

Title	The Borsuk-Ulam theorem and formal group laws
Author(s)	Munkholm, Hans J.; Nakaoka, Minoru
Citation	Osaka Journal of Mathematics. 9(3) P.337-P.349
Issue Date	1972
Text Version	publisher
URL	https://doi.org/10.18910/4982
DOI	10.18910/4982
rights	
Note	

Osaka University Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

Osaka University

THE BORSUK-ULAM THEOREM AND FORMAL GROUP LAWS

HANS J. MUNKHOLM AND MINORU NAKAOKA

(Received October 20, 1971)

Introduction

The present paper is concerned with the following question raised on the classical Borsuk-Ulam theorem : Let G denote a cyclic group of odd order q , and let Σ be a homotopy $(2n+1)$ -sphere on which a free differentiable G -action is given. For any differentiable m -manifold M and any continuous map $f: \Sigma \rightarrow M$, put $A(f) = \{x \in \Sigma \mid f(x) = f(xg) \text{ for all } g \in G\}$. What can be deduced about the covering dimension of $A(f)$?

In response to this question, the authors showed previously that if q is a prime p then $\dim A(f) \geq 2n+1 - (p-1)m$ ([4], [6]). Furthermore, one of the authors showed in [5] that if q is a prime power p^a and M is the Euclidean space \mathbf{R}^m then

$$(0.1) \quad \dim A(f) \geq (2n+1) - (p^a - 1)m - [m(a-1)p^a - (ma+2)p^{a-1} + m + 3].$$

It will be shown in this paper that (0.1) still holds for any differentiable m -manifold M .

The procedure taken in this paper is different from the previous ones, and we shall derive the above result from a general theorem stated in connection with the formal group law for some general cohomology theory.

Assume that there is given a multiplicative cohomology theory h defined on the category of finite CW pairs and satisfying the conditions: i) each complex vector bundle is h -orientable, ii) $h^i(pt) = 0$ for each odd i . Let $F(x, y) \in h(pt)[[x, y]]$ denote the formal group law associated to h , and $[i](x) \in h(pt)[[x]]$ denote the operation of "multiplication by i " for a positive integer i . We shall show that

$$(0.2) \quad \dim A(f) < 2d \quad \Rightarrow \quad x^a \left(\prod_{i=1}^{(q-1)/2} [i](x) \right)^m \in (x^{n+1}, [q](x)) \text{ in } h(pt)[[x]],$$

where (a, b) denotes the ideal generated by a and b .

Take as h the general cohomology theory defined from K -theory. Then it is seen by using elementary algebraic number theory that (0.2) is equivalent to (0.1).

We can also take as h the complex cobordism theory U^* . Since U^* is stronger than K -theory in general, it is expected that sharper result than (0.1) will be obtained from (0.2) applied to $h=U^*$. However we have no method to derive numerical conditions equivalent to (0.2) for $h=U^*$.

In an appendix, we shall prove in the same procedure as above a non-existence theorem for equivariant maps which generalizes the result of Vick [10].

1. The formal group law for a multiplicative cohomology

We recall first some facts on multiplicative cohomology theory (see Dold [3]).

We fix once and for all a multiplicative reduced cohomology theory \tilde{h} defined on the category of finite CW complexes with base point. There is the corresponding multiplicative cohomology theory h defined on the category of finite CW pairs.

Let ξ be a real n -dimensional vector bundle over a finite CW complex B , and denote by $M(\xi)$ the Thom space for ξ . For each $b \in B$ let ξ_b denote the restriction of ξ over b . Then $\tilde{h}(M(\xi_b))$ is a free $h(pt)$ -module on one generator. ξ is said to be h -orientable if there exists $t(\xi) \in \tilde{h}^n(M(\xi))$ such that $t(\xi)|_{M(\xi_b)}$ is a generator of $\tilde{h}(M(\xi_b))$ for each $b \in B$. Such $t(\xi)$ is called an h -orientation or a *Thom class* of ξ . By an h -oriented vector bundle we mean a vector bundle in which an h -orientation is given.

Let $D(\xi)$ (or $S(\xi)$) denote the total space of the disc bundle (or the sphere bundle) associated to ξ , and consider the homomorphism

$$\tilde{h}^n(M(\xi)) = h^n(D(\xi), S(\xi)) \xrightarrow{j^*} h^n(D(\xi)) \xrightarrow[\cong]{p^{*-1}} h^n(B),$$

where j is the inclusion and p is the projection. The image of $t(\xi)$ under this homomorphism is called the *Euler class* of the h -oriented bundle ξ , and is denoted by $e(\xi)$.

