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NOTE ON ALMOST M -INJECTIVES

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Recently, in [2], Harada and Tozaki defined 'almost M -projectives' which are generalized from the concept ' M -projectives' due to Azumaya. In this paper we shall define a dual concept 'almost M -injectives'. In the forthcoming paper [1], we will show several results dual to Harada and Tozaki's ones above. The purpose of this paper is to generalize the following Azumaya's theorem concerning to M -injectives: N is M_1 - and M_2 -injective if and only if N is $M_1 \oplus M_2$ -injective for modules N , M_1 and M_2 , to a case of 'almost M -injectives'. An easy example shows that the theorem can not be modified as the same form.

Throughout this paper, R is an associative ring with identity. Every module is a unitary right R -module. We always use i , i_k and i^k ($k=1, 2, \dots$ or $*$) to denote the inclusion maps. For modules M and N with $N \subseteq M$, we denote by $N \subset M$ and by $N \subset \oplus M$ to mean that M is an essential extension of N and that N is a direct summand of M , respectively. For modules M , N and a homomorphism $f: M \rightarrow N$, $M(f)$ denotes $\{m + f(m) \mid m \in M\}$. For a module M , $\text{unif. dim}(M)$ and $\|M\|$ denote its uniform dimension and composition length, respectively. If for each simple submodule S of M there is a direct summand M' of M such that $S \subset M'$, we say that M is extending for simple modules. For a set T , $|T|$ denotes its cardinal number.

Our main result is the following.

Theorem. *Let U_k be a uniform module of finite composition length for $k=0, 1, 2, \dots, n$. Then the following two conditions are equivalent;*

(1) U_0 is almost $\sum_{k=1}^n \oplus U_k$ -injective.

(2) U_0 is almost U_k -injective for every $k=1, 2, \dots, n$ and if $\text{Soc}(U_0) \approx \text{Soc}(U_k) \approx \text{Soc}(U_l)$ (any $k, l \in \{1, 2, \dots, n\}, k \neq l$) then (i) U_0 is U_k - and U_l -injective or (ii) $U_k \oplus U_l$ is extending for simple modules.

DEFINITION. Let M and N be R -modules. We say that N is almost M -injective if at least one of the following conditions holds for each submodule L of M and each homomorphism $f: L \rightarrow N$:

(1) There exists a homomorphism $\tilde{f}: M \rightarrow N$ such that $\tilde{f} \cdot i = f$,

(2) There exists a non-zero direct summand M_1 of M and a homomorphism $\tilde{f}: N \rightarrow M_1$ such that $\tilde{f} \cdot f = \pi \cdot i$, where $\pi: M \rightarrow M_1$ is a projection of M onto M_1 .

In this definition, for a given diagram:

$$(*) \quad \begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{i} & M \\ & & f \downarrow & & \\ & & N & & \end{array}$$

we call that *the first (respectively, second) case occurs in the diagram(*)* if the condition (1) (respectively, (2)) holds in the diagram.

Lemma A. *Let U be a uniform module and X an indecomposable module. If U is almost X -injective and $\|U\| \geq \|X\|$, U is X -injective.*

Proof. Consider a diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{i} & X \\ & & f \downarrow & & \\ & & U & & \end{array}$$

Assume that the second case occurs in this diagram. Let $\tilde{f}: U \rightarrow X$ be a homomorphism such that $i = \tilde{f} \cdot f$. (Note that X is indecomposable.) Then \tilde{f} is a monomorphism since U is a uniform module, and so $\|U\| \leq \|X\|$. We have $\|U\| = \|X\|$ from the assumption $\|U\| \geq \|X\|$. Therefore \tilde{f} is an isomorphism. Then $f = \tilde{f}^{-1} \cdot i$.

Lemma B. *Let M and N be R -modules. Consider a diagram:*

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{i} & M \\ & & f \downarrow & & \\ & & N & & \end{array}$$

and put $K := \text{Ker}(f)$. Then if the second case occurs in this diagram, there is a proper direct summand M' of M which contains K .

In particular, if $K \subset M$, then the first case occurs.

