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**SOLUTIONS OF AN ALGEBRAIC DIFFERENTIAL
EQUATION OF THE FIRST ORDER
IN A LIOUVILLIAN EXTENSION
OF THE COEFFICIENT FIELD**

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0. Introduction. Let k be an algebraically closed ordinary differential field of characteristic 0, and Ω be a universal extension of k . An element ξ of Ω will be called a *weakly liouvillian element* over k if there exists such an extending chain $L_0 \subset L_1 \subset \cdots \subset L_n$ of differential subfields of Ω that satisfies the following condition:

(i) $L_0 = k$, $L_n \ni \xi$; for each $i(0 \leq i < n)$ we have $L_{i+1} = L_i(t_i)$, where either $t'_i \in L_i$, $t'_i/t_i \in L_i$ or t_i is algebraic over L_i .

If in addition the following condition is satisfied, then ξ is called a *liouvillian element* over k :

(ii) The field of constants of L_n is the same as that of k .

Let F be an algebraically irreducible element of the differential polynomial algebra $k\{y\}$ of the first order. Then, by a theorem due to Kolchin ([1], p. 928) we can prove the following proposition (cf. [1]; Proof of Theorem 3, pp. 930-931):

Suppose that there exists a nonsingular solution of $F=0$ which is a weakly liouvillian element over k . Then, there exists a nonsingular solution of $F=0$ which is a liouvillian element over k .

Let y be a generic point of the general solution of $F=0$ in Ω over k . Then, y is transcendental over k , and $k(y, y')$ is a one-dimensional algebraic function field over k being a differential subfield of Ω . Let K denote $k(y, y')$ and P be a prime divisor of K . Then, the completion K_P of K with respect to P is a differential extension of K and the differentiation gives a continuous mapping from K_P to itself (cf. Rosenlicht[4]). Let τ_1, τ_2 be two prime elements in P . Then, $v_P(\tau'_1) \leq 0$ if and only if $v_P(\tau'_2) \leq 0$.

Theorem. Assume that $v_P(\tau') \leq 0$ for each P , where τ is a prime element in P . Then, any solution of $F=0$ which is a weakly liouvillian element over k is contained in k .

It does not depend on the choice of a generic point y whether our assump-

tion is satisfied or not.

In the section 2 we shall give two examples of $F=0$ to which our Theorem can be applied with success. As a particular case of the first example, we shall obtain the following:

Corollary. *Suppose that every element of k is constant. Then, any non-singular solution of*

$$(y')^2 = (y-a_1)\cdots(y-a_{2m+1}), \quad a_i \in k (1 \leq i \leq 2m+1)$$

is not a weakly liouvillian element over k ; here we assume that $a_i \neq a_j (i \neq j)$ and $m \geq 1$.

REMARK 1. By the valuation theory Rosenlicht [5] obtained a criterion for an algebraic differential equation of order n to have a solution in a liouvillian extension of the coefficient field, and proved our Corollary in the special case where $m=1$.

REMARK 2. Liouville ([2], pp. 536–539) stated the following theorem: Suppose that f is an algebraic function of x, y and that $f_x f_y \neq 0$. Then, any solution of a transcendental equation $\log y = f(x, y)$ is not an elementary transcendental function of x unless it is constant. Our Theorem can not be applied to prove this theorem of Liouville, a differential-algebraic proof of which can be derived from the results obtained by Rosenlicht [4]. In the special case where $f=y/x$, an elementary proof was given by Matsuda [3].

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1. Proof of Theorem. Let Λ be the set of all solutions of $F=0$ that are not contained in k , and Γ be the set of all elements ξ of Λ such that there exists an extending chain $H_0 \subset H_1 \subset \cdots \subset H_m$ of differential subfields of Ω which satisfies the following two conditions:

- (iii) $H_0 = k, H_m \ni \xi$;
- (iv) for each $i (0 \leq i < m)$, H_{i+1} is the algebraic closure of $H_i(t_i)$; here t_i is transcendental over H_i , and either $t'_i \in H_i$, or $t'_i/t_i \in H_i$.

