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## THE BORSUK-ULAM THEOREM AND FORMAL GROUP LAWS

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### Introduction

The present paper is concerned with the following question raised on the classical Borsuk-Ulam theorem : Let  $G$  denote a cyclic group of odd order  $q$ , and let  $\Sigma$  be a homotopy  $(2n+1)$ -sphere on which a free differentiable  $G$ -action is given. For any differentiable  $m$ -manifold  $M$  and any continuous map  $f: \Sigma \rightarrow M$ , put  $A(f) = \{x \in \Sigma \mid f(x) = f(xg) \text{ for all } g \in G\}$ . What can be deduced about the covering dimension of  $A(f)$ ?

In response to this question, the authors showed previously that if  $q$  is a prime  $p$  then  $\dim A(f) \geq 2n+1 - (p-1)m$  ([4], [6]). Furthermore, one of the authors showed in [5] that if  $q$  is a prime power  $p^a$  and  $M$  is the Euclidean space  $\mathbf{R}^m$  then

$$(0.1) \quad \dim A(f) \geq (2n+1) - (p^a - 1)m \\
 - [m(a-1)p^a - (ma+2)p^{a-1} + m + 3].$$

It will be shown in this paper that (0.1) still holds for any differentiable  $m$ -manifold  $M$ .

The procedure taken in this paper is different from the previous ones, and we shall derive the above result from a general theorem stated in connection with the formal group law for some general cohomology theory.

Assume that there is given a multiplicative cohomology theory  $h$  defined on the category of finite  $CW$  pairs and satisfying the conditions: i) each complex vector bundle is  $h$ -orientable, ii)  $h^i(pt) = 0$  for each odd  $i$ . Let  $F(x, y) \in h(pt)[[x, y]]$  denote the formal group law associated to  $h$ , and  $[i](x) \in h(pt)[[x]]$  denote the operation of "multiplication by  $i$ " for a positive integer  $i$ . We shall show that

$$(0.2) \quad \dim A(f) < 2d \quad \Rightarrow \\
 x^a \left( \prod_{i=1}^{(q-1)/2} [i](x) \right)^m \in (x^{n+1}, [q](x)) \text{ in } h(pt)[[x]],$$

where  $(a, b)$  denotes the ideal generated by  $a$  and  $b$ .

Take as  $h$  the general cohomology theory defined from  $K$ -theory. Then it is seen by using elementary algebraic number theory that (0.2) is equivalent to (0.1).

We can also take as  $h$  the complex cobordism theory  $U^*$ . Since  $U^*$  is stronger than  $K$ -theory in general, it is expected that sharper result than (0.1) will be obtained from (0.2) applied to  $h=U^*$ . However we have no method to derive numerical conditions equivalent to (0.2) for  $h=U^*$ .

In an appendix, we shall prove in the same procedure as above a non-existence theorem for equivariant maps which generalizes the result of Vick [10].

**1. The formal group law for a multiplicative cohomology**

We recall first some facts on multiplicative cohomology theory (see Dold [3]).

We fix once and for all a multiplicative reduced cohomology theory  $\tilde{h}$  defined on the category of finite CW complexes with base point. There is the corresponding multiplicative cohomology theory  $h$  defined on the category of finite CW pairs.

Let  $\xi$  be a real  $n$ -dimensional vector bundle over a finite CW complex  $B$ , and denote by  $M(\xi)$  the Thom space for  $\xi$ . For each  $b \in B$  let  $\xi_b$  denote the restriction of  $\xi$  over  $b$ . Then  $\tilde{h}(M(\xi_b))$  is a free  $h(pt)$ -module on one generator.  $\xi$  is said to be  $h$ -orientable if there exists  $t(\xi) \in \tilde{h}^n(M(\xi))$  such that  $t(\xi)|_{M(\xi_b)}$  is a generator of  $\tilde{h}(M(\xi_b))$  for each  $b \in B$ . Such  $t(\xi)$  is called an  $h$ -orientation or a *Thom class* of  $\xi$ . By an  $h$ -oriented vector bundle we mean a vector bundle in which an  $h$ -orientation is given.

Let  $D(\xi)$  (or  $S(\xi)$ ) denote the total space of the disc bundle (or the sphere bundle) associated to  $\xi$ , and consider the homomorphism

$$\tilde{h}^n(M(\xi)) = h^n(D(\xi), S(\xi)) \xrightarrow{j^*} h^n(D(\xi)) \xrightarrow[\cong]{p^{*-1}} h^n(B),$$

where  $j$  is the inclusion and  $p$  is the projection. The image of  $t(\xi)$  under this homomorphism is called the *Euler class* of the  $h$ -oriented bundle  $\xi$ , and is denoted by  $e(\xi)$ .