Proof. Since the second case happens, we have a direct decomposition $M = M' \oplus M''$ and $\tilde{f}: N \rightarrow M'$ for which the diagram:

$$\begin{array}{ccccc}
 0 & \longrightarrow & L & \xrightarrow{i} & M=M' \oplus M'' \\
 & & \downarrow f & & \downarrow \pi' : \text{projection} \\
 & & N & \xrightarrow{\tilde{f}} & M'
 \end{array}$$

is commutative. Then $\pi'(K) = \pi' \cdot i(K) = \tilde{f} \cdot f(K) = 0$, and $K \subseteq \text{Ker}(\pi') = M'' \subset \oplus M$. Since $M' \neq 0$, M'' is a proper direct summand.

Now we prepare for Lemma C below. Let N, M_1 and M_2 be modules, and put $M := M_1 \oplus M_2$. Consider a diagram:

$$(1) \quad \begin{array}{ccccc}
 0 & \longrightarrow & L & \xrightarrow{i} & M \\
 & & \downarrow f & & \\
 & & N & &
 \end{array}$$

From this diagram we induce the following for $k=1, 2$:

$$(2-k) \quad \begin{array}{ccccc}
 0 & \longrightarrow & L_k & \xrightarrow{i_k} & M_k \\
 & & \downarrow f|_{L_k} & & \\
 & & N & &
 \end{array}$$

where $L_k := L \cap M_k$. Moreover when the first case occurs in both diagrams (2-1) and (2-2) (let $\tilde{f}_k: M_k \rightarrow N$ be homomorphisms such that $f|_{L_k} = \tilde{f}_k \cdot i_k$ for $k=1, 2$), we shall consider the following for $k=1, 2$:

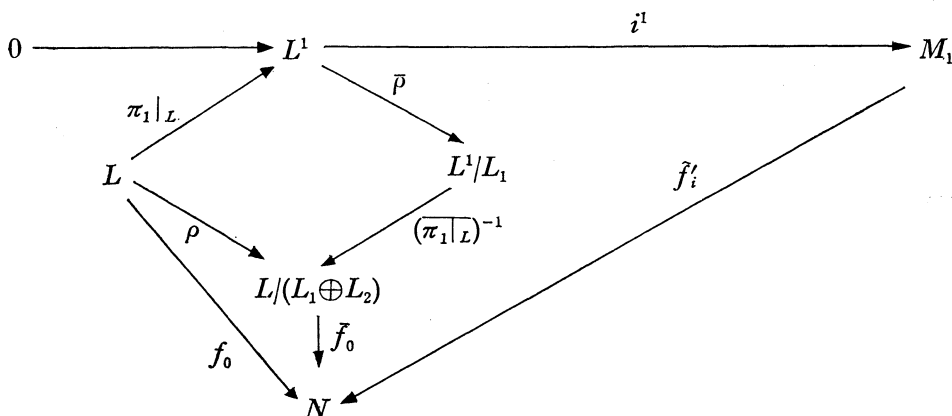
$$(3-k) \quad \begin{array}{ccccc}
 0 & \longrightarrow & L^k & \xrightarrow{i^k} & M_k \\
 & & \downarrow f'_k & & \\
 & & N & &
 \end{array}$$

where, letting $\pi_k: M (= M_1 \oplus M_2) \rightarrow M_k$ be the projection, $L^k := \pi_k(L)$ and the homomorphisms f'_k is defined as follows: Put $f_0 := f - (\sum_{k=1}^2 \tilde{f}_k \cdot \pi_k|_L): L \rightarrow N$. Since $f_0(L_1 \oplus L_2) = 0$ (from the definition of \tilde{f}_k), the canonical map $\tilde{f}_0: L/(L_1 \oplus L_2) \rightarrow N$ is induced. We let $f'_k: L^k \rightarrow N$ be the composite map: $L^k \xrightarrow{\text{natural epi.}} L^k/L_k \xrightarrow{\text{natural iso.}} L/(L_1 \oplus L_2) \xrightarrow{\tilde{f}_0} N$.

