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On the Homology Group of Branched Cyclic Covering Spaces of Links

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Let k be a knot in 3-sphere S^3 and let $\mathfrak{M}_g(k)$ be the g-fold cyclic covering space of S^3, branched along k. By the use of the Alexander polynomial $\Delta(t)$ the 1-dimensional Betti number of $\mathfrak{M}_g(k)$ was calculated by L. Goeritz [2] and the product of the 1-dimensional torsion numbers has been calculated by R. H. Fox [1].

Now let $\mathfrak{L}$ be a link in 3-sphere. Then we can naturally define the g-fold cyclic covering space of S^3, branched along $\mathfrak{L}$ (see Section 2). The purpose of this paper is to calculate the product of the 1-dimensional torsion numbers and the 1-dimensional Betti number of this space. These will be done by the use of the ∇-polynomial defined by one of the authors of this paper [3] and the results are similar to the cases of knots (see Theorems 1, 2, and 3). The proof will be done similarly to [4] in the case of the product of torsion numbers and to [2] in the case of the Betti number.

Professor R. H. Fox kindly pointed out to us that the case of the product of torsion numbers is already proved in the thesis of J. P. Mayberry [5]. As J. P. Mayberry did not use the ∇-polynomial, his result is apparently different to that of ours. But these are essentially equivalent.

The calculation of the fundamental group of the complementary domain of the link represented in Sections 3 and 4 are due to G. Torres [8]. It is contained in this paper only for convenience of readers.

1. In this section we shall prove a lemma with respect to the determinant.

Let the $n\times n$ matrix X be

$$\left\|\begin{matrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & & & \ddots \\ T & 0 & \cdots & 0 \end{matrix}\right\|$$

and the $n\times n$ matrix E be the unit matrix. By the simple calculation we have

$$X^m = \left\| \begin{array}{c|c} \overbrace{\quad 0 \quad}^{m} & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \\ \hline \begin{matrix} T & & \\ & \ddots & \\ & & T \end{matrix} & 0 \end{array} \right\| \quad \text{for} \quad 0 \leq m < n ,$$

$$X^n = \left\| \begin{matrix} T & & 0 \\ & \ddots & \\ 0 & & T \end{matrix} \right\| = TE$$

and

$$X^m = T^k X^{m-nk} \qquad \text{for} \quad m > n , \tag{$*$}$$

where $X^0 = E$, $T^0 = 1$ and k is the greatest positive integer such that $m - nk$ is the non-negative integer.

Let $f(x)$ be the polynomial $a_0 + a_1 x + \cdots + a_m x^m$, where $m = kn + l$ ($k \geq 0$, $0 \leq l < n$). Substituting the matrix polynomial $f(X)$ by ($*$), we have

$$f(X) \equiv F(X) = A_0 E + A_1 X + \cdots + A_{n-1} X^{n-1} ,$$

where

$$\begin{aligned} A_0 &= a_0 + a_n T + \cdots + a_{nk} T^k \\ &\vdots \\ A_l &= a_l + a_{n+l} T + \cdots + a_{nk+l} T^k \\ A_{l+1} &= a_{l+1} + a_{n+l+1} T + \cdots + a_{n(k-1)+l+1} T^{k-1} \\ &\vdots \\ A_{n-1} &= a_{n-1} + a_{n+n-1} T + \cdots + a_{n(k-1)+n-1} T^{k-1} . \end{aligned}$$

Then

$$\det F(X) = \begin{vmatrix} A_0 & A_1 & \cdots & A_{n-1} \\ A_{n-1} T & A_0 & \cdots & A_{n-2} \\ \vdots & \ddots & \ddots & \vdots \\ A_1 T & A_2 T \cdots A_{n-1} T & & A_0 \end{vmatrix} .$$

Therefore we have

$$\det F(X) = \prod_{j=0}^{n-1} F(\omega_j \sqrt[n]{T}) ,$$

where ω_j ($j = 0, 1, \cdots, n-1$) runs the n-th root of unity. Since it is easily shown that $F(\omega_j \sqrt[n]{T}) = f(\omega_j \sqrt[n]{T})$, we have the following

Lemma 1. *Let X be the $n \times n$ matrix* $\left\| \begin{matrix} 0 & 1 \cdots\cdots 0 \\ \vdots \ddots & \ddots \quad \vdots \\ \vdots & \ddots \quad 1 \\ 0 & \ddots \\ T & 0 \cdots\cdots 0 \end{matrix} \right\|$ *and $f(x)$ the* *polynomial* $a_0+a_1x+\cdots+a_mx^m$. *Then* $\det f(X)=\prod_{j=0}^{n-1} f(\omega_j \sqrt[n]{T})$, *where* ω_j $(j=0, 1, \cdots, n-1)$ *runs the n-th roots of unity.*

2. Let $\mathfrak{L}$ be an oriented link of multiplicity μ in the 3-dimensional sphere S^3 and let $F(S^3-\mathfrak{L})$ be the fundamental group of $S^3-\mathfrak{L}$. If h is an element of the fundamental group $F(S^3-\mathfrak{L})$, we shall denote the linking number of h and $\mathfrak{L}$ by link $(h, \mathfrak{L})$. Now let

$$F_g(S^3-\mathfrak{L}) = \{h\,;\ h\in F(S^3-\mathfrak{L}),\ \text{link}\,(h, \mathfrak{L})\equiv 0\ (\text{mod}\ g)\}\ ,$$

where g is a positive integer. Then $F_g(S^3-\mathfrak{L})$ is a normal subgroup of $F(S^3-\mathfrak{L})$. Hence there exists uniquely the g-fold cyclic covering space $\tilde{\mathfrak{M}}_g(\mathfrak{L})$ of $S^3-\mathfrak{L}$, whose fundamental group is isomorphic to $F_g(S^3-\mathfrak{L})$. Then, we can define naturally the g-fold cyclic covering space $\mathfrak{M}_g(\mathfrak{L})$ of S^3, branched along $\mathfrak{L}$. $\mathfrak{M}_g(\mathfrak{L})$ is a closed 3-dimensional manifold without boundary for each g.

Now, we shall give a geometrical image of the g-fold cyclic covering space $\mathfrak{M}_g(\mathfrak{L})$ of S^3 branched along $\mathfrak{L}$. It is a natural generalization of the cyclic covering space, branched along a knot, defined by H. Seifert [6], [7].

Let $\mathfrak{F}$ be a non-singular orientable surface with the boundary $\mathfrak{L}$ in S^3 [6]. Consider that $\mathfrak{F}$ be two leaves of surfaces $\mathfrak{F}^1$ and $\mathfrak{F}^2$ which have the same boundary $\mathfrak{L}$. Now, let S_1^3, S_2^3, $\cdots$ and S_g^3 be 3-spheres homeomorphic to S^3 and let $\mathfrak{F}_1^1$, $\mathfrak{F}_1^2$ in S_1^3, $\mathfrak{F}_2^1$, $\mathfrak{F}_2^2$ in S_2^3, $\cdots$ and $\mathfrak{F}_g^1$, $\mathfrak{F}_g^2$ in S_g^3 be homeomorphic image of $\mathfrak{F}^1$ and $\mathfrak{F}^2$ in S^3. The closed 3-manifold $\mathfrak{M}_g'(\mathfrak{L})=S_1^3\cup S_2^3\cup\cdots\cup S_g^3$ which is constructed by identifying $\mathfrak{F}_1^2$ and $\mathfrak{F}_2^1$, $\mathfrak{F}_2^2$ and $\mathfrak{F}_3^1$, $\cdots$, $\mathfrak{F}_g^2$ and $\mathfrak{F}_1^1$ respectively, is just the g-fold cyclic covering space of S^3, branched along $\mathfrak{L}$.

In fact, over each point of S^3, except for $\mathfrak{L}$, there are g points of $\mathfrak{M}_g'(\mathfrak{L})$ and over each point of $\mathfrak{L}$ there is only one point of $\mathfrak{M}_g'(\mathfrak{L})$. And for each closed curve of $\mathfrak{M}_g'(\mathfrak{L})$, which does not intersect $\mathfrak{L}$, there corresponds a closed curve of S^3 whose linking number with $\mathfrak{L}$ is a multiple of g. Therefore the fundamental group of $\mathfrak{M}_g'(\mathfrak{L})-\mathfrak{L}$ is isomorphic to the given normal subgroup $F_g(S^3-\mathfrak{L})$. Since the cyclic covering space of $S^3-\mathfrak{L}$, which has $F_g(S^3-\mathfrak{L})$ as its fundamental group, is determined uniquely [7], $\mathfrak{M}_g'(\mathfrak{L})-\mathfrak{L}$ is homeomorphic to $\tilde{\mathfrak{M}}_g(\mathfrak{L})$. Hence, $\mathfrak{M}_g'(\mathfrak{L})$ is a g-fold cyclic covering space of S^3, branched along $\mathfrak{L}$.

3. Let $\mathfrak{L}$ be an oriented link of multiplicity μ in S^3 and $L_1, L_2, \cdots,$ of L_μ be components of $\mathfrak{L}$. The Alexander polynomial $\Delta(t_1, t_2, \cdots, t_\mu)$ of $\mathfrak{L}$ is defined by R. H. Fox [1]. Let $\Delta(t)=\Delta(t, t, \cdots, t)$, where $t=t_1=t_2=\cdots=t_\mu$. One of the authers of this paper has defined the ∇-polynomial $\nabla(t)$ as

follows [3]:

$$\nabla(t) = \Delta(t)/(1-t)^{\mu-2} \qquad \text{for} \quad \mu \geq 2$$

and

$$\nabla(t) = \Delta(t) \qquad \text{for} \quad \mu = 1 .$$

Let $\mathfrak{F}$ be an orientable surface of genus h in S^3 whose boundary is $\mathfrak{L}$. Consider a Seifert projection of $\mathfrak{L}$ [8] (Fig. 1). We shall compute the fundamental group $F(S^3-\mathfrak{L})$ of $S^3-\mathfrak{L}$ by the use of this projection.

