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COLOCAL PAIRS IN PERFECT RINGS

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Our main aim of the present note is to provide several sufficient conditions for a colocal module L over a left or right perfect ring A to be injective. Also, by developing the previous works [8] and [5], we will extend recent results of Baba [1, Theorems 1 and 2] to left perfect rings and provide simple proofs of them.

Throughout this note, rings are associative rings with identity and modules are unitary modules. For a ring A we denote by $\operatorname{Mod} A$ (resp. $\operatorname{Mod} A^{\mathrm{op}}$) the category of left (resp. right) A-modules, where A^{op} denotes the opposite ring of A. Sometimes, we use the notation ${}_AL$ (resp. L_A) to signify that the module L considered is a left (resp. right) A-module. For a module L, we denote by $\operatorname{soc}(L)$ the socle, by $\operatorname{rad}(L)$ the Jacobson radical, by $E(L)$ an injective envelope and by $\ell(L)$ the composition length of L. For a subset X of a right module L_A and a subset M of A, we set $l_X(M) = \{x \in X | xM = 0\}$ and $r_M(X) = \{a \in M | Xa = 0\}$. Also, for a subset X of A and a subset M of a left module ${}_AL$ we set $l_X(M) = \{a \in X | aM = 0\}$ and $r_M(X) = \{x \in M | Xx = 0\}$. We abbreviate the ascending (resp. descending) chain condition as the ACC (resp. DCC).

Recall that a module L is called colocal if it has simple essential socle. We call a bimodule ${}_HU_R$ colocal if both ${}_HU$ and U_R are colocal. Let A be a semiperfect ring with Jacobson radical J. Let L_A be a colocal module with $H = \operatorname{End}_A(L_A)$ and $f \in A$ a local idempotent with $\operatorname{soc}(L_A) \cong fA/fJ$. In case L_A has finite Loewy length, we will show that L_A is injective if and only if ${}_HLf_{fAf}$ is a colocal bimodule and $M = r_{Af}(l_L(M))$ for every submodule M of Af_{fAf}. Also, in case A is left or right perfect and $\ell(Af/r_{Af}(L)_{fAf}) < \infty$, we will show that the following are equivalent: (1) L_A is injective; (2) ${}_HLf_{fAf}$ is a colocal bimodule and $r_{Af}(L) = 0$; and (3) ${}_HLf_{fAf}$ is a colocal bimodule and $M = r_{Af}(l_L(M))$ for every submodule M of Af_{fAf}.

Recall that a module L_A is called M-injective if for any submodule N of M_A every $\theta : N_A \to L_A$ can be extended to some $\phi : M_A \to L_A$. Dually, a module L_A is called M-projective if for any factor module N of M_A every $\theta : L_A \to N_A$ can be lifted to some $\phi : L_A \to M_A$. In case L is L-injective (resp. L-projective), L is called quasi-injective (resp. quasi-projective). Let A be a left perfect ring with Jacobson radical J and $e, f \in A$ local idempotents. Assume $\ell(Af/r_{Af}(eA)_{fAf}) < \infty$. Then we will show that eA_A is quasi-injective with $\operatorname{soc}(eA_A) \cong fA/fJ$ if and only if ${}_AE = E({}_AAe/Je)$ is quasi-projective with ${}_AE/JE \cong Af/Jf$ (cf. [1, Theorem 1]).

We call a pair (eA, Af) of a right ideal eA and a left ideal Af in A a colocal pair if $e, f \in A$ are local idempotents and ${}_{eAe}eAf_{fAf}$ is a colocal bimodule. We will see that $\ell({}_{eAe}eA/l_{eA}(Af)) = \ell(Af/r_{Af}(eA)_{fAf})$ for every colocal pair (eA, Af) in A. In case $\ell({}_{eAe}eA/l_{eA}(Af)) = \ell(Af/r_{Af}(eA)_{fAf}) < \infty$, a colocal pair (eA, Af) in A is called finite. Let A be a left perfect ring with Jacobson radical J and $e, f_1, f_2, \cdots, f_n \in A$ local idempotents. Put $E = E({}_AAe/Je)$. Assume (eA, Af_i) is a finite colocal pair in A for all $1 \leq i \leq n$. Then we will show that $\operatorname{soc}(eA_A) \cong \bigoplus_{i=1}^n f_iA/f_iJ$ if and only if ${}_AE/JE \cong \bigoplus_{i=1}^n Af_i/Jf_i$ (cf. [1, Theorem 2]).

Following Harada [4], we call a module L_A M-simple-injective if for any submodule N of M_A every $\theta : N_A \to L_A$ with $\operatorname{Im}\theta$ simple can be extended to some $\phi : M_A \to L_A$. In case L is L-simple-injective, L is called simple-quasi-injective. We will show that a left perfect ring A is left artinian if A satisfies the ascending chain condition on annihilator right ideals and eA_A is simple-quasi-injective for every local idempotent $e \in A$.

1. Preliminaries

In this section, we collect several basic results which we need in later sections. We refer to Bass [2] for perfect rings.

Lemma 1.1. *Let A be a left or right perfect ring and $f \in A$ an idempotent. Assume $\ell(Af_{fAf}) < \infty$. Then ${}_AAf$ has finite Loewy length.*

Proof. Denote by J the Jacobson radical of A. Consider first the case of A being left perfect. Since the descending chain $Af \supset Jf \supset \cdots$ terminates, there exists $n \geq 1$ such that $J^nf = J^{n+1}f$. Thus $J^nf = 0$. Assume next that A is right perfect. Then, since the ascending chain $\operatorname{soc}({}_AAf) \subset \operatorname{soc}^2({}_AAf) \subset \cdots$ terminates, there exists $n \geq 1$ such that $\operatorname{soc}^n({}_AAf) = Af$. Thus $J^nf = J^n(\operatorname{soc}^n({}_AAf)) = 0$. □

Lemma 1.2. *Let $e \in A$ be an idempotent. Then for a module $L \in \operatorname{Mod} A$ with $r_L(eA) = 0$ the following hold.*

(1) *If ${}_AL$ is simple, so is ${}_{eAe}eL$.*
(2) ${}_{eAe}eE({}_AL) \cong E({}_{eAe}eL)$.
(3) *The canonical homomorphism ${}_AE({}_AL) \to {}_A\operatorname{Hom}_{eAe}(eA, eE({}_AL)), x \mapsto (a \mapsto ax)$, is an isomorphism.*

Proof. (1) See e.g. [5, Lemma 1.1].
(2) See e.g. [5, Lemmas 1.2 and 1.3].
(3) See e.g. [5, Lemma 1.3]. □

Recall that a module L_A is called M-injective if for any submodule N of M_A every $\theta : N_A \to L_A$ can be extended to some $\phi : M_A \to L_A$. Dually, a module L_A

is called M-projective if for any factor module N of M_A every $\theta : L_A \to N_A$ can be lifted to some $\phi : L_A \to M_A$. In case L is L-injective (resp. L-projective), L is called quasi-injective (resp. quasi-projective).

Lemma 1.3 ([6, Theorem 1.1]). *Let $L \in \operatorname{Mod} A^{\mathrm{op}}$ and put $H = \operatorname{End}_A(E(L_A))$. Then L_A is quasi-injective if and only if $HL = L$. In particular, if L_A is quasi-injective, then we have a surjective ring homomorphism $\rho_L : \operatorname{End}_A(E(L_A)) \to \operatorname{End}_A(L_A)$, $h \mapsto h|_L$.*

The equivalence (1) $\Leftrightarrow$ (2) of the next lemma is due to Wu and Jans [11, Propositions 2.1, 2.2 and 2.4].

Lemma 1.4 ([11]). *Let A be a left perfect ring. Then for a module $L \in \operatorname{Mod} A$ the following are equivalent.*

(1) *${}_AL$ is indecomposable quasi-projective.*

(2) *There exist a local idempotent $f \in A$ and a two-sided ideal I of A such that ${}_AL \cong Af/If$.*

(3) *There exists a local idempotent $f \in A$ such that ${}_AL \cong Af/l_A(L)f$.*

Proof. (1) $\Rightarrow$ (2). By [11, Proposition 2.4] there exists an epimorphism $\pi : {}_AAf \to {}_AL$ with $f \in A$ a local idempotent. Put $K = \operatorname{Ker} \pi$. Then by [11, Proposition 2.2] $KfAf = K$ and ${}_AL \cong Af/If$ with $I = KfA$ a two-sided ideal of A.

