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## CONSTRUCTION OF PURE CUBIC FIELDS WITH LARGE 2-CLASS GROUPS

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### Introduction

In [5], the author showed the existence of infinitely many pure number fields of any given odd degree  $n(\geq 3)$  whose ideal class groups have 2-rank at least  $3\Delta_n$ , where  $\Delta_n$  is the number of divisors of  $n$  which are smaller than  $n$ . This result was approached via Diophantine equations of the type  $Y^2=X^n+D$ , which was also applied to the research on “ $n$ -rank” of the ideal class groups of quadratic fields (Yamamoto [9], Craig [1], [2]). Particularly, in case  $n=3$ , Craig gave one of the ways to generate the elliptic curves given by  $Y^2=4X^3+D$  possessing suitably many integral points, and utilized it to obtain a precise result on 3-rank of the ideal class groups of quadratic fields.

The aim of the present paper is to apply Craig's elliptic curves to the proof of a 2-rank theorem on pure cubic fields. We shall namely show the following

**Theorem.** *There exist infinitely many pure cubic fields whose ideal class groups have 2-rank at least 6.*

This theorem is stronger than the above result of [5] in case  $n=3$ , because  $\Delta_3=1$ . The proof is given by the method largely due to Craig [2]. To check some results of Craig and to calculate the various numerical values, one can make effective use of an electric computer. In particular, the computer algebra system REDUCE was suited to our purpose (cf. [3]). REDUCE has been implemented on the M680H computer at the Computer Centre University of Tokyo, and able to be utilized at the Gakushuin University Computer Centre (GUCC), by way of the computer network N1. The calculations in Section 3 were carried out by FORTRAN on the COSMO800 III computer at GUCC. The author would like to thank the staff of GUCC for their assistance given to him during the work for the paper. Thanks are also due to Professor H. Wada who kindly supplied valuable information about calculations of larger numbers.

### 1. Craig's elliptic curves

In this section, we summarize Craig's result on the integer solutions of  $Y^2=4X^3+D$ . For details, refer to [2].

We start with the symmetric polynomial in  $X, Y, Z$ ;

$$\begin{aligned} D(X, Y, Z) &= (X^2+Y^2+Z^2)-2(XY+YZ+ZX) \\ &= (-X+Y+Z)^2-4YZ = (X-Y+Z)^2-4ZX = (X+Y-Z)^2-4XY. \end{aligned}$$

From these three expressions, we obtain the three points

$$(yz, -x^3+y^3+z^3), (zx, x^3-y^3+z^3), (xy, x^3+y^3-z^3)$$

on the curve  $Y^2=4X^3+D(x^3, y^3, z^3)$  over  $\mathbf{Q}(x, y, z)$ . In order to find curves of this type having many integral points, Craig studied the simultaneous equations

$$(1) \quad \begin{cases} x_0z_0 = x_1z_1, & x_0^3 - y_0^3 + z_0^3 = -(x_1^3 - y_1^3 + z_1^3), \\ x_0y_0 = x_2y_2, & x_0^3 + y_0^3 - z_0^3 = -(x_2^3 + y_2^3 - z_2^3), \end{cases}$$

which yield immediately

$$D(x_0^3, y_0^3, z_0^3) = D(x_1^3, y_1^3, z_1^3) = D(x_2^3, y_2^3, z_2^3).$$

He gave the solution as follows; For  $\alpha, \beta, \gamma$  such that

$$\alpha + \beta + \gamma = 0,$$

let

$$(2) \quad \phi = \begin{vmatrix} \alpha - \beta & \gamma - \alpha \\ \beta - \gamma & \alpha - \beta \end{vmatrix},$$

$$(3) \quad \begin{bmatrix} L & U \\ M & V \\ N & W \end{bmatrix} = \begin{bmatrix} 1+3\alpha\phi & \phi^2+3\alpha \\ 1+3\beta\phi & \phi^2+3\beta \\ 1+3\gamma\phi & \phi^2+3\gamma \end{bmatrix},$$

$$(4) \quad \left\{ \begin{array}{l} K = M+V+3\phi \\ \lambda = -W(2L^2-LW+W^2) \\ \mu/K = (N+U)^2+W(M-V) \\ \nu = L^2(N+U)-KW(M-V) \\ a = -LW(L-W)(N-U) \\ Kb = -\lambda(3\phi K-LU)-\mu KN \\ c/L = 3\phi(-\lambda/W)+U(-a/LW) \\ N\mu b_1 = 3\phi\lambda a - L\mu b \\ K\nu c_1 = W\lambda a - K\mu b_1. \end{array} \right.$$

Then

$$\begin{bmatrix} x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} = \begin{bmatrix} \lambda a & \mu b & \nu c \\ \lambda c & \mu b_1 & \nu a \\ \lambda b & \mu a & \nu c_1 \end{bmatrix}$$

gives a solution of the equations (1).

