, \alpha_{i_{j+1}}) = -1$. Therefore it is sufficient to deal with the remaining 15 cases. In the cases (k, j, l) = (6, 4, 6), (6, 5, 7), (6, 13, 15), (6, 14, 16), (3, 21, 23), (3, 22, 24), we can take

$$w_I w_0 = s_1 s_2 s_3 s_4 s_6 s_5 s_4 s_3 s_2 s_1 s_7 s_6 s_4 s_5 s_3 s_4 s_6 s_7 s_2 s_3 s_4 s_5 s_6 s_4 s_3 s_2 s_1$$

with p = 4, 6, 13, 15, 21, 23, and in the cases (k, j, l) = (7, 6, 11), (7, 7, 12), (7, 8, 13), (7, 9, 14), (7, 10, 19), we can take

$$w_1w_0 = s_1s_2s_3s_4s_5s_6s_7s_4s_6s_3s_4s_2s_3s_1s_2s_5s_4s_6s_7s_3s_4s_6s_5s_4s_3s_2s_1$$

with p = 6, 8, 10, 12, 14, and in the cases (k, j, l) = (2, 14, 19), (2, 16, 20), (2, 17, 21), (2, 18, 22), we can take

$$w_1w_0 = s_1s_2s_3s_4s_5s_6s_4s_3s_2s_1s_7s_6s_4s_5s_3s_2s_4s_3s_6s_4s_7s_6s_5s_4s_3s_2s_1$$

with p = 15, 17, 19, 21.

We can show the following similarly to the case E_6 . We omit the details.

Lemma 3.2. For $k \in I$, $j \in \Lambda$, we have

$$ad(F_k)Y_j = \begin{cases} Y_l \text{ if there exists } (k, j, l) \in \mathcal{B}, \\ 0 \text{ otherwise,} \end{cases}$$
$$ad(E_k)Y_j = \begin{cases} Y_l \text{ if there exists } (k, l, j) \in \mathcal{B}, \\ 0 \text{ otherwise.} \end{cases}$$

The $U_q(l_1)$ -module $\bigoplus_{j \in \Lambda} \mathbb{C}(q)Y_j$ is an irreducible highest weight module with highest weight vector Y_1 and lowest weight vector Y_{27} . Hence, for any $1 \le m \le 26$, there exists a sequence $((k_1, n'_1, n_1), \ldots, (k_s, n'_s, n_s))$ of \mathcal{B} satisfying $n_1 = 27$, $n'_s = m$, $n'_j = n_{j+1}$ $(1 \le j \le s - 1)$.

Next we shall consider relations among the elements Y_i . We can write

$$Y_i Y_j = \sum_{\substack{s \le t \\ \beta_i + \beta_j = \beta_s + \beta_t}} a_{s,t}^{i,j} Y_s Y_t \quad (a_{s,t}^{i,j} \in \mathbb{C}(q))$$

for i > j (see [4]). Hence if $\beta_i + \beta_j$ does not have another decomposition $\beta + \beta'$ ($\beta, \beta' \in \Delta^+ \setminus \Delta_I, \beta_i + \beta_j = \beta + \beta'$) then we have $Y_i Y_j = a_{i,j} Y_j Y_i$ for some $a_{i,j} \in \mathbb{C}(q)$. Set $\delta = 2\varpi_1 = 3\alpha_1 + 4\alpha_2 + 5\alpha_3 + 6\alpha_4 + 3\alpha_5 + 4\alpha_6 + 2\alpha_7$, where ϖ_1 is the fundamental weight corresponding to α_1 . We denote a set of weights of the 27-dimensional irreducible highest weight l_I -module $J_{C_1}^0$ with highest weight $-\beta_1 - \beta_{10}$ by Γ . Set $\gamma_n = \delta - \beta_n$ ($n \in \Lambda$), and we have $\Gamma = \{-\gamma_n \mid n \in \Lambda\}$. For $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ a weight $\beta + \beta'$ has another decomposition if and only if we have $-(\beta + \beta') \in \Gamma$. For each $n \in \Lambda$ there