The following facts are easily proved:

(1.1) If there is a bundle map $f: \xi \rightarrow \xi'$ and ξ' is h -oriented, then ξ is h -oriented so that $f^*: h(B') \rightarrow h(B)$ preserves the Euler classes.

(1.2) If ξ_1 and ξ_2 are h -oriented, then the Whitney sum $\xi_1 \oplus \xi_2$ is h -oriented so that $e(\xi_1 \oplus \xi_2) = e(\xi_1)e(\xi_2)$.

(1.3) If ξ has a non-zero cross section, then $e(\xi) = 0$.

The classical Leray-Hirsch theorem on fiberings can be generalized to the multiplicative theory h , and so we have the Thom isomorphism

$$\Phi : h(B) \cong \tilde{h}(M(\xi))$$

given by $\Phi(\alpha) = \alpha \cdot t(\xi)$. As a consequence, the Gysin exact sequence

$$\dots \rightarrow h^{i-1}(S(\xi)) \rightarrow h^{i-n}(B) \xrightarrow{e(\xi)} h^i(B) \xrightarrow{p^*} h^i(S(\xi)) \rightarrow \dots$$

holds.

A complex vector bundle ξ is called h -orientable if the real form $\xi_{\mathbb{R}}$ is h -orientable. Let η_n denote the canonical complex line bundle over the complex n -dimensional projective space CP^n . Throughout this section the following will be assumed:

(1.4) For each n , η_n is h -oriented so that the homomorphism $h(CP^{n+1}) \rightarrow h(CP^n)$ preserves the Euler classes.

It follows from this assumption that any complex line bundle ξ over a finite CW complex is h -oriented so that the homomorphism $f^* : h(B') \rightarrow h(B)$ induced by every bundle map $f : \xi \rightarrow \xi'$ preserves the Euler classes.

We can prove

(1.5) The algebra $h(CP^n)$ is a truncated polynomial algebra over $h(pt)$:

$$h(CP^n) = h(pt)[e(\eta_n)]/(e(\eta_n)^{n+1}).$$

(1.6) Put $e(\eta_m)_1 = p_1^* e(\eta_m)$ and $e(\eta_n)_2 = p_2^* e(\eta_n)$ for the projections $p_1 : CP^m \times CP^n \rightarrow CP^m$ and $p_2 : CP^m \times CP^n \rightarrow CP^n$. Then the isomorphism

$$h(CP^m \times CP^n) = h(pt)[e(\eta_m)_1, e(\eta_n)_2]/(e(\eta_m)_1^{m+1}, e(\eta_n)_2^{n+1})$$

holds.

For a CW complex X with finite skelta, we define $h(X)$ as the inverse limit with respect to skelta :

$$h(X) = \varprojlim h(X^n).$$

Then, for the infinite dimensional projective space CP^∞ , the following result is obtained from (1.5) and (1.6).

(1.7) $h(CP^\infty)$ and $h(CP^\infty \times CP^\infty)$ are rings of formal power series :

$$h(CP^\infty) = h(pt)[[x]], \quad h(CP^\infty \times CP^\infty) = h(pt)[[x_1, x_2]],$$

where x, x_1, x_2 are the elements defined by $e(\eta_n), e(\eta_n)_1, e(\eta_n)_2$ respectively.

Let η denote the canonical line bundle over CP^∞ , and consider the external tensor product $\eta \hat{\otimes} \eta$ which is a complex line bundle over $CP^\infty \times CP^\infty$. Let $\mu : CP^\infty \times CP^\infty \rightarrow CP^\infty$ be a classifying map for $\eta \hat{\otimes} \eta$ which is cellular, and put

$$\mu^*(x) = \sum_{i,j \geq 0} a_{ij} x_1^i x_2^j \quad (a_{ij} \in h^{2(1-i-j)}(pt))$$

for $\mu^* : h(CP^\infty) \rightarrow h(CP^\infty \times CP^\infty)$. Then we obtain easily

(1.8) For the tensor product $\xi_1 \otimes \xi_2$ of any complex line bundles ξ_1 and ξ_2

over a finite CW complex,

$$e(\xi_1 \otimes \xi_2) = \sum_{i,j \geq 0} a_{ij} e(\xi_1)^i e(\xi_2)^j$$

holds.

