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## *Remarks on the Postulates of Metric Groups*

By Zen'ichiro KOSHIBA

### § 1. Introduction

Let  $E$  denote a topological space, which is an abstract group at the same time. But we do not mean by  $E$  a topological group in the ordinary sense. In this note we shall discuss about some relations among the following postulates under the condition of metric completeness, or under that of metric local compactness:

- (1)      *If  $\lim x_n = x$ , then  $\lim x_n y = xy$ ,*
- (2)      *if  $\lim y_n = y$ , then  $\lim xy_n = xy$ ,*
- (3)      *if  $\lim x_n = x$  and  $\lim y_n = y$ , then  $\lim x_n y_n = xy$ ,*
- (4)      *if  $\lim x_n = u$  and  $\lim x_n^{-1} = v$ , then  $u^{-1} = v$ ,*
- (5)      *if  $\lim x_n = x$ , then  $\lim x_n^{-1} = x^{-1}$ .*

Our results are the following two theorems:

**Theorem I:** *If  $E$  is a metric complete group, then the property (3) can be deduced from (1) and (2).*

**Theorem II:** *If  $E$  is a metric locally compact group, then the property (5) can be deduced from (1) and (2), and  $E$  is a metric locally compact group in the ordinary sense.*

BANACH gave in his postumas note<sup>1)</sup> a theorem that a metric complete group satisfying (1), (2) and (4) has the property (5). From it and Theorem I follows Theorem II (even in the case of metric completeness instead of metric locally compactness) as their logical consequence. But his proof in the non-separable case is not evident for us. In the following, let " $e$ " denote the unit element of the group  $E$ ,  $V_e$  or  $W_e$  spherical neighborhood of  $e$ ,  $S_r(x)$  the spherical neighborhood of  $x (\in E)$  with radius  $r$ , and  $d(x, y)$  the distance between  $x$  and  $y$ .  $\bar{S}_r(x)$  means the closure of  $S_r(x)$ .

### § 2. Proof of Theorem I.

Before the proof of theorem I we shall prove the following:

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1) Remarques sur les groupes et les corps métriques, *Studia Math.* 10 (1948), p. 178.

**Lemma 1:** For each element  $x_0$  of  $E$  and a natural number  $k$ , there exist an open set  $G$  and  $V_e$  such that

$$(6) \quad V_e \cdot G \subset \bar{S}_{k-1}(x_0)^{2)}.$$

**Proof.** It is evident from (1) that for each element  $y$  of  $\bar{S}_{(2k)-1}(x_0)$  there exists a  $W_e$  such that

$$(7) \quad W_e \cdot y \subset \bar{S}_{(2k)-1}(y).$$

Let  $r(y)$  denote the radius of  $W_e$  and  $A_i$  denote the set of all elements  $y$  of  $\bar{S}_{(2k)-1}(x_0)$  such that there exists a  $W_e$  as (7) whose radius  $r(y)$  satisfies the inequality  $(i+1)^{-1} \leq r(y) \leq i^{-1}$ <sup>3)</sup>. Then by the definition of  $A_i$  and (7) we have

$$\bar{S}_{(2k)-1}(x_0) = \bigcup_{i=1}^{\infty} A_i.$$

It is easily seen that  $\bar{S}_{(2k)-1}(x_0)$  is a set of the second category. Then if the closedness of  $A_i$  is proved, there must be some  $i_0$  such that  $A_{i_0}$  contains an open set  $G$ . This  $G$  is a desired one in (6) and as  $V_e$  there we may choose  $V_e, i_0^{-1}$  i. e. the neighborhood of  $e$  with radius  $i_0^{-1}$ .

The closedness of  $A_i$  can be proved as follows: Suppose that  $y_n \in A_i$  and  $\lim y_n = y_0$ , then  $y_0 \in \bar{S}_{(2k)-1}(x_0)$  and  $(i+1)^{-1} \leq r(y_n) \leq i^{-1}$ . Without loss of generality we can suppose  $\lim r(y_n) = r$ , for  $\{r(y_n)\}$  is bounded. Then we have  $W_e \cdot y_0 \subset \bar{S}_{(2k)-1}(y_0)$ , if we choose  $W_e$  with radius  $r$ . In fact, if it was  $xy_0 \notin \bar{S}_{(2k)-1}(y_0)$  for some  $x$  of  $W_e$ , then

$$(8) \quad d(xy_0, y_0) > (2k)^{-1}.$$

But, since  $\lim r(y_n) = r$ , we have  $x \in W_e, r(y_n)$  (the neighborhood of  $e$  with radius  $r(y_n)$ ) for sufficiently large  $n$ . Hence by (7) we get for sufficiently large  $n$

$$(9) \quad d(xy_n, y_n) < (2k)^{-1}.$$

As  $d(x, x')$  is a continuous function on  $E \times E$ , we obtain from (2) and (9)

$$d(xy_0, y_0) \leq (2k)^{-1},$$

which contradicts with (8). Lemma 1 is thus proved.