Lemma C. Assume that N be almost M_1 - and M_2 -injective. Consider a

diagram (1) and induce the above diagrams. If the first case occurs in both diagrams (2-1) and (2-2) and does in either (3-1) or (3-2), then so does in the diagram (1).

Proof. We say that the first case occurs in the diagram (3-1). Let $f'_1: M_1 \rightarrow N$ be a homomorphism such that $f'_1 = \tilde{f}'_1 \cdot i^1$. The diagram (3-1) induces the following commutative diagram:



where ρ is the canonical epimorphism.

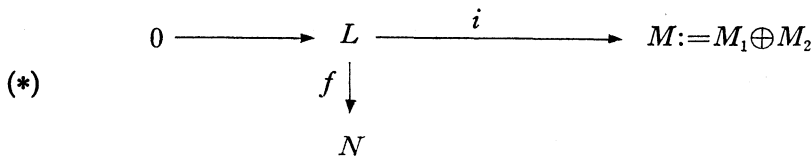
Then, note that $\pi_k|_L = \pi_k \cdot i$,

$$\begin{aligned}
 f &= f_0 + \left(\sum_{k=1}^2 \tilde{f}_k \cdot \pi_k|_L \right) \\
 &= \tilde{f}'_1 \cdot i^1 \cdot (\pi_1|_L) + \left(\sum_{k=1}^2 \tilde{f}_k \cdot \pi_k|_L \right) \\
 &= (\tilde{f}'_1 \cdot i^1 + \tilde{f}_1) \cdot (\pi_1|_L) + \tilde{f}_2 \cdot (\pi_2|_L) \\
 &= \{(\tilde{f}'_1 \cdot i^1 + \tilde{f}_1) \cdot \pi_1 + \tilde{f}_2 \cdot \pi_2\} \cdot i.
 \end{aligned}$$

Put $\tilde{f} := (\tilde{f}'_1 + \tilde{f}_1) \cdot \pi_1 + \tilde{f}_2 \cdot \pi_2$. \tilde{f} is a homomorphism from M to N satisfying $f = \tilde{f} \cdot i$. So the first case occurs in the diagram (1).

Corollary 1. [Azumaya] Let N, M_1 and M_2 be modules. If N is M_1 - and M_2 -injective, then N is $M_1 \oplus M_2$ -injective.

Corollary 2. Let N, M_1 and M_2 be modules, and let N be almost M_1 - and M_2 -injective. Consider a diagram:



and put $K := \text{Ker}(f)$. Then if $K \subset M$, the first case occurs in the diagram (*).

Proof. Induce the diagrams (2- k) ($k=1, 2$) from the diagram (*). Since $K \subset M$ and $K \subseteq L \subseteq M, L \subset M$. And $\text{Ker}(f|_{L_k}) = K \cap L_k \subset M \cap L_k = L_k = M_k \cap L \subset M_k \cap M = M_k$. Therefore Lemma B shows that the first case occurs in the diagrams (2- k). So the diagrams (3- k) for $k=1, 1$ are induced. Since $L_k \subseteq \text{Ker}(f|_k)$ and $L_k \subset M_k, \text{Ker}(f|_k) \subset M_k$. Therefore the first case also occurs in these diagrams. Thus a desired homomorphism exists in the diagram (*) from Lemma C.

Proof of Theorem. (1) \Rightarrow (2): The first condition of (1) holds by [1], Lemma 9. We shall show the remainder condition. To show this, assume that $\text{Soc}(U_0) \approx \text{Soc}(U_1) \approx \text{Soc}(U_2)$ and let U_0 be not U_1 -injective. Let us find a direct decomposition $U_1 \oplus U_2 = V_1 \oplus V_2$ such that $(\text{Soc}(U_1))(g) \subseteq V_1$ and $V_2 \neq 0$ for each isomorphism $g: \text{Soc}(U_1) \rightarrow \text{Soc}(U_2)$. Since V_1 is a uniform module, this means that $U_1 \oplus U_2$ is extending for simple modules.

