Nishitani, T.
Osaka J. Math.
21 (1984), 843-850

A NOTE ON REDUCED FORMS OF EFFECTIVELY HYPERBOLIC OPERATORS AND ENERGY INTEGRALS

TATSUO NISHITANI

(Received May 2, 1983)

1. Introduction

Let P be the principal symbol of a hyperbolic differential operator. At a double characteristic point of P, its Taylor expansion begins with the quadratic form in the cotangent bundle. The coefficient matrix of the Hamiltonian system associated with this quadratic form is called the fundamental (or Hamilton) matrix. If the fundamental matrix has non-zero real eigenvalues, P is said to be effectively hyperbolic operator ([1], [2]).

Ivrii and Petkov conjectured in [2] that C^∞ Cauchy problem for effectively hyperbolic operators is well posed for any lower order term; that is effectively hyperbolic operator is strongly hyperbolic.

In this note, in §2, we reduce effectively hyperbolic operators of second order to certain standard forms by homogeneous canonical transformations. Since we are concerned with the Cauchy problem, we shall use only homogeneous canonical transformations which do not depend on the time and its dual variables. In §3, for some simple but essential examples, we indicate how the standard forms relate to the energy integrals which assure the strong hyperbolicity.

The detailed proofs of deriving the energy estimates for effectively hyperbolic operators of the standard forms will be appear elsewhere.

Denote $x^{(p)}=(x_p, \cdots, x_d)$, $\xi^{(p)}=(\xi_p, \cdots, \xi_d)$, $x=x^{(0)}$, $\xi=\xi^{(0)}$, $0\leq p\leq d$, and consider

$$P(x, \xi) = \xi_0^2-Q(x, \xi^{(1)}),$$

where $Q(x, \xi^{(1)})$ is defined in a conic neighborhood of $(0, \bar{\xi}^{(1)})$, non-negative and homogeneous of degree 2 in $\xi^{(1)}$.

Let $(0, \bar{\xi})$ be a double characteristic point of $P(x, \xi)$. That is $dP(x, \xi)$ vanishes at $(0, \bar{\xi})$. This is the same thing as $\bar{\xi}=(0, \bar{\xi}^{(1)})$, $Q(0, \bar{\xi}^{(1)})=0$. Denote by $F_P(x, \xi)$ the fundamental matrix evaluated at (x, ξ) (for the precise definition, see [2]). In the following, $\{,\}$ denotes the Poisson bracket. The standard forms are the followings.

Theorem 1.1. *Assume that $F_P(0, \hat{\xi})$ has non-zero real eigenvalues. Then, in a conic neighborhood of $(0, \hat{\xi}^{(1)})$, there exists a homogeneous canonical transformation in T^*R^d taking $(0, \hat{\xi}^{(1)})$ to $(0, \hat{\xi}^{(1)})$ under which $Q(x, \xi^{(1)})$ is transformed to $(1.1)_p$ with $(1.1)'_p$ or $(1.2)_p$ with $(1.2)'_p$ and $(1.2)''_p$.*

$$(1.1)_p \quad \sum_{i=1}^{p} (x_{i-1}-x_i)^2 q_i(x, \xi^{(1)})+\sum_{i=1}^{p} \xi_i^2 r_i(x, \xi^{(1)})+\{(x_p-\phi_p(x^{(p+1)}, \xi^{(p+1)}))^2+ +\psi_p(x^{(p+1)}, \xi^{(p+1)})\} q_{p+1}(x, \xi^{(1)}),$$

$$(1.1)'_p \quad \{\phi_p, \{\phi_p, \psi_p\}\}(0, \hat{\xi}^{(p+1)}) = 0, \qquad 0\leq p\leq d-1,$$

$$(1.2)_p \quad \sum_{i=1}^{p} (x_{i-1}-x_i)^2 q_i(x, \xi^{(1)})+\sum_{i=1}^{p} \xi_i^2 r_i (x, \xi^{(1)})+g_p(x^{(p)}, \xi^{(p+1)}) r_{p+1}(x, \xi^{(1)})$$

$$(1.2)'_p \quad \{\xi_p, \{\xi_p, g_p\}\} (0, \hat{\xi}^{(p+1)}) = 0$$

$$(1.2)''_p \quad \sum_{i=1}^{p} r_i(0, \hat{\xi}^{(1)})^{-1}>1, \ 1\leq p\leq d-1,$$

where $q_i(x, \xi^{(1)})$, $r_i(x, \xi^{(1)})$ are positive, homogeneous of degree 2, 0 respectively, ψ_p, g_p are non-negative, vanishing at $(0, \hat{\xi}^{(p+1)})$, homogeneous of degree 0, 2 respectively and ϕ_p is homogeneous of degree 0.