Contracting the center disk of $\mathfrak{F}$ to a point P and the bands to lines (Fig. 2), we have the graph G consisting of $\{a_k\}$. Let us denote by a_{i1}, $\cdots$, a_{ij_i} $(i=1, 2, \cdots, 2h+\mu-1)$ the arcs of the projection of a_i such as these in the usual Wirtinger's method.

To each arc a_{ij} there correspond two arcs x_{ij} and x'_{ij} of the projection of $\mathfrak{L}$, where x_{ij} is the arc which has the same orientation to that of a_{ij} and lies on the right side of it, and x'_{ij} is the arc which has the opposite orientation to that of a_{ij} and lies on the left side of it (Fig. 3).

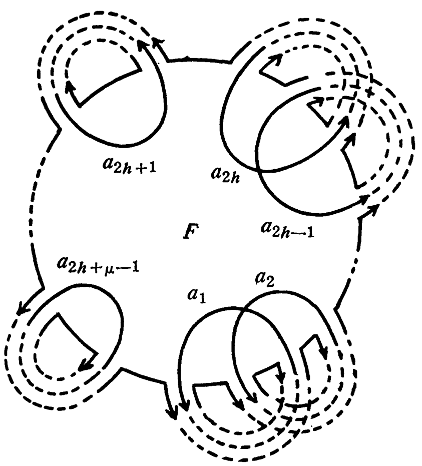

Fig. 1.

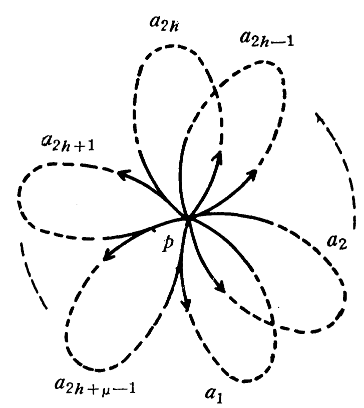

Fig. 2.

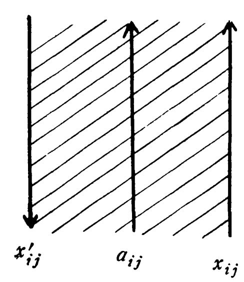

Fig. 3.

Let us also denote by x_{ij} and x'_{ij} ($i=1, \cdots, 2h+\mu-1$, $j=1, \cdots, j_i$) the generators of $F(S^3-\mathfrak{L})$ which correpond to the arcs x_{ij} and x'_{ij} respectively.

For each crossing of a_p over a_i we have two defining relations R_{ij} and S_{ij} of the following forms:

$$\begin{aligned} R_{ij} &= (x'^{-1}_{pq}x_{pq})^{\varepsilon} x_{ij} (x'^{-1}_{pq}x_{pq})^{-\varepsilon} x^{-1}_{i,j+1}, \\ S_{ij} &= (x'^{-1}_{pq}x_{pq})^{\varepsilon} x'_{ij} (x'^{-1}_{pq}x_{pq})^{-\varepsilon} x'^{-1}_{i,j+1}, \end{aligned} \quad \begin{pmatrix} j=1, 2, \cdots, j_i-1 \\ i=1, 2, \cdots, 2h+\mu-1 \end{pmatrix}$$

where $\varepsilon=+1$ when a_p crosses a_i from left to right and $\varepsilon=-1$ when a_p crosses a_i from right to left.

Besides these relations we have the relations:

$$\begin{aligned} T'_{2l-1} &= x'_{2l-1,1}x^{-1}_{2l,1} \\ T'_{2l} &= x'_{2l,1}x'^{-1}_{2l-1,j_{2l-1}} \\ T_{2l-1} &= x_{2l-1,j_{2l-1}}x'^{-1}_{2l,j_{2l}} \\ T_{2l} &= x_{2l,j_{2l}}x^{-1}_{2l+1,1} \end{aligned} \qquad (l=1, 2, \cdots, h)$$

and

$$\begin{aligned} Q_t &= x_{t,j_t}x^{-1}_{t+1,1} && (t=2h+1, \cdots, 2h+\mu-2) \\ Q'_{t'} &= x'_{t',1}x'^{-1}_{t',j_{t'}} && (t'=2h+1, \cdots, 2h+\mu-1). \end{aligned}$$

Then the group $F(S^3-\mathfrak{L})$ has the presentation

$$\{x_{ij}, x'_{ij} : R_{ij}, S_{ij}, T_{2l-1}, T_{2l}, T'_{2l-1}, T'_{2l}, Q_t, Q'_{t'}\}$$

where

$$\begin{cases} i = 1, 2, \cdots, 2h+\mu-1 \\ j = 1, 2, \cdots, j_i-1 \\ l = 1, 2, \cdots, h \\ t = 2h+1, \cdots, 2h+\mu-2 \\ t'= 2h+1, \cdots, 2h+\mu-1. \end{cases}$$

Note that $Q_{2h+\mu-1}$ is already eliminated, for it is the consequence of the other relations.

Let us introduce generators t, s_{ij} and s'_{ij} ($i=1, 2, \cdots, 2h+\mu-1$, $j=1, 2, \cdots, j_i$) defined by means of the relation $W_{ij}=t^{-1}x_{ij}s^{-1}_{ij}$, $W'_{ij}=t^{-1}x'_{ij}S'^{-1}_{ij}$ and $W=t^{-1}x_{2h+\mu-1,j_{2h+\mu-1}}$. Using W_{ij} and W'_{ij}, we obtain from R_{ij}, S_{ij}, T_{2l-1}, T_{2l}, T'_{2l-1}, T'_{2l}, Q_t, $Q'_{t'}$, W_{ij}, W'_{ij}, and W the relations

$$\begin{aligned} \tilde{R}_{ij} &= (s'^{-1}_{pq}s_{pq})^{\varepsilon} ts_{ij} (s'^{-1}_{pq}s_{pq})^{-\varepsilon} s^{-1}_{i,j+1}t^{-1}, \\ \tilde{S}_{ij} &= (s'^{-1}_{pq}s_{pq})^{\varepsilon} ts'_{ij} (s'^{-1}_{pq}s_{pq})^{-\varepsilon} s'^{-1}_{i,j+1}t^{-1}, \end{aligned}$$

$$(i=1, 2, \cdots, 2h+\mu-1,\ j=1, 2, \cdots, j_i-1)$$

$$\left.\begin{aligned}\tilde{T}'_{2l-1} &= s'_{2l-1,1}s_{2l,1}^{-1}\,, \\ \tilde{T}'_{2l} &= s'_{2l,1}s'^{-1}_{2l-1,\,j_{2l-1}}\,, \\ \tilde{T}_{2l-1} &= s_{2l-1,\,j_{2l-1}}s'^{-1}_{2l,\,j_{2l}}\,, \\ \tilde{T}_{2l} &= s_{2l,\,j_{2l}}s_{2l+1,1}^{-1}\,,\end{aligned}\right\}\quad (l=1,2,\cdots,h)$$

$$\tilde{Q}_t = s_{t,\,j_t}s_{t+1,1}^{-1}\,, \qquad (t=2h+1,\cdots,2h+\mu-2)$$

$$\tilde{Q}'_{t'} = s'_{t',1}s'^{-1}_{t',\,j_{t'}}\,, \qquad (t'=2h+1,\cdots,2h+\mu-1)$$

$$\tilde{W} = s_{2h+\mu-1,\,j_{2h+\mu-1}}\,.$$

Since generators x_{ij} and x'_{ij} appear only in the relations W_{ij}, W'_{ij} and W, we can eliminate the generators x_{ij} and x'_{ij} and the relations W_{ij}, W'_{ij} and W $(i=1, 2, \cdots, 2h+\mu-1,\ j=1, 2, \cdots, j_i)$.

Now we must remark that the linking numbers of the generators s_{ij}, s'_{ij} with the link $\mathfrak{L}$ are all equal to zero.

Then we have the following presentation of the group $F(S^3-\mathfrak{L})$

$$\{s_{ij},\ s'_{ij},\ t\ ;\ \tilde{R}_{ij},\ \tilde{S}_{ij},\ \tilde{T}_{2l-1},\ \tilde{T}_{2l},\ \tilde{T}'_{2l-1},\ \tilde{T}'_{2l},\ \tilde{Q}_t,\ \tilde{Q}'_{t'},\ \tilde{W}\}\,.$$

Now we compute the fundamental group $F(\mathfrak{M}_g(\mathfrak{L})-\mathfrak{L})$ of $\mathfrak{M}_g(\mathfrak{L})-\mathfrak{L}$, where $\mathfrak{M}_g(\mathfrak{L})$ is the g-fold cyclic covering space of S^3, branched along $\mathfrak{L}$. Let $s_{ij\alpha}=t^\alpha s_{ij}t^{-\alpha}$ and $s'_{ij\alpha}=t^\alpha s'_{ij}t^{-\alpha}$ $(\alpha=0, 1, 2, \cdots, g-1)$ and $T=t^g$. Then it is easily shown that $s_{ij\alpha}$, $s'_{ij\alpha}$ and T represent simple closed curves in $\mathfrak{M}_g(\mathfrak{L})-\mathfrak{L}$ and that the generators of $F(\mathfrak{M}_g(\mathfrak{L})-\mathfrak{L})$ consist of these $s_{ij\alpha}$, $s'_{ij\alpha}$ $(i=1, 2, \cdots, 2h+\mu-1,\ j=1, 2, \cdots, j_i,\ \alpha=0, 1, 2, \cdots, g-1)$ and T.