Consider now a power series $F(x,y)$ with coefficients in $h(pt)$, which is defined by

$$F(x,y) = \sum_{i,j \geq 0} a_{ij} x^i y^j$$

with a_{ij} above. Then it follows that $F(x,y)$ is a formal group law over $h(pt)$, i.e. the identities

$$\begin{aligned} F(x, 0) &= x, F(x, y) = F(y, x), \\ F(x, F(y, z)) &= F(F(x, y), z) \end{aligned}$$

hold. For each integer $i \geq 1$, let $[i](x) \in h[[x]]$ denote the operation of "multiplication by i " for the formal group, i.e.

$$[1](x) = x, \quad [i](x) = F([i-1](x), x).$$

Since the formula in (1.8) is rewritten as

$$e(\xi_1 \otimes \xi_2) = F(e(\xi_1), e(\xi_2)),$$

for the i -fold tensor product $\xi^i = \xi \otimes \dots \otimes \xi$ we have

$$e(\xi^i) = [i](e(\xi)).$$

Given a positive integer q , let G denote a cyclic group of order q . Define a G -action on the standard $(2n+1)$ -sphere $S^{2n+1} = \{(z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1} \mid \sum_i |z_i|^2 = 1\}$ by

$$(z_0, \dots, z_n) g_0 = (z_0 \exp 2\pi\sqrt{-1}/q, \dots, z_n \exp 2\pi\sqrt{-1}/q),$$

where g_0 is the generator of G . This yields a principal G -bundle $\rho'_n : S^{2n+1} \rightarrow L^n(q)$ over the lens space $L^n(q)$. Let L denote a 1-dimensional complex G -module given by $c \cdot g_0 = c \exp 2\pi\sqrt{-1}/q$, and consider the associated complex line bundle $\rho_n = \rho'_n \times_{\mathbb{C}} L$. For the canonical projection $\pi : L^n(q) \rightarrow CP^n$ we have $\rho_n = \pi^*(\eta_n)$, and hence $e(\rho_n)^{n+1} = 0$ holds.

Proposition 1. *Let $P(x) \in h(pt)[[x]]$. Then the element $P(e(\rho_n))$ of $h(L^n(q))$ is zero if and only if $P(x)$ is in the ideal generated by x^{n+1} and $[q](x)$.*

Proof. Consider the q -fold tensor product η_n^q of η_n . As is observed in [9],

the total space $S(\eta_n^q)$ of the sphere bundle associated to η_n^q is homeomorphic with $L^n(q)$. Therefore we have the Gysin sequence

$$\dots \rightarrow h^{i-2}(CP^n) \xrightarrow{\cdot e(\eta_n^q)} h^i(CP^n) \xrightarrow{\pi^*} h^i(L^n(q)) \rightarrow \dots$$

Since $e(\eta_n^q) = [q](e(\eta_n))$, the desired result follows from the above sequence and (1.5).

2. The element $s^*(\theta)$

As in § 1, let G denote a cyclic group of order q . We shall assume in the following that q is *odd*.

For any space X , let XG denote the product of q copies of X . Writing its elements as $\sum_{g \in G} x_g g$, a G -action on XG is given by

$$\left(\sum_{g \in G} x_g g\right) \cdot h = \sum_{g \in G} x_g h^{-1} g \quad (h \in G).$$

We denote by ΔX the diagonal in XG .

Let Σ be a homotopy $(2n+1)$ -sphere (which is a differentiable manifold), and assume that there is given a free differentiable G -action on Σ . We denote by Σ_G the orbit space.

Let M be a differentiable manifold, and consider the diagonal action on $\Sigma \times MG$ whose orbit space is denoted by $\Sigma \times_g MG$. $\Sigma \times \Delta M$ is an invariant submanifold of the G -manifold $\Sigma \times MG$, and its orbit space is regarded as $\Sigma_G \times \Delta M$. We denote by ν the normal bundle of $\Sigma_G \times \Delta M$ in $\Sigma \times_g MG$. This is a real $m(q-1)$ -dimensional vector bundle.

Choose a point $y_0 \in M$, and identify Σ_G with a subspace $\Sigma_G \times y_0 G$ ($y_0 G = \sum_g y_0 g$) of $\Sigma_G \times \Delta M$.

