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<td><strong>Author(s)</strong></td>
<td>Masuda, Kayo</td>
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<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 38(3) P.501–P.506</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>2001-09</td>
</tr>
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<td><strong>Text Version</strong></td>
<td>publisher</td>
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<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/4968">https://doi.org/10.18910/4968</a></td>
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<td><strong>DOI</strong></td>
<td>10.18910/4968</td>
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Masuda, K.
Osaka J. Math.
38 (2001), 501–506

MODULI OF ALGEBRAIC $SL_3$-VECTOR BUNDLES OVER ADJOINT REPRESENTATION

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(Received June 21, 1999)

1. Introduction and result

Let $G$ be a reductive complex algebraic group and $P$ a complex $G$-module. We consider algebraic $G$-vector bundles over $P$. An algebraic $G$-vector bundle $E$ over $P$ is an algebraic vector bundle $p: E \to P$ together with a $G$-action such that the projection $p$ is $G$-equivariant and the action on the fibers is linear. We assume that $G$ is non-abelian since every $G$-vector bundle over $P$ is isomorphic to a trivial $G$-bundle $P \times Q \to P$ for a $G$-module $Q$ when $G$ is abelian by Masuda-Moser-Petrie [12].

We denote by $\text{VEC}_G(P, Q)$ the set of equivariant isomorphism classes of algebraic $G$-vector bundles over $P$ whose fiber over the origin is a $G$-module $Q$. The isomorphism class of a $G$-vector bundle $E$ is denoted by $[E]$. The set $\text{VEC}_G(P, Q)$ is a pointed set with a distinguished class $[Q]$ where $Q$ is the trivial $G$-bundle $P \times Q$, and can be non-trivial when the dimension of the algebraic quotient space $P//G$ is greater than 0 ([15], [2], [13], [11]). In fact, Schwarz ([15], cf. Kraft-Schwarz [5]) showed that $\text{VEC}_G(P, Q)$ is isomorphic to an additive group $\mathbb{C}^p$ for a nonnegative integer $p$ determined by $P$ and $Q$ when $\dim P//G = 1$. When $\dim P//G \geq 2$, $\text{VEC}_G(P, Q)$ is not necessarily finite-dimensional. In fact, $\text{VEC}_G(P \otimes \mathbb{C}^m, Q) \cong (\mathbb{C}[y_1, \ldots, y_m])^p$ for a $G$-module $P$ with one-dimensional quotient $[9]$. Furthermore, Mederer [14] showed that $\text{VEC}_G(P, Q)$ can contain a space of uncountably-infinite dimension. He considered the case where $G$ is a dihedral group $D_m = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ and $P$ is a two-dimensional $G$-module $V_p$, on which $\mathbb{Z}/m\mathbb{Z}$ acts with weights $p$ and $-p$ and the generator of $\mathbb{Z}/2\mathbb{Z}$ acts by interchanging the weight spaces. Mederer showed that $\text{VEC}_{D_m}(V_1, V_1)$ is isomorphic to $\Omega^1_{\mathcal{C}}$, which is the universal Kähler differential module of $\mathcal{C}$ over $\mathcal{Q}$.

In this article, we show that under some conditions there exists a surjection from $\text{VEC}_G(P, Q)$ to $\text{VEC}_{D_m}(V_1, V_1) \cong \Omega^1_{\mathcal{C}}$. It is induced by taking a $H$-fixed point set $E^H$ for $[E] \in \text{VEC}_G(P, Q)$ where $H$ is a reductive subgroup of $G$ (cf. Proposition 2.3). In particular, we obtain the first example of a moduli space of uncountably-infinite dimension for a connected group.
Theorem 1.1. Let \( G = \text{SL}_3 \) and let \( \mathfrak{sl}_3 \) be the Lie algebra with adjoint action. Then for any \( G \)-module \( R \), there exists a surjection from \( \text{VEC}_G(\mathfrak{sl}_3 \oplus R, \mathfrak{sl}_3) \) onto \( \Omega^1_\mathbb{C} \). Hence \( \text{VEC}_G(\mathfrak{sl}_3 \oplus R, \mathfrak{sl}_3) \) contains an uncountably-infinite dimensional space.