In particular, putting  $[\alpha, \beta, \gamma] = [0, T, -T]$ , we may consider that the parameters appearing above and  $x_0, \dots, z_2$  are rational functions in one variable  $T$ . In fact, we can write them down by a tedious calculation or an electric computer (see Appendix A). Let  $D(T) = D(x_0^3, y_0^3, z_0^3)$  and  $A_i(T), B_i(T)$  ( $1 \leq i \leq 6$ ) be as in the table below.

$i$	$A_i$	$B_i$
1	$x_0 z_0 = x_1 z_1$	$x_0^3 - y_0^3 + z_0^3 = -x_1^3 + y_1^3 - z_1^3$
2	$y_2 z_2$	$-x_2^3 + y_2^3 + z_2^3$
3	$x_1 y_1$	$x_1^3 + y_1^3 - z_1^3$
4	$x_0 y_0 = x_2 y_2$	$x_0^3 + y_0^3 - z_0^3 = -x_2^3 - y_2^3 + z_2^3$
5	$y_0 z_0$	$-x_0^3 + y_0^3 + z_0^3$
6	$y_1 z_1$	$-x_1^3 + y_1^3 + z_1^3$

For  $1 \leq i \leq 6$ ,  $A_i(T)$  and  $B_i(T)$  are all polynomials in  $\mathbf{Z}[T]$ . Thus we obtain the family of the elliptic curves;

$$Y^2 = 4X^3 + D(t), \quad \text{for } t \in \mathbf{Z},$$

having the six integral points  $P_i = (A_i(t), B_i(t))$  ( $1 \leq i \leq 6$ ).

In Craig [2], the last two points  $P_5, P_6$  are not used, since the classes of ideals of a quadratic field corresponding to three points, for example,  $P_1, P_4, P_5$  satisfy an identical relation (see [1] pp. 451). However, in the proof of our theorem, we can take full advantage of these six points.

We now state, as lemmas, a few properties of the polynomials  $A_i(T), B_i(T)$  and  $D(T)$  required later. The first lemma is shown by patient modular calculations.

**Lemma 1.** *For any rational integer  $t$ , we have*

(5)  $B_i(t) \not\equiv 0 \pmod{3} \quad (1 \leq i \leq 6),$

(6)  $D(t) \equiv 1 \pmod{4},$

(7)  $A_i(t) \equiv 0 \pmod{4} \quad (1 \leq i \leq 6), \text{ if } t \equiv 0 \pmod{4}.$

**Lemma 2.**  *$A_i(T)$  and  $B_i(T)$  are relatively prime in  $\mathbf{Q}[T]$ , for  $1 \leq i \leq 6$ .*

**Proof.** For  $1 \leq i \leq 4$ , the assertions are shown in [2] pp. 391–396. The others can be verified from them without difficulty.

**Lemma 3.**  $D(T)$  has at least one root whose multiplicity is prime to 3.

**Proof.** Since  $D(T) = 1 + \dots - 2^2 3^{72} T^{141}$  (see [2] pp. 390), it is easy to see that  $D(T)$  cannot be equal to  $rE(T)^3$  for any  $r \in \mathbf{Q}$ , and  $E(T) \in \mathbf{Q}[T]$ . The lemma follows from this obviously. (In fact, it has been proved in [2] that there are at least three simple roots of  $D(T)$ , which shall not be required in our article.)

**REMARK.** Lemmas 2 and 3 are immediate consequences of the irreducibility of  $D(T)$  which can be shown by a computer calculation. Indeed, the factorizer of REDUCE reveals the fact that  $D(T)$  modulo 79 is irreducible in  $(\mathbf{Z}/79\mathbf{Z})[T]$ .