Suppose that there exists in Λ a weakly liouvillian element over k . Then, Γ is not empty. To each element ξ of Γ we can correspond a positive integer $n(\xi)$ which satisfies the following two conditions:

- (v) There exists a chain $H_0 \subset \cdots \subset H_{n(\xi)}$ which satisfies the two conditions (iii) and (iv) with $m=n(\xi)$;
- (vi) for any chain $I_0 \subset \cdots \subset I_m$ satisfying the two conditions (iii) and (iv) with $H_i = I_i$, we have $m \geq n(\xi)$.

Take an element η of Γ such that

$$(1) \quad n(\eta) = \min \{n(\xi); \xi \in \Gamma\},$$

and let $H_0 \subset \dots \subset H_{n(\eta)}$ be a chain which satisfies the two conditions (iii), (iv) with $\xi = \eta$ and $m = n(\eta)$. For convenience we represent $n(\eta)$ by m , H_{m-1} by N and t_{m-1} by t . Then, η is a transcendental element over N satisfying $F=0$. The equation is algebraically irreducible over N , since it is so over an algebraically closed field k . Let M_1 and M_2 denote $N(\eta, \eta')$ and $N(\eta, \eta', t)$ respectively. They are one-dimensional algebraic function fields over N , being differential subfields of H_m . Since t is transcendental over N , there exists a prime divisor Q of M_2 such that $\nu_Q(t) < 0$. Restricting ν_Q to M_1 we have a valuation of M_1 over N belonging to a certain prime divisor S of M_1 , because M_2 is an algebraic extension of M_1 of finite degree. The completion N_2 of M_2 with respect to Q is a differential extension field of the completion N_1 of M_1 with respect to S . We have $t = \rho^{-d}$ for a prime element ρ in Q , where $d > 0$. Let σ be a prime element in S . Then, in N_2 we have

$$\sigma = a_0\rho^e + a_1\rho^{e+1} + \dots \quad (a_0 \neq 0);$$

here $a_i \in N (i \geq 0)$ and $e > 0$. Hence, $\nu_S(\sigma') > 0$ if $\nu_Q(\rho') > 0$. Let us prove that $\nu_Q(\rho') > 0$. First suppose that $t' = b$ and $b \in N$. Then,

$$b = -d\rho'\rho^{-d-1}.$$

Secondly suppose that $t'/t = c$ and $c \in N$. Then,

$$c\rho^{-d} = -d\rho'\rho^{-d-1}.$$

In any case we have $\nu_Q(\rho') > 0$. Therefore, $\nu_S(\sigma') > 0$. We shall show that it leads us to a contradiction.

First suppose that $\nu_S(\eta) < 0$. Then, restricting ν_S to $k(\eta, \eta')$ we have a normalized valuation of $k(\eta, \eta')$ over k belonging to a certain prime divisor P of $k(\eta, \eta')$, since k is algebraically closed. The completion of $k(\eta, \eta')$ with respect to P is a differential subfield of N_1 . A prime element τ in P is a prime element in S . By our assumption, $\nu_S(\tau') \leq 0$. This is a contradiction. Secondly suppose that $\nu_S(\eta - \alpha) > 0$ with an element α of k . Then, we also meet a contradiction. Lastly suppose that $\nu_S(\eta - \beta) > 0$ with an element β of N which is not contained in k . Then, by a theorem of Rosenlicht [5], β is a solution of $F=0$: In fact, we have

$$\eta = \beta + b_1\sigma + b_2\sigma^2 + \dots \quad (b_i \in N, i \geq 1)$$

in N_1 and

$$\eta' = \beta' + (b_1'\sigma + b_2'\sigma^2 + \dots) + \sigma'(b_1 + 2b_2\sigma + \dots).$$

Because of $\nu_S(\sigma') > 0$, $\nu_S(\eta' - \beta') > 0$. Hence, $F(\beta, \beta') = 0$. Since $\beta \in N$, β

is an element of Γ and $n(\beta) < n(\eta)$. This inequality contradicts the assumption (1). Therefore, there does not exist in Λ any weakly liouvillian element over k .