At present, \( G \)-vector bundles over \( P \) are not yet classified for general \( G \)-modules \( P \) with \( \dim P/\!/G \geq 2 \) (cf. [10]). Theorem 1.1 suggests that the moduli space \( \text{VEC}_G(P, Q) \) is huge when \( \dim P/\!/G \geq 2 \).

I am thankful to M. Miyanishio for his help and encouragement. I thank the referees for giving advice to the previous version of this paper.

2. Proof of Theorem 1.1

Let \( G \) be a reductive algebraic group and let \( P \) and \( Q \) be \( G \)-modules. Let \( \pi_P : P \to P/\!/G \) be the algebraic quotient map. By Luna’s slice theorem [6], there is a finite stratification of \( P/\!/G = \bigcup_i V_i \) into locally closed subvarieties \( V_i \) such that \( \pi_P|_{\pi_P^{-1}(V_i)} : \pi_P^{-1}(V_i) \to V_i \) is a \( G \)-fiber bundle (in the étale topology) and the isotropy groups of closed orbits in \( \pi_P^{-1}(V_i) \) are all conjugate to a fixed reductive subgroup \( H_i \).

The unique open dense stratum of \( P/\!/G \), which we denote by \( U \), is called the principal stratum and the corresponding isotropy group, which we denote by \( H \), is called a principal isotropy group. We denote by \( \text{VEC}_G(P, Q)_0 \) the subset of \( \text{VEC}_G(P, Q) \) consisting of elements which are trivial over \( \pi_P^{-1}(U) \) and \( \pi_P^{-1}(V) \) for \( V := P/\!/G - U \).

When \( \dim P/\!/G = 1 \), it is known that \( \text{VEC}_G(P, Q) = \text{VEC}_G(P, Q)_0 \) ([15], [5]). We assume that the dimension of \( Y := P/\!/G \) is greater than 1 and the ideal of \( V \) is principal. We denote by \( \mathcal{O}(P) \) the \( \mathbb{C} \)-algebra of regular functions on \( P \) and by \( \mathcal{O}(P)^G \) the subalgebra of \( G \)-invariants of \( \mathcal{O}(P) \). Let \( f \) be a polynomial in \( \mathcal{O}(Y) = \mathcal{O}(P)^G \) such that the ideal \( (f) \) defines \( V \).

Lemma 2.1. Let \( [E] \in \text{VEC}_G(P, Q)_0 \). Then \( E \) is trivial over \( P_h := \{ x \in P \mid h(x) \neq 0 \} \) where \( h \) is a polynomial in \( \mathcal{O}(Y) \) such that \( h - 1 \in (f) \).

Proof. Since \( E|_{\pi_P^{-1}(V)} \) is, by the assumption, isomorphic to a trivial bundle, it follows from the Equivariant Nakayama Lemma [1] that the trivialization \( E|_{\pi_P^{-1}(V)} \to \pi_P^{-1}(V) \times Q \) extends to a trivialization over a \( G \)-stable open neighborhood \( \tilde{U} \) of \( \pi_P^{-1}(V) \). Let \( \tilde{V} \) be the complement of \( \tilde{U} \) in \( P \). Since \( \tilde{V} \) is a \( G \)-invariant closed set, \( \pi_P(\tilde{V}) \) is closed in \( Y \) [4]. Note that \( V \cap \pi_P(\tilde{V}) = \emptyset \) since \( \pi_P^{-1}(V) \cap \tilde{V} = \emptyset \). Let \( a \subset \mathcal{O}(Y) \) be the ideal which defines \( \pi_P(\tilde{V}) \). Then \( (f) + a \ni 1 \) since \( V \cap \pi_P(\tilde{V}) = \emptyset \). Hence there exists an \( h \in a \) such that \( h - 1 \in (f) \). Since \( P_h \subset \tilde{U} \), \( E \) is trivial over \( P_h \).