## 2. Unramified extensions

We now consider the pure cubic fields

$$K_t = \mathbf{Q}(\theta(t)) \quad \text{where} \quad 4\theta(t)^3 = D(t),$$

for  $t \in \mathbf{Z}$ . Note that  $K_t$  is actually pure cubic because  $D(t)/4$  is not a cube by (6). Let

$$L_t = K_t(\sqrt{\theta(t) + A_1(t)}, \dots, \sqrt{\theta(t) + A_6(t)}).$$

Then we have

**Lemma 4.** Suppose, for  $1 \leq i \leq 6$ ,  $A_i(t)$  and  $B_i(t)$  are relatively prime in the ring  $\mathbf{Z}[2^{-1}]$  and  $t \equiv 0 \pmod{4}$ . Then  $L_t/K_t$  is unramified at all primes of  $K_t$ .

**Proof.** We write simply  $A_i$  instead of  $A_i(t)$ , and so on. It suffices to prove that the quadratic extension  $K_t(\sqrt{\theta + A_i})/K_t$  is unramified for each  $i$ . First, since  $\theta^3 + A_i^3 = (B_i/2)^2 > 0$  and consequently  $\theta + A_i$  is totally positive, all infinite primes of  $K_t$  are unramified. Next, let  $\mathfrak{p}$  be any prime ideal of  $K_t$  prime to 2.  $\mathfrak{p}$  is unramified for  $K_t(\sqrt{\theta + A_i})$  if it does not divide  $\theta + A_i$ . Assume that  $\theta + A_i$  is divisible by  $\mathfrak{p}$ . Then we have

$$(B_i/2)^2 = \theta^3 + A_i^3 = (\theta + A_i)(\theta^2 - \theta A_i + A_i^2) \equiv 0 \pmod{\mathfrak{p}}.$$

If  $\theta^2 - \theta A_i + A_i^2 \equiv 0 \pmod{\mathfrak{p}}$ , then  $B_i \equiv 3A_i \equiv 0 \pmod{\mathfrak{p}}$  which contradicts (5) or the assumption of the lemma. Therefore

$$\text{ord}_{\mathfrak{p}}(\theta + A_i) = \text{ord}_{\mathfrak{p}}(\theta^3 + A_i^3) = \text{ord}_{\mathfrak{p}}((B_i/2)^2) \equiv 0 \pmod{2}.$$

This implies that  $\mathfrak{p}$  is unramified for  $K_t(\sqrt{\theta + A_i})$ . Lastly, as  $\text{ord}_2(D/4) = -2$

by (6), there is a unique prime ideal  $\mathfrak{q}$  of  $K_t$  lying above 2 which is totally ramified for  $K_t/\mathbb{Q}$ . We have

$$\text{ord}_{\mathfrak{q}}(2\theta) = \text{ord}_{\mathfrak{q}}(2) + \text{ord}_{\mathfrak{q}}(\theta) = 3 - 2 = 1.$$

Thus, by (6), (7) and the assumption  $t \equiv 0 \pmod{4}$ , we have

$$(2\theta)^2(\theta + A_i) \equiv 4\theta^3 = D \equiv 1 \pmod{4}.$$

Hence  $\mathfrak{q}$  is unramified for  $K_t(\sqrt{(2\theta)^2(\theta + A_i)}) = K_t(\sqrt{\theta + A_i})$ . This completes the proof.

### 3. A specific example

After Craig [2], we will give a specific numerical example and utilize it to infer the existence of infinitely many fields mentioned in the theorem.

We will examine the case  $t = -1$ , that is,  $[\alpha, \beta, \gamma] = [0, -1, 1]$ . By (2)–(4), we obtain the following values;

$$\begin{aligned} \phi &= 3, \\ \begin{bmatrix} L & U \\ M & V \\ N & W \end{bmatrix} &= \begin{bmatrix} 1 & 9 \\ -8 & 6 \\ 10 & 12 \end{bmatrix}, \\ \left\{ \begin{array}{l} K = 7 \\ \lambda = -1608 = -2^3 \cdot 3 \cdot 67 \\ \mu/K = 193 \text{ (prime)} \\ \nu = 1195 = 5 \cdot 239 \\ a = 132 = 2^2 \cdot 3 \cdot 11 \\ Kb = -7738 = -2 \cdot 53 \cdot 73 \\ c = 1107 = 3^3 \cdot 41 \\ \mu b_1 = -41687 \text{ (prime)} \\ K\nu c_1 = -2255263 \text{ (prime)}. \end{array} \right. \end{aligned}$$