(2) $\Rightarrow$ (1). Since ${}_{A/I}Af/If \cong {}_{A/I}(A/I)f$ is projective, ${}_AAf/If$ is quasi -projective.

(2) $\Rightarrow$ (3). Since $If = l_A(Af/If)f$, ${}_AL \cong Af/l_A(L)f$.

(3) $\Rightarrow$ (2). Obvious. □

Recall that an object L of an abelian category $\mathcal{A}$ in which arbitrary direct products exist is called linearly compact if for any inverse system of epimorphisms $\{\pi_\lambda : L \to L_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{A}$ the induced morphism $\varprojlim \pi_\lambda : L \to \varprojlim L_\lambda$ is epic. In case $\mathcal{A} = \operatorname{Mod} A$, there is an equivalent definition of linear compactness. Recall that, for a family of submodules $\{L_\lambda\}_{\lambda \in \Lambda}$ in a module ${}_AL$, a system of congruences $\{x \equiv x_\lambda \bmod L_\lambda\}_{\lambda \in \Lambda}$ is said to be finitely solvable if for any nonempty finite subset F of Λ there exists $x_F \in L$ such that $x_F \equiv x_\lambda \bmod L_\lambda$ for all $\lambda \in F$, and to be solvable if there exists $x_0 \in L$ such that $x_0 \equiv x_\lambda \bmod L_\lambda$ for all $\lambda \in \Lambda$.

For the benefit of the reader, we include a proof of the following.

Proposition 1.5. *For a module $L \in \operatorname{Mod} A$ the following are equivalent.*

(1) *${}_AL$ is linearly compact.*

(2) *For any family of submodules $\{L_\lambda\}_{\lambda \in \Lambda}$ in ${}_AL$, every finitely solvable system of congruences $\{x \equiv x_\lambda \bmod L_\lambda\}_{\lambda \in \Lambda}$ is solvable.*

Proof. (1) $\Rightarrow$ (2). Let $\{L_\lambda\}_{\lambda\in\Lambda}$ be a family of submodules in L and $\{x \equiv x_\lambda \bmod L_\lambda\}_{\lambda\in\Lambda}$ a finitely solvable system of congruences. Denote by $\phi_\lambda : L \to L/L_\lambda$ the canonical epimorphism for each $\lambda \in \Lambda$ and set $\phi : L \to \prod_{\lambda\in\Lambda} L/L_\lambda$, $x \mapsto (\phi_\lambda(x))$. Put $\hat{x} = (\phi_\lambda(x_\lambda)) \in \prod_{\lambda\in\Lambda} L/L_\lambda$. We claim that $\hat{x} \in \operatorname{Im}\phi$. Let $\mathcal{F}$ be the directed set of nonempty finite subsets of Λ. For each $F \in \mathcal{F}$, denote by $p_F : \prod_{\lambda\in\Lambda} L/L_\lambda \to \prod_{\lambda\in F} L/L_\lambda$ the projection and put $\hat{x}_F = p_F(\hat{x}) \in \prod_{\lambda\in F} L/L_\lambda$ and $X_F = (p_F \circ \phi)^{-1}(A\hat{x}_F)$. Note that for any $F \in \mathcal{F}$, since $\{x \equiv x_\lambda \bmod L_\lambda\}_{\lambda\in\Lambda}$ is finitely solvable, $p_F \circ \phi : L \to \prod_{\lambda\in F} L/L_\lambda$ induces an epimorphism $\varphi_F : X_F \to A\hat{x}_F$. For each $F \in \mathcal{F}$, take a push-out of $\varphi_F : X_F \to A\hat{x}_F$ along with the inclusion $X_F \to L$:

$$\begin{array}{ccc} X_F & \longrightarrow & L \\ {\scriptstyle\varphi_F}\downarrow & & \downarrow{\scriptstyle\pi_F} \\ A\hat{x}_F & \longrightarrow & Y_F. \end{array}$$

Then we get an inverse system of epimorphisms $\{\pi_F : L \to Y_F\}_{F\in\mathcal{F}}$. Also, since $\varprojlim$ is left exact, we get a pull-back square

$$\begin{array}{ccc} \varprojlim X_F & \longrightarrow & L \\ {\scriptstyle\varprojlim \varphi_F}\downarrow & & \downarrow{\scriptstyle\varprojlim \pi_F} \\ \varprojlim A\hat{x}_F & \longrightarrow & \varprojlim Y_F. \end{array}$$

Since L is linearly compact, $\varprojlim \pi_F$ is epic, so is $\varprojlim \varphi_F$. Note that $\varprojlim X_F \xrightarrow{\sim} \bigcap_{F\in\mathcal{F}} X_F$. Also, $\varprojlim p_F : \prod_{\lambda\in\Lambda} L/L_\lambda \to \varprojlim \prod_{\lambda\in F} L/L_\lambda$ is an isomorphism and hence induces an isomorphism $A\hat{x} \xrightarrow{\sim} \varprojlim A\hat{x}_F$. It follows that $\phi(\bigcap_{F\in\mathcal{F}} X_F) = A\hat{x}$. Thus $\hat{x} \in \operatorname{Im}\phi$.

(2) $\Rightarrow$ (1). Let $\{\pi_\lambda : L \to L_\lambda\}_{\lambda\in\Lambda}$ be an inverse system of epimorphisms in $\operatorname{Mod} A$. We may consider $\varprojlim L_\lambda$ as a submodule of $\prod_{\lambda\in\Lambda} L_\lambda$. Let $(x_\lambda) \in \varprojlim L_\lambda$ and for each $\lambda \in \Lambda$ choose $y_\lambda \in L$ with $\pi_\lambda(y_\lambda) = x_\lambda$. Then, since for any nonempty finite subset F of Λ there exists $\lambda_0 \in \Lambda$ such that $\lambda_0 \geq \lambda$ for all $\lambda \in F$, the system of congruences $\{x \equiv y_\lambda \bmod \operatorname{Ker}\pi_\lambda\}_{\lambda\in\Lambda}$ is finitely solvable and thus solvable. Hence $\varprojlim \pi_\lambda : L \to \varprojlim L_\lambda$ is an epimorphism. □

Let ${}_HU_R$ be a bimodule and $K \in \operatorname{Mod} R^{\mathrm{op}}$. For a pair of a subset X of $(K_R)^*$ and a subset M of K_R, we set $r_M(X) = \{a \in M | h(a) = 0 \text{ for all } h \in X\}$ and $l_X(M) = \{h \in X | h(a) = 0 \text{ for all } a \in M\}$, where $(\ )^* = \operatorname{Hom}_R(-, {}_HU_R)$.

The next lemma is due essentially to [7, Lemma 4].

Lemma 1.6. *Let ${}_HU_R$ be a bimodule and $K \in \operatorname{Mod} R^{\mathrm{op}}$ a module such that U_R is K-injective. Assume $X = l_{K^*}(r_K(X))$ for every submodule X of $(K_R)^*$. Then $(K_R)^*$ is linearly compact.*

Proof. Let $\{\pi_\lambda : K^* \to X_\lambda\}_{\lambda\in\Lambda}$ be an inverse system of epimorphisms in $\operatorname{Mod} H$. For $\lambda \in \Lambda$, put $Y_\lambda = \operatorname{Ker}\pi_\lambda$ and $M_\lambda = r_K(Y_\lambda)$, and let $j_\lambda : M_\lambda \to K$ be the inclusion. Then for each $\lambda \in \Lambda$, since $\operatorname{Ker} j_\lambda^* \cong l_{K^*}(M_\lambda) = Y_\lambda$, and since $j_\lambda^* : K^* \to M_\lambda^*$ is epic, there exists an isomorphism $\phi_\lambda : M_\lambda^* \to X_\lambda$ with $\pi_\lambda = \phi_\lambda \circ j_\lambda^*$. Since $\varinjlim j_\lambda$ is monic, $\varprojlim j_\lambda^* \cong (\varinjlim j_\lambda)^*$ is epic. Also, $\varprojlim \phi_\lambda$ is an isomorphism. Thus $\varprojlim \pi_\lambda = (\varprojlim \phi_\lambda) \circ (\varprojlim j_\lambda^*)$ is epic. □