We define an affine scheme \( \tilde{Y} = \text{Spec} \tilde{A} \) by

\[
\tilde{A} = \{ h_1/h_2 \mid h_1, h_2 \in \mathcal{O}(Y), h_2 - 1 \in (f) \}.
\]
Set $\tilde{Y}_f := Y_f \times_Y \tilde{Y}$, $\tilde{P} := \tilde{Y} \times P$ and $\tilde{P}_f := \tilde{Y}_f \times_Y P$. The group of morphisms from $P$ to $M := \text{GL}(Q)$ is denoted by $\text{Mor}(P, M)$ or $M(P)$. The group $G$ acts on $M$ by conjugation and on $M(P)$ by $(g \mu)(\chi) = g \cdot (\mu(g^{-1})\chi)$ for $g \in G$, $\chi \in P$, $\mu \in M(P)$. The group of $G$-invariants of $M(P)$ is denoted by $\text{Mor}(P, M)^G$ or $M(P)^G$. Let $[E] \in \text{VEC}_G(P, Q)_0$. Then by definition of $\text{VEC}_G(P, Q)_0$, $E$ has a trivialization over $\pi_p^{-1}(U) = P_f$. By Lemma 2.1, $E$ has a trivialization also over an open neighborhood of $\pi_p^{-1}(V)$, i.e., $P_h$ for some $h \in \mathcal{O}(Y)$ with $h - 1 \in (f)$. Hence, $E$ is isomorphic to a $G$-vector bundle obtained by gluing two trivial $G$-vector bundles $P_f \times Q$ and $P_h \times Q$ over $P_{fh}$. Note that the transition function of $E$ is an element of $M(P_{fh})^G \subset M(\tilde{P}_f)^G$.

Conversely, if $\phi \in M(\tilde{P}_f)^G$ is given, then $\phi \in M(P_{fh})^G$ for some $h \in \mathcal{O}(Y)$ with $h - 1 \in (f)$ and we obtain a $G$-vector bundle $[E] \in \text{VEC}_G(P, Q)_0$ by gluing together trivial bundles $P_f \times Q$ and $P_h \times Q$ by $\phi$. Since $[E]$ is determined by the transition function $\phi \in M(P_{fh})^G$ up to automorphisms of trivial $G$-bundles $P_f \times Q$ and $P_h \times Q$, we have a bijection to a double coset (cf. [8, 3.4])

$$\text{VEC}_G(P, Q)_0 \cong M(P_f)^G \backslash M(\tilde{P}_f)^G / M(\tilde{P})^G.$$  

The inclusion $P^H \hookrightarrow P$ induces an isomorphism $P^H \backslash N(H) \cong P^H / G$ where $N(H)$ is the normalizer of $H$ in $G$. The stratification of $P^H / G$ coincides with the one induced by $P^H \backslash N(H)$ [7]. Set $W := N(H)/H$. When we consider $P^H$ as a $W$-module, we denote it by $B$. Let $L := \text{GL}(Q)^H$. By an observation similar to the case of $\text{VEC}_G(P, Q)_0$, we have

$$\text{VEC}_{N(H)}(P^H, Q)_0 \cong L(B_f)^W \backslash L(\tilde{B}_f)^W / L(\tilde{B})^W.$$  

Let $\beta : M(P)^G \to L(B)^W$ be the restriction map. We say $P$ has generically closed orbits if $\pi_p^{-1}(\xi)$ for any $\xi \in Y_f$ consists of a closed orbit, i.e. $\pi_p^{-1}(\xi) \cong G/H$. When $P$ has generically closed orbits, $P_f = GP_f^H$. Hence $M(P_f)^G = \text{Mor}(GP_f^H, GL(Q))^G \cong L(B_f)^W$, i.e. $\beta$ is an isomorphism over $Y_f$.