With the same procedure as in [2] pp. 383–386, we can show that  $A_i(-1)$  and  $B_i(-1)$  are relatively prime for all  $1 \leq i \leq 6$  (cf. Appendix B). Let  $R_i$  be the resultant of  $A_i(T)$  and  $B_i(T)$  ( $1 \leq i \leq 6$ ). By Lemma 2,  $R_i$  is a non-zero rational integer. Let  $M_1$  be the product of all odd prime factors of  $\prod_{i=1}^6 R_i$ .

**Lemma 5.** *If  $t \equiv -1 \pmod{M_1}$ , then  $A_i(t)$  and  $B_i(t)$  are relatively prime in  $\mathbb{Z}[2^{-1}]$  for  $1 \leq i \leq 6$ .*

Proof. If an odd prime  $l$  divides both of  $A_i(t)$  and  $B_i(t)$ , then we have

$R_i \equiv 0 \pmod{l}$  thus  $M_1 \equiv 0 \pmod{l}$ . Therefore, if  $t \equiv -1 \pmod{M_1}$ , then  $A_i(-1) \equiv B_i(-1) \equiv 0 \pmod{l}$  which is a contradiction.

Next we are concerned with the independency of  $\theta(t) + A_i(t)$  ( $1 \leq i \leq 6$ ) in  $K_t^\times / K_t^{\times 2}$ . Let  $p$  be a prime for which  $D(-1)/4$  is a cubic residue modulo  $p$ . We define  $c_i(p) \in \mathbf{Z}/2\mathbf{Z}$  as

$$\left( \frac{r + A_i(-1)}{p} \right) = (-1)^{c_i(p)},$$

where  $r^3 \equiv D(-1)/4 \pmod{p}$  and  $(-)$  is the quadratic residue symbol. It should be noted that  $c_i(p)$  is dependent on the choice of  $r$ . We will fix  $r$  suitably for each  $p$ . A computer search gives the following table;

$p$	$D/4$	$r$	$c_1$	$c_2$	$c_3$	$c_4$	$c_5$	$c_6$
17	10	3	1	0	1	1	1	1
19	12	10	0	1	1	0	1	1
23	4	3	0	1	0	0	1	0
31	1	25	1	0	0	0	0	0
37	1	1	0	1	0	0	0	1
41	16	10	0	0	0	0	0	1

The second and third columns are given modulo  $p$ . Let  $[p_1, \dots, p_6] = [17, 19, 23, 31, 37, 41]$  and  $M_2 = \prod_{j=1}^6 p_j$ . We need the fact that  $\det(c_i(p_j)) \neq 0$  in  $\mathbf{Z}/2\mathbf{Z}$ , which is easily verified.

**Lemma 6.** *If  $t \equiv -1 \pmod{M_2}$ , then  $\theta(t) + A_1(t), \dots, \theta(t) + A_6(t)$  are independent in  $K_t^\times / K_t^{\times 2}$ .*

Proof. Suppose  $t \equiv -1 \pmod{M_2}$ . It is not difficult to show that there exist a prime ideal  $\mathfrak{p}_j$  of  $K_t$  such that

$$\left( \frac{\theta(t) + A_i(t)}{\mathfrak{p}_j} \right) = (-1)^{c_i(p_j)} \quad (1 \leq j \leq 6).$$

Assume

$$\prod_{i=1}^6 (\theta(t) + A_i(t))^{a_i} \in K_t^{\times 2}$$

for some  $a_i \in \mathbf{Z}/2\mathbf{Z}$ . Considering this modulo  $\mathfrak{p}_j$ , we have

$$\sum_{i=1}^6 c_i(p_j) a_i = 0 \quad (1 \leq j \leq 6).$$

As  $\det(c_i(p_j)) \neq 0$ , we see  $a_i = 0$  for all  $1 \leq i \leq 6$ . This proves the lemma.