Let $\lambda' : \Sigma \rightarrow \Sigma_G$ denote the principal G -bundle defined by the G -action on Σ , and consider the associated complex line bundle $\lambda = \lambda' \times_g L$.

Proposition 2. *The normal bundle ν has a complex structure for which*

$$i^*(\nu) = m(\lambda \oplus \lambda^2 \oplus \dots \oplus \lambda^{(q-1)/2})$$

holds, where $i : \Sigma_G \rightarrow \Sigma_G \times \Delta M$ is the inclusion.

Proof. If $\nu_1 : N_1 \rightarrow \Delta M$ denote the normal G -vector bundle of ΔM in MG , then we have $\nu = id \times_g \nu_1 : \Sigma \times_g N_1 \rightarrow \Sigma_G \times \Delta M$. Therefore it suffices to prove that there exists a G -equivariant complex structure on ν_1 with the fiber over

y_0G being $m(L \oplus \dots \oplus L^{(q-1)/2})$.

To prove this, let IG be defined by the exact sequence of real G -modules

$$0 \rightarrow \Delta R \rightarrow RG \rightarrow IG \rightarrow 0.$$

View this as a sequence of real G -vector bundles over a point, and identify ΔM with $M \times \mathfrak{pt} = M$ in the obvious way. Then we have the exact sequence

$$0 \rightarrow \tau M \hat{\otimes} \Delta R \rightarrow \tau M \hat{\otimes} RG \rightarrow \tau M \hat{\otimes} IG \rightarrow 0$$

of real G -vector bundles over M , where τM denotes the tangent bundle over M . Since $\tau(MG) = (\tau M)G$, an equivariant isomorphism

$$\beta : \tau(MG)|_{\Delta M} \rightarrow \tau M \hat{\otimes} RG$$

can be given by

$$\beta(\sum_g v_g g) = \sum_g v_g \otimes g \quad (v_g \in \tau_y(M), y \in M).$$

Since $\sum v_g g$ is in $\tau(\Delta M)$ if and only if all v_g are equal, β maps $\tau(\Delta M)$ onto $\tau M \hat{\otimes} \Delta R$. Thus it holds that $\nu_1 \cong \tau M \hat{\otimes} IG$ as real G -vector bundles. From elementary representation theory of groups, it follows that IG is the real form of $L \oplus \dots \oplus L^{(q-1)/2}$. This gives ν_1 its complex structure, and we get

$$\begin{aligned} (\nu_1)_{y_0} &= \tau_{y_0} M \otimes (L \oplus \dots \oplus L^{(q-1)/2}) \\ &= R^m \otimes (L \oplus \dots \oplus L^{(q-1)/2}) = m(L \oplus \dots \oplus L^{(q-1)/2}) \end{aligned}$$

as desired. This completes the proof.

As in § 1, let h be a given multiplicative cohomology theory. In the following we shall assume the following conditions:

(2.1) every complex vector bundle of any dimension is h -orientable.

(2.2) $h^{odd}(\mathfrak{pt}) = 0$.

Assuming that M is closed, consider the normal bundle ν . Then, by Proposition 2 and (2.1), we have a Thom class $t(\nu) \in \tilde{h}^{m(q-1)}(M(\nu))$ and the corresponding Euler class $e(\nu) \in h^{m(q-1)}(\Sigma_G \times \Delta M)$ such that

$$\begin{aligned} (2.3) \quad i^*e(\nu) &= e(m(\lambda \oplus \lambda^2 \oplus \dots \oplus \lambda^{(q-1)/2})) \\ &= \left(\prod_{i=1}^{(q-1)/2} [i](e(\lambda)) \right)^m. \end{aligned}$$

As usual we shall regard the total space N of ν as a tubular neighborhood of $\Sigma_G \times \Delta M$ in $\Sigma_G \times MG$. Then we can identify $\tilde{h}(M(\nu))$ with $h(\Sigma_G \times MG)$,

$\Sigma \times_a MG - N$) canonically. Let

$$\theta \in h^{m(a-1)}(\Sigma \times_a MG)$$

be the image of the Thom class $t(\nu)$ under the homomorphism $l^* : h(\Sigma \times_a MG, \Sigma \times_a MG - N) \rightarrow h(\Sigma \times_a MG)$ induced by the inclusion. We have immediately

(2.4) For the homomorphism $j^* : h(\Sigma \times_a MG) \rightarrow h(\Sigma_G \times \Delta M)$ induced by the inclusion, $j^*(\theta) = e(\nu)$ holds.