Let $[E] \in \text{VEC}_G(P, Q)_0$. The $H$-fixed point set $E^H$ is equipped with a $W$-vector bundle structure over $B$. The fiber of $E^H$ over the origin is a $W$-module $Q^H$. Hence there is a map

$$r_H : \text{VEC}_G(P, Q) \ni [E] \mapsto [E^H] \in \text{VEC}_W(B, Q^H)_0.$$  

Note that $r_H$ factors through $\text{VEC}_{N(H)}(P^H, Q)_0$ since the restricted bundle $[E]_{P^H}$ fixes to a Whitney sum of trivial $H$-bundles $[E]_{P^H}^H = E^H$. Note also that $r_H$ maps $\text{VEC}_G(P, Q)_0$ to $\text{VEC}_W(B, Q^H)_0$.

**Lemma 2.2.** Suppose that $P$ has generically closed orbits. Then

$$r_H : \text{VEC}_G(P, Q)_0 \to \text{VEC}_W(B, Q^H)_0.$$
is surjective.

Proof. By the above statement, it is sufficient to show that the restriction map
\[ \text{res} : \text{VEC}_G(P, Q) \ni [E] \mapsto [E]_{\mu^H} \in \text{VEC}_{N(H)}(P^H, Q) \] is surjective. Note that the map \( \text{res} \) coincides with the map on double cosets induced by \( \beta : M(P)^G \to L(B)^W; \)
\[ M(P_f)^G \backslash M(\hat{P}_f)^G / M(\hat{P})^G \to L(B_f)^W \backslash L(\hat{B}_f)^W / L(\hat{B})^W. \]

Let \( [E] \in \text{VEC}_{N(H)}(P^H, Q) \) and let \( \phi \in L(B_{fh})^W \), where \( h \in \mathcal{O}(Y) \) such that \( h - 1 \in (f) \), be the transition function corresponding to \( E \). Since \( \beta \) is an isomorphism over \( Y_f \), \( \phi \in L(B_{fh})^W \) extends to \( \tilde{\phi} \in M(P_{fh})^G \). The \( G \)-vector bundle \( \tilde{E} \) obtained by glueing trivial bundles over \( P_f \) and \( P_h \) by \( \tilde{\phi} \) is mapped to \( E \) by \( \text{res} \). \( \square \)

**Remark.** It seems that the restriction \( r_H : \text{VEC}_G(P, Q) \to \text{VEC}_W(B, Q^H) \) is not necessarily surjective, though the author does not know any counterexamples. Every \( G \)-vector bundle over a \( G \)-module is locally trivial [3], however, it seems difficult that a set of transition functions of a \( W \)-vector bundle over \( B \) with fiber \( Q^H \) extends to a set of transition functions of some \( G \)-vector bundle over \( P \) with fiber \( Q \); some conditions seem to be needed so that the restriction \( M(X)^G \to L(X^H)^W \) is surjective for a \( G \)-stable open set \( X \) of \( P \) such that \( X \not\subseteq \pi_{p\cdot}(U) \) (cf. [17, III,11]).

For any reductive subgroup \( K \) of \( G \), we can construct a map \( r_K \) similarly;
\[ r_K : \text{VEC}_G(P, Q) \ni [E] \mapsto [E^K] \in \text{VEC}_W(P^K, Q^K) \]
where \( W_K := N(K)/K. \) Assume that \( W_K \) contains a subgroup isomorphic to \( D_3 \) and that \( P^K \) and \( Q^K \) contain \( V_1 \) as \( D_3 \)-modules, say, as \( D_3 \)-modules \( P^K = V_1 \oplus P' \) and \( Q^K = V_1 \oplus Q' \) for \( D_3 \)-modules \( P' \) and \( Q' \). Restricting the group \( W_K \) to \( D_3 \), we have a map
\[ \text{VEC}_W(P^K, Q^K) \to \text{VEC}_{D_3}(V_1 \oplus P', V_1 \oplus Q'). \]