REMARK. For  $t \in \mathbf{Z}$ , let  $E_t$  be the elliptic curve defined by  $Y^2 = 4X^3 + D(t)$ , and  $E_t(\mathbf{Q})$  be the set of  $\mathbf{Q}$ -rational points of  $E_t$ . Then there is an injective homomorphism

$$E_t(\mathbf{Q})/2E_t(\mathbf{Q}) \rightarrow K_t^\times / K_t^{\times 2}$$

given by

$$(x, y) \rightarrow \theta(t) + x.$$

(See [7] Chap. X.) Therefore the above lemma implies that, for infinitely many rational integers  $t$ , the six points  $(A_i(t), B_i(t))$  ( $1 \leq i \leq 6$ ) are independent in  $E_t(\mathbf{Q})/2E_t(\mathbf{Q})$ , and consequently,

$$\text{rank } E_t(\mathbf{Q}) \geq 6.$$

On the other hand, Kihara [4] showed a result of this type using the duplication formula on elliptic curves, for the case  $[\alpha, \beta, \gamma] = [-t, t, 0]$ . Kihara's argument can be applied to the proof of the independency of  $\theta(t) + A_i(t)$  ( $1 \leq i \leq 6$ ) in  $K_t^\times / K_t^{\times 2}$ .

#### 4. Proof of Theorem

We are now ready to prove the theorem. Let  $t$  be a rational integer such that

$$(8) \quad \begin{cases} t \equiv 0 & (\text{mod } 4), \\ t \equiv -1 & (\text{mod } M_0), \end{cases}$$

where  $M_0$  is the least common multiple of  $M_1$  and  $M_2$ . Then, by Lemmas 4, 5 and 6,  $L_t$  is an unramified abelian extension of  $K_t$  with Galois group isomorphic to the elementary 2-abelian group of rank 6. Hence  $K_t$  is a pure cubic field whose ideal class group has 2-rank at least 6.

Lastly, we must make sure of existence of infinitely many such fields. From Lemma 3, we can take a root  $\tau$  of  $D(T)$  with multiplicity  $m$  prime to 3. Write

$$D(T) = (T - \tau)^m D_0(T).$$

It is well known that there exist infinitely many prime ideals of  $\mathbf{Q}(\tau)$  of degree 1 which are unramified for  $\mathbf{Q}(\tau)/\mathbf{Q}$ . We choose a prime ideal  $\mathfrak{p}$  of this kind such that

$$\text{ord}_{\mathfrak{p}}(6M_0) = \text{ord}_{\mathfrak{p}}(\tau) = \text{ord}_{\mathfrak{p}}(D_0(\tau)) = 0.$$

Then it is possible to find a rational integer  $t$  satisfying in addition to (8) also the congruence;



$$t \equiv \tau + p \pmod{p^2},$$

where  $p$  is the rational prime contained in  $\mathfrak{p}$ . Since

$$\text{ord}_p(t - \tau) = \text{ord}_p(p) = 1 \quad \text{and} \quad \text{ord}_p(D_0(t)) = \text{ord}_p(D_0(\tau)) = 0,$$

we have

$$\text{ord}_p(D(t)) = m \not\equiv 0 \pmod{3}.$$

Hence  $p$  is ramified for  $K_t$ . The various choices of  $\mathfrak{p}$  prove our assertion.

### Appendix

A. The following explicit expressions in the case  $[\alpha, \beta, \gamma] = [0, T, -T]$  are obtained, in an instant, by means of REDUCE.

$$\phi = 3T^2,$$

$$\begin{bmatrix} L & U \\ M & V \\ N & W \end{bmatrix} = \begin{bmatrix} 1 & 9T^4 \\ 1+9T^3 & 3T+9T^4 \\ 1-9T^3 & -3T+9T^4 \end{bmatrix},$$

$$K = 1+3T+9T^2+9T^3+9T^4,$$

$$\lambda = 6T+9T^2+27T^3-18T^4-54T^5-243T^6+81T^8+729T^9-729T^{12},$$

$$\mu/K = 1-3T+9T^2-18T^3+81T^6-81T^7,$$

$$\nu = 1+3T-9T^3-27T^4+81T^6+81T^7-486T^9+729T^{12},$$

$$a = 3T+9T^2-36T^4-162T^5-81T^6+81T^7+648T^8+486T^9-729T^{11} \\ -729T^{12},$$

$$Kb = -1-3T-18T^2-72T^3-288T^4-675T^5-1215T^6-567T^7+1701T^8 \\ +7047T^9+9477T^{10}+8019T^{11}-3645T^{12}-13122T^{13}-19683T^{14} \\ -13122T^{15}-6561T^{16},$$