Now let  $G(k, x_0)$  be any but a definit non-empty open set which satisfies the relation (6) for some  $V_e$  but for fixed  $x_0$  and  $k$ . Let  $G_0$  be

2) For two subsets  $A$  and  $B$  of the group  $E$  we mean by  $A \cdot B$  the set of all elements  $xy$  such that  $x \in A, y \in B$ .

3) We can assume that  $0 < r(y) \leq 1$  for all  $y \in \bar{S}_{(2k)-1}(x_0)$ .

defined by

$$G_0 = \bigcap_{k=1}^{\infty} \bigcup_{x \in E} G(k, x),$$

then we have:

**Lemma 2.**  *$G_0$  is a set of the second category. Hence, it is not empty.*

*Proof.* It is evident that  $G(k) = \bigcup_{x \in E} G(k, x)$  is an open set.  $G(k)$  is also everywhere dense in  $E$  for each fixed  $k$ , for we have by the definition of  $G(k, x)$ ,  $G(k', x) \subset S_{k'-1}(x)$ ,  $0 \neq G(k', x) \subset G(k, x)$  for all  $k' > k$ . Then  $E - G_0 = \bigcup_{k=1}^{\infty} (E - G(k))$  is a set of the first category, since  $E - G(k)$  is closed and non-dense. Therefore  $G_0$  is a set of the second category. Lemma 2 is thus proved.

*Proof of Theorem I.* First suppose that  $\lim x_n = x$ ,  $\lim y_n = y$  and  $y \in G_0$ . For an arbitrary positive number  $\varepsilon$  there exist a  $k_0$  such that  $2/k_0 < \varepsilon/2$  and  $N_0$  such that

$$(10) \quad d(y_n, y) < \varepsilon/2 \quad \text{for sufficiently large } n.$$

Since  $y \in G_0$ , there exists a  $G(k_0, x_0)$  containing  $y$  and a  $V_\varepsilon$ .

$$(11) \quad V_\varepsilon \cdot G(k_0, x_0) \subset S_{k_0-1}(x_0).$$

And since  $G(k_0, x_0)$  is open, we have

$$(12) \quad y_n \in G(k_0, x_0) \quad \text{for sufficiently large } n.$$

Furthermore  $\lim x^{-1}x_n = e$  by (2), then we have

$$(13) \quad x^{-1}x_n \in V_\varepsilon \quad \text{for sufficiently large } n.$$

Then, since both  $x^{-1}x_n y_n$  and  $y_n$  are contained in  $S_{k_0-1}(x_0)$  by (11), (12), and (13), we have

$$(14) \quad d(x^{-1}x_n y_n, y_n) < 2/k_0 < \varepsilon/2 \quad \text{for sufficiently large } n.$$

Hence from (10) and (14)

$$d(x^{-1}x_n y_n, y) \leq d(x^{-1}x_n y_n, y_n) + d(y_n, y) < \varepsilon$$

for sufficiently large  $n$ , which says that  $\lim x^{-1}x_n y_n = y$ . Then by (2),  $\lim x_n y_n = xy$ .

Now for the case  $y \notin G_0$  we can take  $z_0$  such that  $yz_0 \in G_0$  since  $G_0$  is not empty. By (1),  $\lim y_n z_0 = yz_0$  ( $yz_0 \in G_0$ ), then we get by the first part of the proof that  $\lim x_n y_n z_0 = xy z_0$ . Therefore again by (1),  $\lim x_n y_n = xy$ . Thus the proof is completed.

This theorem holds also when  $E$  is metric locally compact.

### § 3. Proof of Theorem II.

Before the proof of Theorem II we shall prove the following:

**Lemma 3:** *For each element  $x_0$  of  $E$ , every neighborhood  $U(x_0)$  of  $x_0$ , and a natural number  $k$ , there exist a non-empty open set  $G'$  contained in  $U(x_0)$  and a natural number  $i_0$  such that*

$$(15) \quad y \cdot V_{k^{-1}} \supset S_{i_0^{-1}}(y) \quad \text{for all } y \in G',$$

where  $V_r$  is a spherical neighborhood of  $e$  with radius  $r$ .

*Proof.* We may assume that  $\bar{V}_{k^{-1}}$  is compact for all natural number  $k$ . Let us take  $U'(x_0)$  (neighborhood of  $x_0$ ) and  $r(>0)$  such that

$$\bar{U}'(x_0) \subset U(x_0) \quad \text{and} \quad r < k^{-1},$$

then  $\bar{U}'(x_0)$  is a set of the second category and  $V_r$  is compact.