REMARK 1.1. The condition $(1.2)''_p$ is closely related with the energy integrals, see §3.

2. Proof of Theorem 1.1

First, we shall prove the following lemma which also will be useful to study the standard forms for non effectively hyperbolic operators.

Lemma 2.1. *Let $\partial_0^2 Q(0, \hat{\xi}^{(1)})>0$. Then $Q(x, \xi^{(1)})$ is transformed to $(1.1)_p$ with $(1.1)'_p$ $(0\leq p\leq d-1)$ or $(1.2)_p$ with $(1.2)'_p$ $(1\leq p\leq d-1)$, by a local homogeneous canonical transformation in T^*R^d which takes $(0, \hat{\xi}^{(1)})$ to $(0, \hat{\xi}^{(1)})$.*

Proof. From the Malgrange preparation theorem, we get

$$Q(x, \xi^{(1)}) = \{(x_0-\phi_0(x^{(1)}, \xi^{(1)}))^2+\psi_0(x^{(1)}, \xi^{(1)})\} q_1(x, \xi^{(1)}),$$

where ϕ_0, ψ_0 are homogeneous of degree 0 with $\psi_0\geq 0$ and q_1 is positive, homogeneous of degree 2. This is just $(1.1)_0$.

Now we assume that $(1.1)'_{p-1}$ is not satisfied. Set $X_p(x^{(p)}, \xi^{(p)})=\phi_{p-1}(x^{(p)}, \xi^{(p)})$. Then it follows that $d\phi_{p-1}$ and $\sum_{j=p}^{d} \xi_j\, dx_j$ are linearly independent at $(0, \hat{\xi}^{(p)})$. In fact, if $d\phi_{p-1}$ were to be proportional to $\sum_{j=p}^{d} \xi_j\, dx_j$ at $(0, \hat{\xi}^{(p)})$, taking into account that $d\psi_{p-1}(0, \hat{\xi}^{(p)})=0$, the Euler's identity would give (since ψ_{p-1} is homogeneous of degree 0),

$$\{\phi_{p-1}, \{\phi_{p-1}, \psi_{p-1}\}\}(0, \hat{\xi}^{(p)}) = 0.$$

This contradicts to our assumption. Thus following proposition 3.11 in Melrose [3], we can construct a homogeneous canonical transformation $[X_j(x^{(p)}, \xi^{(p)}), \Xi_j(x^{(p)}, \xi^{(p)})]_{j=p}^d$ so that

(2.1) $X_j(0, \hat{\xi}^{(p)}) = 0, p \le j \le d, \Xi_j(0, \hat{\xi}^{(p)}) = 0, p \le j \le d-1, \Xi_d(0, \hat{\xi}^{(p)}) \neq 0$.

After having done this transformation, remarking that

$$(\partial^2 \psi_{p-1}/\partial \xi_p^2)(0, \hat{\xi}^{(p)}) \neq 0, \hat{\xi}^{(p)} = (0, \cdots, 0, \hat{\xi}_d), \hat{\xi}_d \neq 0,$$

the Malgıange preparation theorem gives that

$$\psi_{p-1}(x^{(p)}, \xi^{(p)}) = \{(\xi_p - h_p(x^{(p)}, \xi^{(p+1)}))^2 + k_p(x^{(p)}, \xi^{(p+1)})\}\, b_p(x^{(p)}, \xi^{(p)}),$$

where b_p is positive, homogeneous of degree -2, k_p is non-negative, homogeneous of degree 2 and h_p is homogeneous of degree 1. Take

$$\Xi_p(x^{(p)}, \xi^{(p)}) = \xi_p - h_p(x^{(p)}, \xi^{(p+1)}),\ X_p(x^{(p)}, \xi^{(p)}) = x_p.$$