Corollary 1.7. *Let A be a left or right perfect ring. Assume A_A is injective and $I = l_A(r_A(I))$ for every left ideal I of A. Then A is quasi-Frobenius.*

Proof. It follows by Lemma 1.6 that ${}_AA$ is linearly compact. Thus by [10, Propositions 2.9 and 2.12] A is left noetherian. □

2. Bilinear maps into colocal bimodules

In this section, as further preliminaries, we modify results of [8, Section 1]. For a left H-module ${}_HL$, a right R-module K_R and an H-R-bimodule ${}_HU_R$, we call a map $\varphi : {}_HL \times K_R \to {}_HU_R$ H-R-bilinear if $K_R \to U_R$, $a \mapsto \varphi(x,a)$, is R-linear for every $x \in L$ and ${}_HL \to {}_H\operatorname{Hom}_R(K_R, {}_HU_R)$, $x \mapsto (a \mapsto \varphi(x,a))$, is H-linear.

Throughout this section, $\varphi : {}_HL \times K_R \to {}_HU_R$ is a fixed H-R -bilinear map. For a pair of a subset X of L and a subset M of K we set $r_M(X) = \{a \in M | \varphi(x,a) = 0 \text{ for all } x \in X\}$ and $l_X(M) = \{x \in X | \varphi(x,a) = 0 \text{ for all } a \in M\}$. We denote by $\mathcal{A}_l(L,K)$ the lattice of submodules X of ${}_HL$ with $X = l_L(r_K(X))$ and by $\mathcal{A}_r(L,K)$ the lattice of submodules M of K_R with $M = r_K(l_L(M))$.

REMARKS (see e.g. [3, Part I] for details). (1) Let X be a subset of L. Then $\varphi(X, r_K(X)) = 0$ implies $X \subset l_L(r_K(X))$ and thus $r_K(l_L(r_K(X))) \subset r_K(X)$. Also, $\varphi(l_L(r_K(X)), r_K(X)) = 0$ implies $r_K(X) \subset r_K(l_L(r_K(X)))$. Thus $r_K(X) = r_K(l_L(r_K(X)))$ and $r_K(X) \in \mathcal{A}_r(L,K)$.

(2) Let X be a subset of L. For any $Y \in \mathcal{A}_l(L,K)$ with $X \subset Y$, $l_L(r_K(X)) \subset l_L(r_K(Y)) = Y$. Thus $l_L(r_K(X))$ is the smallest module in $\mathcal{A}_l(L,K)$ containing X.

(3) Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of submodules of ${}_HL$. For any $\mu \in \Lambda$, since $\bigcap_{\lambda\in\Lambda} X_\lambda \subset X_\mu \subset \sum_{\lambda\in\Lambda} X_\lambda$, $r_K(\sum_{\lambda\in\Lambda} X_\lambda) \subset r_K(X_\mu) \subset r_K(\bigcap_{\lambda\in\Lambda} X_\lambda)$. Thus $r_K(\sum_{\lambda\in\Lambda} X_\lambda) \subset \bigcap_{\lambda\in\Lambda} r_K(X_\lambda)$ and $\sum_{\lambda\in\Lambda} r_K(X_\lambda) \subset r_K(\bigcap_{\lambda\in\Lambda} X_\lambda)$. Let $a \in \bigcap_{\lambda\in\Lambda} r_K(X_\lambda)$. Since $\varphi(X_\lambda, a) = 0$ for all $\lambda \in \Lambda$, and since ${}_HL \to {}_HU$, $x \mapsto \varphi(x,a)$, is H-linear, $\varphi(\sum_{\lambda\in\Lambda} X_\lambda, a) = 0$ and $a \in r_K(\sum_{\lambda\in\Lambda} X_\lambda)$. Thus $r_K(\sum_{\lambda\in\Lambda} X_\lambda) = \bigcap_{\lambda\in\Lambda} r_K(X_\lambda)$.

(4) Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of submodules of ${}_HL$ with the $X_\lambda \in \mathcal{A}_l(L,K)$. Then by (3) $\bigcap_{\lambda\in\Lambda} X_\lambda = \bigcap_{\lambda\in\Lambda} l_L(r_K(X_\lambda)) = l_L(\sum_{\lambda\in\Lambda} r_K(X_\lambda))$. Thus $r_K(\bigcap_{\lambda\in\Lambda} X_\lambda) = r_K(l_L(\sum_{\lambda\in\Lambda} r_K(X_\lambda)))$ and by (2) $r_K(\bigcap_{\lambda\in\Lambda} X_\lambda)$ is the smallest module in $\mathcal{A}_r(L,K)$ containing $\sum_{\lambda\in\Lambda} r_K(X_\lambda)$, so that $r_K(\bigcap_{\lambda\in\Lambda} X_\lambda) = \sum_{\lambda\in\Lambda} r_K(X_\lambda)$ whenever $\sum_{\lambda\in\Lambda} r_K(X_\lambda) \in \mathcal{A}_r(L,K)$.

(5) We have an anti-isomorphism of lattices $\mathcal{A}_l(L, K) \to \mathcal{A}_r(L, K)$, $X \mapsto r_K(X)$. In particular, $\mathcal{A}_l(L, K)$ satisfies the ACC (resp. DCC) if and only if $\mathcal{A}_r(L, K)$ satisfies the DCC (resp. ACC).

Recall that a module is called colocal if it has simple essential socle. We call a bimodule ${}_HU_R$ colocal if both ${}_HU$ and U_R are colocal modules.

Lemma 2.1. *Let ${}_HU_R$ be a colocal bimodule. Then* $\mathrm{soc}({}_HU) = \mathrm{soc}(U_R)$.

Proof. Since $\mathrm{soc}({}_HU)$ is a subbimodule of ${}_HU_R$, $\mathrm{soc}(U_R) \subset \mathrm{soc}({}_HU)$. Similarly, $\mathrm{soc}({}_HU) \subset \mathrm{soc}(U_R)$. Thus $\mathrm{soc}({}_HU) = \mathrm{soc}(U_R)$. □

Throughout the rest of this section, ${}_HU_R$ is assumed to be a colocal bimodule with ${}_HS_R = \mathrm{soc}({}_HU) = \mathrm{soc}(U_R)$, and $(\ )^*$ denotes both the U-dual functors.

Lemma 2.2. *The following hold.*

(1) *The canonical ring homomorphisms $H \to \mathrm{End}_R(S_R)$ and $R \to \mathrm{End}_H({}_HS)^{\mathrm{op}}$ are surjective.*

(2) *$({}_HS)^* \cong S_R$ and $(S_R)^* \cong {}_HS$.*

Proof. (1) Let $0 \neq u \in S$. Then $S = Hu = uR$. For any $h \in \mathrm{End}_R(S_R)$, $h(u) = au$ for some $a \in H$ and $h(ub) = h(u)b = (au)b = a(ub)$ for all $b \in R$. Thus the canonical ring homomorphism $H \to \mathrm{End}_R(S_R)$ is surjective. Similarly, the canonical ring homomorphism $R \to End_H({}_HS)^{\mathrm{op}}$ is surjective.