Take an isomorphism $g': \text{Soc}(U_2) \rightarrow \text{Soc}(U_0)$ and consider the following diagram:

$$\begin{array}{ccccc}
 0 & \longrightarrow & \text{Soc}(U_1) \oplus \text{Soc}(U_2) & \xrightarrow{i} & U_1 \oplus U_2 \\
 (\otimes) & & \downarrow f & & \\
 & & U_0 & &
 \end{array}$$

where $f(s_1 + s_2) = g'(s_2 - g(s_1))$ for any s_k in $\text{Soc}(U_k)$ ($k=1, 2$). Then note that $\text{Ker}(f) = (\text{Soc}(U_1))(g)$.

The assumption that U_0 is almost $\sum_{k=1}^n U_k$ -injective induces that U_0 is almost $U_1 \oplus U_2$ -injective by [1], Lemma 9. If the first case occurs in this diagram, let $\tilde{f}: U_1 \oplus U_2 \rightarrow U_0$ be a homomorphism such that $f = \tilde{f} \cdot i$, then $\tilde{f}|_{U_1}: U_1 \rightarrow U_0$ is not a monomorphism since $\|U_1\| > \|U_0\|$ by Lemma A and the assumption that U_0 is not U_1 -injective. Therefore $f(\text{Soc}(U_1)) = \tilde{f}(\text{Soc}(U_1)) = 0$. But, by the definition of f , we see $f(\text{Soc}(U_1)) \neq 0$. This is a contradiction. So the second case occurs in the diagram (\otimes). Hence, by Lemma B, we have a direct decomposition $U_1 \oplus U_2 = V_1 \oplus V_2$ such that $(\text{Soc}(U_1))(g) \subseteq V_1$ and $V_2 \neq 0$.

(2) \Rightarrow (1): We shall show this implication by induction on n . Take a diagram:

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 & & K & & \\
 & & \downarrow & & \\
 (*) & 0 & \longrightarrow & L & \xrightarrow{i} & U := \sum_{k=1}^n \oplus U_k \\
 & & & \downarrow f & & \\
 & & & U_0 & &
 \end{array}$$

We may assume that $L \subset U$, since, otherwise, there is a submodule L' of U such that $L \oplus L' \subset U$. Then consider the following diagram:

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 & & K \oplus L' & & \\
 & & \downarrow & & \\
 0 & \longrightarrow & L \oplus L' & \xrightarrow{i'} & U \\
 & & \downarrow \tilde{f} & & \\
 & & U_0 & &
 \end{array}$$

where the homomorphism $\tilde{f}: L \oplus L' \rightarrow U_0$ is defined as $\tilde{f}(x+x') = f(x)$ for any $x \in L$ and $x' \in L'$. If the first case occurs in this diagram, let $\tilde{f}: U \rightarrow U_0$ be a homomorphism such that $\tilde{f} = \tilde{f} \cdot i'$, then $\tilde{f} \cdot i = \tilde{f} \cdot (i'|_L) = \tilde{f}|_L = f$. The first case also occurs in the original diagram (*). On the other hand, if the second case occurs in this diagram, let $0 \neq U' \subset \oplus U$, $p: U \rightarrow U'$ be a projection and $\tilde{f}: U_0 \rightarrow U'$ be a homomorphism such that $p \cdot i' = \tilde{f} \cdot \tilde{f}$, then $p \cdot i = p \cdot (i'|_L) = \tilde{f} \cdot (\tilde{f}|_L) = \tilde{f} \cdot f$. The second case also occurs in the diagram (*).

Now assume that the first case does not occur in this diagram. And we will show that the second case occurs in it.

If $K \subset U$, the first case occurs in the diagram (*) by Corollary 2, a contradiction. Hence $K \not\subset U$. Then we may assume that $K \cap \text{Soc}(U_1) = 0$. Since U is a finite direct sum of uniform modules, we can take a maximal $\{k \in \{2, 3, \dots,$

$n\} | \text{Soc}(U_k) \subseteq K\} |$ among $|\{k \in \{2, 3, \dots, n\} | \text{Soc}(U'_k) \subseteq K\} |$ related to the direct decomposition of U into uniform modules U'_k such that $K \cap \text{Soc}(U'_1) = 0$.

Now we denote its direct decomposition by $\sum_{k=1}^n U_k$.