Furthermore, the natural inclusion \( V_1 \to V_1 \oplus P' \) induces a surjection
\[ \text{VEC}_{D_3}(V_1 \oplus P', V_1 \oplus Q') \to \text{VEC}_{D_3}(V_1, V_1 \oplus Q'). \]

By taking a composite of the maps \( r_K \), (1) and (2), we obtain a map \( \Phi_K : \text{VEC}_G(P, Q) \to \text{VEC}_{D_3}(V_1, V_1 \oplus Q') \). By Mederer [14], \( \text{VEC}_{D_3}(V_1, V_1 \oplus Q') \cong \Omega_1^1 \), where \( \Omega_1^1 \) is a subspace of \( \Omega_1^2 \), but unfortunately, \( \Omega_1^1 \) is not known so far except when \( Q' = \{0\} \). When \( Q' = \{0\} \), i.e. \( Q^K \cong V_1 \) as a \( D_3 \)-module, we have a map
\[ \Phi_K : \text{VEC}_G(P, Q) \to \text{VEC}_{D_3}(V_1, V_1) \cong \Omega_1^1. \]
In the case where \( K = H \) and \( N(H)/H \cong D_3 \), the map \( \Phi_H \) constructed as above can be surjective.

**Proposition 2.3.** Let \( H \) be a principal isotropy group of \( P \) and let \( N(H)/H \cong D_3 \). Suppose that \( P \) has generically closed orbits. If \( P^H \) contains \( V_1 \) as a \( D_3 \)-module and \( Q^H \cong V_1 \) as a \( D_3 \)-module, then the map

\[
\Phi_H: \text{VEC}_G(P, Q) \to \Omega^1_C
\]

is surjective.

**Proof.** The assertion follows from Lemma 2.2 and the fact that \( \text{VEC}_{D_3}(V_1, V_1) = \text{VEC}_{D_3}(V_1, V_1) \cong \Omega^1_C \) \[14\].

The condition on the fiber \( Q \) in Proposition 2.3 is rather strict. However, by Proposition 2.3, we obtain the first example of a moduli space of uncountably-infinite dimension for a connected group \( G \).

**Proof of Theorem 1.1.** Let \( G = SL_3 \) and let \( \mathfrak{sl}_3 \) be the Lie algebra with adjoint action. A principal isotropy group of \( \mathfrak{sl}_3 \) is a maximal torus \( T \cong (\mathbb{C}^*)^2 \) and \( \mathfrak{sl}_3^T \) is the Lie algebra \( t \) of \( T \). \( N(T)/T \) is the Weyl group which is isomorphic to the symmetric group \( S_3 \cong D_3 \) and \( \mathfrak{sl}_3^T = t \cong V_1 \) as a \( D_3 \)-module. The algebraic quotient space is \( \mathfrak{sl}_3//G \cong t//S_3 \cong \mathbb{A}^2 \). The complement of the principal stratum in \( \mathfrak{sl}_3//G \cong \mathbb{A}^2 \) is defined by \( y^2 - x^3 = 0 \). The general fiber of the quotient map \( \mathfrak{sl}_3 \rightarrow \mathfrak{sl}_3//G \) is isomorphic to \( G/T \) and \( \mathfrak{sl}_3 \) has generically closed orbits. Applying Proposition 2.3 to the case where \( P = \mathfrak{sl}_3 \) and \( Q = \mathfrak{sl}_3 \), we obtain a surjection \( \text{VEC}_G(\mathfrak{sl}_3, \mathfrak{sl}_3) \rightarrow \Omega^1_C \). Since there is a surjection \( \text{VEC}_G(\mathfrak{sl}_3 \oplus R, \mathfrak{sl}_3 \oplus R) \rightarrow \text{VEC}_G(\mathfrak{sl}_3, \mathfrak{sl}_3) \) induced by the inclusion \( \mathfrak{sl}_3 \rightarrow \mathfrak{sl}_3 \oplus R \) for any \( G \)-module \( R \), Theorem 1.1 follows.

**References**


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