Given a continuous map $f : \Sigma \rightarrow M$, define a continuous map $s : \Sigma_G \rightarrow \Sigma \times_a MG$ by

$$s(xG) = (x, \sum_g f(xg^{-1})g)G.$$

For the projection $p : \Sigma \times_a MG \rightarrow \Sigma_G$, $p \circ s$ is the identity.

Proposition 3. For the homomorphism $s^* : h(\Sigma \times_a MG) \rightarrow h(\Sigma_G)$ and the homomorphism $i^* : h(\Sigma_G \times \Delta M) \rightarrow h(\Sigma_G)$, we have

$$s^*(\theta) = i^*(e(\nu)).$$

Proof. It is easily seen that there exist a continuous map $f_1 : \Sigma \rightarrow M$ and an open set V of Σ satisfying the following conditions: i) f is homotopic to f_1 , ii) V is homeomorphic to \mathbf{R}^{2n+1} , iii) $f_1(\Sigma - V) = y_0$, iv) $xg \notin \bar{V}$ for any $g \neq 1$ and any $x \in \bar{V}$, where \bar{V} denotes the closure of V . Define $s_1 : \Sigma_G \rightarrow \Sigma \times_a MG$ from f_1 as s was defined from f , then s and s_1 are homotopic. Let $(MG)_1$ denote the subspace of MG consisting of points with at most one coordinate $\neq y_0$. Then $(MG)_1$ is an invariant subspace of the G -space MG , and the orbit space $\Sigma \times_a (MG)_1$ contains $s_1(\Sigma_G)$. Since $\Sigma - V$ is contractible, there exists a homotopy $\psi_t : (\bar{V}, \partial V) \rightarrow (\Sigma, \Sigma - V)$ such that ψ_0 is the inclusion and $\psi_1(\partial V) = x_0 \in \partial V$, where $\partial V = \bar{V} - V$. Put $V_G = \pi(V)$ for the projection $\pi : \Sigma \rightarrow \Sigma_G$. Consider now the following commutative diagram:

$$\begin{array}{ccc} \Sigma_G & \xrightarrow{s_1} & \Sigma \times_a (MG)_1 \\ \downarrow j_2 & & \downarrow j_1 \\ (\Sigma_G, \Sigma_G - V_G) & \xrightarrow{s_1} & (\Sigma \times_a (MG)_1, \Sigma_G \times y_0 G) \end{array}$$

where j_1, j_2 are the inclusions.

We have

$$h^{m(q-1)}(\Sigma_G, \Sigma_G - V_G) = \widehat{h}^{m(q-1)}(S^{2n+1}) = h^{m(q-1)-(2n+1)}(pt) = 0$$

by (2.2). Therefore

$$s_1^* \circ i_1^* : h^{m(q-1)}(\Sigma \times (MG)_1, \Sigma_G \times y_0 G) \rightarrow h^{m(q-1)}(\Sigma_G)$$

is trivial.

Next consider the commutative diagram

$$\begin{array}{ccc}
 h(\Sigma \times_{\mathcal{G}} MG) & \xrightarrow{j^*} & h(\Sigma_G \times \Delta M) \\
 \downarrow s^* & \searrow i_2^* & \downarrow i^* \\
 & h(\Sigma \times_{\mathcal{G}} (MG)_1) & \\
 \downarrow s_1^* & \nearrow p^* & \searrow i_1^* \\
 h(\Sigma_G) = h(\Sigma_G) & = & h(\Sigma_G \times y_0 G)
 \end{array}$$

where i_1, i_2 are the inclusions. Putting $\theta' = p^* i_1^* i_2^*(\theta) - i_2^*(\theta)$, we have

$$s_1^*(\theta') = i^* i^*(\theta) - s^*(\theta) = i^*(e(\nu)) - s^*(\theta)$$

by (2.4), and $i_1^*(\theta') = 0$. Therefore θ' is in the image of $j_1^* : h^{m(q-1)}(\Sigma \times_{\mathcal{G}} (MG)_1, \Sigma_G \times y_0 G) \rightarrow h^{m(q-1)}(\Sigma \times_{\mathcal{G}} (MG)_1)$, and hence $s_1^*(\theta') = 0$ by the fact proved above. Thus we have $i^*(e(\nu)) = s^*(\theta)$.