Since $K \cap \text{Soc}(U_1) = 0$, there is a homomorphism $g: K^* \rightarrow U_1$ with $K = K^*$ (g) where $U_* := \sum_{k=2}^n U_k$, $\pi_*: U (= U_1 \oplus U_*) \rightarrow U_*$ is the projection and $K^* := \pi_*$ (K). Put $K_* := K \cap U_*$. Then we have the following two cases:

case A: $K_* \subset U_*$.

case B: $K_* \not\subset U_*$.

Note that $\text{Soc}(U_k) \subseteq K^*$ for any $k \in \{2, 3, \dots, n\}$. Because, if $\text{Soc}(U_k) \not\subseteq K^*$, then $f(\text{Soc}(U_1) \oplus \text{Soc}(U_k))$ is a direct sum of two simple submodules of U_0 , a contradiction. Therefore, if $\text{Soc}(U_k) \not\subseteq K_*$ for some $k \in \{2, 3, \dots, n\}$, the socle of K^*/K_* , which is isomorphic to $\text{Soc}(U_1)$ via g , is $(\text{Soc}(U_k) \oplus K_*)/K_* (\approx \text{Soc}(U_k))$. So $\text{Soc}(U_1) \approx \text{Soc}(U_k)$. On the other hand, f induces $\text{Soc}(U_1) \approx \text{Soc}(U_0)$ since $K \cap \text{Soc}(U_1) = 0$. Hence $\text{Soc}(U_0) \approx \text{Soc}(U_1) \approx \text{Soc}(U_k)$.

In case B, if $\text{Soc}(U_2) \not\subseteq K$, we have either the following two properties by assumption:

$\langle 2-i \rangle$ U_0 is U_1 - and U_2 -injective.

$\langle 2-ii \rangle$ $U_1 \oplus U_2$ is extending for simple modules.

Assume that $\langle 2-ii \rangle$ occurs. We have a direct decomposition $U_1 \oplus U_2 = V_1 \oplus V_2$ such that $V_2 \supset (\text{Soc}(U_2)) (g|_{\text{Soc}(U_2)})$. Then $V_1 \neq 0$ and $V_1 \cap K = 0$. Because, if $V_1 \cap K \neq 0$, $\text{unif. dim}((V_1 \oplus V_2) \cap K) = 2$ since $(\text{Soc}(U_2)) (g|_{\text{Soc}(U_2)}) \subseteq K$. But $U_1 \cap K = 0$ induces $\text{unif. dim}((U_1 \oplus U_2) \cap K) \leq 1$, a contradiction. On the other hand, $\text{Soc}(V_2) = (\text{Soc}(U_2)) (g|_{\text{Soc}(U_2)}) \subseteq K$. Therefore, we have a new direct decomposition $U = V_1 \oplus V_2 \oplus (\sum_{k=3}^n U_k)$ such that $K \cap \text{Soc}(V_1) = 0$. Then, since $\text{Soc}(U_2) \not\subseteq K$ and $\text{Soc}(V_2) \subseteq K$, the existence of this direct decomposition gives us a contradiction to the maximality of $|\{k \in \{2, 3, \dots, n\} | \text{Soc}(U_k) \subseteq K\} |$. Consequently, if $\text{Soc}(U_2) \not\subseteq K$, $\langle 2-i \rangle$ only occurs, i.e. U_0 is U_1 - and U_2 -injective.

Taking the same argument for U_3, U_4, \dots, U_n in order, we may assume that U_0 is U_1 -, U_2 -, \dots and U_m -injective and $\text{Soc}(U_{m+1}), \text{Soc}(U_{m+2}), \dots$ and $\text{Soc}(U_n) \subseteq K$ for some $m \geq 2$. (Since we are considering the case B, $m \geq 2$.) Put $M_1 := U_1 \oplus U_2 \oplus \dots \oplus U_m$ and $M_2 := U_{m+1} \oplus U_{m+2} \oplus \dots \oplus U_n$ and consider the diagrams (2-1), (2-2) and (3-1) with respect to the direct decomposition $U = M_1 \oplus M_2$. Then, using Corollary 1 inductively, the first case occurs in both diagrams (2-1) and (3-1). On the other hand, in the diagram (2-2), $\text{Ker}(f|_{L \cap M_2}) = K \cap M_2 \subset M_2$ since $\text{Soc}(M_2) = \sum_{k=m+1}^n \text{Soc}(U_k) \subseteq K_*$. So the second case does not occur in the

diagram (2-2) by Lemma B. The first case occurs in it since U_0 is almost M_2 -injective by the inductive assumption. Then the first case occurs in the diagram (\otimes) by Lemma C, a contradiction.