It is clear that $\{\Xi_p, X_p\} = 1$. Moreover the differentials $\sum_{j=p}^d \xi_j\, dx_j$, $d\Xi_p$, dX_p are linearly independent at $(0, \hat{\xi}^{(p)})$. Indeed, if there were to be a dependence relation

$$\sum_{j=p}^d \xi_j\, dx_j = \alpha d\Xi_p + \beta dX_p \quad \text{at} \quad (0, \hat{\xi}^{(p)}),$$

then applying this to H_{X_p}, the Hamilton vector field of X_p, would give $\alpha=0$, hence

$$\sum_{j=p}^d \xi_j\, dx_j = \beta dX_p.$$

But this gives a contradiction because $\hat{\xi}^{(p)} = (0, \cdots, 0, \hat{\xi}_d)$, $dX_p = dx_p$ and $p \le d-1$. Again, from proposition 3.11 in [3], one can extend Ξ_p, X_p to a homogeneous canonical transformation $[X_j(x^{(p)}, \xi^{(p)}), \Xi_j(x^{(p)}, \xi^{(p)})]_{j=p}^d$ satisfying (2.1). Since $0 = \{\xi_j, x_p\} = \{\xi_j, X_p\} = -\partial \xi_j/\partial \Xi_p$, $p+1 \le j \le d$, $0 = \{x_j, x_p\} = \{x_j, X_p\} = -\partial x_j/\partial \Xi_p$, $p \le j \le d$, it follows that $\xi_j(X^{(p)}, \Xi^{(p)})$ $(p+1 \le j \le d)$ and $x_j(X^{(p)}, \Xi^{(p)})$ $(p \le j \le d)$ do not depend on Ξ_p. Thus we get $(1.2)_p$ with $g_p(x^{(p)}, \xi^{(p+1)}) \ge 0$, $r_{p+1} > 0$ being homogeneous degree 2, 0 respectively.

Finally, assume that $(1.2)'_p$ does not hold. Then one can write

$$g_p(x^{(p)}, \xi^{(p+1)}) = \{(x_p - \phi_p(x^{(p+1)}, \xi^{(p+1)}))^2 + \psi_p(x^{(p+1)}, \xi^{(p+1)})\}\, a_p(x^{(p)}, \xi^{(p+1)}),$$

with ϕ_p, ψ_p which are homogeneous of degree 0 and $\psi_p \ge 0$, where a_p is positive, homogeneous of degree 2. This yields $(1.1)_p$. Therefore, the induction on p proves this lemma.

We proceed to the proof of theorem 1.1. If $\partial_0^2 Q(0, \bar{\xi}^{(1)}) = 0$, it is easily seen that $F_P(0, \bar{\xi})$ has only pure imaginary eigenvalues (cf. [1], [2]). Then we may suppose that $\partial_0^2 Q(0, \xi^{(1)})>0$. Applying lemma 2.1, $Q(x, \xi^{(1)})$ is reduced to $(1.1)_p$ with $(1.1)_p'$ $(0\leq p\leq d-1)$ or $(1.2)_p$ with $(1.2)_p'$ $(1\leq p\leq d-1)$.

We note that the fundamental matrix is transformed to a similar matrix by a canonical transformation. Therefore to prove theorem 1.1, it suffices to show that an operator P with Q of type $(1.1)_p$ with $(1.1)_p'$ is in fact effectively hyperbolic and an operator P with Q of type $(1.2)_p$ with $(1.2)_p'$ is effectively hyperbolic if and only if $(1.2)_p''$ holds. The following two propositions are easily verified.

Proposition 2.1. *Let*

$$\hat{P} = \xi_0^2-\sum_{i=1}^{p} q_i(x_{i-1}-x_i)^2-\sum_{i=1}^{p-1} r_i\, \xi_i^2,\ q_i>0,\ r_i>0\,.$$

Then we have

$$\det F_{\hat{P}} = -\Big(\prod_{j=1}^{p} 4q_j\Big) \Big(\prod_{j=1}^{p-1} r_j\Big)\,.$$

Here $\hat{P}(x, \xi)$ is considered as a function of $(x_0, \cdots, x_{p-1}, \xi_0, \cdots, \xi_{p-1})$.