The following facts are easily proved:

(1.1) If there is a bundle map  $f: \xi \rightarrow \xi'$  and  $\xi'$  is  $h$ -oriented, then  $\xi$  is  $h$ -oriented so that  $f^*: h(B') \rightarrow h(B)$  preserves the Euler classes.

(1.2) If  $\xi_1$  and  $\xi_2$  are  $h$ -oriented, then the Whitney sum  $\xi_1 \oplus \xi_2$  is  $h$ -oriented so that  $e(\xi_1 \oplus \xi_2) = e(\xi_1)e(\xi_2)$ .

(1.3) If  $\xi$  has a non-zero cross section, then  $e(\xi) = 0$ .

The classical Leray-Hirsch theorem on fiberings can be generalized to the multiplicative theory  $h$ , and so we have the Thom isomorphism

$$\Phi : h(B) \cong \tilde{h}(M(\xi))$$

given by  $\Phi(\alpha) = \alpha \cdot t(\xi)$ . As a consequence, the Gysin exact sequence

$$\dots \rightarrow h^{i-1}(S(\xi)) \rightarrow h^{i-n}(B) \xrightarrow{e(\xi)} h^i(B) \xrightarrow{p^*} h^i(S(\xi)) \rightarrow \dots$$

holds.

A complex vector bundle  $\xi$  is called  $h$ -orientable if the real form  $\xi_{\mathbb{R}}$  is  $h$ -orientable. Let  $\eta_n$  denote the canonical complex line bundle over the complex  $n$ -dimensional projective space  $CP^n$ . Throughout this section the following will be assumed:

(1.4) For each  $n$ ,  $\eta_n$  is  $h$ -oriented so that the homomorphism  $h(CP^{n+1}) \rightarrow h(CP^n)$  preserves the Euler classes.

It follows from this assumption that any complex line bundle  $\xi$  over a finite  $CW$  complex is  $h$ -oriented so that the homomorphism  $f^* : h(B') \rightarrow h(B)$  induced by every bundle map  $f : \xi \rightarrow \xi'$  preserves the Euler classes.

We can prove

(1.5) The algebra  $h(CP^n)$  is a truncated polynomial algebra over  $h(pt)$  :

$$h(CP^n) = h(pt)[e(\eta_n)]/(e(\eta_n)^{n+1}).$$

(1.6) Put  $e(\eta_m)_1 = p_1^* e(\eta_m)$  and  $e(\eta_n)_2 = p_2^* e(\eta_n)$  for the projections  $p_1 : CP^m \times CP^n \rightarrow CP^m$  and  $p_2 : CP^m \times CP^n \rightarrow CP^n$ . Then the isomorphism

$$h(CP^m \times CP^n) = h(pt)[e(\eta_m)_1, e(\eta_n)_2]/(e(\eta_m)_1^{m+1}, e(\eta_n)_2^{n+1})$$

holds.

For a  $CW$  complex  $X$  with finite skelta, we define  $h(X)$  as the inverse limit with respect to skelta :

$$h(X) = \varprojlim h(X^n).$$

Then, for the infinite dimensional projective space  $CP^\infty$ , the following result is obtained from (1.5) and (1.6).

(1.7)  $h(CP^\infty)$  and  $h(CP^\infty \times CP^\infty)$  are rings of formal power series :

$$h(CP^\infty) = h(pt)[[x]], \quad h(CP^\infty \times CP^\infty) = h(pt)[[x_1, x_2]],$$

where  $x, x_1, x_2$  are the elements defined by  $e(\eta_n), e(\eta_n)_1, e(\eta_n)_2$  respectively.

Let  $\eta$  denote the canonical line bundle over  $CP^\infty$ , and consider the external tensor product  $\eta \hat{\otimes} \eta$  which is a complex line bundle over  $CP^\infty \times CP^\infty$ . Let  $\mu : CP^\infty \times CP^\infty \rightarrow CP^\infty$  be a classifying map for  $\eta \hat{\otimes} \eta$  which is cellular, and put

$$\mu^*(x) = \sum_{i,j \geq 0} a_{ij} x_1^i x_2^j \quad (a_{ij} \in h^{2(1-i-j)}(pt))$$

for  $\mu^* : h(CP^\infty) \rightarrow h(CP^\infty \times CP^\infty)$ . Then we obtain easily

(1.8) For the tensor product  $\xi_1 \otimes \xi_2$  of any complex line bundles  $\xi_1$  and  $\xi_2$

over a finite CW complex,

$$e(\xi_1 \otimes \xi_2) = \sum_{i,j \geq 0} a_{i,j} e(\xi_1)^i e(\xi_2)^j$$

holds.