On the other hand, since the linking numbers of the relation words $\tilde{R}_{ij}$, $\tilde{S}_{ij}$, $\tilde{T}_{2l-1}$, $\tilde{T}_{2l}$, $\tilde{T}'_{2l-1}$, $\tilde{T}'_{2l}$, $\tilde{Q}_t$, $\tilde{Q}'_{t'}$, and $\tilde{W}$ with $\mathfrak{L}$ are all equal to zero, $\tilde{R}_{ij\alpha}=t^\alpha\tilde{R}_{ij}t^{-\alpha}$, $\tilde{S}_{ij\alpha}=t^\alpha\tilde{S}_{ij}t^{-\alpha}$, $\tilde{T}_{2l-1,\alpha}=t^\alpha\tilde{T}_{2l-1}t^{-\alpha}$, $\tilde{T}_{2l,\alpha}=t^\alpha\tilde{T}_{2l}t^{-\alpha}$, $\tilde{T}'_{2l-1,\alpha}=t^\alpha\tilde{T}'_{2l-1}t^{-\alpha}$, $\tilde{T}'_{2l,\alpha}=t^\alpha\tilde{T}'_{2l}t^{-\alpha}$, $\tilde{Q}_{t,\alpha}=t^\alpha\tilde{Q}_t t^{-\alpha}$, $\tilde{Q}'_{t',\alpha}=t^\alpha\tilde{Q}'_{t'}t^{-\alpha}$ and $\tilde{W}_\alpha=t^\alpha\tilde{W}t^{-\alpha}$ are expressible by words which consist of at most $s_{ij\alpha}$, $s'_{ij\alpha}$ and T, and they are the relation of $F(\mathfrak{M}_g(\mathfrak{L})-\mathfrak{L})$. Therefore we have the following presentation of $F(\mathfrak{M}_g(\mathfrak{L})-\mathfrak{L})$:

$$\{s_{ij\alpha},\ s'_{ij\alpha},\ T\ ;\ \tilde{R}_{ij\alpha}\tilde{S}_{ij\alpha}\tilde{T}_{2l-1,\alpha}\tilde{T}_{2l,\alpha}\tilde{T}'_{2l-1,\alpha}\tilde{T}'_{2l,\alpha}\tilde{Q}_{t,\alpha}\tilde{Q}'_{t',\alpha}\tilde{W}_\alpha\}$$

$$\left\{\begin{aligned} i &= 1, 2, \cdots, 2h+\mu-1 \\ j &= 1, 2, \cdots, j_i-1 \\ l &= 1, 2, \cdots, h \\ t &= 2h+1, \cdots, 2h+\mu-2 \\ t' &= 2h+1, \cdots, 2h+\mu-1 \\ \alpha &= 0, 1, 2, \cdots, g-1\,. \end{aligned}\right.$$

From this presentation we can compute the fundamental group

$F(\mathfrak{M}_g(\mathfrak{L}))$ of $\mathfrak{M}_g(\mathfrak{L})$. Namely, to add $\mathfrak{L}$ to $\mathfrak{M}_g(\mathfrak{L})-\mathfrak{L}$ induces the new relations, so-called the branch relations [1], $B_k=(x'_{k,j_k})^g$ $(k=2h+1, \cdots, 2h+\mu-1)$ and $T=1$. From the relations W'_{k,j_k} $(k=2h+1, \cdots, 2h+\mu-1)$ we have $B_k=(ts'_{k,j_k})^g=s'_{k,j_k,1}\cdot s'_{k,j_k,2}\cdot\cdots\cdot s'_{k,j_k,g-1}\cdot s'_{k,j_k,0}$.

Therefore the presentation of $F(\mathfrak{M}_g(\mathfrak{L}))$ is

$$\{s_{ij\alpha},\ s'_{ij\alpha},\ T;\ \tilde{R}_{ij\alpha},\ \tilde{S}_{ij\alpha},\ \tilde{T}_{2l-1,\alpha},\ \tilde{T}_{2l\alpha},\ \tilde{T}'_{2l-1,\alpha},\ \tilde{T}'_{2l,\alpha},\ \tilde{Q}_{t,\alpha},\ \tilde{Q}'_{t',\alpha},\ \tilde{W}_\alpha,\ B_k,\ T\}$$

$$\left|\begin{array}{l} i = 1, 2, \cdots, 2h+\mu-1 \\ j = 1, 2, \cdots, j_i-1 \\ l = 1, 2, \cdots, h \\ t = 2h+1, \cdots, 2h+\mu-2 \\ t' = 2h+1, \cdots, 2h+\mu-1 \\ k = 2h+1, \cdots, 2h+\mu-1 \\ \alpha = 0, 1, 2, \cdots, g-1\,. \end{array}\right.$$

4. In this section we shall obtain the Alexander matrix of $\mathfrak{L}$ from the presentation of $F(S^3-\mathfrak{L})$.

Replace the multiplication by the addition and put

$$jt\pm s-jt = \pm t^j s\,,$$

where s is any generator of the form s_{ij} or s'_{ij}. Furthermore suppose that the addition is commutative. Then, for each of the relations of $F(S^3-\mathfrak{L})$ we have the following relations:

$$\begin{array}{lll}
\bar{R}_{ij} & = \varepsilon(1-t)(s_{pq}-s'_{pq})+ts_{ij}-ts_{i,j+1} & \left(\begin{array}{l} i=1, 2, \cdots, 2h+\mu-1, \\ j=1, 2, \cdots, j_i-1 \end{array}\right) \\
\bar{S}_{ij} & = \varepsilon(1-t)(s_{pq}-s'_{pq})+ts'_{ij}-ts'_{i,j+1} & \\
\bar{T}'_{2l-1} & = s'_{2l,1}-s_{2l,1} & \\
\bar{T}'_{2l} & = s'_{2l,1}-s'_{2l-1,j_{2l-1}} & (l=1, 2, \cdots, h) \\
\bar{T}_{2l-1} & = s_{2l-1,j_{2l-1}}-s'_{2l,j_{2l}} & \\
\bar{T}_{2l} & = s_{2l,j_{2l}}-s_{2l+1,1} & \\
\bar{Q}_t & = s_{t,j_t}-s_{t+1,1} & (t=2h+1, \cdots, 2h+\mu-2) \\
\bar{Q}'_{t'} & = s'_{t',1}-s'_{t',j_{t'}} & (t'=2h+1, \cdots, 2h+\mu-1) \\
\bar{W} & = s_{2h+\mu-1,j_{2h+\mu-1}}\,. &
\end{array}$$

Let us introduce generators a_{ij} defined by means of the relations

$\bar{A}_{ij}=s_{ij}-s'_{ij}-a_{ij}$ $(i=1, 2, \cdots, 2h+\mu-1,\ j=1, 2, \cdots, j_i)$. Using $\bar{A}_{ij}$, we obtain from $\bar{R}_{ij}$, $\bar{S}_{ij}$ the relations

$$\begin{aligned}\bar{R}'_{ij} &= \varepsilon(1-t)a_{pq}+ts_{ij}-ts_{i,j+1}\\ \bar{S}'_{ij} &= \varepsilon(1-t)a_{pq}+ts'_{ij}-ts'_{i,j+1}\,.\end{aligned}$$

Since $\bar{R}_{ij}$ and $\bar{S}_{ij}$ are the consequences of $\bar{A}_{ij}$, $\bar{R}'_{ij}$ and $\bar{S}'_{ij}$, we can eliminate $\bar{R}_{ij}$ and $\bar{S}_{ij}$ from the relations. Furthermore we obtain from $\bar{R}'_{ij}$, $\bar{S}'_{ij}$ and $\bar{A}_{ij}$ the relations