In case A. Let $\pi_1: U(=\sum_{k=1}^n \oplus U_k) \rightarrow U_1$ be the projection and put $K^1 := \pi_1(K)$. For each direct decomposition of U into uniform modules U'_k in which the case A occurs and $U'_1 \cap K = 0$, we obtain such K^1 . Since $\|K^1\|$ is finite, we can take a minimal $\|K^1\|$ among $\|K^1\|$ related to the direct decomposition $\sum_{k=1}^n \oplus U'_k$ and we denote its direct decomposition by $\sum_{k=1}^n \oplus U_k$.

The special case $K^1 = 0$ may occur. We shall first consider this case. From $K^1 = 0$ it follows that $K \subseteq U_*$. There are two monomorphisms: $L_1 \oplus (L_*/K) \xrightarrow{\text{natural}} L/K \xrightarrow{\text{induced from } f} U_0$ and U_0 is uniform. Since $L \subset U$ and hence $L_1 \neq 0$, and so $L_* = K$. Put $L^1 := \pi_1(L)$. since $\text{Ker}(\pi_1|_L) = L_* = K = \text{Ker}(f)$, there is a homomorphism $f': L^1 \rightarrow U_0$ such that $f = f' \cdot (\pi_1|_L)$. So consider the following diagram:

$$\begin{array}{ccccc}
 0 & \longrightarrow & L^1 & \xrightarrow{i^1} & U_1 \\
 & & f' \downarrow & & \\
 & & U_0 & &
 \end{array}$$

From the assumption that U_0 is almost U_1 -injective, the first case or the second occurs in this diagram. Assume that the first case occurs and let $\tilde{f}': U_1 \rightarrow U_0$ be a homomorphism such that $f' = \tilde{f}' \cdot i^1$, put $\tilde{f} := \tilde{f}' \cdot \pi_1: U \rightarrow U_0$, then $f = f' \cdot (\pi_1|_L) = \tilde{f}' \cdot i^1 \cdot (\pi_1|_L) = \tilde{f}' \cdot \pi_1 \cdot i = \tilde{f}' \cdot i$ in the diagram (\otimes) , i.e. the first case also occurs in the diagram (\otimes) , a contradiction. So the second case occurs. Let $\tilde{f}': U_0 \rightarrow U_1$ be a homomorphism such that $i^1 = \tilde{f}' \cdot f'$. Then $\tilde{f}' \cdot f = \tilde{f}' \cdot f' \cdot (\pi_1|_L) = i^1 \cdot (\pi_1|_L) = \pi_1 \cdot i$ in the diagram (\otimes) , i.e. the second case also occurs in the diagram (\otimes) .

In the case $K^1 \neq 0$. Consider the diagrams (2-1), (2-*) and (3-*) from the diagram (\otimes) with respect to the direct decomposition $U = U_1 \oplus U_*$.

[Claim. 1] The first case occurs in the diagram (2-*). Otherwise the second case occurs in it by inductive assumption. So there is a proper direct summand of U_* which contains K_* by Lemma B, i.e. $K_* \not\subseteq U_*$, since $\text{Ker}(f|_{L_*}) = K_*$. Then the case B occurs with respect to the direct decomposition $U = U_1 \oplus U_*$, a contradiction.

[Claim. 2] The first case occurs in the diagram (3-*). Otherwise the second case occurs in it by inductive assumption. So there is a proper direct

summand of U which contains L_* by Lemma B, i.e. $L_* \not\subseteq U_*$, since $L_* \subseteq \text{Ker}(f'_*)$. So $K_* \not\subseteq U_*$ for $K_* \subseteq L_*$. Therefore the case B also occurs, a contradiction.