(2) Let $\pi : R_R \to S_R$ be an epimorphism. We have a monomorphism $\mu : (S_R)^* \to {}_HU$ such that $\mu(h) = (\pi^*(h))(1)$ for $h \in (S_R)^*$. Put $u = \pi(1)$. Then $\mu(h) = h(u) \in S$ for all $h \in (S_R)^*$ and $\mathrm{Im}\,\mu = {}_HS$, so that $(S_R)^* \cong {}_HS$. Similarly, $({}_HS)^* \cong S_R$. □

Lemma 2.3. *Let $N \subset M$ be submodules of K_R with $N = r_K(l_L(N))$ and M/N_R simple. Then the following hold.*

(1) *$M/N \cong S_R$ and $l_L(N)/l_L(M) \cong (M/N)^* \cong {}_HS$.*

(2) *$M = r_K(l_L(M))$.*

Proof. (1) Since $M \neq N = r_K(l_L(N))$, $l_L(M) \subset l_L(N)$ with $l_L(N)/l_L(M) \neq 0$. Let $j : N_R \to M_R$ be the inclusion. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & l_L(M) & \longrightarrow & L & \longrightarrow & M^* \\ & & \downarrow & & \| & & \downarrow{\scriptstyle j^*} \\ 0 & \longrightarrow & l_L(N) & \longrightarrow & L & \longrightarrow & N^*. \end{array}$$

Thus $0 \neq l_L(N)/l_L(M)$ embeds in $\operatorname{Ker} j^* \cong (M/N)^*$. Hence $(M/N)^* \neq 0$, so that $M/N \cong S_R$ and by Lemma 2.2(2) $(M/N)^* \cong {}_HS$.

(2) Since $l_L(M) \subset l_L(N)$ with $l_L(N)/l_L(M)$ simple, one can apply the part (1) to conclude that $r_K(l_L(M))/r_K(l_L(N))$ is simple. Thus, since $r_K(l_L(N)) = N \subset M \subset r_K(l_L(M))$ with both M/N and $r_K(l_L(M))/r_K(l_L(N))$ simple, it follows that $M = r_K(l_L(M))$. □

Lemma 2.4. *Let M be a submodule of K_R with $r_K(L) \subset M$ and $\ell(M/r_K(L)_R) < \infty$. Then the following hold.*

(1) *Every composition factor of $M/r_K(L)_R$ is isomorphic to S_R.*

(2) $M = r_K(l_L(M))$.

Proof. Since $r_K(L) = r_K(l_L(r_K(L)))$, Lemma 2.3 enables us to make use of induction on $\ell(M/r_K(L)_R)$. □

Lemma 2.5 ([8, Lemma 1.3]). $\ell({}_HL/l_L(K)) = \ell(K/r_K(L)_R)$.

Proof. By symmetry we may assume $\ell({}_HL/l_L(K)) < \infty$. Let $l_L(K) = L_0 \subset L_1 \subset \cdots \subset L_n = L$ be a chain of submodules of ${}_HL$ with the L_{i+1}/L_i simple. Then by Lemma 2.3 we get a chain of submodules $r_K(L) = r_K(L_n) \subset \cdots \subset r_K(L_1) \subset r_K(L_0) = K$ in K_R with the $r_K(L_i)/r_K(L_{i+1})$ simple. □

Lemma 2.6. *Assume R is left perfect. Then the following are equivalent.*

(1) $\ell(K/r_K(L)_R) < \infty$.

(2) *$\mathcal{A}_r(L, K)$ satisfies both the ACC and the DCC.*

(3) *$\mathcal{A}_r(L, K)$ satisfies the ACC.*

Proof. (1) ⇒ (2) ⇒ (3). Obvious.

(3) ⇒ (1). It follows by Lemma 2.4 that there exists a maximal element K_0 in the set of submodules M of K_R with $r_K(L) \subset M$ and $\ell(M/r_K(L)_R) < \infty$. We claim $K_0 = K$. Otherwise, there exists a submodule M of K_R with $K_0 \subset M$ and M/K_0 simple, a contradiction. □

3. Simple-injective colocal modules

Throughout the rest of this note, A stands for a ring with Jacobson radical J. For any pair of a right module L_A and a left ideal K of A, we have a canonical bilinear map ${}_HL \times K_R \to {}_HLK_R$, $(x, a) \mapsto xa$, where $H = \operatorname{End}_A(L_A)$ and $R = \operatorname{End}_A({}_AK)^{\mathrm{op}}$, so that, in case ${}_HLK_R$ is a colocal bimodule, we can apply results of the preceding section.

Lemma 3.1. *Let $L \in \operatorname{Mod} A^{\mathrm{op}}$ be a colocal module and $f \in A$ a local idempotent with $\operatorname{soc}(L_A) \cong fA/fJ$. Then the following hold.*
(1) $l_L(Af) = 0$.
(2) $l_L(If) = l_L(I)$ *for every right ideal I of A.*
(3) Lf_{fAf} *is colocal with* $\operatorname{soc}(Lf_{fAf}) = \operatorname{soc}(L_A)f$.

Proof. (1) For any $0 \neq x \in L$, since $\operatorname{soc}(L_A) \subset xA$, $0 \neq \operatorname{soc}(L_A)f \subset xAf$ and thus $x \notin l_L(Af)$.

(2) We have $l_L(I) \subset l_L(If)$. For any $x \in l_L(If)$, since $xIAf = xIf = 0$, by the part (1) $xI \subset l_L(Af) = 0$ and $x \in l_L(I)$. Thus $l_L(If) \subset l_L(I)$.

(3) Let $0 \neq x \in \operatorname{soc}(L_A)f$. For any $0 \neq y \in Lf$, since $xA \subset yA$, $xfAf = xAf \subset yAf = yfAf$. Thus Lf_{fAf} is colocal and $\operatorname{soc}(Lf_{fAf}) = \operatorname{soc}(L_A)f$. □

Lemma 3.2. *Let $L \in \operatorname{Mod} A^{\mathrm{op}}$ and $f \in A$ a local idempotent. Then the following are equivalent.*
(1) L_A *is colocal with* $\operatorname{soc}(L_A) \cong fA/fJ$.
(2) Lf_{fAf} *is colocal and* $l_L(Af) = 0$.

Proof. (1) ⇒ (2). By (3) and (1) of Lemma 3.1.

(2) ⇒ (1). Since by Lemma 1.2(2) $E(L_A)f_{fAf} \cong E(Lf_{fAf}) \cong E(fAf/fJf_{fAf}) \cong E(fA/fJ_A)f_{fAf}$, by Lemma 1.2(3) $E(L_A) \cong \operatorname{Hom}_{fAf}(Af, E(L_A)f)_A \cong \operatorname{Hom}_{fAf}(Af, E(fA/fJ_A)f)_A \cong E(fA/fJ_A)$. Thus L_A is colocal with $\operatorname{soc}(L_A) \cong fA/fJ$. □

Corollary 3.3. *Let $e, f \in A$ be local idempotents. Then the following are equivalent.*
(1) $eA/l_{eA}(Af)_A$ *is colocal with* $\operatorname{soc}(eA/l_{eA}(Af)_A) \cong fA/fJ$.
(2) eAf_{fAf} *is colocal.*

Proof. Put $L = eA/l_{eA}(Af)_A$. Then $l_L(Af) = 0$ and, since $l_{eA}(Af)f = 0$, $Lf_{fAf} \cong eAf_{fAf}$. Thus Lemma 3.2 applies. □

Following Harada [4], we call a module L_A M-simple-injective if for any submodule N of M_A every $\theta : N_A \to L_A$ with $\operatorname{Im}\theta$ simple can be extended to some $\phi : M_A \to L_A$. In case L is L-simple-injective, L is called simple-quasi-injective.

Lemma 3.4. *Let $L \in \operatorname{Mod} A^{\mathrm{op}}$ be a colocal module and put $H = \operatorname{End}_A(L_A)$. Let $f \in A$ be a local idempotent with $\operatorname{soc}(L_A) \cong fA/fJ$. Then the following hold.*
(1) *If L_A is A-simple-injective, then $M = r_{Af}(l_L(M))$ for every submodule M of Af_{fAf}.*
(2) *If ${}_HLf_{fAf}$ is a colocal bimodule and $M = r_{Af}(l_L(M))$ for every submodule M of Af_{fAf}, then L_A is A-simple-injective.*

Proof. (1) Let M be a submodule of Af_{fAf} and put $N = r_{Af}(l_L(M))$. We claim $M = N$. Suppose otherwise. Note first that $l_L(N) = l_L(M)$. Since $(NA/MA)f \cong N/M \neq 0$, there exist right ideals K, I of A such that $MA \subset K \subset I \subset NA$ and $I/K \cong fA/fJ \cong \mathrm{soc}(L_A)$. Then we have $\theta : I_A \to L_A$ with $\mathrm{Im}\,\theta = \mathrm{soc}(L_A)$ and $\mathrm{Ker}\,\theta = K$. Let $\mu : I_A \to A_A$ be the inclusion. There exists $\phi : A_A \to L_A$ with $\phi \circ \mu = \theta$. Then $\phi(1)I = \phi(I) = \theta(I) \neq 0$ and $\phi(1)K = \phi(K) = \theta(K) = 0$. Thus $\phi(1) \in l_L(K)$ and $\phi(1) \notin l_L(I)$. Since $l_L(N) = l_L(NA) \subset l_L(I) \subset l_L(K) \subset l_L(MA) = l_L(M)$, $l_L(K) \neq l_L(I)$ implies $l_L(M) \neq l_L(N)$, a contradiction.