$$r_{ij} = ta_{ij}-ta_{i,j+1}\equiv a_{i,j}-a_{i,j+1}\,.$$

From r_{ij} $(j=1, 2, \cdots, j_j-1)$ we obtain

$$a_{i,1} = a_{i,2} = \cdots = a_{i,j_i}\,.$$

Here let us introduce generators a_i defined by $a_i=a_{i1}=\cdots=a_{i,j_i}$ $(i=1, 2, \cdots, 2h+\mu-1)$. Then we can eliminate the relations r_{ij} and the genertors a_{ij} $(i=1, 2, \cdots, 2h+\mu-1,\ j=1, 2, \cdots, j_i)$. Now, let us rewrite a_{ij} to a_i in the remaining relations. Moreover we can eliminate $\bar{S}'_{ij}$, since $\bar{S}'_{ij}$ is a consequence of $\bar{R}'_{ij}$ and $\bar{A}_{ij}$. And since s'_{ij} $(i=1, 2, \cdots, 2h+\mu-1,\ 1<j<j_i)$ appears only in $\bar{A}_{ij}$, we can eliminate $\bar{A}_{ij}$ and the generators s'_{ij} $(i=1, 2, \cdots, 2h+\mu-1,\ 1<j<j_i)$. Furthermore the generator s_{i2} appears only in $\bar{R}'_{i1}=\varepsilon_{i1}(1-t)a_{p_{i1}}+ts_{i1}-ts_{i2}$ and $\bar{R}'_{i2}=\varepsilon_{i2}(1-t)a_{p_{i2}}+ts_{i2}-ts_{i3}$ and from $\bar{R}'_{i1}$ and $\bar{R}'_{i2}$ we obtain $\bar{R}''_{i1}=(1-t)(\varepsilon_{i1}a_{p_{i1}}+\varepsilon_{i2}a_{p_{i2}})+ts_{i1}-ts_{i3}$. Therefore we can eliminate $\bar{R}'_{i1}$ and $\bar{R}'_{i2}$ and the generator s_{i2}. Similarly the generator s_{i3} appears only in $\bar{R}''_{i1}$ and $\bar{R}'_{i3}=\varepsilon_{i3}(1-t)a_{p_{i3}}+ts_{i3}-ts_{i4}$ and from $\bar{R}''_{i1}$ and $\bar{R}'_{i3}$ we obtain $\bar{R}'''_{i1}=(1-t)\,(\varepsilon_{i1}a_{p_{i1}}+\varepsilon_{i2}a_{p_{i2}}+\varepsilon_{i3}a_{p_{i3}})+ts_{i1}-ts_{i4}$, therefore we can eliminate $\bar{R}''_{i1}$ and $\bar{R}'_{i3}$ and the generator s_{i3}. By the iteration of this process, introducing

$$\bar{R}_i = (1-t)\,(\varepsilon_{i1}a_{p_{i1}}+\varepsilon_{i2}a_{p_{i2}}+\cdots+\varepsilon_{i,j_i-1}a_{p_{i,j_i-1}})+ts_{i,1}-ts_{i,j_i} \qquad (i=1, 2, \cdots, 2h+\mu-1)\,,$$

we can eliminate the relations $\bar{R}'_{ij}$ $(i=1, 2, \cdots, 2_h+\mu-1,\ j=1, 2, \cdots, j_i-1)$ and the generators s_{ij} $(i=1, 2, \cdots, 2h+\mu-1,\ j=2, \cdots, j_i-1)$.

If we set $\omega_i(a)=\sum_{k=1}^{j_i-1}\varepsilon_{ik}a_{p_{ik}}$, then

$$\bar{R}_i = (1-t)\omega_i(a) + ts_{i1} - ts_{ij_i}.$$

Therefore we have the generators s_{i1}, s_{ij_i}, a_i $(i=1, 2, \cdots, 2h+\mu-1)$ and the relations $\bar{R}_i$, $\bar{A}_{i1}$, $\bar{A}_{ij_i}$ $(i=1, 2, \cdots, 2h+\mu-1)$ $\bar{T}_{2l-1}$, $\bar{T}_{2l}$, $\bar{T}'_{2l-1}$, $\bar{T}'_{2l}$ $(l=1, 2, \cdots, h)$, $\bar{Q}_t$ $(t=2h+1, \cdots, 2h+\mu-2)$, $\bar{Q}'_{t'}$ $(t'=2h+1, \cdots, 2h+\mu-1)$ and $\bar{W}$.

Since the generator $s'_{2l-1,1}$ appears only in the relation $\bar{T}'_{2l-1}$ and $\bar{A}_{2l-1,1}$, from $\bar{T}'_{2l-1}$ and $\bar{A}_{2l-1,1}$ we obtain

$$\bar{U}_{2l-1} = s_{2l-1,1} - s_{2l,1} - a_{2l-1}.$$

Therefore we can eliminate the relation $\bar{A}_{2l-1,1}$, $\bar{T}'_{2l-1}$ and the generator $s'_{2l-1,1}$. Similarly from $\bar{T}_{2l-1}$ and $\bar{A}_{2lj_{2l}}$ we obtain

$$\bar{U}_{2l} = s_{2l-1,j_{2l-1}} - s_{2l,j_{2l}} + a_{2l}$$

and from $\bar{T}'_{2l}$, $\bar{A}_{2l,1}$ and $\bar{A}_{2l-1,j_{2l-1}}$ we obtain

$$\bar{V}_{2l} = s_{2l,1} - a_{2l} - s_{2l-1,j_{2l-1}} + a_{2l-1}$$

and form $\bar{Q}'_{t'}$, $\bar{A}_{t',1}$ and $\bar{A}_{t'j_{t'}}$ we obtain

$$\bar{P}_{t'} = s_{t',1} - s_{t',j_{t'}}.$$

Therefore we have the generators

$$a_i,\ s_{i1},\ s_{ij_i} \qquad (i=1, \cdots, 2h+\mu-1)$$

and the relations

$$\begin{aligned}
\bar{R}_i &= (1-t)\omega_i(a) + ts_{i1} - ts_{i,j_i},\\
\bar{U}_{2l-1} &= s_{2l-1,1} - s_{2l,1} - a_{2l-1},\\
\bar{U}_{2l} &= s_{2l-1,j_{2l-1}} - s_{2l,j_{2l}} + a_{2l},\\
\bar{V}_{2l} &= s_{2l,1} - a_{2l} - s_{2l-1,j_{2l-1}} + a_{2l-1},\\
\bar{T}_{2l} &= s_{2l,j_{2l}} - s_{2l+1,1},\\
\bar{Q}_t &= s_{tj_t} - s_{t+1,1},\\
\bar{P}_{t'} &= s_{t',1} - s_{t'j_{t'}},\\
\bar{W} &= s_{2h+\mu-1,j_{2h+\mu-1}}.
\end{aligned}$$

$$\begin{cases} i &= 1, 2, \cdots, 2h+\mu-1 \\ l &= 1, 2, \cdots, h \\ t' &= 2h+1, \cdots, 2h+\mu-1 \\ t &= 2h+1, \cdots, 2h+\mu-2\,. \end{cases}$$

From $\bar{R}_{t'}$ ($t'=2h+1, \cdots, 2h+\mu-1$) and $\bar{P}_{t'}$ we obtain

$$\bar{R}_{t'} = (1-t)\,\omega_{t'}(a)\,,$$

and from $\bar{R}_{2l-1}$ ($l=1, \cdots, h$), $\bar{V}_{2l}$ and $\bar{U}_{2l-1}$ we obtain

$$\bar{R}_{2l-1} = (1-t)\,\omega_{2l-1}(a)+ta_{2l}\,,$$

and from $\bar{R}_{2l}$ ($l=1, \cdots, h$), $\bar{V}_{2l}$ and $\bar{U}_{2l}$ we obtain

$$\bar{R}_{2l} = (1-t)\,\omega_{2l}(a)-ta_{2l-1}\,.$$

From these relations and generators we have the following relation matrix

$$M(t) = \begin{array}{c|c|c|l} & a_i & s_{i1} \quad s_{ij_i} & \\ \hline \bar{R}_i & A(t) & 0 & \}\ 2h+\mu-1 \\ \hline \begin{array}{c} \bar{U}_{2l-1} \\ \bar{V}_{2l} \\ \bar{U}_{2l} \\ \bar{T}_{2l} \\ \bar{P}_{t'} \\ \bar{Q}_t \\ \bar{W} \end{array} & C & B & \}\ 2(2h+\mu-1) \\ \hline & \underbrace{\qquad\qquad}_{2h+\mu-1} & \underbrace{\qquad\qquad}_{2(2h+\mu-1)} & \end{array}$$

The submatrix B of $M(t)$, corresponding to the generators s_{i1}, s_{ij_i} ($i=1, 2, \cdots, 2h+\mu-1$) and the relations $\bar{U}_{2l-1}$, $\bar{V}_{2l}$, $\bar{U}_{2l}$, $\bar{T}_{2l}$, $\bar{P}_{t'}$, $\bar{Q}_t$, $\bar{W}$ ($l=1, 2, \cdots, h$, $t'=2h+1, \cdots, 2h+\mu-1$, $t=2h+1, \cdots, 2h+\mu-2$), is the following:

$$
B = \begin{array}{c||cccccccc|cccc||}
 & S_{11} & S_{21} & S_{1j_1} & S_{2j_2}\cdots\cdots S_{2h-1,1} & S_{2h,1} & S_{2h-1,j_{2h-1}} & S_{2h,j_{2h}} & S_{2h+1,1} & S_{2h+1,j_{2h+1}} & \cdots\cdots\cdots\cdots S_{2h+\mu-1,1} & S_{2h+\mu-1,j_{2h+\mu-1}} & \\
\hline
\bar{U}_1 & 1 & -1 & 0 & 0\cdots\cdots\cdots 0 & 0 & 0 & 0 & & & & \\
\bar{V}_2 & 0 & 1 & -1 & 0\cdots\cdots\cdots 0 & 0 & 0 & 0 & & & & \\
\bar{U}_2 & 0 & 0 & 1 & -1\cdots\cdots\cdots 0 & 0 & 0 & 0 & & & & \\
\bar{T}_2 & 0 & 0 & 0 & 1\cdots\cdots\cdots 0 & 0 & 0 & 0 & & & 0 & \\
\vdots & \vdots & \vdots & \vdots & \vdots\ \ddots\ \vdots & \vdots & \vdots & \vdots & & & & \\
\bar{U}_{2h-1} & 0 & 0 & 0 & 0\cdots\cdots\cdots 1 & -1 & 0 & 0 & & & & \\
\bar{V}_{2h} & 0 & 0 & 0 & 0\cdots\cdots\cdots 0 & 1 & -1 & 0 & & & & \\
\bar{U}_{2h} & 0 & 0 & 0 & 0\cdots\cdots\cdots 0 & 0 & 1 & -1 & & & & \\
\bar{T}_{2h} & 0 & 0 & 0 & 0\cdots\cdots\cdots 0 & 0 & 0 & 1 & -1 & & & \\
\hline
\bar{P}_{2h+1} & & & & & & & & 1 & -1 & \cdots\cdots\cdots\cdots 0 & 0 \\
\bar{Q}_{2h+1} & & & & & & & & 0 & 1 & -1\cdots\cdots\cdots 0 & 0 \\
\vdots & & & & 0 & & & & \vdots & \vdots\ \ddots & \ddots\ \vdots & \vdots \\
\bar{Q}_{2h+\mu-2} & & & & & & & & 0 & 0\cdots\cdots\cdots\ddots & \cdots -1 & 0 \\
\bar{P}_{2h+\mu-1} & & & & & & & & 0 & 0\cdots\cdots\cdots\cdots & 1 & -1 \\
\bar{W} & & & & & & & & 0 & 0\cdots\cdots\cdots\cdots & 0 & 1 \\
\hline
\end{array}
$$

