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<th>Solutions of an algebraic differential equation of the first order in a Liouvillian extension of the coefficient field</th>
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<td>Author(s)</td>
<td>Ōtsubo, Shūji</td>
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<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 16(1) P.289–P.293</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1979</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/4944">https://doi.org/10.18910/4944</a></td>
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<td>DOI</td>
<td>10.18910/4944</td>
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SOLUTIONS OF AN ALGEBRAIC DIFFERENTIAL
EQUATION OF THE FIRST ORDER
IN A LIOUVILLIAN EXTENSION
OF THE COEFFICIENT FIELD

SHUJI OTSUBO

(Received November 30, 1977)

0. Introduction. Let \( k \) be an algebraically closed ordinary differential
field of characteristic 0, and \( \Omega \) be a universal extension of \( k \). An element
\( \xi \) of \( \Omega \) will be called a weakly liouvillian element over \( k \) if there exists such an ex-
tending chain \( L_0 \subset L_1 \subset \cdots \subset L_n \) of differential subfields of \( \Omega \) that satisfies the
following condition:

(i) \( L_n = k, L_n \ni \xi \); for each \( i(0 \leq i < n) \) we have \( L_{i+1} = L_i(t_i) \), where either
\( t_i' \in L_i, \ t_i' t_i \in L_i \) or \( t_i \) is algebraic over \( L_i \).

If in addition the following condition is satisfied, then \( \xi \) is called a liouvillian
element over \( k \):

(ii) The field of constants of \( L_n \) is the same as that of \( k \).

Let \( F \) be an algebraically irreducible element of the differential polynomial
algebra \( k \{ y \} \) of the first order. Then, by a theorem due to Kolchin ([1], p. 928)
we can prove the following proposition (cf. [1]; Proof of Theorem 3, pp. 930-
931):

Suppose that there exists a nonsingular solution of \( F = 0 \) which is a weakly
liouvillian element over \( k \). Then, there exists a nonsingular solution of \( F = 0 \)
which is a liouvillian element over \( k \).

Let \( y \) be a generic point of the general solution of \( F = 0 \) in \( \Omega \) over \( k \). Then,
\( y \) is transcendental over \( k \), and \( k(y, y') \) is a one-dimensional algebraic function
field over \( k \) being a differential subfield of \( \Omega \). Let \( K \) denote \( k(y, y') \) and \( P \) be
a prime divisor of \( K \). Then, the completion \( K_P \) of \( K \) with respect to \( P \) is a dif-
ferential extension of \( K \) and the differentiation gives a continuous mapping
from \( K_P \) to itself (cf. Rosenlicht[4]). Let \( \tau_1, \tau_2 \) be two prime elements in
\( P \). Then, \( v_\tau(\tau_i') \leq 0 \) if and only if \( v_\tau(\tau_i) \leq 0 \).

Theorem. Assume that \( v_\tau(\tau') \leq 0 \) for each \( P \), where \( \tau \) is a prime element
in \( P \). Then, any solution of \( F = 0 \) which is a weakly liouvillian element over \( k \)
is contained in \( k \).

It does not depend on the choice of a generic point \( y \) whether our assump-
tion is satisfied or not.

In the section 2 we shall give two examples of $F=0$ to which our Theorem can be applied with success. As a particular case of the first example, we shall obtain the following:

**Corollary.** Suppose that every element of $k$ is constant. Then, any non-singular solution of

$$(y')^2 = (y-a_i)\cdots(y-a_{2m+1}), \quad a_i \in k (1 \leq i \leq 2m+1)$$

is not a weakly liouvillian element over $k$; here we assume that $a_i \neq a_j (i \neq j)$ and $m \geq 1$.

**Remark 1.** By the valuation theory Rosenlicht [5] obtained a criterion for an algebraic differential equation of order $n$ to have a solution in a liouvillian extension of the coefficient field, and proved our Corollary in the special case where $m=1$.

**Remark 2.** Liouville ([2], pp. 536–539) stated the following theorem: Suppose that $f$ is an algebraic function of $x,y$ and that $f_x f_y \neq 0$. Then, any solution of a transcendental equation $\log y = f(x,y)$ is not an elementary transcendental function of $x$ unless it is constant. Our Theorem can not be applied to prove this theorem of Liouville, a differential-algebraic proof of which can be derived from the results obtained by Rosenlicht [4]. In the special case where $f=y/x$, an elementary proof was given by Matsuda [3].