Proposition 2.2. *Let*

$$\hat{P} = \xi_0^2-\sum_{i=1}^{p} q_i(x_{i-1}-x_i)^2-\sum_{i=1}^{p} r_i\, \xi_i^2,\ q_i>0,\ r_i>0\,.$$

Then we have

$$\det(\lambda+F_{\hat{P}}) = \Phi(\lambda, q_i, r_i) = \lambda^2\, \psi(\lambda, q_i, r_i),\ \psi(0, q_i, r_i) = \\ = -\Big(\prod_{j=1}^{p} 4q_j\Big) \Big(\prod_{j=1}^{p} r_j\Big) \Big(\sum_{j=1}^{p} r_j^{-1}-1\Big)\,.$$

Here $\hat{P}(x, \xi)$ is considered as a function of $(x_0, \cdots, x_p, \xi_0, \cdots, \xi_p)$.

First we consider the case when Q is of the form $(1.1)_p$ with $(1.1)_p'$. We denote by r_i, q_i the value of $r_i(x, \xi^{(1)})$, $q_i(x, \xi^{(1)})$ at $(0, \hat{\xi}^{(1)})$. Since ϕ_p, ψ_p depend only on $(x^{(p+1)}, \xi^{(p+1)})$ and $\{\phi_p, \{\phi_p, \psi_p\}\}\,(0, \hat{\xi}^{(p+1)})=0$, it follows that

$$\det(\lambda+F_P(0, \hat{\xi})) = \det(\lambda+F_{\hat{P}}) \det(\lambda+F_E)\,,$$

where

$$\hat{P} = \xi_0^2-\sum_{i=1}^{p} q_i(x_{i-1}-x_i)^2-\sum_{i=1}^{p} r_i\, \xi_i^2-q_{p+1}\, x_p^2$$

is considered as a function of $(x_0, \cdots, x_p, \xi_0, \cdots, \xi_p)$ and E is a non-negative quadratic form in $(x^{(p+1)}, \xi^{(p+1)})$. By proposition 2.1, we get

$$\Phi(\lambda) = \det(\lambda+F_{\hat{P}}) = \lambda^{2p+2}+\cdots+\Phi(0),\ \Phi(0) = -\Big(\prod_{j=1}^{p+1} 4q_j\Big) \Big(\prod_{j=1}^{p} r_j\Big)<0\,.$$

This shows that $\Phi(\lambda)=0$ has non-zero real roots.

Next we consider the case when Q has the form $(1.2)_p$ with $(1.2)'_p$. Put

$$\hat{P} = \xi_0^2-\sum_{i=1}^{p} q_i(x_{i-1}-x_i)^2-\sum_{i=1}^{p} r_i\,\xi_i^2,\ E = -g_p(x^{(p)}, \xi^{(p+1)})\, r_{p+1}(x, \xi^{(1)})\,.$$

From the non-negativity of g_p, taking that $(\partial^2 g_p/\partial x_p^2)\,(0, \hat{\xi}^{(p+1)})=0$ into account, it follows that

$$\partial^2 g_p/\partial x_\mu \partial x_p = 0,\ p\leq\mu\leq d,\ \partial^2 g_p/\partial \xi_\mu \partial x_p = 0,\ p+1\leq\mu\leq d \quad \text{at} \quad (0, \hat{\xi}^{(p+1)})\,.$$

Then the same reasoning as before shows that

$$\det(\lambda+F_P(0, \hat{\xi})) = \det(\lambda+F_{\hat{P}})\det(\lambda+F_E(0, \hat{\xi}))\,,$$

where $\hat{P}$, E are considered as functions of $(x_0, \cdots, x_p, \xi_0, \cdots, \xi_p)$, $(x^{(p+1)}, \xi^{(p+1)})$ respectively. From proposition 2.2, we get

$$\det(\lambda+F_{\hat{P}}) = \lambda^2\psi(\lambda, q_i, r_i)\,,\quad \psi(\lambda, q_i, r_i) = \lambda^{2p}+\cdots+\psi(0, q_i, r_i)\,,$$

$$\psi(0, q_i, r_i)=-(\prod_{j=1}^{p} 4q_j)\,(\prod_{j=1}^{p} r_j)\,\{\sum_{j=1}^{p} r_j^{-1}-1\}\,.$$