exist exactly five pairs $(i, j) \in \Lambda^2$ such that i < j, $\beta_i + \beta_j = \gamma_n$. We denote them by (i_1^n, j_1^n) , (i_2^n, j_2^n) , (i_3^n, j_3^n) , (i_4^n, j_4^n) , $(i_5^n, j_5^n) \in \Lambda^2$ where $i_5^n < i_4^n < i_3^n < i_2^n < i_1^n$, $j_1^n < j_2^n < j_3^n < j_4^n < j_5^n$, and i_1^n , j_1^n satisfy the following condition (\mathbf{P}_1^+) or (\mathbf{P}_1^-) . Set $\mathbf{B}(n) = (i_5^n, i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n, j_5^n) \in \Lambda^{10} \ (n \in \Lambda).$ Then we have $\mathbf{B}(1) = (10, 19, 20, 21, 23, 22, 24, 25, 26, 27), \mathbf{B}(2) = (9, 14, 16, 17, 23, 18, 24, 25, 26, 27),$ $\mathbf{B}(3) = (8, 13, 15, 17, 21, 18, 22, 25, 26, 27), \quad \mathbf{B}(4) = (7, 12, 15, 16, 20, 18, 22, 24, 26, 27),$ $\mathbf{B}(5) = (6, 11, 15, 16, 20, 17, 21, 23, 26, 27), \quad \mathbf{B}(6) = (5, 12, 13, 14, 19, 18, 22, 24, 25, 27),$ $\mathbf{B}(7) = (4, 11, 13, 14, 19, 17, 21, 23, 25, 27), \mathbf{B}(8) = (3, 11, 12, 14, 19, 16, 20, 23, 24, 27),$ $\mathbf{B}(9) = (2, 11, 12, 13, 19, 15, 20, 21, 22, 27), \quad \mathbf{B}(10) = (1, 11, 12, 13, 14, 15, 16, 17, 18, 27),$ $\mathbf{B}(11) = (5, 7, 8, 9, 10, 18, 22, 24, 25, 26),$ $\mathbf{B}(12) = (4, 6, 8, 9, 10, 17, 21, 23, 25, 26),$ $\mathbf{B}(13) = (3, 6, 7, 9, 10, 16, 20, 23, 24, 26),$ $\mathbf{B}(14) = (2, 6, 7, 8, 10, 15, 20, 21, 22, 26),$ $\mathbf{B}(15) = (3, 4, 5, 9, 10, 14, 19, 23, 24, 25),$ $\mathbf{B}(16) = (2, 4, 5, 8, 10, 13, 19, 21, 22, 25),$ $\mathbf{B}(17) = (2, 3, 5, 7, 10, 12, 19, 20, 22, 24),$ $\mathbf{B}(18) = (2, 3, 4, 6, 10, 11, 19, 20, 21, 23),$ $\mathbf{B}(19) = (1, 6, 7, 8, 9, 15, 16, 17, 18, 26),$ $\mathbf{B}(20) = (1, 4, 5, 8, 9, 13, 14, 17, 18, 25),$ $\mathbf{B}(21) = (1, 3, 5, 7, 9, 12, 14, 16, 18, 24),$ $\mathbf{B}(22) = (1, 3, 4, 6, 9, 11, 14, 16, 17, 23),$ $\mathbf{B}(23) = (1, 2, 5, 7, 8, 12, 13, 15, 18, 22),$ $\mathbf{B}(24) = (1, 2, 4, 6, 8, 11, 13, 15, 17, 21),$ $\mathbf{B}(25) = (1, 2, 3, 6, 7, 11, 12, 15, 16, 20),$ $\mathbf{B}(26) = (1, 2, 3, 4, 5, 11, 12, 13, 14, 19),$ $\mathbf{B}(27) = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10).$

For $n \in \Lambda$ we denote the set $\{i_5^n, i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n, j_5^n\}$ by $|\mathbf{B}(n)|$. For any $i, j \in \Lambda$ there exists $n \in \Lambda$ satisfying $i, j \in |\mathbf{B}(n)|$.