3. Generalization of Borsuk-Ulam theorem

Let Σ be as in §2, and let $f : \Sigma \rightarrow M$ be a continuous map to a differentiable m -manifold. Put

$$A(f) = \{x \in \Sigma \mid f(x) = f(xg) \text{ for any } g \in G\}.$$

In this section we shall consider the covering dimension of $A(f)$.

For the image $A(f)_G = \pi(A(f))$, we have $\dim A(f) = \dim A(f)_G$.

Proposition 4. *Assume that M is closed. Then $\dim A(f) < 2d$ implies*

$$e(d\lambda)s^*(\theta) = 0.$$

Proof. Since $\dim A(f)_G \leq 2d - 1$, it follows that $d\lambda$ has a non-zero cross section over $A(f)_G$ (see [5], Lemma 2). By standard facts on extension of cross section, this cross section extends to a non-zero cross section over the closure \bar{W} of some neighborhood W of $A(f)_G$ in Σ_G . Here we may assume that \bar{W} is

a finite CW complex, and that $s(\Sigma_G - W) \subset \Sigma \times_G MG - N$ by taking N small. We have then $e(d\lambda | \bar{W}) = 0$, and so $e(d\lambda)$ is in the image of $l_1^* : h(\Sigma_G, \bar{W}) \rightarrow h(\Sigma_G)$ induced by the inclusion.

On the other hand, it follows from the commutative diagram

$$\begin{CD} h(\Sigma \times_G MG, \Sigma \times_G MG - N) @>l^*>> h(\Sigma \times_G MG) \\ @V s^* VV @VV s^* V \\ h(\Sigma_G, \Sigma_G - W) @>l_2^*>> h(\Sigma_G) \end{CD}$$

(l, l_2 : inclusions) that $s^*(\theta)$ is in the image of l_2^* .

Therefore $e(d\lambda) \cdot s^*(\theta)$ is in the image of the homomorphism $h(\Sigma_G, \bar{W} \cup (\Sigma_G - W)) = h(\Sigma_G, \Sigma_G) \rightarrow h(\Sigma_G)$, and hence we have the desired result.

We shall now prove the main theorem.

Theorem 1. *Let G be a cyclic group of odd order q , and Σ be a homotopy $(2n+1)$ -sphere on which a free differentiable G -action is given. Let M be a differentiable m -manifold. Assume that there exists a continuous map $f : \Sigma \rightarrow M$ with $\dim A(f) < 2d$. Then, for any multiplicative cohomology theory h defined on the category of finite CW pairs and satisfying the conditions (2.1), (2.2), it holds that*

$$x^d \left(\prod_{i=1}^{(q-1)/2} [i](x) \right)^m \in h(pt)[[x]]$$

is contained in the ideal generated by x^{n+1} and $[q](x)$.

Proof. Recall that any differentiable m -manifold is regarded as an increasing union of compact differentiable m -manifold, and that any differentiable m -manifold with boundary is contained in a differentiable m -manifold without boundary. Since Σ is connected and compact, it follows from these facts that we may assume M to be closed without loss of generality.

Then, in virtue of (2.3), Propositions 3 and 4, we have

$$\begin{aligned} & e(\lambda)^d \left(\prod_{i=1}^{(q-1)/2} [i](e(\lambda)) \right)^m \\ &= e(d\lambda) \cdot i^* e(v) = e(d\lambda) \cdot s^*(\theta) = 0. \end{aligned}$$

Since ρ_n is a principal G -bundle whose base space is $(2n+1)$ -dimensional CW complex, and since λ' is a $(2n+1)$ -universal principal G -bundle, there is a bundle map of ρ_n to λ . Hence the last equation implies

$$e(\rho_n)^d \left(\prod_{i=1}^{(q-1)/2} [i](e(\rho_n)) \right)^m = 0.$$

From this and Proposition 1 we have the desired result.

As typical examples of the multiplicative cohomology theory satisfying the conditions in Theorem 1, we have the classical integral cohomology theory $H^*(\ ; \mathbf{Z})$, the Grothendieck-Atiyah-Hirzebruch periodic cohomology theory $K^*(\)$ of K -theory, and the complex cobordism theory $U^*(\)$ obtained from the Milnor spectrum MU (see [2]).