From [2], the equation $\psi(\lambda, q_i, r_i)=0$ has only pure imaginary roots except for at most one simple real root μ ($\neq 0$) and for $-\mu$, for any $q_i, r_i>0$. Since $\psi(\lambda, q_i, r_i)$ depends continuously on q_i, r_i, in order that the equation $\psi(\lambda, q_i, r_i)=0$ has a non-zero real root, it is necessary and sufficient that $\psi(0, q_i, r_i)$ is negative. Taking into account that $\det(\lambda+F_E)=0$ has only pure imaginary roots (since E is non-negative), $F_P(0, \hat{\xi})$ has a non-zero real eigen value if and only if the condition $(1.2)''_p$ holds.

These facts prove theorem 1.1.

3. Energy integrals

As a simple example, we shall indicate some connections between the condition $(1.2)''_p$ and the energy integrals. Let us consider the following operator,

$$P = \partial_0^2-\sum_{i=0}^{p-1} q_i(x_i-x_{i+1})^2\,\partial_d^2-\sum_{i=1}^{p} r_i\,\partial_i^2\,,$$

where $1\leq p<d$, $q_i>0$ $(0\leq i\leq p-1)$, $r_i>0$ $(1\leq i\leq p-1)$, $r_p\geq 0$. We assume that P is effectively hyperbolic. If $r_p>0$, the effective hyperbolicity means that $\sum_{i=1}^{p} r_i^{-1}>1$.

Here, we note that the condition $\sum_{i=1}^{p} r_i^{-1}>1$ $(r_i>0)$, is equivalent to the existence of real numbers $\{\varepsilon_i\}_{i=1}^{p}$ such that

$$(3.1)\qquad \sum_{i=1}^{p} \varepsilon_i^2 r_i<1,\ \sum_{i=1}^{p} \varepsilon_i = 1 .$$

Taking (3.1) into account, we use the following weight (or separating) function,

$$Y(x) = x_0-\sum_{i=1}^{p} \varepsilon_i x_i ,$$

with $\{\varepsilon_i\}_{i=1}^{p}$ in (3.1) if $r_p>0$ and $\varepsilon_j=0$, $1\leq j\leq p-1$, $\varepsilon_p=1$ if $r_p=0$. Then the integration by parts gives that

$$(3.2)\quad -2Re\int_{\Omega^-} Y(x)^n Pu\cdot Y(x)^n\overline{\partial_0 u}dx = 2n\int_{\Omega^-} Y(x)^{2n-1}|\partial_0 u|^2dx+$$
$$+2n\int_{\Omega^-} Y(x)^{2n-1}\{\sum_{i=0}^{p-1} q_i(x_i-x_{i+1})^2\}|\partial_d u|^2dx+$$
$$+2n\int_{\Omega^-} Y(x)^{2n-1}\{\sum_{i=1}^{p} r_i|\partial_i u|^2\}dx+2q_0\int_{\Omega^-} Y(x)^{2n}(x_0-x_1)|\partial_d u|^2dx+$$
$$+2n\int_{\Omega^-} Y(x)^{2n-1}\{\sum_{i=1}^{p} \varepsilon_i r_i(\partial_i u\cdot\overline{\partial_0 u}+\overline{\partial_i u}\cdot\partial_0 u)\}dx .$$

for $u\in C_0^\infty(R^{d+1})$, where $\Omega^\pm=\{x\in R^{d+1};\ Y(x)\gtrless 0\}$, and n is a positive integer. We shall estimate the last term of the right hand of (3.2). Let $r_p>0$, then by the Cauchy–Schwarz inequality, it follows that

$$2n\sum_{i=1}^{p} \varepsilon_i r_i(\partial_i u\cdot\overline{\partial_0 u}+\overline{\partial_i u}\cdot\partial_0 u)\leq 2n\delta\sum_{i=1}^{p} r_i|\partial_i u|^2+2n\delta^{-1}(\sum_{i=1}^{p} \varepsilon_i^2 r_i)|\partial_0 u|^2 ,$$

with $\delta>0$. From (3.1), we can take δ so that

$$\tilde{\delta} = \delta^{-1}(\sum_{i=1}^{p} \varepsilon_i^2 r_i)<1,\ \delta<1 .$$