Thus, we only have either the following two cases:

- i) The first case occurs in the diagrams (2-1), (2-*) and (3-*).
- ii) The second case occurs in the diagram (2-1) and the first case does in the diagram (2-*).

In case i), the first case occurs in the diagram (※) by Lemma C, a contradiction. So we consider the case ii).

Since the second case occurs in the diagram (2-1), there exists a homomorphism $p: U_0 \rightarrow U_1$ such that $i_1 = p \cdot (f|_{L_1})$, i.e. $p \cdot (f|_{L_1}) = 1_{L_1}$. Since the first case occurs in the diagram (2-*), there exists a homomorphism $q: U_* \rightarrow U_0$ such that $f|_{L_*} = q \cdot i_*$, i.e. $q|_{L_*} = f|_{L_*}$. Put $g' := p \cdot q: U_* \rightarrow U_1$ and $X := g^{-1}(\text{Soc}(U_1))$. Then $U_*(-g') \supseteq X(g|_X)$. Because, since $x + g(x) \in K$ for any $x \in X$, $0 = f(x + g(x)) = f(x) + f \cdot g(x)$, i.e. $f(x) = -f \cdot g(x)$. (Note that $L \subset U$ induces $\text{Soc}(U) \subseteq L_1$. So $g(x) \in L$, and $x \in L$ since $x + g(x) \in K \subseteq L$. Hence $f \cdot g(x)$ and $f(x)$ are defined.)

$$\begin{aligned} \text{Therefore } g'(x) &= p \cdot q(x) \\ &= p \cdot f(x) && (q|_{L_*} = f|_{L_*}) \\ &= p \cdot (-f \cdot g(x)) \\ &= -g(x) && (p \cdot (f|_{L_1}) = 1_{L_1}) \end{aligned}$$

Hence $U_*(-g') \supseteq X(g|_X)$. Then $U_*(-g') \supseteq X(g|_X) \cong K_*(g|_{K_*}) = K_*$ since $X \cong K_*$. (We are considering the case $K^1 \neq 0$. So $g \neq 0$. And we have $X \cong K_*$.)

Now we consider the direct decomposition $U = U_1 \oplus (\sum_{k=2}^n U_k(-g'|_{U_k}))$. Put $K'_* := K \cap U_*(-g')$. Then $K'_* \cong K_*$ since $U_*(-g') \cong K_*$. So $K'_* \subset U_*(-g')$ for $K_* \subset U_*$. Hence the case A occurs in this direct decomposition. Let $\pi'_1: U (= U_1 \oplus U_*(-g')) \rightarrow U_1$ be the projection and put $K'^1 := \pi'_1(K)$. Then $\|K'^1\| < \|K^1\|$, since $K'_* \cong K_*$ induces $\|K'_*\| > \|K_*\|$ and $(\|K\| =) \|K^1\| + \|K_*\| = \|K'^1\| + \|K'_*\|$. Therefore the direct decomposition $U = U_1 \oplus (\sum_{k=2}^n U_k(-g'|_{U_k}))$ give us a contradiction to the minimality of $\|K^1\|$.

As a consequence, taking an adequate direct decomposition of U , the special case $K^1 = 0$ occurs in the case A.

DEFINITION. Let R be a right artinian ring. We say that R is *right Co-Nakayama* if every indecomposable injective right R -module E is uniserial (i.e. E has a unique composition series.).

Corollary. *The following two conditions are equivalent:*

- (1) R is right Co-Nakayama.

(2) For any uniform modules U^i and $U_j (i=1, \dots, m; j=1, \dots, n)$ of finite composition length, $\bigoplus_{i=1}^m U^i$ is almost $\bigoplus_{j=1}^n U_j$ -injective if U^i is almost U_j -injective for all i and j . (i.e. The almost injectivity among uniform modules of finite composition length is closed under finite direct sums.)