Now let us consider about the word $\omega_i(a)$. We can easily show that the coefficient of $a_j(i \neq j)$, which appears in $\omega_i(a)$, is the number v_{ij} which is equal to the number of times that a_j crosses over a_i from left to right minus the number of times that a_j crosses over a_i from right to left [6], [8]. Therefore, the submatrix $A(t)$ of $M(t)$, corresponding to the generators a_i $(i=1, 2, \cdots, 2h+\mu-1)$ and the relations $\bar{R}_i$ $(i=1, 2, \cdots, 2h+\mu-1)$, is of the form on page 343.

Similarly the submatrix C of $M(t)$, corresponding to the generators a_i $(i=1, 2, \cdots, 2h+\mu-1)$ and the relations $\bar{U}_{2l-1}$, $\bar{V}_{2l}$, $\bar{U}_{2l}$, $\bar{T}_{2l}$, $\bar{P}_{t'}$, $\bar{Q}_t$, $\bar{W}$ $(l=1, 2, \cdots, h,\ \ t'=2h+1, \cdots, 2h+\mu-1,\ \ t=2h+1, \cdots, 2h+\mu-2)$ is the following:

$C =$

	a_1	a_2 ⋯⋯ a_{2h-1}	a_{2h}	a_{2h+1} ⋯⋯ $a_{2h+\mu-1}$
$\bar{U}_1$	-1	$0 \cdots\cdots 0$	0	
$\bar{V}_2$	1	$-1 \cdots\cdots 0$	0	
$\bar{U}_2$	1	$0 \cdots\cdots 0$	0	
$\bar{T}_2$	0	$0 \cdots\cdots 0$	0	
$\vdots$		$\ddots$		0
$\bar{U}_{2h-1}$	0	$0 \cdots\cdots -1$	0	
$\bar{V}_{2h}$	0	$0 \cdots\cdots 1$	-1	
$\bar{U}_{2h}$	0	$0 \cdots\cdots 1$	0	
$\bar{T}_{2h}$	0	$0 \cdots\cdots 0$	0	
$\bar{P}_{2h+1}$				
$\bar{Q}_{2h+1}$				
$\vdots$		0		0
$\bar{Q}_{2h+\mu-2}$				
$\bar{P}_{2h+\mu-1}$				
$\bar{W}$				

Then, the matrix $M(t)$ is the Alexander matrix of the oriented link $\mathfrak{L}$ of multiplicity μ.

5. In this section we shall obtain the relation matrix of the 1-dim. homology group of $\mathfrak{M}_g(\mathfrak{L})$. Firstly let r_1, r_2 and r_3 be relations with the forms

$A(t) =$

	a_1	$a_2 \cdots\cdots a_{2h-1}$	a_{2h}	$a_{2h+1} \cdots\cdots a_{2h+\mu-1}$
$\bar{R}_1$	$v_{11}(1-t)$,	$v_{12}(1-t)+t \cdots v_{1,2h-1}(1-t)$,	$v_{1,2h}(1-t)$	$u_{1,2h+1}(1-t) \cdots\cdots v_{1,2h+\mu-1}(1-t)$
$\bar{R}_2$	$v_{21}(1-t)-t$,	$v_{22}(1-t) \cdots\cdots v_{2,2h-1}(1-t)$,	$v_{2,2h}(1-t)$	$v_{2,2h+1}(1-t) \cdots\cdots v_{2,2h+\mu-1}(1-t)$
$\vdots$	$\vdots$	$\vdots \quad \ddots \quad \vdots$	$\vdots$	$\vdots \quad \ddots \quad \vdots$
$\bar{R}_{2h-1}$	$v_{2h-1,1}(1-t)$,	$v_{2h-1,2}(1-t) \cdots v_{2h-1,2h-1}(1-t)$,	$v_{2h-1,2h}(1-t)+t$	$v_{2h-1,2h+1}(1-t) \cdots v_{2h-1,2h+\mu-1}(1-t)$
$\bar{R}_{2h}$	$v_{2h,1}(1-t)$,	$v_{2h,2}(1-t) \cdots\cdots v_{2h,2h-1}(1-t)-t$,	$v_{2h,2h}(1-t)$	$v_{2h,2h+1}(1-t) \cdots\cdots v_{2h,2h+\mu-1}(1-t)$
$\bar{R}_{2h+1}$	$v_{2h+1,1}(1-t)$,	$v_{2h+1,2}(1-t) \cdots v_{2h+1,2h-1}(1-t)$,	$v_{2h+1,2h}(1-t)$	$v_{2h+1,2h+1}(1-t) \cdots v_{2h+1,2h+\mu-1}(1-t)$
$\vdots$	$\vdots$	$\vdots \quad \ddots \quad \vdots$	$\vdots$	$\vdots \quad \ddots \quad \vdots$
$\bar{R}_{2h+\mu-1}$	$v_{2h+\mu-1,1}(1-t)$,	$v_{2h+\mu-1,2}(1-t) \cdot\cdot v_{2h+\mu-1,2h-1}(1-t)$,	$v_{2h+\mu-1,2h}(1-t)$	$v_{2h+\mu-1,2h+1}(1-t) \cdot\cdot v_{2h+\mu-1,2h+\mu-1}(1-t)$

	$a_1 \cdots\cdots a_{2h}$	$a_{2h+1} \cdots\cdots a_{2h+\mu-1}$	$s_{i,1} \quad s_{i,j_i} \quad (i=1, 2, \cdots, 2h)$	$s_{2h+1,1} \quad s_{2h+1,j_{2h+1}}$	$\cdots\cdots\cdots$	$s_{2h+\mu-1,1} \quad s_{2h+\mu-1,j_{2h+\mu-1}}$
$\bar{B}_{2h+1}$		$-e$		$0 \quad e$		
		$\ddots \quad 0$			$\ddots$	0
$\vdots$	0	$\ddots$	0	0	$\ddots$	
		$0 \quad \ddots$			$\ddots$	
$\bar{B}_{2h+\mu-1}$		$-e$				$0 \quad e$

$$r_1 = vsw\,,$$
$$r_2 = vtst^{-1}w\,,$$
$$r_3 = vswts^{-1}t^{-1}u\,,$$

where s is a generator and u, v, w are words which consist of generators except s. Making use of the abridged notation (see the first of the section 4), these can be written in additional forms

$$\bar{r}_1 = \bar{v}+s+\bar{w}$$
$$\bar{r}_2 = \bar{v}+ts+\bar{w}$$
$$\bar{r}_3 = \bar{v}+(1-t)s+\bar{w}+\bar{u}\,.$$

Now let $r_{i\beta}=t^{\beta}r_it^{-\beta}$ ($\beta=0, 1, \cdots, g-1$), $T=t^g$ and $s_\beta=t^\beta st^{-\beta}$ ($\beta=0, 1, \cdots, g-1$). Then

$$r_{1,\beta} = v_\beta t^\beta st^{-\beta}w_\beta = v_\beta s_\beta w_\beta$$
$$r_{2,\beta} = v_\beta t^{\beta+1}st^{-(\beta+1)}w_\beta = v_\beta s_{\beta+1}w_\beta \quad (\beta=0, 1, \cdots, g-2)$$
$$r_{3,\beta} = v_\beta t^\beta st^{-\beta}w_\beta t^{\beta+1}s^{-1}t^{-(\beta+1)}u_\beta = v_\beta s_\beta w_\beta s_{\beta+1}^{-1}u_\beta$$

and

$$r_{1,g-1} = v_{g-1}s_{g-1}w_{g-1}$$
$$r_{2,g-1} = v_{g-1}Ts_0T^{-1}w_{g-1}$$
$$r_{3,g-1} = v_{g-1}s_{g-1}w_{g-1}Ts_0^{-1}T^{-1}u_{g-1}\,,$$

where v_β, w_β and u_β ($\beta=0, \cdots, g-1$) are suitably transformed words from v, w and u respectively. Therefore, again by the use of the abridged notation

$$jT\pm s-jT = \pm T^js\,,$$

we have

$$\bar{r}_{1,\beta} = \bar{v}_\beta+s_\beta+\bar{w}_\beta$$
$$\bar{r}_{2,\beta} = \bar{v}_\beta+s_{\beta+1}+\bar{w}_\beta \qquad (\beta=0, 1, 2, \cdots, g-2)$$
$$\bar{r}_{3,\beta} = \bar{v}_\beta+s_\beta-s_{\beta+1}+\bar{w}_\beta+\bar{u}_\beta$$

and

$$\bar{r}_{1,g-1} = \bar{v}_{g-1}+s_{g-1}+\bar{w}_{g-1}$$
$$\bar{r}_{2,g-1} = \bar{v}_{g-1}+Ts_0+\bar{w}_{g-1}$$
$$\bar{r}_{3,g-1} = \bar{v}_{g-1}+s_{g-1}-Ts_0+\bar{w}_{g-1}+\bar{u}_{g-1}\,.$$