$$c = 18T^2+27T^3+90T^4+27T^5-81T^6-567T^7-405T^8-243T^9+729T^{10} \\ +729T^{11}+729T^{12},$$

$$\mu b_1 = y_1 \text{ (see below),}$$

$$K\nu c_1 = Kz_2 \text{ (see below).}$$

The following expressions are given with coefficients which are factorized into prime factors;

$$x_0 = 2 \cdot 3^2 T^2 + 3^4 T^3 + 2 \cdot 3^4 T^4 - 3^3 T^5 - 2^2 \cdot 3^4 \cdot 5 T^6 - 3^5 \cdot 17 T^7 - 2^2 \cdot 3^4 \cdot 19 T^8 \\ + 3^5 \cdot 31 T^9 + 3^8 \cdot 5 T^{10} + 3^6 \cdot 97 T^{11} + 3^6 \cdot 13 T^{12} - 3^7 \cdot 53 T^{13} - 2^2 \cdot 3^8 \cdot 13 T^{14} \\ - 3^{11} T^{15} + 3^8 \cdot 29 T^{16} + 3^9 \cdot 43 T^{17} + 2 \cdot 3^{10} \cdot 5 T^{18} - 2 \cdot 3^{10} T^{19} - 2 \cdot 3^{12} T^{20} \\ - 3^{11} \cdot 5 T^{21} + 3^{12} T^{23} + 3^{12} T^{24},$$