Consider now a power series  $F(x,y)$  with coefficients in  $h(pt)$ , which is defined by

$$F(x,y) = \sum_{i,j \geq 0} a_{i,j} x^i y^j$$

with  $a_{i,j}$  above. Then it follows that  $F(x,y)$  is a formal group law over  $h(pt)$ , i.e. the identities

$$\begin{aligned} F(x, 0) &= x, F(x, y) = F(y, x), \\ F(x, F(y, z)) &= F(F(x, y), z) \end{aligned}$$

hold. For each integer  $i \geq 1$ , let  $[i](x) \in h[[x]]$  denote the operation of "multiplication by  $i$ " for the formal group, i.e.

$$[1](x) = x, \quad [i](x) = F([i-1](x), x).$$

Since the formula in (1.8) is rewritten as

$$e(\xi_1 \otimes \xi_2) = F(e(\xi_1), e(\xi_2)),$$

for the  $i$ -fold tensor product  $\xi^i = \xi \otimes \dots \otimes \xi$  we have

$$e(\xi^i) = [i](e(\xi)).$$

Given a positive integer  $q$ , let  $G$  denote a cyclic group of order  $q$ . Define a  $G$ -action on the standard  $(2n+1)$ -sphere  $S^{2n+1} = \{(z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1} \mid \sum_i |z_i|^2 = 1\}$  by

$$(z_0, \dots, z_n) g_0 = (z_0 \exp 2\pi\sqrt{-1}/q, \dots, z_n \exp 2\pi\sqrt{-1}/q),$$

where  $g_0$  is the generator of  $G$ . This yields a principal  $G$ -bundle  $\rho'_n : S^{2n+1} \rightarrow L^n(q)$  over the lens space  $L^n(q)$ . Let  $L$  denote a 1-dimensional complex  $G$ -module given by  $c \cdot g_0 = c \exp 2\pi\sqrt{-1}/q$ , and consider the associated complex line bundle  $\rho_n = \rho'_n \times_{\sigma} L$ . For the canonical projection  $\pi : L^n(q) \rightarrow CP^n$  we have  $\rho_n = \pi^*(\eta_n)$ , and hence  $e(\rho_n)^{n+1} = 0$  holds.

**Proposition 1.** *Let  $P(x) \in h(pt)[[x]]$ . Then the element  $P(e(\rho_n))$  of  $h(L^n(q))$  is zero if and only if  $P(x)$  is in the ideal generated by  $x^{n+1}$  and  $[q](x)$ .*

*Proof.* Consider the  $q$ -fold tensor product  $\eta_n^q$  of  $\eta_n$ . As is observed in [9],

the total space  $S(\eta_n^q)$  of the sphere bundle associated to  $\eta_n^q$  is homeomorphic with  $L^n(q)$ . Therefore we have the Gysin sequence

$$\dots \rightarrow h^{i-2}(CP^n) \xrightarrow{\cdot e(\eta_n^q)} h^i(CP^n) \xrightarrow{\pi^*} h^i(L^n(q)) \rightarrow \dots$$

Since  $e(\eta_n^q) = [q](e(\eta_n))$ , the desired result follows from the above sequence and (1.5).

**2. The element  $s^*(\theta)$**

As in § 1, let  $G$  denote a cyclic group of order  $q$ . We shall assume in the following that  $q$  is *odd*.

For any space  $X$ , let  $XG$  denote the product of  $q$  copies of  $X$ . Writing its elements as  $\sum_{g \in G} x_g g$ , a  $G$ -action on  $XG$  is given by

$$\left(\sum_{g \in G} x_g g\right) \cdot h = \sum_{g \in G} x_g h^{-1} g \quad (h \in G).$$

We denote by  $\Delta X$  the diagonal in  $XG$ .

Let  $\Sigma$  be a homotopy  $(2n+1)$ -sphere (which is a differentiable manifold), and assume that there is given a free differentiable  $G$ -action on  $\Sigma$ . We denote by  $\Sigma_G$  the orbit space.