(2) Let I be a nonzero right ideal of A and $\mu : I_A \to A_A$ the inclusion. Let $\theta : I_A \to L_A$ with $\mathrm{Im}\,\theta = \mathrm{soc}(L_A)$ and put $K = \mathrm{Ker}\,\theta$. Since by Lemma 1.2(1) $If/Kf_{fAf} \cong (I/K)f_{fAf}$ is simple, by Lemma 2.3(1) so is ${}_H l_L(Kf)/l_L(If)$. Let $a \in If$ with $a \notin Kf$. Then, since $l_L(Kf)a \neq 0$ and $l_L(If)a = 0$, ${}_H l_L(Kf)a$ is simple. Thus by Lemmas 2.1 and 3.1(3) $l_L(Kf)a = \mathrm{soc}(Lf_{fAf}) = \mathrm{soc}(L_A)f$, so that $\theta(a) = \theta(af) = \theta(a)f = xa$ for some $x \in l_L(Kf)$. Define $\phi : A_A \to L_A$ by $1 \mapsto x$. Then, since by Lemma 3.1(2) $x \in l_L(Kf) = l_L(K)$, and since $I = K + aA$, we have $\phi \circ \mu = \theta$. □

Lemma 3.5. *Let $L \in \mathrm{Mod}\,A^{\mathrm{op}}$ be a colocal module and put $H = \mathrm{End}_A(L_A)$. Let $f \in A$ be a local idempotent with $\mathrm{soc}(L_A) \cong fA/fJ$. Then the following hold.*

(1) *If L_A is simple-quasi-injective, then ${}_H Lf_{fAf}$ is a colocal bimodule and $l_L(Af) = 0$.*

(2) *If L_A is A-simple-injective, then $r_{Af}(L) = 0$ and $r_A(L/LJ_A) \subset l_A(\mathrm{soc}({}_A Af))$.*

Proof. (1) By Lemma 3.2 Lf_{fAf} is colocal and $l_L(Af) = 0$. Let $0 \neq x \in \mathrm{soc}(L_A)f$. We claim that $x \in Hy$ for all $0 \neq y \in Lf$. Note that $r_{fA}(x) = fJ$. Let $0 \neq y \in Lf$. Then $r_{fA}(y) \subset fJ = r_{fA}(x)$ and we have $\theta : yA_A \to xA_A = \mathrm{soc}(L_A)$, $ya \mapsto xa$. Let $\mu : \mathrm{soc}(L_A) \to L_A$ and $\nu : yA_A \to L_A$ be inclusions. There exists $h \in H$ with $h \circ \nu = \mu \circ \theta$, so that $x = h(y) \in Hy$. Thus ${}_H Lf$ is colocal.

(2) By Lemma 3.4(1) $r_{Af}(L) = r_{Af}(l_L(0)) = 0$. Next, let $a \in r_A(L/LJ)$. Since $La \subset LJ$, $La(\mathrm{soc}({}_A Af)) \subset LJ(\mathrm{soc}({}_A Af)) = 0$. Thus $a(\mathrm{soc}({}_A Af)) \subset r_{Af}(L) = 0$ and $a \in l_A(\mathrm{soc}({}_A Af))$. □

Lemma 3.6 ([5, Lemma 3.3]). *Let $L \in \mathrm{Mod}\,A^{\mathrm{op}}$ be a simple-quasi-injective module with $\mathrm{soc}(L_A) \neq 0$. Assume $\mathrm{End}_A(L_A)$ is a local ring. Then $\mathrm{soc}(L_A)$ is simple.*

Proof. Let S be a simple submodule of $\mathrm{soc}(L_A)$. Suppose to the contrary that $S \neq \mathrm{soc}(L_A)$. Let $\pi : \mathrm{soc}(L_A) \to S_A$ be a projection and $\mu : \mathrm{soc}(L_A) \to L_A$, $\nu : S_A \to L_A$ inclusions. There exists $\phi : L_A \to L_A$ with $\phi \circ \mu = \nu \circ \pi$. Since π is not monic, ϕ is not an isomorphism. Thus $\phi \in \mathrm{rad}\,\mathrm{End}_A(L_A)$ and $(\mathrm{id}_L - \phi)$ is a unit in $\mathrm{End}_A(L_A)$, so that for $0 \neq x \in S$, since $\phi(x) = \pi(x) = x$, $(\mathrm{id}_L - \phi)(x) = 0$ and $x = 0$, a contradiction. □

4. Injectivity of colocal modules

In this section, by extending the previous results [8, Theorems 2.7, 2.8 and Proposition 2.9], we provide several sufficient conditions for a colocal module over a left or right perfect ring A to be injective.

Lemma 4.1 ([5, Lemma 3.4]). *Let A be a semiperfect ring and $L \in \operatorname{Mod} A^{\mathrm{op}}$ an A-simple-injective colocal module of finite Loewy length. Then L_A is injective.*

Proof. Let I be a right ideal of A and $\mu : I_A \to A_A$ the inclusion. Let $\theta : I_A \to L_A$. We make use of induction on the Loewy length of $\theta(I)$ to show the existence of $\phi : A_A \to L_A$ with $\theta = \phi \circ \mu$. Let $n = \min\{k \geq 0 | \theta(I)J^k = 0\}$. We may assume $n > 0$. Since $\operatorname{soc}(L_A)$ is simple, $\operatorname{soc}(L_A) = \theta(I)J^{n-1} = \theta(IJ^{n-1})$. Let μ_1 and θ_1 denote the restrictions of μ and θ to IJ^{n-1}, respectively. Then $\operatorname{Im}\theta_1 = \operatorname{soc}(L_A)$ and there exists $\phi_1 : A_A \to L_A$ with $\phi_1 \circ \mu_1 = \theta_1$. Since $(\theta - \phi_1 \circ \mu)(I)J^{n-1} = 0$, by induction hypothesis there exists $\phi_2 : A_A \to L_A$ with $\phi_2 \circ \mu = \theta - \phi_1 \circ \mu$. Thus $\theta = (\phi_1 + \phi_2) \circ \mu$. □

Thorem 4.2. *Let A be a semiperfect ring. Let $L \in \operatorname{Mod} A^{\mathrm{op}}$ be a colocal module of finite Loewy length and put $H = \operatorname{End}_A(L_A)$. Let $f \in A$ be a local idempotent with $\operatorname{soc}(L_A) \cong fA/fJ$. Then the following are equivalent.*

(1) *L_A is injective.*

(2) *${}_HLf_{fAf}$ is a colocal bimodule and $M = r_{Af}(l_L(M))$ for every submodule M of Af_{fAf}.*

Proof. (1) ⇒ (2). By Lemmas 3.5(1) and 3.4(1).

(2) ⇒ (1). By Lemmas 3.4(2) and 4.1. □

Corollary 4.3. *Let A be a semiperfect ring. Let $L \in \operatorname{Mod} A^{\mathrm{op}}$ be a colocal module of finite Loewy length and put $H = \operatorname{End}_A(L_A)$. Let $f \in A$ be a local idempotent with $\operatorname{soc}(L_A) \cong fA/fJ$. Assume ${}_HLf_{fAf}$ is a colocal bimodule and $M = r_{Af}(l_L(M))$ for every submodule M of Af_{fAf} with $r_{Af}(L) \subset M$. Then L_A is quasi-injective.*

Proof. Put $I = r_A(L)$. Then by Theorem 4.2 $L_{A/I}$ is injective, so that L_A is quasi-injective. □

Thorem 4.4. *Let A be a left or right perfect ring. Let $L \in \operatorname{Mod} A^{\mathrm{op}}$ be a colocal module and put $H = \operatorname{End}_A(L_A)$. Let $f \in A$ be a local idempotent with $\operatorname{soc}(L_A) \cong fA/fJ$. Assume $\ell(Af/r_{Af}(L)_{fAf}) < \infty$. Then the following are equivalent.*

(1) *L_A is injective.*

(2) *${}_HLf_{fAf}$ is a colocal bimodule and $r_{Af}(L) = 0$.*

(3) ${}_HLf_{fAf}$ *is a colocal bimodule and* $M = r_{Af}(l_L(M))$ *for every submodule* M *of* Af_{fAf}.