As is well known, $H^i(pt; \mathbf{Z}) = \mathbf{Z}$ ($i=0$), $=0$ ($i \neq 0$) and the formal group law for $H^*(\ ; \mathbf{Z})$ is given by $F(x, y) = x + y$. Hence the conclusion in Theorem 1 for $h = H^*(\ ; \mathbf{Z})$ is stated that

$$\left(\frac{q-1}{2}\right)^m x^{d+m(q-1)/2} \in \mathbf{Z}[x]$$

is contained in the ideal generated by x^{n+1} and qx . From this we obtain the following result.

(3.1) If q is an odd prime, for any continuous map $f : \Sigma \rightarrow M$ we have $\dim A(f) \geq 2n - m(q-1)$.

REMARK. The conclusion in (3.1) is strengthened to $\dim A(f) \geq 2n + 1 - m(q-1)$ (see [4], [6]).

For $K^*(\)$ it is known that $K^{even}(pt) = \mathbf{Z}$, $K^{odd}(pt) = 0$ and the formal group law is given by $F(x, y) = x + y + xy$ (see [1]). Therefore the conclusion in Theorem 1 for $h = K^*(\)$ is stated that

$$x^d \left(\prod_{i=1}^{(q-1)/2} ((x+1)^i - 1)\right)^m \in \mathbf{Z}[x]$$

is contained in the ideal generated by x^{n+1} and $(x+1)^q - 1$. Putting $y = x + 1$ this is restated that

$$(y-1)^d \left(\prod_{i=1}^{(q-1)/2} (y^i - 1)\right)^m \in \mathbf{Z}[y]$$

is contained in the ideal generated by $(y-1)^{n+1}$ and $y^q - 1$. If q is an odd prime power p^a , it can be proved by making use of elementary algebraic number theory that the above statement is equivalent to

$$d \geq n + p^{a-1} - am(p^a - p^{a-1})/2$$

(see [5], p. 453). Thus theorem 1 implies the following theorem containing (3.1) and being a generalization of the main result in [5].

Theorem 2. *If q is an odd prime power p^a , for any continuous map $f : \Sigma \rightarrow M$ we have*

$$\begin{aligned} \dim A(f) \geq & 2n + 1 - (p^a - 1)m \\ & - [m(a-1)p^a - (ma + 2)p^{a-1} + m + 3]. \end{aligned}$$

For $U^*(\quad)$, it is known that $U^*(pt)$ is a polynomial ring over \mathbf{Z} with one generator of degree $-2i$ for each positive integer i , and that the formal group law for $U^*(\quad)$ is given by

$$F(x, y) = g^{-1}(g(x) + g(y))$$

with $g(x) = \sum_{i \geq 0} \frac{[CP^i]}{i+1} x^{i+1} \in U^*(pt)[[x]] \otimes \mathbf{Q}$, where \mathbf{Q} is the ring of rational numbers (see [1], [7]). However we can not deduce numerical conditions equivalent to the conclusion in Theorem 1 for $h = U^*(\quad)$.

Appendix

In this appendix we shall show a generalization of a result due to Vick [10].

For any positive integer r , let $T_r : S^{2n+1} \rightarrow S^{2n+1}$ denote the fixed point free transformation of period r given by

$$T_r(z_1, \dots, z_{n+1}) = (z_1 \exp 2\pi\sqrt{-1}/r, \dots, z_n \exp 2\pi\sqrt{-1}/r).$$

Then a fixed point free transformation $\bar{T}_p : L^n(q) \rightarrow L^n(q)$ of period p on the lens space $L^n(q)$ is induced by $T_{pq} : S^{2n+1} \rightarrow S^{2n+1}$.

Theorem 3. *Suppose that there exists an equivariant map f of $(L^n(q), \bar{T}_p)$ to (S^{2m+1}, T_p) . Then, for any multiplicative cohomology theory h defined on the category of finite CW pairs and satisfying (1.4), it holds that $([q](x))^{m+1} \in h(pt)[[x]]$ is contained in the ideal generated by x^{n+1} and $[pq](x)$.*