Let  $M$  be a differentiable manifold, and consider the diagonal action on  $\Sigma \times MG$  whose orbit space is denoted by  $\Sigma \times_g MG$ .  $\Sigma \times \Delta M$  is an invariant submanifold of the  $G$ -manifold  $\Sigma \times MG$ , and its orbit space is regarded as  $\Sigma_G \times \Delta M$ . We denote by  $\nu$  the normal bundle of  $\Sigma_G \times \Delta M$  in  $\Sigma \times_g MG$ . This is a real  $m(q-1)$ -dimensional vector bundle.

Choose a point  $y_0 \in M$ , and identify  $\Sigma_G$  with a subspace  $\Sigma_G \times y_0 G$  ( $y_0 G = \sum_g y_0 g$ ) of  $\Sigma_G \times \Delta M$ .

Let  $\lambda' : \Sigma \rightarrow \Sigma_G$  denote the principal  $G$ -bundle defined by the  $G$ -action on  $\Sigma$ , and consider the associated complex line bundle  $\lambda = \lambda' \times_g L$ .

**Proposition 2.** *The normal bundle  $\nu$  has a complex structure for which*

$$i^*(\nu) = m(\lambda \oplus \lambda^2 \oplus \dots \oplus \lambda^{(q-1)/2})$$

*holds, where  $i : \Sigma_G \rightarrow \Sigma_G \times \Delta M$  is the inclusion.*

*Proof.* If  $\nu_1 : N_1 \rightarrow \Delta M$  denote the normal  $G$ -vector bundle of  $\Delta M$  in  $MG$ , then we have  $\nu = id \times_g \nu_1 : \Sigma \times_g N_1 \rightarrow \Sigma_G \times \Delta M$ . Therefore it suffices to prove that there exists a  $G$ -equivariant complex structure on  $\nu_1$  with the fiber over

$y_0G$  being  $m(L \oplus \dots \oplus L^{(q-1)/2})$ .

To prove this, let  $IG$  be defined by the exact sequence of real  $G$ -modules

$$0 \rightarrow \Delta R \rightarrow RG \rightarrow IG \rightarrow 0.$$

View this as a sequence of real  $G$ -vector bundles over a point, and identify  $\Delta M$  with  $M \times \mathfrak{pt} = M$  in the obvious way. Then we have the exact sequence

$$0 \rightarrow \tau M \hat{\otimes} \Delta R \rightarrow \tau M \hat{\otimes} RG \rightarrow \tau M \hat{\otimes} IG \rightarrow 0$$

of real  $G$ -vector bundles over  $M$ , where  $\tau M$  denotes the tangent bundle over  $M$ . Since  $\tau(MG) = (\tau M)G$ , an equivariant isomorphism

$$\beta : \tau(MG)|_{\Delta M} \rightarrow \tau M \hat{\otimes} RG$$

can be given by

$$\beta(\sum_g v_g g) = \sum_g v_g \otimes g \quad (v_g \in \tau_y(M), y \in M).$$

Since  $\sum v_g g$  is in  $\tau(\Delta M)$  if and only if all  $v_g$  are equal,  $\beta$  maps  $\tau(\Delta M)$  onto  $\tau M \hat{\otimes} \Delta R$ . Thus it holds that  $\nu_1 \cong \tau M \hat{\otimes} IG$  as real  $G$ -vector bundles. From elementary representation theory of groups, it follows that  $IG$  is the real form of  $L \oplus \dots \oplus L^{(q-1)/2}$ . This gives  $\nu_1$  its complex structure, and we get

$$\begin{aligned} (\nu_1)_{y_0} &= \tau_{y_0} M \otimes (L \oplus \dots \oplus L^{(q-1)/2}) \\ &= R^m \otimes (L \oplus \dots \oplus L^{(q-1)/2}) = m(L \oplus \dots \oplus L^{(q-1)/2}) \end{aligned}$$

as desired. This completes the proof.

As in § 1, let  $h$  be a given multiplicative cohomology theory. In the following we shall assume the following conditions:

(2.1) every complex vector bundle of any dimension is  $h$ -orientable.

(2.2)  $h^{odd}(\mathfrak{pt}) = 0$ .