On the other hand, from $\sum_{i=1}^{p} \varepsilon_i=1$, we have $|Y(x)(x_0-x_1)|\leq c\sum_{i=0}^{p-1} q_i(x_i-x_{i+1})^2$, with some $c>0$. Hence the right hand side of (3.2) is estimated from below by

$$2n(1-\tilde{\delta})\int_{\Omega^-} Y(x)^{2n-1}|\partial_0 u|^2dx+(2n-c)\int_{\Omega^-} Y(x)^{2n-1}\{\sum_{i=0}^{p-1} q_i(x_i-x_{i+1})^2\}|\partial_d u|^2 dx+$$
$$+2n(1-\delta)\int_{\Omega^-} Y(x)^{2n-1}\{\sum_{i=1}^{p} r_i|\partial_i u|^2\} dx .$$

In the case $r_p=0$, the last term of the right hand side of (3.2) is equal to zero, and we have the following estimate from below,

$$2n\int_{\Omega^-} Y(x)^{2n-1}|\partial_0 u|^2 dx+(2n-c)\int_{\Omega^-} Y(x)^{2n-1}\{\sum_{i=0}^{p-1} q_i(x_i-x_{i+1})^2\}|\partial_d u|^2 dx+$$
$$+2n\int_{\Omega^-} Y(x)^{2n-1}\{\sum_{i=1}^{p-1} r_i|\partial_i u|^2\} dx .$$

Now let us consider the first order term $B\partial_d$, $B\in C$. From the following two inequalities

$$2Re\int_{\Omega^-} Y(x)^n B\partial_d u\cdot Y(x)^n\overline{\partial_0 u}\,dx\le 2\delta_1^{-1}\,n^{-1}|B|^2\int_{\Omega^-} Y(x)^{2n+1}|\partial_d u|^2\,dx+$$
$$+\delta_1 n\int_{\Omega^-} Y(x)^{2n-1}|\partial_0 u|^2\,dx+\delta_1 n^3\int_{\Omega^-} Y(x)^{2n-3}|u|^2\,dx\,,$$
$$n(n-2)\int_{\Omega^-} Y(x)^{2n-3}|u|^2\,dx\le\int_{\Omega^-} Y(x)^{2n-1}|\partial_0 u|^2\,dx\,,$$

it follows that

$$2Re\int_{\Omega^-} Y(x)^n B\partial_d u\cdot Y(x)^n\overline{\partial_0 u}\,dx\le 2\delta_1^{-1}\,n^{-1}|B|^2\int_{\Omega^-} Y(x)^{2n+1}|\partial_d u|^2\,dx+$$
$$+3\delta_1 n\int_{\Omega^-} Y(x)^{2n-1}|\partial_0 u|^2\,dx,\quad n\ge 4\,.$$

Then using the inequality $Y(x)^2\le\hat{c}\sum_{i=0}^{p-1} q_i(x_i-x_{i+1})^2$, we get finally

$$(3.3)\quad 2Re\int_{\Omega^-} Y(x)^n B\partial_d\, u\cdot Y(x)^n\overline{\partial_0 u}\,dx\le 2\delta_1^{-1}\,n^{-1}\,\hat{c}|B|^2\int_{\Omega^-} Y(x)^{2n-1}\times$$
$$\times\{\sum_{i=0}^{p-1} q_i(x_i-x_{i+1})^2\}\,|\partial_d u|^2\,dx+3\delta_1 n\int_{\Omega^-} Y(x)^{2n-1}|\partial_0 u|^2\,dx\,.$$

Now we take δ_1, n so that $2(1-\delta)-3\delta_1>0$ and $n(2n-c)>2\delta_1^{-1}\,\hat{c}|B|^2$, then one can absorb the first order term $B\partial_d$.