The author wishes to express his sincere gratitude to Dr. M. Matsuda who presented this problem and gave kind advices, and to Mr. K. Nishioka for fruitful discussions with him.

1. **Proof of Theorem.** Let $\Lambda$ be the set of all solutions of $F=0$ that are not contained in $k$, and $\Gamma$ be the set of all elements $\xi$ of $\Lambda$ such that there exists an extending chain $H_0 \subset H_1 \subset \cdots \subset H_m$ of differential subfields of $\Omega$ which satisfies the following two conditions:

(iii) $H_0 = k, H_m \ni \xi$;

(iv) for each $i (0 \leq i < m)$, $H_{i+1}$ is the algebraic closure of $H_i(t_i)$; here $t_i$ is transcendental over $H_i$, and either $t'_i \in H_i$ or $t'_i/t_i \in H_i$.

Suppose that there exists in $\Lambda$ a weakly liouvillian element over $k$. Then, $\Gamma$ is not empty. To each element $\xi$ of $\Gamma$ we can correspond a positive integer $n(\xi)$ which satisfies the following two conditions:

(v) There exists a chain $H_0 \subset \cdots \subset H_{n(\xi)}$ which satisfies the two conditions (iii) and (iv) with $m=n(\xi)$;

(vi) for any chain $I_0 \subset \cdots \subset I_m$ satisfying the two conditions (iii) and (iv) with $H_i = I_i$ we have $m \geq n(\xi)$.

Take an element $\eta$ of $\Gamma$ such that
(1) \[ n(\eta) = \min \{ n(\xi); \xi \in \Gamma \}, \]
and let \( H_0 \subset \cdots \subset H_{n(\eta)} \) be a chain which satisfies the two conditions (iii), (iv) with \( \xi = \eta \) and \( m = n(\eta) \). For convenience we represent \( n(\eta) \) by \( m \), \( H_{m-1} \) by \( N \) and \( t_{m-1} \) by \( t \). Then, \( \eta \) is a transcendental element over \( N \) satisfying \( F=0 \).

The equation is algebraically irreducible over \( N \), since it is so over an algebraically closed field \( k \). Let \( M_1 \) and \( M_2 \) denote \( N(\eta, \eta') \) and \( N(\eta, \eta', t) \) respectively. They are one-dimensional algebraic function fields over \( N \), being differential subfields of \( H_m \). Since \( t \) is transcendental over \( N \), there exists a prime divisor \( Q \) of \( M_2 \) such that \( v_Q(p') < 0 \). Restricting \( v \) to \( M_1 \) we have a valuation of \( M_1 \) over \( N \) belonging to a certain prime divisor \( S \) of \( k(\eta, \eta') \), since \( \Lambda \) is algebraically closed. The completion \( M_2 \) with respect to \( Q \) is a differential extension field of the completion \( M_1 \) with respect to \( S \). We have \( t = \rho^{-d} \) for a prime element \( \rho \) in \( Q \), where \( d > 0 \). Let \( \sigma \) be a prime element in \( S \). Then, in \( M_2 \) we have
\[ \sigma = a_0 \rho^e + a_1 \rho^{e+1} + \cdots \quad (a_0 \neq 0); \]
here \( a_i \in N(i \geq 0) \) and \( e > 0 \). Hence, \( v_\sigma(\sigma') > 0 \) if \( v_\sigma(\rho') > 0 \). Let us prove that \( v_\sigma(\sigma') > 0 \). First suppose that \( t' = b \) and \( b \in N \). Then,
\[ b = -d \rho^d \rho^{-d-1}. \]
Secondly suppose that \( t'/t = c \) and \( c \in N \). Then,
\[ c \rho^{-d} = -d \rho^d \rho^{-d-1}. \]
In any case we have \( v_\sigma(\rho') > 0 \). Therefore, \( v_\sigma(\sigma') > 0 \). We shall show that it leads us to a contradiction.