Assuming that  $M$  is closed, consider the normal bundle  $\nu$ . Then, by Proposition 2 and (2.1), we have a Thom class  $t(\nu) \in \tilde{h}^{m(q-1)}(M(\nu))$  and the corresponding Euler class  $e(\nu) \in h^{m(q-1)}(\Sigma_G \times \Delta M)$  such that

$$\begin{aligned} (2.3) \quad i^* e(\nu) &= e(m(\lambda \oplus \lambda^2 \oplus \dots \oplus \lambda^{(q-1)/2})) \\ &= \left( \prod_{i=1}^{(q-1)/2} [i](e(\lambda)) \right)^m. \end{aligned}$$

As usual we shall regard the total space  $N$  of  $\nu$  as a tubular neighborhood of  $\Sigma_G \times \Delta M$  in  $\Sigma \times_g MG$ . Then we can identify  $\tilde{h}(M(\nu))$  with  $h(\Sigma \times_g MG)$ ,

$\Sigma \times_a MG - N$ ) canonically. Let

$$\theta \in h^{m(a-1)}(\Sigma \times_a MG)$$

be the image of the Thom class  $t(\nu)$  under the homomorphism  $l^* : h(\Sigma \times_a MG, \Sigma \times_a MG - N) \rightarrow h(\Sigma \times_a MG)$  induced by the inclusion. We have immediately

(2.4) For the homomorphism  $j^* : h(\Sigma \times_a MG) \rightarrow h(\Sigma_G \times \Delta M)$  induced by the inclusion,  $j^*(\theta) = e(\nu)$  holds.

Given a continuous map  $f : \Sigma \rightarrow M$ , define a continuous map  $s : \Sigma_G \rightarrow \Sigma \times_a MG$  by

$$s(xG) = (x, \sum_g f(xg^{-1})g)G.$$

For the projection  $p : \Sigma \times_a MG \rightarrow \Sigma_G$ ,  $p \circ s$  is the identity.

**Proposition 3.** For the homomorphism  $s^* : h(\Sigma \times_a MG) \rightarrow h(\Sigma_G)$  and the homomorphism  $i^* : h(\Sigma_G \times \Delta M) \rightarrow h(\Sigma_G)$ , we have

$$s^*(\theta) = i^*(e(\nu)).$$

Proof. It is easily seen that there exist a continuous map  $f_1 : \Sigma \rightarrow M$  and an open set  $V$  of  $\Sigma$  satisfying the following conditions: i)  $f$  is homotopic to  $f_1$ , ii)  $V$  is homeomorphic to  $\mathbf{R}^{2n+1}$ , iii)  $f_1(\Sigma - V) = y_0$ , iv)  $xg \notin \bar{V}$  for any  $g \neq 1$  and any  $x \in \bar{V}$ , where  $\bar{V}$  denotes the closure of  $V$ . Define  $s_1 : \Sigma_G \rightarrow \Sigma \times_a MG$  from  $f_1$  as  $s$  was defined from  $f$ , then  $s$  and  $s_1$  are homotopic. Let  $(MG)_1$  denote the subspace of  $MG$  consisting of points with at most one coordinate  $\neq y_0$ . Then  $(MG)_1$  is an invariant subspace of the  $G$ -space  $MG$ , and the orbit space  $\Sigma \times_a (MG)_1$  contains  $s_1(\Sigma_G)$ . Since  $\Sigma - V$  is contractible, there exists a homotopy  $\psi_t : (\bar{V}, \partial V) \rightarrow (\Sigma, \Sigma - V)$  such that  $\psi_0$  is the inclusion and  $\psi_1(\partial V) = x_0 \in \partial V$ , where  $\partial V = \bar{V} - V$ . Put  $V_G = \pi(V)$  for the projection  $\pi : \Sigma \rightarrow \Sigma_G$ . Consider now the following commutative diagram:

$$\begin{array}{ccc} \Sigma_G & \xrightarrow{s_1} & \Sigma \times_a (MG)_1 \\ \downarrow j_2 & & \downarrow j_1 \\ (\Sigma_G, \Sigma_G - V_G) & \xrightarrow{s_1} & (\Sigma \times_a (MG)_1, \Sigma_G \times y_0 G) \end{array}$$

where  $j_1, j_2$  are the inclusions.



We have

$$h^{m(q-1)}(\Sigma_G, \Sigma_G - V_G) = \widehat{h}^{m(q-1)}(S^{2n+1}) = h^{m(q-1)-(2n+1)}(pt) = 0$$

by (2.2). Therefore

$$s_1^* \circ i_1^* : h^{m(q-1)}(\Sigma \times (MG)_1, \Sigma_G \times y_0 G) \rightarrow h^{m(q-1)}(\Sigma_G)$$

is trivial.