Proof. (1) \Rightarrow (2): If $\text{Soc}(U_k) \approx \text{Soc}(U_l)$, $U_k \oplus U_l$ is extending for simple modules by (1). Then U^i is almost $\bigoplus_{j=1}^n U_j$ -injective for any $i \in \{1, \dots, m\}$ since the condition in Theorem holds. Give a diagram:

$$\begin{array}{ccccc}
 0 & \longrightarrow & L & \xrightarrow{i} & \bigoplus_{j=1}^n U_j \\
 & & \downarrow f & & \\
 & & \bigoplus_{i=1}^m U^i & &
 \end{array}$$

Let $p_i: \bigoplus_{i=1}^m U^i \rightarrow U^i$ be projections ($i=1, \dots, m$). Consider the following diagrams for $i=1, \dots, m$:

$$\begin{array}{ccccc}
 0 & \longrightarrow & L & \xrightarrow{i} & \bigoplus_{j=1}^n U_j \\
 (\#-i) & & \downarrow p_i \cdot f & & \\
 & & U^i & &
 \end{array}$$

If the first case occurs in all diagrams (#-i), let $\tilde{f}_i: \bigoplus_{j=1}^n U_j \rightarrow U^i$ with $p_i \cdot f = \tilde{f}_i \cdot i$, $f = (\bigoplus_{i=1}^m \tilde{f}_i) \cdot i$, i.e. the first case occurs in the given diagram. If the second case occurs in a diagram (#-r) ($r \in \{1, \dots, m\}$), let U' be a direct summand of $\bigoplus_{j=1}^n U_j$, $\pi: \bigoplus_{j=1}^n U_j \rightarrow U'$ be a projection and $\tilde{f}_r: U' \rightarrow U^r$ be a homomorphism such that $\pi \cdot i = \tilde{f}_r \cdot p_r \cdot f$, the second case occurs in the given diagram. Therefore $\bigoplus_{i=1}^m U^i$ is almost $\bigoplus_{j=1}^n U_j$ -injective.

(2) \Rightarrow (1): Claim. For each uniform module U , $U/\text{Soc}(U)$ is also uniform.

First we show this claim. Let M_1 and M_2 be submodules of U with $\|M_i\| = 2 (i=1, 2)$. Then $\text{Soc}(U)$ is almost M_1 - and M_2 -injective but neither M_1 -nor M_2 -injective. $M := M_1 \oplus M_2$ is extending for simple modules by (2) and Theorem. Let $1_s: \text{Soc}(U) \rightarrow \text{Soc}(U)$ be the identity map. There is an isomorphism $f: M_1 \rightarrow M_2$ such that $f|_{\text{Soc}(U)} = 1_s$, by [3], Corollary 8. Let $1: U \rightarrow U$ and $i_2: M_2 \rightarrow U$ be the identity map and the inclusion map, respectively. Put $g := 1|_{M_1}$

$-i_2 \cdot f: M_1 \rightarrow U$. Since $f|_{\text{Soc}(U)} = 1_s$, $g(\text{Soc}(U)) = 0$. And so $g(M_1) \subseteq \text{Soc}(U)$ for $\|M_1\| = 2$. $M_1 = 1(M_1) = (i_2 \cdot f + g)(M_1) \subseteq i_2 \cdot f(M_1) + g(M_1) \subseteq M_2 + \text{Soc}(U) = M_2$. Hence $M_1 = M_2$, i.e. $U/\text{Soc}(U)$ is uniform.

Let E be an injective indecomposable module. Since R is right artinian, $J^n = 0$ for some n . Hence E has the finite socle series:

$$0 = S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_m = E$$

for some $m \leq n$, where S_i is the left annihilator of J^i for each i . Then apply inductively the above claim to this series to see that S_i/S_{i-1} is simple for each $i \in \{1, \dots, m\}$, whence the assertion follows.

EXAMPLE. There is an example which shows that the Azumaya's Theorem is not able to be extended without an additional condition.

Let K be a field and

$$R = \begin{pmatrix} K & 0 & K \\ & K & K \\ 0 & & K \end{pmatrix}$$

Then, $e_{33}R$ is almost $e_{11}R$ - and $e_{22}R$ -injective, but not almost $e_{11}R \oplus e_{22}R$ -injective, where e_{kk} are matrix units.

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