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SOLUTIONS OF AN ALGEBRAIC DIFFERENTIAL EQUATION OF THE FIRST ORDER IN A LIOUVILLIAN EXTENSION OF THE COEFFICIENT FIELD

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0. Introduction. Let k be an algebraically closed ordinary differential field of characteristic 0, and Ω be a universal extension of k. An element ξ of Ω will be called a *weakly liouvillian element* over k if there exists such an extending chain $L_0 \subset L_1 \subset \cdots \subset L_n$ of differential subfields of Ω that satisfies the following condition:

(i) $L_0 = k$, $L_n \ni \xi$; for each $i (0 \leqq i < n)$ we have $L_{i+1} = L_i(t_i)$, where either $t'_i \in L_i$, $t'_i / t_i \in L_i$ or t_i is algebraic over L_i.

If in addition the following condition is satisfied, then ξ is called a *liouvillian element* over k:

(ii) The field of constants of L_n is the same as that of k.

Let F be an algebraically irreducible element of the differential polynomial algebra $k\{y\}$ of the first order. Then, by a theorem due to Kolchin ([1], p. 928) we can prove the following proposition (cf. [1]; Proof of Theorem 3, pp. 930–931):

Suppose that there exists a nonsingular solution of $F=0$ which is a weakly liouvillian element over k. Then, there exists a nonsingular solution of $F=0$ which is a liouvillian element over k.

Let y be a generic point of the general solution of $F=0$ in Ω over k. Then, y is transcendental over k, and $k(y, y')$ is a one-dimensional algebraic function field over k being a differential subfield of Ω. Let K denote $k(y, y')$ and P be a prime divisor of K. Then, the completion K_P of K with respect to P is a differential extension of K and the differentiation gives a continuous mapping from K_P to itself (cf. Rosenlicht[4]). Let τ_1, τ_2 be two prime elements in P. Then, $\nu_P(\tau'_1) \leqq 0$ if and only if $\nu_P(\tau'_2) \leqq 0$.

Theorem. *Assume that $\nu_P(\tau') \leqq 0$ for each P, where τ is a prime element in P. Then, any solution of $F=0$ which is a weakly liouvillian element over k is contained in k.*

It does not depend on the choice of a generic point y whether our assumption is satisfied or not.

In the section 2 we shall give two examples of $F=0$ to which our Theorem can be applied with success. As a particular case of the first example, we shall obtain the following:

Corollary. *Suppose that every element of k is constant. Then, any non-singular solution of*

$$(y')^2=(y-a_1)\cdots(y-a_{2m+1}), \quad a_i\in k(1\leqq i\leqq 2m+1)$$

is not a weakly liouvillian element over k; here we assume that $a_i\neq a_j(i\neq j)$ and $m\geqq 1$.

REMARK 1. By the valuation theory Rosenlicht [5] obtained a criterion for an algebraic differential equation of order n to have a solution in a liouvillian extension of the coefficient field, and proved our Corollary in the special case where $m=1$.

REMARK 2. Liouville ([2], pp. 536–539) stated the following theorem: Suppose that f is an algebraic function of x, y and that $f_x f_y\neq 0$. Then, any solution of a transcendental equation $\log y=f(x, y)$ is not an elementary transcendental function of x unless it is constant. Our Theorem can not be applied to prove this theorem of Liouville, a differential-algebraic proof of which can be derived from the results obtained by Rosenlicht [4]. In the special case where $f=y/x$, an elementary proof was given by Matsuda [3].

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1. Proof of Theorem. Let Λ be the set of all solutions of $F=0$ that are not contained in k, and Γ be the set of all elements ξ of Λ such that there exists an extending chain $H_0\subset H_1\subset\cdots\subset H_m$ of differential subfields of Ω which satisfies the following two conditions:

(iii) $H_0=k$, $H_m\ni\xi$;

(iv) for each $i(0\leqq i<m)$, H_{i+1} is the algebraic closure of $H_i(t_i)$; here t_i is transcendental over H_i, and either $t_i'\in H_i$ or $t_i'/t_i\in H_i$.

Suppose that there exists in Λ a weakly liouvillian element over k. Then, Γ is not empty. To each element ξ of Γ we can correspond a positive integer $n(\xi)$ which satisfies the following two conditions:

(v) There exists a chain $H_0\subset\cdots\subset H_{n(\xi)}$ which satisfies the two conditions (iii) and (iv) with $m=n(\xi)$;

(vi) for any chain $I_0\subset\cdots\subset I_m$ satisfying the two conditions (iii) and (iv) with $H_i=I_i$ we have $m\geqq n(\xi)$.

Take an element η of Γ such that

$$(1) \qquad n(\eta) = \min \{n(\xi);\ \xi \in \Gamma\} ,$$

and let $H_0 \subset \cdots \subset H_{n(\eta)}$ be a chain which satisfies the two conditions (iii), (iv) with $\xi=\eta$ and $m=n(\eta)$. For convenience we represent $n(\eta)$ by m, H_{m-1} by N and t_{m-1} by t. Then, η is a transcendental element over N satisfying $F=0$. The equation is algebraically irreducible over N, since it is so over an algebraically closed field k. Let M_1 and M_2 denote $N(\eta, \eta')$ and $N(\eta, \eta', t)$ respectively. They are one-dimensional algebraic function fields over N, being differential subfields of H_m. Since t is transcendental over N, there exists a prime divisor Q of M_2 such that $\nu_Q(t)<0$. Restricting ν_Q to M_1 we have a valuation of M_1 over N belonging to a certain prime divisor S of M_1, because M_2 is an algebraic extension of M_1 of finite degree. The completion N_2 of M_2 with respect to Q is a differential extension field of the completion N_1 of M_1 with respect to S. We have $t=\rho^{-d}$ for a prime element ρ in Q, where $d>0$. Let σ be a prime element in S. Then, in N_2 we have