Proof. (1) $\Rightarrow$ (2). By Lemma 3.5.

(2) $\Rightarrow$ (3). By Lemma 2.4.

(3) $\Rightarrow$ (1). By Lemma 3.4(2) L_A is A-simple-injective. Note that $r_{Af}(L) = r_{Af}(l_L(0)) = 0$. Thus $\ell(Af_{fAf}) < \infty$ and by Lemma 1.1 $J^n f = 0$ for some $n \geq 1$, so that $LJ^n Af = LJ^n f = 0$ and by Lemma 3.1(1) $LJ^n \subset l_L(Af) = 0$. Hence by Lemma 4.1 L_A is injective. □

Corollary 4.5. *Let* A *be a left or right perfect ring. Let* $L \in \operatorname{Mod} A^{\mathrm{op}}$ *be a colocal module and put* $H = \operatorname{End}_A(L_A)$. *Let* $f \in A$ *be a local idempotent with* $\operatorname{soc}(L_A) \cong fA/fJ$. *Assume* ${}_HLf_{fAf}$ *is a colocal bimodule and* $\ell(Af/r_{Af}(L)_{fAf}) < \infty$. *Then* L_A *is quasi-injective.*

Proof. Put $I = r_A(L)$. Then $r_{Af/If}(L) = 0$ and by Theorem 4.4 $L_{A/I}$ is injective, so that L_A is quasi-injective. □

Proposition 4.6. *Let* A *be a left or right perfect ring. Let* $L \in \operatorname{Mod} A^{\mathrm{op}}$ *be a colocal module and put* $H = \operatorname{End}_A(L_A)$. *Let* $f \in A$ *be a local idempotent with* $\operatorname{soc}(L_A) \cong fA/fJ$. *Then the following are equivalent.*

(1) L_A *is injective and* $X = l_L(r_{Af}(X))$ *for every submodule* X *of* ${}_HL$.

(2) ${}_HLf_{fAf}$ *is a colocal bimodule,* $r_{Af}(L) = 0$ *and* $\ell(Af_{fAf}) < \infty$.

Proof. (1) $\Rightarrow$ (2). By Lemma 3.5(1) ${}_HLf_{fAf}$ is a colocal bimodule, and by Lemma 3.5(2) $r_{Af}(L) = 0$. It remains to show $\ell(Af_{fAf}) < \infty$. Put $K_n = Af(fJf)^n$ for $n \geq 0$. We claim $\ell(K_n/K_{n+1\,fAf}) < \infty$ for all $n \geq 0$. Let $n \geq 0$. Note that by Lemma 3.4(1) the lattice of submodules of Af_{fAf} is anti-isomorphic to the lattice of submodules of ${}_HL$. Thus $\ell(K_n/K_{n+1\,fAf}) = \ell({}_Hl_L(K_{n+1})/l_L(K_n))$. Also, since $\operatorname{rad}(K_n/K_{n+1\,fAf}) = 0$, ${}_Hl_L(K_{n+1})/l_L(K_n)$ is semisimple. For any submodule X of ${}_HL$, since $r_{Af}(X) = r_A(X)f$, by Lemma 3.1(2) $X = l_L(r_{Af}(X)) = l_L(r_A(X)f) = l_L(r_A(X))$. Thus by Lemma 1.6 ${}_HL \cong \operatorname{Hom}_A(A_A, {}_HL_A)$ is linearly compact, so is ${}_Hl_L(K_{n+1})/l_L(K_n)$ by [10, Proposition 2.2]. Hence by [10, Lemma 2.3] $\ell(K_n/K_{n+1\,fAf}) = \ell({}_Hl_L(K_{n+1})/l_L(K_n)) < \infty$. Since $\ell(fJf/(fJf)^2_{fAf}) \leq \ell(K_1/K_{2\,fAf}) < \infty$, by [9, Lemma 11] fAf is right artinian. Then $\ell(K_0/K_{1\,fAf}) < \infty$ implies $\ell(Af_{fAf}) < \infty$.

(2) $\Rightarrow$ (1). By Theorem 4.4 L_A is injective. Since by Lemma 3.1(1) $l_L(Af) = 0$, by Lemma 2.5 $\ell({}_HL) = \ell(Af_{fAf}) < \infty$ and thus by Lemma 2.4 $X = l_L(r_{Af}(X))$ for every submodule X of ${}_HL$. □

5. Colocal pairs

We call a pair (eA, Af) of a right ideal eA and a left ideal Af in A a colocal pair if $e, f \in A$ are local idempotents and ${}_{eAe}eAf_{fAf}$ is a colocal bimodule. Note that by Lemma 2.5 $\ell({}_{eAe}eA/l_{eA}(Af)) = \ell(Af/r_{Af}(eA)_{fAf})$ for every colocal pair (eA, Af) in A. In case $\ell({}_{eAe}eA/l_{eA}(Af)) = \ell(Af/r_{Af}(eA)_{fAf}) < \infty$, a colocal pair (eA, Af) in A is called finite.

In [5], a pair (eA, Af) of a right ideal eA and a left ideal Af in A is called an i-pair if $e, f \in A$ are local idempotents, eA_A is colocal with $\mathrm{soc}(eA_A) \cong fA/fJ$ and ${}_AAf$ is colocal with $\mathrm{soc}({}_AAf) \cong Ae/Je$.

Lemma 5.1. *Let $e, f \in A$ be local idempotents. Then the following are equivalent.*

(1) *(eA, Af) is an i-pair in A.*
(2) *(eA, Af) is a colocal pair in A with $l_{eA}(Af) = 0$ and $r_{Af}(eA) = 0$.*

Proof. (1) $\Rightarrow$ (2). By (1) and (3) of Lemma 3.1.
(2) $\Rightarrow$ (1). By Corollary 3.3. □

The equivalence (1) $\Leftrightarrow$ (2) of the next lemma has been established in [5, Theorem 3.7]. Here we provide another proof of the implication (2) $\Rightarrow$ (1) which does not appeal to Morita duality.

Lemma 5.2 ([5, Theorem 3.7]). *Let (eA, Af) be an i-pair in a left or right perfect ring A. Then the following are equivalent.*

(1) *(eA, Af) is finite.*
(2) *Both eA_A and ${}_AAf$ are injective.*
(3) *eA_A is injective and ${}_AAf$ is A-simple-injective.*

Proof. (1) $\Rightarrow$ (2). By Theorem 4.4.
(2) $\Rightarrow$ (3). Obvious.
(3) $\Rightarrow$ (1). It follows by Lemma 3.4(1) that $X = l_{eA}(r_{Af}(X))$ for every submodule X of ${}_{eAe}eA$. Thus by Proposition 4.6 $\ell(Af_{fAf}) < \infty$. □

Lemma 5.3. *Let (eA, Af) be a finite colocal pair in a left or right perfect ring A. Then the following hold.*

(1) *$eA/l_{eA}(Af)_A$ is a quasi-injective colocal module with $\mathrm{soc}(eA/l_{eA}(Af)_A) \cong fA/fJ$.*
(2) *If $r_{Af}(eA) = 0$, then $E(fA/fJ_A) \cong eA/l_{eA}(Af)$, so that $E(fA/fJ_A)$ is quasi-projective and $eA/l_{eA}(Af)_A$ is injective.*

Proof. Put $I = l_A(Af)$ and $L = eA/eI_A$. Then $l_{eA}(Af) = eI$ and $l_L(Af) = 0$. Note that, since $If = 0$, $Lf_{fAf} \cong eAf_{fAf}$. Thus by Lemma 3.2 L_A is colocal with $\mathrm{soc}(L_A) \cong fA/fJ$. Since $Lf_{fAf} \cong eAf_{fAf}$ and $H = \mathrm{End}_A(L_A) \cong eAe/eIe$, ${}_HLf_{fAf}$ is a colocal bimodule. Note also that $\ell(Af/r_{Af}(L)_{fAf}) = \ell(Af/r_{Af}(eA)_{fAf}) < \infty$.