On the Alexander matrix of $\mathfrak{L}$ in S^3 the terms corresponding to the generator s and the relations $\bar{r}_1$, $\bar{r}_2$, $\bar{r}_3$ are the following:

	s
$\bar{r}_1$	1
$\bar{r}_2$	t
$\bar{r}_3$	$1-t$

and on the Alexander matrix of $\mathfrak{L}$ in $\mathfrak{M}_g(\mathfrak{L})$ the terms corresponding to the generators $s_0, s_1, \cdots, s_{g-1}$ and the relations $\bar{r}_{i0}, \bar{r}_{i1}, \cdots, \bar{r}_{i,g-1}$ $(i=1, 2, 3)$ are the following:

	s_0	s_1	$s_2 \cdots\cdots s_{g-1}$
$\bar{r}_{1,0}$	1	0	$0 \cdots\cdots 0$
$\bar{r}_{1,1}$	0	1	$0 \cdots\cdots 0$
$\bar{r}_{1,2}$	0	0	$1 \cdots\cdots 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots \ \ddots \ \vdots$
$\bar{r}_{1,g-1}$	0	0	$0 \cdots\cdots 1$
$\bar{r}_{2,0}$	0	1	$0 \cdots\cdots 0$
$\bar{r}_{2,1}$	0	0	$1 \cdots\cdots 0$
$\bar{r}_{2,2}$	0	0	$0 \ \ddots \ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots \ \ddots \ \ddots \ 1$
$\bar{r}_{2,g-1}$	T	0	$0 \cdots\cdots 0$
$\bar{r}_{3,0}$	1	-1	$0 \cdots\cdots 0$
$\bar{r}_{3,1}$	0	1	$-1 \cdots\cdots 0$
$\bar{r}_{3,2}$	0	0	$1 \ \ddots \ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots \ \ddots \ \ddots \ -1$
$\bar{r}_{3,g-1}$	$-T$	0	$0 \cdots\cdots 1$

Therefore the relation matrix of the latter is the matrix which substitute 1 and t by E and X respectively in the relation matrix of the former, where E is the $g \times g$ unit matrix and X is the $g \times g$ matrix

$$\left\| \begin{matrix} 0 & 1 \cdots 0 \\ \vdots \ \ddots & \ddots \ \vdots \\ \vdots & \ddots \ 1 \\ \vdots & \ddots \\ T \cdots\cdots & \cdots 0 \end{matrix} \right\| .$$

Since the relations of $\mathfrak{F}(S^3-\mathfrak{L})$ and $\mathfrak{F}(\mathfrak{M}_g(\mathfrak{L})-\mathfrak{L})$ have the above properties, we have easily the Alexander matrix $M'(T)$ of the $\mathfrak{F}(\mathfrak{M}_g(\mathfrak{L})-\mathfrak{L})$ as the following:

$$M'(T) = \left\| \begin{array}{c|c} A'(T) & 0 \\ \hline C' & B' \end{array} \right\| \begin{array}{l} \} \ g(2h+\mu-1) \\ \} \ 2g(2h+\mu-1) \end{array}$$
$$\underbrace{}_{g(2h+\mu-1)} \ \underbrace{}_{2g(2h+\mu-1)} \qquad ,$$

where $A'(T)$, B', C' are the submatrices substituting E and X for 1 and t of the submatrices $A(t)$, B, C of $M(t)$.

Next we compute the relation matrix of the 1-dim. homology group $H(\mathfrak{M}_g(\mathfrak{L}))$ of $\mathfrak{M}_g(\mathfrak{L})$, changing the multiplication to the addition, which is commutative, in the branch relations B_i $(i=2h+1, \cdots, 2h+\mu-1)$. Namely, we have $\tilde{B}'_i=\sum_{\beta=0}^{g-1} s'_{i,j_i,\beta}$. From $\tilde{B}'_i$ and $\tilde{A}_{i,j_i,\beta}=s_{i,j_i,\beta}-s'_{i,j_i,\beta}-a_{i,\beta}$ $(\beta=0, 1, \cdots, g-1)$ we obtain

$$\bar{B}_i = \sum_{\beta=0}^{g-1}(s_{i,j_i,\beta}-a_{i,\beta}), \qquad (i=2h+1, \cdots, 2h+\mu-1).$$

Let the $(\mu-1)\times 3(2h+\mu-1)$ matrix D be of the form on page 343, and D' the $(\mu-1)\times 3g(2h+\mu-1)$ matrix substituting by e of D the $1\times g$ matrix $(1, \cdots, 1)$, and let the $\{3g(2h+\mu-1)+\mu-1\}\times\{3g(2h+\mu-1)\}$ matrix $\bar{M}'(T)$ be the following:

$$\bar{M}'(T) = \left\| \begin{array}{c|c} A'(T) & 0 \\ \hline C' & B' \\ \hline \multicolumn{2}{c}{D'} \end{array} \right\| .$$

If we put $T=1$ in the $\bar{M}'(T)$, then the resulting matrix $\bar{M}'(1)$ is the relation matrix of the 1-dim. homology group of $\mathfrak{M}_g(\mathfrak{L})$.

6. In this section we shall investigate the 1-dim. torsion numbers of the g-fold cyclic covering space $\mathfrak{M}_g(\mathfrak{L})$ branched along $\mathfrak{L}$. In the previous section we have proved that the relation matrix of the 1-dim. homology group of $\mathfrak{M}_g(\mathfrak{L})$ is $(3g(2h+\mu-1)+\mu-1)\times(3g(2h+\mu-1))$ matrix $\bar{M}'(1)$.

By an elementary transformation on a matrix we mean one of the following transformations:

I) Multiplication ± 1 to a row (or column)

II) Multiplication of an arbitrary real number to a row (or a column), followed by the addition of this row (or column) to another row (or column).

It is easily shown that by elementary transformations on $\bar{M}'(1)$ the value of the $3g(2h+\mu-1)$ minor determinant is invariant up to the factor ± 1.

Put

$$\bar{M}(t)=\left\|\begin{array}{c|c} A(t) & 0 \\ \hline C & B \\ \hline \multicolumn{2}{c}{D} \end{array}\right\|.$$

Suppose that a matrix $\bar{N}(t)$ is obtained from $\bar{M}(t)$ by a finite number of elementary transformations, and a matrix $\bar{N}'(T)$ is obtained from $\bar{N}(t)$ substituting 1 and t by E and X. Then we can easily show that $\bar{N}'(1)$ can be obtained from $\bar{M}'(1)$ by a finite number of elementary transformations.

Now we consider the $(3(2h+\mu-1)+\mu-1)\times(3(2h+\mu-1))$ matrix $\bar{M}(t)$.

First of all, by additions of suitable multiples of the $(2h+1)$-th row of $\bar{M}(t)$ to the other rows of the submatrix $A(t)$ of $\bar{M}(t)$, all of the $(2h+1)$-th column of $A(t)$ except $(1-t)v_{2h+1,2h+1}$, which may qe described by $(1-t)\alpha_1$ from now on, may be made equal to 0. Next, by additions of suitable multiples of the $(2h+2)$-th row of $\bar{M}(t)$ to the other row of $A(t)$, all of the $(2h+2)$-the column of $A(t)$ except $(1-t)\left(v_{2h+2,2h+2}-\frac{v_{2h+2,2h+1}}{v_{2h+1,2h+1}}\right)$, which may be described by $(1-t)\alpha_2$ from now on, may be made equal to 0. By repeating this process, we can transform our matrix $\bar{M}(t)$ into the matrix whose submatrices B, C and D remain invariant and whose submatrix $A_1(t)$ corresponding to $A(t)$ satisfies that every elements of the $(2h+j)$-th column $(j=1, 2, \cdots, \mu-1)$ except for the diagonal one $(1-t)\alpha_j$ are all equal to 0.

Furthermore, by addition of suitable multiples of the $(2h+j)$-th column to the non-zero other columns of $A_1(t)$ all of the $(2h+j)$-th row except $(1-t)\alpha_j$ may be made equal to $0\,(j=1, 2, \cdots, \mu-1)$.

Then we obtain the matrix of $\bar{M}_1(t)$ whose submatrix B and C remain invariant and whose submatrix $A_2(t)$ corresponding to $A_1(t)$ satisfies that every elements in the $(2h+j)$-th row and in the $(2h+j)$-th column except for $(1-t)\alpha_j$ are all equal to $0\,(j=1, 2, \cdots, \mu-1)$ and whose submatrix D_1 corresponding to D satisfies that the n-th column $(n>2h)$ remains invariant.