For $(k, n', n) \in \mathcal{B}$ and $m \in \{1, 2, 3, 4, 5\}$, we have either (\mathbf{P}_m^+) $(\beta_{i_m^n}, \alpha_k) = 0, \ i_m^{n'} = i_m^n, (\beta_{j_m^n}, \alpha_k) = -1, \ \beta_{j_m^{n'}} = \beta_{j_m^n} + \alpha_k$ or (\mathbf{P}_m^-) $(\beta_{i_m^n}, \alpha_k) = -1, \ \beta_{i_m^{n'}} = \beta_{i_m^n} + \alpha_k, \ (\beta_{j_m^n}, \alpha_k) = 0, \ j_m^{n'} = j_m^n.$

Proposition 3.3. For any $i, j \in \Lambda$ satisfying i < j, we have

$$(Q7) Y_{i}Y_{j} = \begin{cases} Y_{j}Y_{i} & \text{if there exists } n \in \Lambda \text{ such that } \{i, j\} = \{i_{1}^{n}, j_{1}^{n}\}, \\ Y_{j_{2}^{n}}Y_{i_{2}^{n}} + (q - q^{-1})Y_{i_{1}^{n}}Y_{j_{1}^{n}} \\ & \text{if there exists } n \in \Lambda \text{ such that } i = i_{2}^{n}, j = j_{2}^{n}, \\ Y_{j_{m}^{n}}Y_{i_{m}^{n}} + qY_{j_{m-1}^{n}}Y_{i_{m-1}^{n}} - q^{-1}Y_{i_{m-1}^{n}}Y_{j_{m-1}^{n}} \\ & \text{if there exist } n \in \Lambda, m \in \{3, 4, 5\} \text{ such that } i = i_{m}^{n}, j = j_{m}^{n}, \\ qY_{j}Y_{i} & \text{otherwise.} \end{cases}$$

Proof. Since there exists $n \in \Lambda$ satisfying $i, j \in |\mathbf{B}(n)|$ for any $i, j \in \Lambda$, it is

sufficient to show

$$Y_{i_1^n} Y_{j_1^n} = Y_{j_1^n} Y_{i_1^n}$$
 (R*n*, 1)

$$(\mathbf{R}n) \begin{cases} Y_{i_m^n} Y_{j_m^n} = Y_{j_m^n} Y_{i_m^n} + q Y_{j_{m-1}^n} Y_{i_{m-1}^n} - q^{-1} Y_{i_{m-1}^n} Y_{j_{m-1}^n} & (2 \le m \le 5) \\ Y_{l_1} Y_{l_2} = q Y_{l_2} Y_{l_1} \end{cases}$$
(Rn, 2)

$$(l_1, l_2 \in |\mathbf{B}(n)|, l_1 < l_2, \{l_1, l_2\} \neq \{i_m^n, j_m^n\} \ (1 \le m \le 5)) \quad (\mathbf{R}n, 3)$$

for $n \in \Lambda$ and $1 \leq m \leq 5$.

When n = 27, the elements Y_i $(1 \le i \le 10)$ satisfy the same relations as those for type D_6 , and hence relations (R27) hold.

Since there exists a sequence $((k_1, n'_1, n_1), \dots, (k_s, n'_s, n_s))$ of \mathcal{B} satisfying $n_1 = 27$, $n'_s = m$, $n'_j = n_{j+1}$ $(1 \le j \le s-1)$ for any $1 \le m \le 26$, it is sufficient to show (Rn') for $(k, n', n) \in \mathcal{B}$ assuming (Rn). This is proved similarly to Proposition 2.3. Details are omitted.

By [4] and Proposition 3.3 we obtain the following:

Theorem 3.4. The formulas (Q7) give fundamental relations for the generator system $\{Y_i\}_{i \in \Lambda}$ of the algebra $A_q = U_q(\mathfrak{n}_l)$.

We shall construct a quantum deformation of the lowest degree part $J_{C_1}^0$ of the defining ideal J_{C_1} and we shall give canonical generators of a quantum deformation of J_{C_1} .