Proposition 3.1. *Assume that P is effectively hyperbolic. Then we have*

$$\int_{\Omega^-} Y(x)^{2n+1}|(P+B\partial_d)u|^2\,dx\ge 2n\delta_2\int_{\Omega^-} Y(x)^{2n-1}|\partial_0 u|^2\,dx+$$
$$+c_2 n\int_{\Omega^-} Y(x)^{2n-1}\,\{\sum_{i=0}^{p-1} q_i(x_i-x_{i+1})^2\}\,|\partial_d u|^2\,dx+$$
$$+2\delta_2 n\int_{\Omega^-} Y(x)^{2n-1}\,\{\sum_{i=1}^{p} r_i|\partial_i u|^2\}\,dx\,,$$

where $n\ge c_3|B|$.

By a similar way, one can obtain the energy estimate in Ω^+ (cf. [4]).

Finally, we consider a simple example corresponding to operators of type $\{(1.1)_p, (1.1)'_p\}$. Let

$$P=\xi_0^2-\sum_{i=0}^{p-1} q_i(x_i-x_{i+1})^2\,\xi_d^2-\sum_{i=1}^{p} r_i\,\xi_i^2-q_p(x_p-\phi(\xi''))^2\,\xi_d^2\,,$$

where $q_i>0$, $r_i>0$, $\xi''=(\xi_{p+1},\cdots,\xi_d)$, $p+1\le d$. Taking the Fourier transform with respect to $x''=(x_{p+1},\cdots,x_d)$, it suffices to consider

$$\tilde{p} = \partial_0^2 + \sum_{i=0}^{p-1} q_i(x_i - x_{i+1})^2\, \xi_d^2 - \sum_{i=1}^{p} r_i\, \partial_i^2 + q_p(x_p - \phi(\xi''))^2\, \xi_d^2 .$$

As a separating function, we take

$$Y(x, \xi) = x_0 - \phi(\xi'') . \tag{3.4}$$

Denote by $\tilde{u}(x', \xi'')$ the partial Fourier transform with respect to x''. Then the same reasoning as before gives that

$$\int_{\omega^-} Y(x, \xi)^{2n+1} |(\tilde{p} + B\xi_d)\, \tilde{u}|^2\, dx' \geq 2n\delta_2 \int_{\omega^-} Y(x, \xi)^{2n-1} |\partial_0 \tilde{u}|^2\, dx' +$$

$$+ c_2 n \int_{\omega^-} Y(x, \xi)^{2n-1} \{\sum_{i=0}^{p-1} q_i(x_i - x_{i+1})^2 + q_p(x_p - \phi(\xi''))^2\} |\xi_d \tilde{u}|^2 dx' +$$

$$+ 2n \int_{\omega^-} Y(x, \xi)^{2n-1} \{\sum_{i=1}^{p} r_i |\partial_i \tilde{u}|^2\}\, dx' ,$$

for $n \geq c_3 |B|$, where $\omega^\pm = \omega^\pm(\xi'') = \{x';\, x_0 - \phi(\xi'') \gtrless 0\}$. (cf. [5]).

References

[1] L. Hörmander: *The Cauchy problem for differential equations with double characteristics*, J. Analyse Math. **32** (1977), 118–196.

[2] V. Ja. Ivrii and V.M. Petkov: *Necessary conditions for the Cauchy problem for non-strictly hyperbolic equations to be well-posed*, Russian Math. Surveys **29** (1974), 1–70.

[3] R.B. Melrose: *Equivalence of glancing hypersurfaces*, Invent. Math. **37** (1976), 165–191.

[4] T. Nishitani: *The Cauchy problem for weakly hyperbolic equations of second order*, Comm. Partial Differential Equations **5** (1980), 1273–1296.

[5] T. Nishitani: *Sur les opérateurs fortement hyperboliques qui ne dépendent que du temps*, Séminaire sur les équations aux dérivées partielles hyperboliques et holomorphes, Univ. de Paris VI, (1981–82).

Department of Mathematics
College of General Education
Osaka University
Toyonaka, Osaka 560
Japan