First suppose that \( v_\sigma(\eta) < 0 \). Then, restricting \( v_\sigma \) to \( k(\eta, \eta') \) we have a normalized valuation of \( k(\eta, \eta') \) over \( k \) belonging to a certain prime divisor \( P \) of \( k(\eta, \eta') \), since \( k \) is algebraically closed. The completion of \( k(\eta, \eta') \) with respect to \( P \) is a differential subfield of \( N_1 \). A prime element \( \tau \) in \( P \) is a prime element in \( S \). By our assumption, \( v_\tau(\tau') \leq 0 \). This is a contradiction. Secondly suppose that \( v_\tau(\eta - \alpha) > 0 \) with an element \( \alpha \) of \( k \). Then, we also meet a contradiction. Lastly suppose that \( v_\sigma(\eta - \beta) > 0 \) with an element \( \beta \) of \( N \) which is not contained in \( k \). Then, by a theorem of Rosenlicht [5], \( \beta \) is a solution of \( F=0 \): In fact, we have
\[ \eta = \beta + b_1 \sigma + b_2 \sigma^2 + \cdots \quad (b_i \in N, i \geq 1) \]
in \( N_1 \) and
\[ \eta' = \beta' + (b'_1 \sigma + b'_2 \sigma^2 + \cdots) + \sigma' (b_1 + 2b_2 \sigma + \cdots). \]
Because of \( v_\sigma(\sigma') > 0, v_\sigma(\eta' - \beta') > 0 \). Hence, \( F(\beta, \beta') = 0 \). Since \( \beta \in N, \beta \)
is an element of $\Gamma$ and $n(\beta)<n(\eta)$. This inequality contradicts the assumption (1). Therefore, there does not exist in $\Lambda$ any weakly liouvillian element over $k$.

2. Examples. Let $k_0$ be an algebraically closed field of characteristic 0. We set $c'=0$ for all elements $c$ of $k_0$.

Example 1. Let us assume that $k=k_0$ and

$$F(y, y') = G(y, y')y^m + H(y),$$

where $m>0$, $G \in k[y, y']$, $H \in k[y]$. We set on $F$ the following conditions:

(vii) $H$ has only simple roots;
(viii) $\deg_y G < m$ and $\deg_y G < \deg_y H$;
(ix) $G(a, y') \neq 0$ for any root $a$ of $H$.

Then, $F$ is algebraically irreducible. Let us set on $F$ one more condition:

(x) $m>1$ and $m+\deg_{y, y'} G < \deg_y H$.

We prove that the assumption of our Theorem is satisfied by $F$. First suppose that $\nu_p(y)<0$. Then, $y=\tau^e$ with $e>0$. If $\nu_p(\tau')>0$, then $\nu_p(y') \geq -e$ and

$$\nu_p(Gy^m) \geq -e(m+\deg_{y, y'} G) > -e \cdot \deg_y H = \nu_p(H).$$

Secondly suppose that $\nu_p(y-a)>0$ for some root $a$ of $H$. Then, $y=a+\tau^e$ with $e>0$. If $\nu_p(\tau')>0$, then $\nu_p(y') \geq e$ and

$$\nu_p(Gy^m) \geq em > e = \nu_p(H).$$

Lastly suppose that $\nu_p(y-b)>0$ with an element $b$ of $k$ different from any root of $H$. If $\nu_p(\tau')>0$, then $\nu_p(y') \geq 1$ and

$$\nu_p(Gy^m) \geq m > 0 = \nu_p(H).$$

In any case we meet a contradiction if it is assumed that $\nu_p(\tau')>0$. Since

$$\frac{\partial F}{\partial y'} = y^{m-1}(mG+y' \partial G/\partial y'),$$

any nonsingular solution of $F=0$ is not a constant. Hence, by our Theorem, any nonsingular solution of $F=0$ is not a weakly liouvillian element over $k$.

Example 2. Let us assume that $k$ is the algebraic closure of the one-dimensional rational function field $k_0(x)$ over $k_0$ with $x'=1$, and that

$$F(y, y') = xy'-y(1-y)^n-x, \ n>0.$$

Then, it can be proved that any element of $k$ does not satisfy $F=0$. We show that the assumption of our Theorem is satisfied by $F$. First suppose that $\nu_p(y)<0$. Then, we have $y=\tau^{-1}$ and
Hence, $\nu_p(\tau') = 1 - n \leq 0$. Secondly suppose that $\nu_p(y - a) > 0$ with an element $a$ of $k$. Then, $y = a + \tau$. Since $a$ can not be a solution of $F = 0$, we have $\nu_p(\tau') \leq 0$. Hence, by our Theorem, any solution of $F = 0$ is not a weakly liouvillian element over $k$.

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**Bibliography**