Set

$$\psi_n = Y_{i_5^n} Y_{j_5^n} - q Y_{i_4^n} Y_{j_4^n} + q^2 Y_{i_3^n} Y_{j_3^n} - q^3 Y_{i_2^n} Y_{j_2^n} + q^4 Y_{i_1^n} Y_{j_1^n}$$

for $n \in \Lambda$, where $\mathbf{B}(n) = (i_5^n, i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n, j_5^n)$. Using the formulas (Rn,1), (Rn,2), we can write

$$\psi_n = Y_{j_5^n} Y_{i_5^n} - q^{-1} Y_{j_4^n} Y_{i_4^n} + q^{-2} Y_{j_3^n} Y_{i_3^n} - q^{-3} Y_{j_2^n} Y_{i_2^n} + q^{-4} Y_{j_1^n} Y_{i_1^n}.$$

Similarly to Lemma 2.5 and Proposition 2.6 we can show the following:

Lemma 3.5. We have

$$ad(F_k)\psi_n = \begin{cases} \psi_{n'} \text{ if there exists } (k, n', n) \in \mathcal{B}, \\ 0 \text{ otherwise,} \end{cases}$$
$$ad(E_k)\psi_n = \begin{cases} \psi_{n'} \text{ if there exists } (k, n, n') \in \mathcal{B}, \\ 0 \text{ otherwise} \end{cases}$$

for $k \in I$, and

$$\mathrm{ad}(K_k)\psi_n=q^{-(\gamma_n,\alpha_k)}\psi_n$$

for $k \in I_0$.

Proposition 3.6. $\sum_{n \in \Lambda} \mathbb{C}(q)\psi_n$ is an irreducible highest weight $U_q(l_1)$ -module with highest weight vector ψ_{27} .

By [4] and Proposition 3.6 we obtain the following:

Theorem 3.7. A quantum deformation of the defining ideal J_{C_1} of the closure of the non-open orbit C_1 is given by the two-sided ideal of A_q generated by $\{\psi_n \mid n \in \Lambda\}$.

Set

$$\varphi = \sum_{n \in \Lambda} (-q)^{|\beta_n| - 1} Y_n \psi_n,$$

where $|\beta| = \sum_{i \in I_0} m_i \ (\beta = \sum_{i \in I_0} m_i \alpha_i).$

Proposition 3.8. $\mathbb{C}(q)\varphi$ is a one-dimensional $U_q(l_1)$ -module.

Proof. By Proposition 3.3 we can check that the coefficient $a_{1,10,27}$ of $Y_1Y_{10}Y_{27}$ in $\varphi = \sum_{i < j < k} a_{ijk}Y_iY_jY_k$ is $1 + q^8 + q^{16}$. Therefore we have $\varphi \neq 0$.

Let $(k, n, n') \in \mathcal{B}$. Then we have $|\beta_{n'}| = |\beta_n| + 1$, $\operatorname{ad}(F_k)Y_n = Y_{n'}$, $\operatorname{ad}(F_k)Y_{n'} = 0$, $\operatorname{ad}(F_k)\psi_{n'} = \psi_n$, $\operatorname{ad}(F_k)\psi_n = 0$, $(\beta_{n'}, \alpha_k) = 1$. Hence $\operatorname{ad}(F_k)(Y_n\psi_n - qY_{n'}\psi_{n'}) = Y_{n'}\psi_n - qq^{-1}Y_{n'}\psi_n = 0$. Therefore we have $\operatorname{ad}(F_k)\varphi = 0$ for any $k \in I$, and similarly we have $\operatorname{ad}(E_k)\varphi = 0$ for any $k \in I$. Since $\gamma_n + \beta_n = \delta$ for any $n \in \Lambda$, we have $\operatorname{ad}(K_k)\varphi = q^{-(\delta,\alpha_k)}\varphi$ for any $k \in I_0$. In particular, we have $\operatorname{ad}(K_k)\varphi = \varphi$ for any $k \in I$, and $\operatorname{ad}(K_1)\varphi = q^{-2}\varphi$.

The element φ is a quantum deformation of the irreducible relative invariant on the prehomogeneous vector space.

Theorem 3.9. A quantum deformation of the defining ideal J_{C_2} of the closure of the non-open orbit C_2 is given by the two-sided ideal of A_q generated by φ .

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