Therefore $\bar{M}_1(t)$ is the following matrix:

$$\bar{M}_1(t) = \begin{array}{r} \\ 2h\left\{\begin{array}{c} \\ \\ \end{array}\right. \\ \\ \\ \\ \\ \mu-1\left\{\begin{array}{c} \\ \\ \end{array}\right. \end{array} \left\| \begin{array}{c|c|c} \overbrace{\quad * \quad}^{2h} & 0 & \\ \cline{1-2} 0 & \begin{matrix} (1-t)\alpha_1 & & 0 \\ & \ddots & \\ 0 & & (1-t)\alpha_{\mu-1} \end{matrix} & 0 \\ \hline \multicolumn{2}{c|}{C} & B \\ \hline * & \begin{matrix} -e & & 0 \\ & \ddots & \\ 0 & & -e \end{matrix} & \begin{matrix} \boxed{o\ e} & & \\ & \ddots & \\ & & \boxed{o\ e} \end{matrix} \end{array} \right\|$$

Substituting 1, t and e of the $\bar{M}_1(t)$ by E, X and $1\times g$ matrix $(1\cdots1)$, we have the following matrix $\bar{M}'_1(T)$;

$$\bar{M}'_1(T) = \left\| \begin{array}{c|c|c}
* & 0 & \\
\hline
0 & \begin{array}{ccc}
\begin{array}{|cccc|} \alpha_1 & -\alpha_1 & & 0 \\ & \ddots & \ddots & \\ 0 & & \ddots & -\alpha_1 \\ -T\alpha_1 & & & \alpha_1 \\ \hline \end{array} & & \\
& \ddots & \\
& & \begin{array}{|cccc|} \hline \alpha_{\mu-1} & -\alpha_{\mu-1} & & 0 \\ & \ddots & \ddots & \\ & & \ddots & -\alpha_{\mu-1} \\ -T\alpha_{\mu-1} & & & \alpha_{\mu-1} \end{array}
\end{array} & 0 \\
\hline
* & 0 & \begin{array}{cccc}
\begin{array}{ccc|ccc} 1 & & & -1 & & \\ & \ddots & & & \ddots & \\ & & 1 & & & -1 \end{array} & & & 0 \\
& \begin{array}{ccc|ccc} 1 & & & -1 & & \\ & \ddots & & & \ddots & \\ & & 1 & & & -1 \end{array} & & \\
& & \ddots & \\
0 & & & \begin{array}{ccc|ccc} 1 & & & -1 & & \\ & \ddots & & & \ddots & \\ & & 1 & & & -1 \\ \hline & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{array}
\end{array} \\
\hline
* & \begin{array}{ccc} -1 \cdots -1 & & \\ & \ddots & \\ & & -1 \cdots\cdots -1 \end{array} & \begin{array}{ccc} 0\cdots 0 \mid 1\cdots 1 & & \\ & \ddots & \\ & & 0\cdots 0 \mid 1\cdots 1 \end{array}
\end{array} \right\| .$$

Now consider the $g\times g$ submatrix

$$\alpha_i(E-X) = \left\| \begin{array}{cccc} \alpha_i & -\alpha_i & & 0 \\ \vdots & \alpha_i & \ddots & \\ \vdots & & \ddots & \ddots \\ \vdots & & & -\alpha_i \\ -T\alpha_i & \cdots & \cdots & \alpha_i \end{array} \right\|$$

for $i=1, 2, \cdots, \mu-1$. Add the g-th column to the $(g-1)$-th column and next add the $(g-1)$-th column to the $(g-2)$-th column, $\cdots$, and add the second column to the first column, then the submatrix is transformed to the following:

$$\begin{Vmatrix} 0 & -\alpha_i & & 0 \\ 0 & 0 & \ddots & \\ \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & 0 & -\alpha_i \\ (1-T)\alpha_i & \alpha_i \cdots\cdots \alpha_i & & \alpha_i \end{Vmatrix}.$$

Moreover, by addition of all rows, except g-th row, to the g-th row we have the following:

$$\begin{Vmatrix} 0 & -\alpha_i & & 0 \\ 0 & 0 & \ddots & \\ \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \ddots & -\alpha_i \\ (1-T)\alpha_i & 0 \cdots\cdots & \cdots\cdots & 0 \end{Vmatrix}.$$

By the above elementary transformations $\bar{M}'(T)$ is transformed to the following matrix $\bar{N}'(T)$:

$$\bar{N}'(T)=\left\Vert\begin{array}{c|c|c}
* & 0 & \\ \hline
0 & \begin{array}{cc} \begin{array}{|ccc|} 0 & -\alpha_1 & 0 \\ \vdots\ \ddots & \ddots & -\alpha_1 \\ (1-T)\alpha_1 & \cdots & 0 \end{array} & 0 \\ & \ddots \\ 0 & \begin{array}{|ccc|} 0 & -\alpha_{\mu-1} & 0 \\ \vdots\ \ddots & \ddots & -\alpha_{\mu-1} \\ (1-T)\alpha_{\mu-1} & \cdots\cdots & 0 \end{array} \end{array} & 0 \\ \hline
* & 0 & \begin{array}{cccc} \begin{array}{|c|c|} 1 \ \ddots \ 1 & -1 \ \ddots \ -1 \end{array} & & & 0 \\ & \begin{array}{|c|c|} 1 \ \ddots \ 1 & -1 \ \ddots \ -1 \end{array} & & \\ & & \ddots & \\ 0 & & & \begin{array}{|c|c|} 1 \ \ddots \ 1 & -1 \ \ddots \ -1 \\ & 1 \ \ddots \ 1 \end{array} \end{array} \\ \hline
* & \begin{array}{cc} \boxed{-g,\ -g+1,\cdots,\ -1} & 0 \\ & \ddots \\ 0 & \boxed{-g,\ -g+1,\cdots,\ -1} \end{array} & \begin{array}{cc} \boxed{0\cdots\cdots 0 \mid 1\cdots\cdots 1} & \\ & \ddots \\ & \boxed{0\cdots\cdots 0 \mid 1\cdots\cdots 1} \end{array}
\end{array}\right\Vert .$$

All of the elements of $(2hg+g)$-th row, $(2hg+2g)$-th row, $\cdots$, and $(2hg+(\mu-1)g)$-th row of $\bar{N}'(1)$, which are obtained by the substitution of T by 1 in $\bar{N}'(T)$, are equal to zero. Eliminating the $(2hg+g)$-th row, the $(2gh+2g)$-th row, $\cdots$, and the $(2hg+(\mu-1)g)$-th row, we obtain a $(3g(2h+\mu-1))\times(3g(2h+\mu-1))$ square matrix $\tilde{M}'(1)$.

Now we can easily show that

$$\det \tilde{M}'(1) = \pm g^{\mu-1} \left(\det M'(T)/\prod_{i=1}^{\mu-1} \alpha_i(1-T)\right)_{T=1}.$$

On the other hand the determinant of Alexander matrix $M(t)$ of link $\mathfrak{L}$ is equal to $\pm t^p(1-t)^{\mu-1}\nabla(t)$, where p is a suitably chosen integer, and by Lemma 1 the determinant of $M'(T)$ is equal to $\pm T^p(1-T)^{\mu-1}\prod_{j=0}^{g-1}\nabla(\omega_j \sqrt[g]{T})$, where $\omega_j\,(j=0, 1, \cdots, g-1)$ is a g-th root of unity.

And the product $\alpha_1\times\alpha_2\times\cdots\times\alpha_{\mu-1}$ is equal to $\nabla(1)$.

In fact, since the determinant of the matrix

$$\left\| \begin{matrix} v_{2h+1,2h+1} \cdots\cdots v_{2h+1,2h+\mu-1} \\ \vdots \qquad\qquad \vdots \\ v_{2h+\mu-1,2h+1} \cdots v_{2h+\mu-1,2h+\mu-1} \end{matrix} \right\|$$

is equal to $\nabla(1)$[3], it is easily seen by the process of elementary transformations of the matrix $\bar{M}(t)$ that the submatrix

$$\left\| \begin{matrix} (1-t)\alpha_1 & & 0 \\ & \ddots & \\ 0 & & (1-t)\alpha_{\mu-1} \end{matrix} \right\|$$

can be transformed from the matrix

$$\left\| \begin{matrix} (1-t)v_{2h+1,2h+1}\cdots\cdots(1-t)v_{2h+1,2h+\mu-1} \\ (1-t)v_{2h+\mu-1,2h+1}\cdots(1-t)v_{2h+\mu-1,2h+\mu-1} \end{matrix} \right\|.$$

Thus we have immediately

$$\det \tilde{M}'(1) = \pm g^{\mu-1}\prod_{j=1}^{g-1}\nabla(\omega_j).$$

Since the $\det \tilde{M}'(1)$ coincides with the product of the 1-dim torsion numbers of $\mathfrak{M}_g(\mathfrak{L})$ we have the following

Theorem 1. *Let $\mathfrak{L}$ be a link of multiplicity μ in 3-sphere and let $\mathfrak{M}_g(\mathfrak{L})$ be the g-fold cyclic covering space branched along $\mathfrak{L}$. If the 1-dimensional Betti number of $\mathfrak{M}_g(\mathfrak{L})$ is equal to zero, i.e. $\prod_{j=1}^{g-1}\nabla(\omega_j)\neq 0$, then the product of the 1-dimensional torsion numbers of $\mathfrak{M}_g(\mathfrak{L})$ is equal to*

$$g^{\mu-1}\prod_{j=1}^{g-1}\nabla(\omega_j),$$

where $\omega_1, \omega_2, \cdots, \omega_{g-1}$ *are the distinct g-th roots of unity except* 1.