Next consider the commutative diagram

$$\begin{array}{ccc}
 h(\Sigma \times_{\mathcal{G}} MG) & \xrightarrow{j^*} & h(\Sigma_G \times \Delta M) \\
 \downarrow s^* & \searrow i_2^* & \downarrow i^* \\
 & h(\Sigma \times_{\mathcal{G}} (MG)_1) & \\
 \swarrow s_1^* & \uparrow p^* & \searrow i_1^* \\
 h(\Sigma_G) = h(\Sigma_G) & = & h(\Sigma_G \times y_0 G)
 \end{array}$$

where  $i_1, i_2$  are the inclusions. Putting  $\theta' = p^* i_1^* i_2^*(\theta) - i_2^*(\theta)$ , we have

$$s_1^*(\theta') = i^* i^*(\theta) - s^*(\theta) = i^*(e(\nu)) - s^*(\theta)$$

by (2.4), and  $i_1^*(\theta') = 0$ . Therefore  $\theta'$  is in the image of  $j_1^* : h^{m(q-1)}(\Sigma \times_{\mathcal{G}} (MG)_1, \Sigma_G \times y_0 G) \rightarrow h^{m(q-1)}(\Sigma \times_{\mathcal{G}} (MG)_1)$ , and hence  $s_1^*(\theta') = 0$  by the fact proved above. Thus we have  $i^*(e(\nu)) = s^*(\theta)$ .

### 3. Generalization of Borsuk-Ulam theorem

Let  $\Sigma$  be as in §2, and let  $f : \Sigma \rightarrow M$  be a continuous map to a differentiable  $m$ -manifold. Put

$$A(f) = \{x \in \Sigma \mid f(x) = f(xg) \text{ for any } g \in G\}.$$

In this section we shall consider the covering dimension of  $A(f)$ .

For the image  $A(f)_G = \pi(A(f))$ , we have  $\dim A(f) = \dim A(f)_G$ .

**Proposition 4.** *Assume that  $M$  is closed. Then  $\dim A(f) < 2d$  implies*

$$e(d\lambda)s^*(\theta) = 0.$$

*Proof.* Since  $\dim A(f)_G \leq 2d - 1$ , it follows that  $d\lambda$  has a non-zero cross section over  $A(f)_G$  (see [5], Lemma 2). By standard facts on extension of cross section, this cross section extends to a non-zero cross section over the closure  $\bar{W}$  of some neighborhood  $W$  of  $A(f)_G$  in  $\Sigma_G$ . Here we may assume that  $\bar{W}$  is

a finite CW complex, and that  $s(\Sigma_G - W) \subset \Sigma \times_G MG - N$  by taking  $N$  small. We have then  $e(d\lambda | \bar{W}) = 0$ , and so  $e(d\lambda)$  is in the image of  $l_1^* : h(\Sigma_G, \bar{W}) \rightarrow h(\Sigma_G)$  induced by the inclusion.

On the other hand, it follows from the commutative diagram

$$\begin{array}{ccc} h(\Sigma \times_G MG, \Sigma \times_G MG - N) & \xrightarrow{l^*} & h(\Sigma \times_G MG) \\ \downarrow s^* & & \downarrow s^* \\ h(\Sigma_G, \Sigma_G - W) & \xrightarrow{l_2^*} & h(\Sigma_G) \end{array}$$

( $l, l_2$  : inclusions) that  $s^*(\theta)$  is in the image of  $l_2^*$ .

Therefore  $e(d\lambda) \cdot s^*(\theta)$  is in the image of the homomorphism  $h(\Sigma_G, \bar{W} \cup (\Sigma_G - W)) = h(\Sigma_G, \Sigma_G) \rightarrow h(\Sigma_G)$ , and hence we have the desired result.

We shall now prove the main theorem.

**Theorem 1.** *Let  $G$  be a cyclic group of odd order  $q$ , and  $\Sigma$  be a homotopy  $(2n+1)$ -sphere on which a free differentiable  $G$ -action is given. Let  $M$  be a differentiable  $m$ -manifold. Assume that there exists a continuous map  $f : \Sigma \rightarrow M$  with  $\dim A(f) < 2d$ . Then, for any multiplicative cohomology theory  $h$  defined on the category of finite CW pairs and satisfying the conditions (2.1), (2.2), it holds that*

$$x^d \left( \prod_{i=1}^{(q-1)/2} [i](x) \right)^m \in h(pt)[[x]]$$

is contained in the ideal generated by  $x^{n+1}$  and  $[q](x)$ .

Proof. Recall that any differentiable  $m$ -manifold is regarded as an increasing union of compact differentiable  $m$ -manifold, and that any differentiable  $m$ -manifold with boundary is contained in a differentiable  $m$ -manifold without boundary. Since  $\Sigma$  is connected and compact, it follows from these facts that we may assume  $M$  to be closed without loss of generality.