(1) By Corollary 4.5 L_A is quasi-injective.

(2) By Theorem 4.4 L_A is injective. Thus, since $\mathrm{soc}(L_A) \cong fA/fJ$, $E(fA/fJ_A) \cong L$. Since $L_{A/I} \cong e(A/I)_{A/I}$ is projective, L_A is quasi-projective. □

Proposition 5.4. *Let (eA, Af) be a colocal pair with $l_{eA}(Af) = 0$ in a left or right perfect ring A. Put $\overline{A} = A/r_A(eA)$. Let $\pi : A \to \overline{A}$ be the canonical ring homomorphism and put $\bar{e} = \pi(e)$, $\bar{f} = \pi(f)$. Then the following are equivalent.*

(1) *(eA, Af) is finite.*

(2) *eA_A is quasi-injective, ${}_{eAe}eA$ is finitely generated and ${}_AAf/r_{Af}(eA)$ is injective.*

(3) *$(\bar{e}\overline{A}, \overline{A}\bar{f})$ is a finite i-pair in $\overline{A}$.*

Proof. Note first that $\overline{A}$ is left or right perfect and $\bar{e}, \bar{f} \in \overline{A}$ are local idempotents. Put $I = r_A(eA)$. Then $eI = 0$ and $If = r_{Af}(eA)$. Thus $\ell({}_{\bar{e}\overline{A}\bar{e}}\bar{e}\overline{A}) = \ell({}_{eAe}eA)$ and, since ${}_{eAe}e\overline{A}f_{fAf} \cong {}_{eAe}eAf_{fAf}$ is a colocal bimodule, $(\bar{e}\overline{A}, \overline{A}\bar{f})$ is a colocal pair in $\overline{A}$.

(1) ⇒ (2). By Lemma 5.3(1) eA_A is quasi-injective, and by Lemma 5.3(2) ${}_AAf/r_{Af}(eA)$ is injective. Also, since $\ell({}_{eAe}eA) < \infty$, ${}_{eAe}eA$ is finitely generated.

(2) ⇒ (3). By [3, Corollary 5.6A] $\bar{e}\overline{A}_{\overline{A}} \cong eA_{\overline{A}}$ is injective. Also, since ${}_A\overline{A}\bar{f} \cong {}_AAf/r_{Af}(eA)$ is injective, so is ${}_{\overline{A}}\overline{A}\bar{f}$. It is obvious that $r_{\overline{A}}(\bar{e}\overline{A}) = 0$. For any $a \in l_{eA}(\overline{A}\bar{f})$, since $aAf \subset If$, $aAf = eaAf \subset eIf = 0$ and $a \in l_{eA}(Af) = 0$. It follows that $l_{\bar{e}\overline{A}}(\overline{A}\bar{f}) = 0$. Thus by Lemmas 5.1 and 5.2 $(\bar{e}\overline{A}, \overline{A}\bar{f})$ is a finite i-pair in $\overline{A}$.

(3) ⇒ (1). Obvious. □

Corollary 5.5. *Let (eA, Af) be an i-pair in a left or right perfect ring A. Then the following are equivalent.*

(1) *(eA, Af) is finite.*

(2) *eA_A is quasi-injective, ${}_{eAe}eA$ is finitely generated and ${}_AAf$ is injective.*

6. Applications of colocal pairs I

In this section, as applications of colocal pairs, we extend recent results of Baba [1, Theorems 1 and 2] to left perfect rings and provide simple proofs of them.

Lemma 6.1. *Let A be a left perfect ring and $e \in A$ a local idempotent. Assume ${}_AE = E({}_AAe/Je)$ is quasi-projective. Then ${}_AE/JE$ is simple and for a local idempotent $f \in A$ with ${}_AE/JE \cong Af/Jf$ the following hold:*

(a) ${}_AE \cong Af/r_{Af}(eA)$;
(b) ${}_{eAe}eAf \cong {}_{eAe}eE$ *is injective; and*
(c) (eA, Af) *is a colocal pair in* A *with* $l_{eA}(Af) = 0$.

Proof. Put $I = l_A(E)$. By Lemma 1.4 there exists a local idempotent $f \in A$ such that ${}_AE \cong Af/If$. We claim $If = r_{Af}(eA)$. Since by Lemma 3.5(2) $eAIf = eIf \subset l_{eA}(E) = 0$, $If \subset r_{Af}(eA)$. Conversely, let $a \in r_{Af}(eA)$. Then $eA(a+If) = 0$ and by Lemma 3.1(1) $(a + If) \in r_{Af/If}(eA) = 0$, so that $a \in If$. Next, since $e(r_{Af}(eA)) = 0$, ${}_{eAe}eE \cong {}_{eAe}e(Af/r_{Af}(eA)) \cong {}_{eAe}eAf$. Thus ${}_{eAe}eAf$ is colocal by Lemma 3.1(3) and injective by Lemma 1.2(2). Also, since $\mathrm{End}_A({}_AAf/If) \cong fAf/fIf$, by Lemma 3.5(1) eAf_{fAf} is colocal. Finally, by Lemma 3.5(2) $l_{eA}(Af) \subset l_{eA}(Af/r_{Af}(eA)) = l_{eA}(E) = 0$. □

Thorem 6.2 (cf. [1, Theorem 1]). *Let* A *be a left perfect ring and* $e, f \in A$ *local idempotents. Put* $E = E({}_AAe/Je)$. *Assume* $\ell(Af/r_{Af}(eA)_{fAf}) < \infty$. *Then the following are equivalent.*
(1) eA_A *is quasi-injective with* $\mathrm{soc}(eA_A) \cong fA/fJ$.
(2) ${}_AE$ *is quasi-projective with* ${}_AE/JE \cong Af/Jf$.
(3) (eA, Af) *is a colocal pair in* A *with* $l_{eA}(Af) = 0$.
(4) ${}_{eAe}eAf$ *is colocal and* $\mathrm{soc}(eA_A) \cong fA/fJ$.

Proof. (1) ⇒ (3). By Lemma 3.5(1).
(3) ⇒ (1). By Lemma 5.3(1).
(2) ⇒ (3). By Lemma 6.1.
(3) ⇒ (2). By Lemma 5.3(2).
(3) ⇒ (4). By Corollary 3.3.
(4) ⇒ (3). By (3) and (1) of Lemma 3.1. □

Lemma 6.3. *Let* (eA, Af) *be a colocal pair in a left or right perfect ring* A. *Put* $E = E({}_AAe/Je)$ *and* $H = \mathrm{End}_A({}_AE)^{\mathrm{op}}$. *Assume* $\mathrm{soc}(eA_A)f \neq 0$. *Then the following hold.*
(1) $\mathrm{soc}(eA_A)fA$ *is the unique simple submodule of* eA_A *which is isomorphic to* fA/fJ_A.
(2) *If* (eA, Af) *is finite, then* ${}_AE_H$ *contains a subbimodule* X *such that* ${}_AX \cong Af/r_A(eA)f$, ${}_{eAe}eX_H$ *is a colocal bimodule,* $\mathrm{soc}(eA_A)fA \cap l_{eA}(X) = 0$ *and* $\ell({}_{eAe}eA/l_{eA}(X)) < \infty$.

Proof. (1) Since $\mathrm{soc}(eA_A)f \neq 0$, eA_A contains a simple submodule $K \cong fA/fJ$. On the other hand, by Corollary 3.3 $eA/l_{eA}(Af)_A$ is colocal with $\mathrm{soc}(eA/l_{eA}(Af)_A) \cong fA/fJ$. Thus K is the unique simple submodule of eA_A which is isomorphic to fA/fJ. It follows that $K = \mathrm{soc}(eA_A)fA$.