7. In this section we shall consider the case that the 1-dim. Betti number of $\mathfrak{M}_g(\mathfrak{L})$ is not zero. In this case we can not yet compute the product of the 1-dim. torsion numbers of $\mathfrak{M}_g(\mathfrak{L})$, but we shall compute the 1-dim. Betti number of $\mathfrak{M}_g(\mathfrak{L})$.

Let $\bar{M}(t)$, $M(t)$, $M'(T)$, $\bar{M}'(T)$, $\bar{N}'(t)$ be the matrices in the previous sections. $\bar{M}'(T)$ is transformed to $\bar{N}'(T)$ by elemetary transformations. We transform the matrix $M'(T)$ by the same elementary transformations. Let us denote this matrix by $N'(T)$. Then $N'(T)$ the matrix obtained from $\bar{N}'(T)$ by the elimination of $3g(2h+\mu-1)+j$-th row ($j=1, 2, \cdots, \mu-1$), i.e. those rows corresponding to the branch relations.

Clearly

$$\text{rank of } \bar{M}'(1) = \text{rank of } \bar{N}'(1)$$

and

$$\text{rank of } M'(1) = \text{rank of } N'(1).$$

Since all of the $2hg+jg$-th row and $(2hg+1+(j-1)g)$-th column ($j=1, 2, \cdots, \mu-1$) of $N'(1)$ are equal to zero, the rank of $N'(1)$ is smaller than $3g(2h+\mu-1)-(\mu-1)$. But in $\bar{N}'(1)$, only one element of $(2hg+1+(j-1)g)$-th column ($j=1, 2, \cdots, \mu-1$) is not zero and this element is in the $(3g(2h+\mu-1)+j)$-th row ($j=1, 2, \cdots, \mu-1$).

Therefore, it is easy to see that

$$\text{rank of } \bar{N}'(1) = \mu-1+\text{rank of } N'(1).$$

Since the 1-dim. Betti number of $\mathfrak{M}_g(\mathfrak{L})$ is equal to the rank of $\bar{M}'(1)$, we are only to compute the rank $M'(1)$.

By the use of $M'(1)=\left\|\begin{array}{c|c} A'(1) & 0 \\ \hline C' & B' \end{array}\right\|$ and rank $(B')=2g(2h+\mu-1)$, we shall compute the rank of $A'(1)$.

By elementary transformations and multiplications of t^p to some rows (or columns), where p is an integer, $A(t)$ is transformed to the following diagonal form $A^*(t)$:

$$\left\|\begin{array}{cccc} e_1(t) & & & \\ & e_2(t) & & 0 \\ & & \ddots & \\ 0 & & & e_{2h+\mu-1}(t) \end{array}\right\|$$

where $e_1(t)$, $e_2(t)$, $\cdots$, and $e_{2h+\mu-1}(t)$ are the elementary divisors of $A(t)$. Since $\det A^*(t)=e_1(t)e_2(t)\cdots e_{2h+\mu-1}(t)=(1-t)^{\mu-1}\nabla(t)\neq 0^{(3)}$, we have

$$e_i(t)\neq 0\,, \qquad (i=1, 2, \cdots, 2h+\mu-1)\,.$$

Let $A'^*(T)$ be the $g(2h+\mu-1)\times g(2h+\mu-1)$ matrix which is obtained from $A^*(t)$ by the substitution of 1 and t by E and X. Then, it is clear that $A'^*(1)$ is transformed from $A'(1)$ by elementary transformations and

$$\text{rank of } A'(1) = \text{rank of } A'^*(1) = \sum_{i=1}^{2h+\mu-1} \text{rank of } e_i(X)_{T=1}\,,$$

where $e_i(X)$ is a $g\times g$ matrix obtained from $e_i(t)$ by the substitution of 1 and t by E and X $(i=1,2,\cdots,2h+\mu-1)$.

Let $e_i(t)=a_0^{(i)}+a_1^{(i)}t+\cdots+a_m^{(i)}t^m$ and $m=gk+l$, where k is a non-negative integer and $0\leq l<m$. Then by $(*)$ of Section 1 we have

$$e_i(X) = b_0^{(i)}E+b_1^{(i)}X+\cdots+b_{g-1}^{(i)}X^{g-1}\,,$$

where

$$\begin{aligned}
b_0^{(i)} &= a_0^{(i)}+a_g^{(i)}T+\cdots+a_{gk}^{(i)}T^k\\
&\vdots\\
b_l^{(i)} &= a_l^{(i)}+a_{g+l}^{(i)}T+\cdots+a_{gk+l}^{(i)}T^k\\
b_{l+1}^{(i)} &= a_{l+1}^{(i)}+a_{g+l+1}^{(i)}T+\cdots+a_{g(k-1)+l+1}^{(i)}T^{k-1}\\
&\vdots\\
b_{g-1}^{(i)} &= a_{g-1}^{(i)}+a_{g+g-1}^{(i)}T+\cdots+a_{g(k-1)+g-1}^{(i)}T^{k-1}
\end{aligned}$$

and

$$e_i(X)_{T=1} = \begin{Vmatrix} b_0^{(i)} & b_1^{(i)} & \cdots\cdots & b_{g-1}^{(i)} \\ b_{g-1}^{(i)} & b_0^{(i)} & \cdots\cdots & b_{g-2}^{(i)} \\ \vdots & \ddots & \ddots & \vdots \\ b_1^{(i)} & \cdots\cdots & b_{g-1}^{(i)} & b_0^{(i)} \end{Vmatrix}\,.$$

Moreover, let $1, \omega_1, \omega_2, \cdots, \omega_{g-1}$ be the g-th roots of unity and let the $g\times g$ matrix W be

$$\begin{Vmatrix} 1 & 1 & 1 & \cdots\cdots & 1 \\ 1 & \omega_1 & \omega_2 & \cdots\cdots & \omega_{g-1} \\ 1 & \omega_1^2 & \omega_2^2 & \cdots\cdots & \omega_{g-1}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_1^{g-1} & \omega_2^{g-1} & \cdots\cdots & \omega_{g-1}^{g-1} \end{Vmatrix}\,.$$

Since W has the non zero determinant, i.e. the rank of W is g, the rank of the product of matrices $e_i(X)_{T=1}\times W$ is equal to the rank of $e_i(X)_{T=1}$. Now

$$e_i(X)_{T=1}\times W=\left\|\begin{matrix}\sum b_j^{(i)} & \sum b_j^{(i)}\omega_1^j & \cdots\cdots & \sum b_j^{(i)}\omega_{g-1}^j \\ \sum b_j^{(i)} & \omega_1^{g-1}\sum b_j^{(i)}\omega_1^j & \cdots\cdots & \omega_{g-1}^{g-1}\sum b_j^{(i)}\omega_{g-1}^j \\ \vdots & \vdots & & \vdots \\ \sum b_j^{(i)} & \omega_1\sum b_j^{(i)}\omega_1^j & \cdots\cdots & \omega_{g-1}\sum b_j^{(i)}\omega_{g-1}^j\end{matrix}\right\|$$

where the summation of j runs from 0 to $g-1$.

If ω_k is a root of the equation $e_i(t)=0$, then ω_k is a root of the equation $b_0^{(i)}+b_1^{(i)}x+\cdots+b_{g-1}^{(i)}x^{g-1}=0$ and all of the k-th column of the matrix $e_i(X)_{T=1}\times W$ are equal to zero. And if ω_k is not a root of the equation $e_i(t)=0$, then all of the k-th column of the matrix $e_i(X)_{T=1}\times W$ are not equal to zero.

From these it is easy to see that, if α_i is the number of different g-th roots of unity which is also the roots of the equation $e_i(t)=0$, then we have

$$\text{rank of } \quad e_i(X)_{T=1}=g-\alpha_i\,.$$

By this equality we have

$$\text{rank of } \quad A^{*\prime}(1)=g(2h+\mu-1)-\sum_{i=1}^{2h+\mu-1}\alpha_i$$

and

$$\text{rank of } \quad M'(1)=3g(2\alpha+\mu-1)-\sum_{i=1}^{2h+\mu-1}\alpha_i\,.$$

From rank of $M'(1)+\mu-1=$ rank of $\bar{M}'(1)$ we have

$$\text{rank of } \bar{M}'(1)=3g(2h+\mu-1)-\{\sum_{i=1}^{2h+\mu-1}\alpha_i-(\mu-1)\}\,.$$

Therefore we have the following

Theorem 2. *Let $\mathfrak{L}$ be a link of multiplicity μ in 3-sphere, let $\mathfrak{M}_g(\mathfrak{L})$ be the g-fold cyclic covering space branched along $\mathfrak{L}$ and let $e_1(t), e_2(t), \cdots, e_{2h+\mu-1}(t)$ be elementary divisors of the matrix $A(t)$ of Section 5. If α_i is the number of distinct g-th roots of unity which are also the roots of the equations $e_i(t)=0$ $(i=1, 2, \cdots, 2h+\mu-1)$, then the 1-dimensional Betti number of $\mathfrak{M}_g(\mathfrak{L})$ is equal to $\sum_{i=1}^{2h+\mu-1}\alpha_i-(\mu-1)$.*

By Theorem 2 and the property of the matrix $A(t)$ we have immediately the following

Theorem 3. *Let $\nabla(t)$ be the ∇-polynomial of the link $\mathfrak{L}$ and let α be the number of the common roots of the equations $t^g-1=0$ and $\nabla(t)=0$. Then the 1-dimensional Betti number of the g-fold cyclic covering space of S^3 branched along the link $\mathfrak{L}$ is not less than α.*

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