Then, in virtue of (2.3), Propositions 3 and 4, we have

$$\begin{aligned} & e(\lambda)^d \left( \prod_{i=1}^{(q-1)/2} [i](e(\lambda)) \right)^m \\ &= e(d\lambda) \cdot i^* e(v) = e(d\lambda) \cdot s^*(\theta) = 0. \end{aligned}$$

Since  $\rho_n$  is a principal  $G$ -bundle whose base space is  $(2n+1)$ -dimensional CW complex, and since  $\lambda'$  is a  $(2n+1)$ -universal principal  $G$ -bundle, there is a bundle map of  $\rho_n$  to  $\lambda$ . Hence the last equation implies

$$e(\rho_n)^d \left( \prod_{i=1}^{(q-1)/2} [i](e(\rho_n)) \right)^m = 0.$$

From this and Proposition 1 we have the desired result.

As typical examples of the multiplicative cohomology theory satisfying the conditions in Theorem 1, we have the classical integral cohomology theory  $H^*(\ ; \mathbf{Z})$ , the Grothendieck-Atiyah-Hirzebruch periodic cohomology theory  $K^*(\ )$  of  $K$ -theory, and the complex cobordism theory  $U^*(\ )$  obtained from the Milnor spectrum  $MU$  (see [2]).

As is well known,  $H^i(pt; \mathbf{Z}) = \mathbf{Z}$  ( $i=0$ ),  $=0$  ( $i \neq 0$ ) and the formal group law for  $H^*(\ ; \mathbf{Z})$  is given by  $F(x, y) = x + y$ . Hence the conclusion in Theorem 1 for  $h = H^*(\ ; \mathbf{Z})$  is stated that

$$\left(\frac{q-1}{2}\right)^m x^{d+m(q-1)/2} \in \mathbf{Z}[x]$$

is contained in the ideal generated by  $x^{n+1}$  and  $qx$ . From this we obtain the following result.

(3.1) If  $q$  is an odd prime, for any continuous map  $f : \Sigma \rightarrow M$  we have  $\dim A(f) \geq 2n - m(q-1)$ .

REMARK. The conclusion in (3.1) is strengthened to  $\dim A(f) \geq 2n + 1 - m(q-1)$  (see [4], [6]).

For  $K^*(\ )$  it is known that  $K^{even}(pt) = \mathbf{Z}$ ,  $K^{odd}(pt) = 0$  and the formal group law is given by  $F(x, y) = x + y + xy$  (see [1]). Therefore the conclusion in Theorem 1 for  $h = K^*(\ )$  is stated that

$$x^d \left(\prod_{i=1}^{(q-1)/2} ((x+1)^i - 1)\right)^m \in \mathbf{Z}[x]$$

is contained in the ideal generated by  $x^{n+1}$  and  $(x+1)^q - 1$ . Putting  $y = x + 1$  this is restated that

$$(y-1)^d \left(\prod_{i=1}^{(q-1)/2} (y^i - 1)\right)^m \in \mathbf{Z}[y]$$

is contained in the ideal generated by  $(y-1)^{n+1}$  and  $y^q - 1$ . If  $q$  is an odd prime power  $p^a$ , it can be proved by making use of elementary algebraic number theory that the above statement is equivalent to

$$d \geq n + p^{a-1} - am(p^a - p^{a-1})/2$$

(see [5], p. 453). Thus theorem 1 implies the following theorem containing (3.1) and being a generalization of the main result in [5].

**Theorem 2.** *If  $q$  is an odd prime power  $p^a$ , for any continuous map  $f : \Sigma \rightarrow M$  we have*

$$\begin{aligned} \dim A(f) &\geq 2n + 1 - (p^a - 1)m \\ &\quad - [m(a-1)p^a - (ma + 2)p^{a-1} + m + 3]. \end{aligned}$$

For  $U^*(\ )$ , it is known that  $U^*(pt)$  is a polynomial ring over  $\mathbf{Z}$  with one generator of degree  $-2i$  for each positive integer  $i$ , and that the formal group law for  $U^*(\ )$  is given by

$$F(x, y) = g^{-1}(g(x) + g(y))$$

with  $g(x) = \sum_{i \geq 0} \frac{[CP^i]}{i+1} x^{i+1} \in U^*(pt)[[x]] \otimes \mathbf{Q}$ , where  $\mathbf{Q}$  is the ring of rational numbers (see [1], [7]). However we can not deduce numerical conditions equivalent to the conclusion in Theorem 1 for  $h = U^*(\ )$ .