$$\sigma = a_0\rho^e + a_1\rho^{e+1} + \cdots \qquad (a_0 \neq 0);$$

here $a_i \in N (i \geqq 0)$ and $e>0$. Hence, $\nu_S(\sigma')>0$ if $\nu_Q(\rho')>0$. Let us prove that $\nu_Q(\rho')>0$. First suppose that $t'=b$ and $b \in N$. Then,

$$b = -d\rho'\rho^{-d-1} .$$

Secondly suppose that $t'/t=c$ and $c \in N$. Then,

$$c\rho^{-d} = -d\rho'\rho^{-d-1} .$$

In any case we have $\nu_Q(\rho')>0$. Therefore, $\nu_S(\sigma')>0$. We shall show that it leads us to a contradiction.

First suppose that $\nu_S(\eta)<0$. Then, restricting ν_S to $k(\eta, \eta')$ we have a normalized valuation of $k(\eta, \eta')$ over k belonging to a certain prime divisor P of $k(\eta, \eta')$, since k is algebraically closed. The completion of $k(\eta, \eta')$ with respect to P is a differential subfield of N_1. A prime element τ in P is a prime element in S. By our assumption, $\nu_S(\tau') \leqq 0$. This is a contradiction. Secondly suppose that $\nu_S(\eta-\alpha)>0$ with an element α of k. Then, we also meet a contradiction. Lastly suppose that $\nu_S(\eta-\beta)>0$ with an element β of N which is not contained in k. Then, by a theorem of Rosenlicht [5], β is a solution of $F=0$: In fact, we have

$$\eta = \beta + b_1\sigma + b_2\sigma^2 + \cdots \qquad (b_i \in N,\ i \geqq 1)$$

in N_1 and

$$\eta' = \beta' + (b_1'\sigma + b_2'\sigma^2 + \cdots) + \sigma'(b_1 + 2b_2\sigma + \cdots) .$$

Because of $\nu_S(\sigma')>0$, $\nu_S(\eta'-\beta')>0$. Hence, $F(\beta, \beta')=0$. Since $\beta \in N$, β

is an element of Γ and $n(\beta)<n(\eta)$. This inequality contradicts the assumption (1). Therefore, there does not exist in Λ any weakly liouvillian element over k.

2. Examples. Let k_0 be an algebraically closed field of characteristic 0. We set $c'=0$ for all elements c of k_0.

Example 1. Let us assume that $k=k_0$ and

$$F(y, y') = G(y, y')y'^m + H(y),$$

where $m>0$, $G\in k[y, y']$, $H\in k[y]$. We set on F the following conditions:

(vii) H has only simple roots;
(viii) $\deg_{y'} G<m$ and $\deg_y G<\deg_y H$;
(ix) $G(a, y')\neq 0$ for any root a of H.

Then, F is algebraically irreducible. Let us set on F one more condition:

(x) $m>1$ and $m+\deg_{y,y'} G<\deg_y H$.

We prove that the assumption of our Theorem is satisfied by F. First suppose that $\nu_P(y)<0$. Then, $y=\tau^{-e}$ with $e>0$. If $\nu_P(\tau')>0$, then $\nu_P(y')\geqq -e$ and

$$\nu_P(Gy'^m)\geqq -e(m+\deg_{y,y'} G)>-e\cdot\deg_y H = \nu_P(H).$$

Secondly suppose that $\nu_P(y-a)>0$ for some root a of H. Then, $y=a+\tau^e$ with $e>0$. If $\nu_P(\tau')>0$, then $\nu_P(y')\geqq e$ and

$$\nu_P(Gy'^m)\geqq em>e = \nu_P(H).$$

Lastly suppose that $\nu_P(y-b)>0$ with an element b of k different from any root of H. If $\nu_P(\tau')>0$, then $\nu_P(y')\geqq 1$ and

$$\nu_P(Gy'^m)\geqq m>0 = \nu_P(H).$$

In any case we meet a contradiction if it is assumed that $\nu_P(\tau')>0$. Since

$$\partial F/\partial y' = y'^{m-1}(mG+y'\partial G/\partial y'),$$

any nonsingular solution of $F=0$ is not a constant. Hence, by our Theorem, any nonsingular solution of $F=0$ is not a weakly liouvillian element over k.

Example 2. Let us assume that k is the algebraic closure of the one-dimensional rational function field $k_0(x)$ over k_0 with $x'=1$, and that

$$F(y, y') = xy'-y(1-y)^n-x, \quad n>0.$$

Then, it can be proved that any element of k does not satisfy $F=0$. We show that the assumption of our Theorem is satisfied by F. First suppose that $\nu_P(y)<0$. Then, we have $y=\tau^{-1}$ and

$$\tau' = -\tau^{1-n}(\tau-1)^n/x-\tau^2 .$$

Hence, $\nu_P(\tau')=1-n\leqq 0$. Secondly suppose that $\nu_P(y-a)>0$ with an element a of k. Then, $y=a+\tau$. Since a can not be a solution of $F=0$, we have $\nu_P(\tau')\leqq 0$. Hence, by our Theorem, any solution of $F=0$ is not a weakly liouvillian element over k.

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