Proof. For a multiple pq of q , let $\rho'(q, pq)$ denote the principal \mathbf{Z}_p -bundle $L^n(q) \rightarrow L^n(pq)$ defined the canonical projection. Corresponding to the standard 1-dimensional complex representation of \mathbf{Z}_p , we have the associated complex line bundle $\rho_n(q, pq)$ on $L^n(pq)$. As is observed in [8], it holds that

$$\rho_n(q, pq) \cong \rho_n(1, pq) \otimes \dots \otimes \rho_n(1, pq) \quad (q\text{-times}).$$

Therefore, if there exists an equivariant map $f : (L^n(q), \bar{T}_p) \rightarrow (S^{2m+1}, T_p)$, then it holds that

$$f^* \rho_m(1, p) \cong \rho_n(1, pq) \otimes \dots \otimes \rho_n(1, pq) \quad (q\text{-times})$$

for the map $f : L^n(pq) \rightarrow L^m(p)$ induced by f .

$$\begin{array}{ccccc} S^{2n+1} & \xrightarrow{\rho'_n(1, q)} & L^n(q) & \xrightarrow{f} & S^{2m+1} \\ & \searrow \rho'_n(1, pq) & \downarrow \rho'_n(q, pq) & & \downarrow \rho'_m(1, p) \\ & & L^n(pq) & \xrightarrow{f} & L^m(p) \end{array}$$

Therefore we have

$$f^*e(\rho_m(1, p)) = [q](e(\rho_n(1, pq)))$$

in $h(L^n(pq))$. Since $e(\rho_m(1, p))^{m+1} = 0$ it holds that

$$([q](e(\rho_n(1, pq))))^{m+1} = 0$$

in $h(L^n(pq))$. This and Proposition 1 prove the desired result.

The conclusion of Theorem 3 applied to $h = K^*(\quad)$ is stated that $((x+1)^q - 1)^{m+1} \in Z[x]$ is contained in the ideal generated by x^{n+1} and $(x+1)^{pq} - 1$. Therefore, the argument similar to the proof of Lemma 1 in [5] proves the following

Theorem 4. *Let p be a prime, and suppose that there exists an equivariant map of $(L^n(q), \bar{T}_p)$ to (S^{2m+1}, T_p) . Then we have*

$$p^a m \geq n,$$

where $q = p^a r$, $(p, r) = 1$.

REMARK 1. This generalizes the result due to Vick [10].

REMARK 2. Shibata [8] proves this result by applying Theorem 3 to $h = U^*(\quad)$.

(added in proof) Since the formal group law for the complex cobordism theory is universal (see [1], [7]), we have the following corollary of Theorem 1 :
For any formal group law over a commutative ring R with unit, it holds that

$$\dim A(f) < 2d \Rightarrow$$

$$x^d \left(\prod_{i=1}^{(q-1)/2} [i](x) \right)^m \subset (x^{n+1}, [q](x)) \text{ in } R[[x]].$$

Similar for Theorem 3. This fact was pointed out by J. Morava.

ODENSE UNIVERSITY, DENMARK

OSAKA UNIVERSITY

References

- [1] J.F. Adams: Quillen's Work on Formal Groups and Complex Cobordism, Lecture notes, Univ. of Chicago, 1970.
- [2] P.E. Conner - E.E. Floyd: The Relation of Cobordism to K -theories, Lecture notes in Math., Springer-Verlag, 1966.
- [3] A. Dold: On General Cohomology, Lecture notes, Aarhus Univ., 1968.
- [4] H.J. Munkholm: Borsuk-Ulam type theorems for proper Z_p -actions on (mod p homology) n -spheres, Math. Scand. **24** (1969), 167-185.

- [5] H.J. Munkholm: *On the Borsuk-Ulam theorem for \mathbb{Z}_p^a actions on S^{2n-1} and maps $S^{2n-1} \rightarrow \mathbb{R}^m$* , Osaka J. Math. **7** (1970), 451–456.
- [6] M. Nakaoka: *Generalizations of Borsuk-Ulam theorem*, Osaka J. Math. **7** (1970), 423–441.
- [7] D. Quillen: *On the formal group laws of unoriented and complex cobordism theory*, Bull. Amer. Math. Soc. **75** (1969), 1293–1298.
- [8] K. Shibata: *Oriented and weakly complex bordism algebra of free periodic maps* (to appear).
- [9] R.E. Stong: *Complex and oriented equivariant bordism*, Proc. Georgia Conference (1969), 291–316.
- [10] J.W. Vick: *An application of K-theory to equivariant maps*, Bull. Amer. Math. Soc. **75** (1969), 1017–1019.