### Appendix

In this appendix we shall show a generalization of a result due to Vick [10].

For any positive integer  $r$ , let  $T_r : S^{2n+1} \rightarrow S^{2n+1}$  denote the fixed point free transformation of period  $r$  given by

$$T_r(z_1, \dots, z_{n+1}) = (z_1 \exp 2\pi\sqrt{-1}/r, \dots, z_n \exp 2\pi\sqrt{-1}/r).$$

Then a fixed point free transformation  $\bar{T}_p : L^n(q) \rightarrow L^n(q)$  of period  $p$  on the lens space  $L^n(q)$  is induced by  $T_{pq} : S^{2n+1} \rightarrow S^{2n+1}$ .

**Theorem 3.** *Suppose that there exists an equivariant map  $f$  of  $(L^n(q), \bar{T}_p)$  to  $(S^{2m+1}, T_p)$ . Then, for any multiplicative cohomology theory  $h$  defined on the category of finite CW pairs and satisfying (1.4), it holds that  $([q](x))^{m+1} \in h(pt)[[x]]$  is contained in the ideal generated by  $x^{n+1}$  and  $[pq](x)$ .*

*Proof.* For a multiple  $pq$  of  $q$ , let  $\rho'(q, pq)$  denote the principal  $\mathbf{Z}_p$ -bundle  $L^n(q) \rightarrow L^n(pq)$  defined the canonical projection. Corresponding to the standard 1-dimensional complex representation of  $\mathbf{Z}_p$ , we have the associated complex line bundle  $\rho_n(q, pq)$  on  $L^n(pq)$ . As is observed in [8], it holds that

$$\rho_n(q, pq) \cong \rho_n(1, pq) \otimes \dots \otimes \rho_n(1, pq) \quad (q\text{-times}).$$

Therefore, if there exists an equivariant map  $f : (L^n(q), \bar{T}_p) \rightarrow (S^{2m+1}, T_p)$ , then it holds that

$$f^* \rho_m(1, p) \cong \rho_n(1, pq) \otimes \dots \otimes \rho_n(1, pq) \quad (q\text{-times})$$

for the map  $f : L^n(pq) \rightarrow L^m(p)$  induced by  $f$ .

$$\begin{array}{ccccc} S^{2n+1} & \xrightarrow{\rho'_n(1, q)} & L^n(q) & \xrightarrow{f} & S^{2m+1} \\ & \searrow \rho'_n(1, pq) & \downarrow \rho'_n(q, pq) & & \downarrow \rho'_m(1, p) \\ & & L^n(pq) & \xrightarrow{f} & L^m(p) \end{array}$$

Therefore we have

$$f^*e(\rho_m(1, p)) = [q](e(\rho_n(1, pq)))$$

in  $h(L^n(pq))$ . Since  $e(\rho_m(1, p))^{m+1} = 0$  it holds that

$$([q](e(\rho_n(1, pq))))^{m+1} = 0$$

in  $h(L^n(pq))$ . This and Proposition 1 prove the desired result.

The conclusion of Theorem 3 applied to  $h = K^*(\ )$  is stated that  $((x+1)^q - 1)^{m+1} \in Z[x]$  is contained in the ideal generated by  $x^{n+1}$  and  $(x+1)^{pq} - 1$ . Therefore, the argument similar to the proof of Lemma 1 in [5] proves the following

**Theorem 4.** *Let  $p$  be a prime, and suppose that there exists an equivariant map of  $(L^n(q), \bar{T}_p)$  to  $(S^{2m+1}, T_p)$ . Then we have*

$$p^a m \geq n,$$

where  $q = p^a r$ ,  $(p, r) = 1$ .

REMARK 1. This generalizes the result due to Vick [10].

REMARK 2. Shibata [8] proves this result by applying Theorem 3 to  $h = U^*(\ )$ .

(added in proof) Since the formal group law for the complex cobordism theory is universal (see [1], [7]), we have the following corollary of Theorem 1 :  
For any formal group law over a commutative ring  $R$  with unit, it holds that

$$\dim A(f) < 2d \Rightarrow$$

$$x^d \left( \prod_{i=1}^{(q-1)/2} [i](x) \right)^m \subset (x^{n+1}, [q](x)) \text{ in } R[[x]].$$

Similar for Theorem 3. This fact was pointed out by J. Morava.

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