2. Examples. Let k_0 be an algebraically closed field of characteristic 0. We set $c' = 0$ for all elements c of k_0 .

EXAMPLE 1. Let us assume that $k = k_0$ and

$$F(y, y') = G(y, y')y'^m + H(y),$$

where $m > 0$, $G \in k[y, y']$, $H \in k[y]$. We set on F the following conditions:

- (vii) H has only simple roots;
- (viii) $\deg_y G < m$ and $\deg_y G < \deg_y H$;
- (ix) $G(a, y') \neq 0$ for any root a of H .

Then, F is algebraically irreducible. Let us set on F one more condition:

- (x) $m > 1$ and $m + \deg_{y,y'} G < \deg_y H$.

We prove that the assumption of our Theorem is satisfied by F . First suppose that $\nu_p(y) < 0$. Then, $y = \tau^{-e}$ with $e > 0$. If $\nu_p(\tau') > 0$, then $\nu_p(y') \geq -e$ and

$$\nu_p(Gy'^m) \geq -e(m + \deg_{y,y'} G) > -e \cdot \deg_y H = \nu_p(H).$$

Secondly suppose that $\nu_p(y - a) > 0$ for some root a of H . Then, $y = a + \tau^e$ with $e > 0$. If $\nu_p(\tau') > 0$, then $\nu_p(y') \geq e$ and

$$\nu_p(Gy'^m) \geq em > e = \nu_p(H).$$

Lastly suppose that $\nu_p(y - b) > 0$ with an element b of k different from any root of H . If $\nu_p(\tau') > 0$, then $\nu_p(y') \geq 1$ and

$$\nu_p(Gy'^m) \geq m > 0 = \nu_p(H).$$

In any case we meet a contradiction if it is assumed that $\nu_p(\tau') > 0$. Since

$$\partial F / \partial y' = y'^{m-1}(mG + y' \partial G / \partial y'),$$

any nonsingular solution of $F = 0$ is not a constant. Hence, by our Theorem, any nonsingular solution of $F = 0$ is not a weakly liouvillian element over k .

EXAMPLE 2. Let us assume that k is the algebraic closure of the one-dimensional rational function field $k_0(x)$ over k_0 with $x' = 1$, and that

$$F(y, y') = xy' - y(1 - y)^n - x, \quad n > 0.$$

Then, it can be proved that any element of k does not satisfy $F = 0$. We show that the assumption of our Theorem is satisfied by F . First suppose that $\nu_p(y) < 0$. Then, we have $y = \tau^{-1}$ and

$$\tau' = -\tau^{1-n}(\tau-1)^n/x-\tau^2.$$

Hence, $v_p(\tau')=1-n\leq 0$. Secondly suppose that $v_p(y-a)>0$ with an element a of k . Then, $y=a+\tau$. Since a can not be a solution of $F=0$, we have $v_p(\tau')\leq 0$. Hence, by our Theorem, any solution of $F=0$ is not a weakly liouvillian element over k .

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Bibliography

- [1] E. R. Kolchin: *Existence theorems connected with the Picard-Vessiot theory of homogeneous linear ordinary differential equations*, Bull. Amer. Math. Soc. **54** (1948), 927-932.
- [2] J. Liouville: *Suite du memoire sur la classification des transcendentes, et sur l'impossibilité d'exprimer les racines de certaines équation en fonction finie explicite des coefficients*, J. Math. Pures Appl. **3** (1838), 523-546.
- [3] M. Matsuda: *Liouville's theorem on a transcendental equation $\log y=y/x$* , J. Math. Kyoto Univ. **16** (1976), 545-554.
- [4] M. Rosenlicht: *On the explicit solvability of certain transcendental equations*, Publ. Math. Inst. HES. **36** (1969), 15-22.
- [5] ———: *An analogue of L'Hospital's rule*, Proc. Amer. Math. Soc. **37** (1973), 369-373.