(2) Put $I = r_A(eA)$. Then $If = r_{Af}(eA)$ and by Lemma 5.3(1) ${}_AAf/If$ is a quasi-injective colocal module with $\mathrm{soc}({}_AAf/If) \cong Ae/Je$. Thus ${}_AE$ contains a submodule $X \cong {}_AAf/If$. Then by Lemma 1.3 $XH \subset X$. Since by Lemma 3.5(1) ${}_{eAe}eE_H$ is a colocal bimodule, so is ${}_{eAe}eX_H$. Also, since $eI = 0$, $\mathrm{soc}(eA_A)fA(Af/If) \cong \mathrm{soc}(eA_A)fAf \neq 0$. Thus $\mathrm{soc}(eA_A)fA \cap l_{eA}(X) = 0$. Finally, since $l_{eA}(X) = l_{eA}(Af)$, $\ell({}_{eAe}eA/l_{eA}(X)) = \ell({}_{eAe}eA/l_{eA}(Af)) < \infty$. □

Lemma 6.4. *Let A be a left perfect ring and $e \in A$ a local idempotent. Put $E = E({}_AAe/Je)$ and $H = \mathrm{End}_A({}_AE)^{\mathrm{op}}$. Assume $\mathrm{soc}(eA_A) \cong \bigoplus_{i=1}^n f_iA/f_iJ$ with the (eA, Af_i) finite colocal pairs in A. Then $f_iA/f_iJ \not\cong f_jA/f_jJ$ for $i \neq j$, $\ell(E_H) = \ell({}_{eAe}eA) < \infty$ and ${}_AE/JE \cong \bigoplus_{i=1}^n Af_i/Jf_i$.*

Proof. By Lemma 6.3(1) $f_iA/f_iJ \not\cong f_jA/f_jJ$ for $i \neq j$. Also, for each $1 \leq i \leq n$, by Lemma 6.3(2) ${}_AE_H$ contains a subbimodule X_i such that ${}_AX_i \cong Af_i/r_A(eA)f_i$, ${}_{eAe}eX_{iH}$ is a colocal bimodule, $\mathrm{soc}(eA_A)f_iA \cap l_{eA}(X_i) = 0$ and $\ell({}_{eAe}eA/l_{eA}(X_i)) < \infty$. Put ${}_AX_H = \sum_{i=1}^n X_i$. Then, by Lemmas 3.1(1) and 2.5 $\ell(X_{iH}) = \ell({}_{eAe}eA/l_{eA}(X_i)) < \infty$ for all $1 \leq i \leq n$, so that $\ell(X_H) < \infty$. Also, since $\mathrm{soc}(eA_A)f_iA \cap l_{eA}(X) = 0$ for all $1 \leq i \leq n$, by Lemma 6.3(1) $\mathrm{soc}(eA_A) \cap l_{eA}(X) = 0$. Thus, since eA_A has essential socle, $l_{eA}(X) = 0$. Since by Lemma 3.5(1) ${}_{eAe}eE_H$ is a colocal bimodule, so is ${}_{eAe}eX_H$. Thus by Lemma 2.5 $\ell({}_{eAe}eA) = \ell(X_H) < \infty$. Since by Lemma 1.3 we have a surjective ring homomorphism $\rho_X : H \to \mathrm{End}_A({}_AX)^{\mathrm{op}}$, $h \mapsto h|_X$, it follows by Theorem 4.4 that ${}_AX$ is injective. Thus $X = E$ and we have an epimorphism $\bigoplus_{i=1}^n Af_i/Jf_i \to {}_AE/JE$. On the other hand, since $f_iA/f_iJ \not\cong f_jA/f_jJ$ for $i \neq j$, it follows by Lemma 3.5(2) that ${}_AE/JE$ has a direct summand which is isomorphic to $\bigoplus_{i=1}^n Af_i/Jf_i$. Thus ${}_AE/JE \cong \bigoplus_{i=1}^n Af_i/Jf_i$. □

Thorem 6.5 (cf. [1, Theorem 2]). *Let A be a left perfect ring and $e, f_1, f_2, \cdots, f_n \in A$ local idempotents. Put $E = E({}_AAe/Je)$. Assume (eA, Af_i) is a finite colocal pair in A for all $1 \leq i \leq n$. Then the following are equivalent.*
(1) $\mathrm{soc}(eA_A) \cong \bigoplus_{i=1}^n f_iA/f_iJ$.
(2) ${}_AE/JE \cong \bigoplus_{i=1}^n Af_i/Jf_i$.

Proof. (1) ⇒ (2). By Lemma 6.4.

(2) ⇒ (1). It follows by Lemmas 3.5(2) and 6.3(1) that $\mathrm{soc}(eA_A)$ is isomorphic to a direct summand of $\bigoplus_{i=1}^n f_iA/f_iJ$. We may assume $\mathrm{soc}(eA_A) \cong \bigoplus_{i=1}^r f_iA/f_iJ$ for some $1 \leq r \leq n$. Then by Lemma 6.4 ${}_AE/JE \cong \bigoplus_{i=1}^r Af_i/Jf_i$, so that $r = n$. □

7. Applications of colocal pairs II

In this section, we provide some other applications of colocal pairs. Recall that a set $\{e_1, \cdots, e_n\}$ of orthogonal local idempotents in a semiperfect ring A is called basic if $(\sum_{i=1}^{n} e_i)A(\sum_{i=1}^{n} e_i)$ is a basic ring of A.

Lemma 7.1 ([5, Lemma 3.5]). *Let A be a semiperfect ring and $\{e_1, \cdots, e_n\}$ a basic set of orthogonal local idempotents in A. Assume every e_iA_A is A-simple-injective and has essential socle. Then there exists a permutation ν of the set $\{1, \cdots, n\}$ such that $(e_iA, Ae_{\nu(i)})$ is an i-pair in A for all $1 \leq i \leq n$.*

Proof. By [5, Lemma 3.5] there exists a mapping $\nu : \{1, \cdots, n\} \to \{1, \cdots, n\}$ such that $(e_iA, Ae_{\nu(i)})$ is an i-pair in A for all $1 \leq i \leq n$. Then by the definition of i-pairs ν is injective. □

Corollary 7.2. *Let A be a left perfect ring. Assume A_A is simple-quasi-injective. Then $E({}_AA)$ and $E(A_A)$ are injective cogenerators in* Mod A *and* Mod A^{op}, *respectively.*

Lemma 7.3. *Let A be a left perfect ring. Assume $\mathcal{A}_r(A, A)$ satisfies the ACC and eA_A is simple-quasi-injective for every local idempotent $e \in A$. Then A is left artinian.*

Proof. It suffices to show that $\ell({}_{eAe}eA) < \infty$ for every local idempotent $e \in A$. Let $e \in A$ be a local idempotent. Since by Lemma 3.6 eA_A is colocal, there exists a local idempotent $f \in A$ with $\mathrm{soc}(eA_A) \cong fA/fJ$. By Lemma 3.5(1) (eA, Af) is a colocal pair in A with $l_{eA}(Af) = 0$. For each $M \in \mathcal{A}_r(eA, Af)$, put $\hat{M} = r_A(l_{eA}(M)) \in \mathcal{A}_r(A, A)$. Then $\hat{M}f = r_{Af}(l_{eA}(M)) = M$ for every $M \in \mathcal{A}_r(eA, Af)$. Thus, for $M, N \in \mathcal{A}_r(eA, Af)$ with $M \subset N$, $\hat{M} \subset \hat{N}$ and $\hat{M} = \hat{N}$ implies $M = \hat{M}f = \hat{N}f = N$. It follows that $\mathcal{A}_r(eA, Af)$ satisfies the ACC. Thus by Lemmas 2.5 and 2.6 $\ell({}_{eAe}eA) = \ell(Af/r_{Af}(eA)_{fAf}) < \infty$. □

Corollary 7.4. *Let A be a left perfect ring. Assume $\mathcal{A}_r(A, A)$ satisfies the ACC and A_A is simple-quasi-injective. Then A is quasi-Frobenius.*

Proof. By Lemma 7.3 A is left artinian. Then it follows by Lemmas 3.6 and 4.1 that A_A is injective. □

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