$$\begin{aligned}
 y_0 &= -1 - 2 \cdot 3^2 T^2 - 3^3 T^3 - 2^2 \cdot 3^2 \cdot 5 T^4 - 3^3 \cdot 5 T^5 - 3^4 \cdot 7 T^6 + 3^4 \cdot 5^2 T^7 \\
 &\quad + 2 \cdot 3^5 \cdot 7 T^8 + 3^5 \cdot 59 T^9 - 3^6 \cdot 5 T^{10} - 2 \cdot 3^6 \cdot 13 T^{11} - 3^6 \cdot 5 \cdot 31 T^{12} \\
 &\quad - 2 \cdot 3^7 \cdot 11 T^{13} + 2^2 \cdot 3^8 T^{14} + 3^8 \cdot 5 \cdot 13 T^{15} + 2^2 \cdot 3^8 \cdot 11 T^{16} + 3^9 \cdot 7 T^{17} \\
 &\quad - 3^{10} \cdot 13 T^{18} - 3^{10} \cdot 11 T^{19} - 3^{12} T^{20} + 3^{12} T^{21} + 3^{12} T^{22} + 3^{12} T^{23}, \\
 z_0 &= 2 \cdot 3^2 T^2 + 3^4 T^3 + 3^2 \cdot 19 T^4 + 3^3 \cdot 5 T^5 - 3^6 T^6 - 3^4 \cdot 29 T^7 - 3^4 \cdot 41 T^8 + 3^7 T^9 \\
 &\quad + 3^6 \cdot 23 T^{10} + 3^6 \cdot 31 T^{11} - 2 \cdot 3^6 T^{12} - 3^7 \cdot 43 T^{13} - 2^4 \cdot 3^8 T^{14} - 3^9 T^{15} \\
 &\quad + 3^8 \cdot 5 \cdot 11 T^{16} + 3^9 \cdot 17 T^{17} + 3^{11} T^{28} - 2^2 \cdot 3^{11} T^{19} - 3^{10} \cdot 11 T^{20} - 3^{12} T^{21} \\
 &\quad + 3^{12} T^{22} + 3^{12} T^{23} + 3^{12} T^{24}, \\
 x_1 &= 2^2 \cdot 3^3 T^3 + 2^2 \cdot 3^4 T^4 + 3^3 \cdot 47 T^5 + 3^4 \cdot 17 T^6 + 3^6 T^7 - 2 \cdot 3^4 \cdot 67 T^8 - 3^5 \cdot 89 T^9 \\
 &\quad - 2^3 \cdot 3^6 \cdot 7 T^{10} + 2^2 \cdot 3^6 \cdot 5 T^{11} + 2 \cdot 3^6 \cdot 61 T^{12} + 2^3 \cdot 3^8 \cdot 5 T^{13} + 3^8 \cdot 19 T^{14} \\
 &\quad - 3^9 \cdot 5 T^{15} - 3^8 \cdot 113 T^{16} - 2^2 \cdot 3^9 \cdot 7 T^{17} - 2^2 \cdot 3^{10} T^{18} + 3^{10} \cdot 17 T^{19} + 3^{11} \cdot 5 T^{20} \\
 &\quad + 2^2 \cdot 3^{11} T^{21} - 3^{12} T^{22} - 3^{12} T^{23} - 3^{12} T^{24}, \\
 y_1 &= 1 + 2 \cdot 3^2 T^2 + 2^2 \cdot 3^2 T^3 + 2 \cdot 3^2 \cdot 19 T^4 + 2 \cdot 3^3 \cdot 19 T^5 + 3^4 \cdot 29 T^6 + 2 \cdot 3^4 \cdot 5 T^7 \\
 &\quad - 2^2 \cdot 3^7 T^8 - 3^5 \cdot 5^3 T^9 - 3^6 \cdot 61 T^{10} + 3^6 \cdot 11 T^{11} + 3^6 \cdot 5 \cdot 37 T^{12} \\
 &\quad + 2 \cdot 3^7 \cdot 5 \cdot 13 T^{13} + 2^2 \cdot 3^8 \cdot 5 T^{14} - 3^9 \cdot 13 T^{15} - 2 \cdot 3^8 \cdot 61 T^{16} - 2^2 \cdot 3^9 \cdot 7 T^{17} \\
 &\quad + 3^{11} T^{18} + 2 \cdot 3^{12} T^{19} + 3^{11} \cdot 5 T^{20} - 3^{12} T^{22} - 3^{12} T^{23}, \\
 z_1 &= 3T + 2 \cdot 3^2 T^2 + 3^3 T^3 - 3^2 \cdot 7 T^4 - 2^4 \cdot 3^3 T^5 - 2 \cdot 3^4 \cdot 5 T^6 + 3^4 \cdot 5 T^7 + 3^4 \cdot 53 T^8 \\
 &\quad + 2 \cdot 3^5 \cdot 17 T^9 - 2 \cdot 3^6 T^{10} - 2^3 \cdot 3^6 \cdot 5 T^{11} - 3^6 \cdot 61 T^{12} + 2 \cdot 3^7 T^{13} + 3^8 \cdot 23 T^{14} \\
 &\quad + 2^3 \cdot 3^9 T^{15} - 3^8 T^{16} - 3^9 \cdot 5^2 T^{17} - 3^{10} \cdot 7 T^{18} + 2 \cdot 3^{10} \cdot 7 T^{20} + 2^2 \cdot 3^{11} T^{21} \\
 &\quad - 3^{12} T^{23} - 3^{12} T^{24}, \\
 Kx_2 &= -2 \cdot 3T - 3^3 T^2 - 2 \cdot 3^4 T^3 - 3^2 \cdot 73 T^4 - 2 \cdot 3^4 \cdot 17 T^5 - 3^4 \cdot 97 T^6 - 2^5 \cdot 3^4 \cdot 7 T^7 \\
 &\quad - 3^5 \cdot 79 T^8 + 2^2 \cdot 3^5 \cdot 17 T^9 + 3^6 \cdot 229 T^{10} + 3^8 \cdot 59 T^{11} + 2 \cdot 3^6 \cdot 373 T^{12} \\
 &\quad - 3^7 \cdot 17 T^{13} - 3^9 \cdot 73 T^{14} - 2 \cdot 3^9 \cdot 7 \cdot 13 T^{15} - 2 \cdot 3^9 \cdot 5 \cdot 17 T^{16} + 3^9 \cdot 5 T^{17} \\
 &\quad + 3^{11} \cdot 47 T^{18} + 2^4 \cdot 3^{10} \cdot 13 T^{19} + 2^3 \cdot 3^{11} \cdot 7 T^{20} - 2 \cdot 3^{12} \cdot 5 T^{21} - 3^{12} \cdot 31 T^{22} \\
 &\quad - 2^3 \cdot 3^{12} \cdot 5 T^{23} - 2 \cdot 3^{12} \cdot 7 T^{24} + 3^{14} T^{25} + 3^{15} T^{26} + 2 \cdot 3^{14} T^{27} + 3^{14} T^{28}, \\
 y_2/K &= 3T - 3^2 T^4 - 2^3 \cdot 3^3 T^5 + 3^4 T^6 - 3^5 T^7 + 2 \cdot 3^4 \cdot 19 T^8 - 3^6 \cdot 5^2 T^{11} - 3^6 T^{12} \\
 &\quad + 2^2 \cdot 3^7 T^{13} + 2^3 \cdot 3^8 T^{14} - 2 \cdot 3^9 T^{16} - 3^{10} T^{17} + 3^{10} T^{19}, \\
 Kz_2 &= -1 - 3T - 3^3 T^2 - 3^2 \cdot 17 T^3 - 2^5 \cdot 3^3 T^4 - 2^4 \cdot 3^3 \cdot 7 T^5 - 2^2 \cdot 3^7 T^6 - 2^3 \cdot 3^4 \cdot 23 T^7 \\
 &\quad - 2^2 \cdot 3^4 \cdot 41 T^8 + 3^7 \cdot 17 T^9 + 2^2 \cdot 3^7 \cdot 17 T^{10} + 3^9 \cdot 17 T^{11} + 3^7 \cdot 149 T^{12} \\
 &\quad - 2 \cdot 3^7 \cdot 11 T^{13} - 3^8 \cdot 11 \cdot 17 T^{14} - 2 \cdot 3^9 \cdot 59 T^{15} - 2 \cdot 3^8 \cdot 211 T^{16} - 3^9 \cdot 5 T^{17} \\
 &\quad + 3^{10} \cdot 71 T^{18} + 3^{10} \cdot 7 \cdot 23 T^{19} + 3^{10} \cdot 5 \cdot 29 T^{20} + 3^{11} \cdot 11 T^{21} - 2^2 \cdot 3^{12} \cdot 5 T^{22} \\
 &\quad - 3^{12} \cdot 31 T^{23} - 2^3 \cdot 3^{13} T^{24} + 2 \cdot 3^{14} T^{26} + 2 \cdot 3^{14} T^{27} + 3^{14} T^{28}.
 \end{aligned}$$