Now let  $A_i'$  be the set of all elements of  $y$  of  $\bar{U}'(x_0)$  such that the relation  $y \cdot \bar{V}_r \supset S_{i^{-1}}(y)$ <sup>4)</sup> is satisfied, then we have

$$(16) \quad \bar{U}'(x_0) = \bigcup_{i=1}^{\infty} A_i'$$

$A_i'$  must be closed. In fact, suppose that  $\lim y_n = y_0$  and  $y_n \in A_i'$ , then  $y_n \cdot \bar{V}_r \supset S_{i^{-1}}(y_n)$ . Let  $x$  be an element of  $S_{i^{-1}}(y_0)$ , then  $d(x, y_n) < i^{-1}$  for sufficiently large  $n$ , therefore  $x$  can be represented in such a way that  $x = y_n v_n$ , where  $v_n \in \bar{V}_r$ . By the compactness of  $\bar{V}_r$  we may assume that  $\lim v_n = v_0 \in \bar{V}_r$ . Then by theorem I we have

$$x = y_0 v_0 \quad \text{where} \quad v_0 \in \bar{V}_r \subset V_{k^{-1}}.$$

Then from (16) and the closedness of  $A_i'$ , there must be some  $i_0$  such that  $A_{i_0}'$  contains an open set  $G'$ , which is a desired one in (15). Lemma 3 is thus proved.

Now let  $G'(k, x_0)$  be any but a definit non-empty open set which satisfies the relation (15) for some  $i_0$ , but for fixed  $x_0$  and  $k$ . Let  $G_0'$  be defined by

$$G_0' = \bigcap_{k=1}^{\infty} \bigcup_{x \in E} G(k, x),$$

then we have:

**Lemma 4:**  $G_0'$  is a set of the second category. Hence, it is not empty.

*Proof.*  $G'(k) = \bigcup_{x \in E} G(k, x)$  is open and everywhere dense in  $E$  for each fixed  $k$ , for there exists a non empty  $G'(k, x) \subset U(x)$  for every  $U(x)$  by Lemma 3. By the same way as in the proof of Lemma 2, we obtain easily that  $G_0'$  is a set of the second category, which was to be proved.

4) This is possible, since  $y \cdot \bar{V}_r$  is an open set containing  $y$  by (2).

*Proof of Theorem II.* By (2), for every neighborhood  $U(x^{-1})$  of  $x^{-1}$ , there exists a neighborhood  $V_{k_0^{-1}}$  of  $e$  such that

$$(17) \quad x^{-1} \cdot V_{k_0^{-1}} \subset U(x^{-1}).$$

Suppose that  $\lim x_n = x$ . Now take an element  $y_0$  of the non-empty  $G'_0$ . Then by (1) and (2) we have  $\lim y_0 x_n x^{-1} = y_0$ . By the definition of  $G'_0 (\ni y_0)$  there exists an open set  $G'(k_0, x_0)$  containing  $y_0$  such that

$$(18) \quad y \cdot V_{k_0^{-1}} \supset S_{i_0^{-1}}(y) \quad (i_0 = i_0(k_0)) \quad \text{for all } y \in G'(k_0, x_0).$$

Since  $G'(k_0, x_0)$  is open, (18) holds for  $y = y_0 x_n x^{-1}$  for sufficiently large  $n$ , i. e.

$$(19) \quad y_0 x_n x^{-1} V_{k_0^{-1}} \supset S_{i_0^{-1}}(y_0 x_n x^{-1}).$$

But since  $i_0^{-1}$  is independent of  $n$  and  $\lim y_0 x_n x^{-1} = y_0$ , we have

$$(20) \quad S_{i_0^{-1}}(y_0 x_n x^{-1}) \ni y_0 \quad \text{for sufficiently large } n.$$

Hence by (19) and (20),  $y_0 x_n x^{-1} V_{k_0^{-1}} \ni y_0$ , from which it follows that  $x_n^{-1} \in x^{-1} V_{k_0^{-1}}$ , then by (17)  $x_n^{-1} \subset U(x^{-1})$ . From this and Theorem I follows that  $E$  is a metric locally compact group in the ordinary sense.

#### § 4. Remark

In the general case without metric, the elements-convergence in the five postulates (1)-(5) should be replaced by suitable statements in the term of neighborhood: for example, (1) must be replaced by the following:

(1') If  $xy = z$ , then for an arbitrary neighborhood  $U(z)$  of  $z$ , there exists a neighborhood  $U(x)$  of  $x$  such that  $U(x) \cdot y \subset U(z)$ .

In such a general case as this we can give an example of the completely regular space for which Theorem I does not hold, as follows:

Let  $E = R^2$  denote a plane i. e, the set of all pairs of real numbers. The group-composition is defined by the ordinary vector-addition. We introduce topology into  $R^2$  by the definition of neighborhoods  $U(z)$  of  $z$  such that  $U(z) = S - A_\alpha$ , where  $S$  is a sphere of centre  $z = (x, y)$ , and  $A_\alpha$  is the set of all  $w = (u, v)$  which satisfies the inequality  $\left| \tan^{-1} \frac{v-y}{u-x} \right| \leq \alpha < \alpha_0 \left( < \frac{\pi}{2} \right)$ ,  $\alpha_0$  being a fixed constant.

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