**B.** To check quickly that  $(A_i(-1), B_i(-1))=1$  ( $1 \leq i \leq 6$ ), one can use the Euclidian algorithm, directly for  $A_i(-1)$  and  $B_i(-1)$ , which is carried out on an

electric computer. Another method is to factorize them into primes. Our assertion is verified by the following complete factorizations;

$$\begin{array}{ll}
 A_1 = -2^5 \cdot 3^5 \cdot 5 \cdot 11 \cdot 41 \cdot 67 \cdot 239 & B_1 = 401 \cdot 1229 \cdot 2153 \cdot 5311935749 \\
 A_2 = -2^2 \cdot 3 \cdot 11 \cdot 193 \cdot 2255263 & B_2 = -750719 \cdot 7518218706311 \\
 A_3 = 2^3 \cdot 3^4 \cdot 41 \cdot 67 \cdot 41687 & B_3 = -11 \cdot 23304629 \cdot 22017793601 \\
 A_4 = 2^6 \cdot 3^2 \cdot 11 \cdot 53 \cdot 67 \cdot 73 \cdot 193 & B_4 = -5 \cdot 61 \cdot 18542335623945929 \\
 A_5 = -2 \cdot 3^3 \cdot 5 \cdot 41 \cdot 53 \cdot 73 \cdot 193 \cdot 239 & B_5 = -29 \cdot 34701175163759747 \\
 A_6 = -2^2 \cdot 3 \cdot 5 \cdot 11 \cdot 239 \cdot 41687 & B_6 = 139 \cdot 40605300259955467
 \end{array}$$

Lastly, we append the factorization of  $D=D(-1)$ ;

$$D = 185478257 \cdot 171752102638681035930180903617 .$$

These factorizations were performed in consulting of Riesel [6] or Wada [8].

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