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ALGEBRAIC SUM OF THE IMAGE SETS FOR A RANDOM STRING PROCESS

ZHENLONG CHEN

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Abstract

Let $\{u_t(x)\colon t \geq 0,\ x \in \mathbb{R}\}$ be a random string taking values in $\mathbb{R}^d$. It is specified by the following stochastic partial differential equation,

$$\frac{\partial u_t(x)}{\partial t} = \frac{\partial^2 u_t(x)}{\partial x^2} + \dot{W},$$

where $\dot{W}(x, t)$ is two-parameter white noise. The objective of the present paper is to study the fractal properties of the algebraic sum of the image sets for the random string process $\{u_t(x)\colon t \geq 0,\ x \in \mathbb{R}\}$. We obtain the Hausdorff and packing dimensions of the algebraic sum of the image sets of the string. We also consider the existence of the local times of the process $\{u_s(y) - u_t(x)\colon s, t \geq 0\colon x, y \in \mathbb{R}\}$, and find the Hausdorff and packing dimensions of the level sets for the process $\{u_s(y) - u_t(x)\colon s, t \geq 0; x, y \in \mathbb{R}\}$.

1. Introduction

Consider the following model of a random string first introduced by Funaki (1983):

$$\frac{\partial u_t(x)}{\partial t} = \frac{\partial^2 u_t(x)}{\partial x^2} + \dot{W}, \tag{1.1}$$

where $\dot{W}(x, t)$ is a space-time white noise in $\mathbb{R}^d$ and $\{u_t(x)\colon t \geq 0,\ x \in \mathbb{R}\}$ is a continuous $\mathbb{R}^d$ valued process. The components $\dot{W}_1(x, t), \dots, \dot{W}_d(x, t)$ of the vector noise $\dot{W}(x, t)$ are independent space-time white noise, which are generalized Gaussian processes with covariance given by

$$\mathbb{E}[\dot{W}_j(x, t)\dot{W}_j(y, s)] = \delta(x - y)\delta(t - s).$$

That is, for every $1 \leq j \leq d$, $W_j(f)$ is a random field indexed by functions $f \in \mathbb{L}^2([0, \infty) \times \mathbb{R})$, and for two such test functions $f, g \in \mathbb{L}^2([0, \infty) \times \mathbb{R})$ we have

$$\mathbb{E}[W_j(f)W_j(g)] = \int_0^\infty \int_{\mathbb{R}} f(t, x)g(t, x)\, dx\, dt.$$

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Therefore, $W_j(f)$ can be represented as

$$W_j(f) = \int_0^\infty \int_{\mathbb{R}} f(t, x) W_j(dx\, dt).$$

We suppose that the noise is adapted with respect to a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, where $\mathcal{F}$ is complete and the filtration $\{\mathcal{F}_t,\ t \geq 0\}$ is right continuous, in that $W_j(f)$ is $\mathcal{F}_t$-measurable whenever f is supported on $[0, t] \times \mathbb{R}$.

Recall from Mueller and Tribe (2002) that a solution of (1.1) is defined as an $\mathcal{F}_t$-adapted, continuous random field $\{u_t(x): t \geq 0,\ x \in \mathbb{R}\}$ with values in $\mathbb{R}^d$ satisfying properties:

(i) $u_0(\cdot) \in \varepsilon_{\exp}$ almost surely and is adapted to $\mathcal{F}_0$, where $\varepsilon_{\exp} = \bigcup_{\lambda>0} \varepsilon_\lambda$ and

$$\varepsilon_\lambda = \{f \in C(\mathbb{R}, \mathbb{R}^d) : |f(x)| \exp(-\lambda|x|) \to 0 \text{ as } |x| \to \infty\};$$

(ii) For every $t > 0$, there exists $\lambda > 0$ such that $u_s(\cdot) \in \varepsilon_\lambda$ for all $s \leq t$, almost surely;

(iii) For every $t > 0$ and $x \in \mathbb{R}$, the following Green's function representation holds

$$u_t(x) = \int_{\mathbb{R}} G_t(x - y) u_0(y)\, dy + \int_0^t G_{t-r}(x - y) W(dy\, dr). \tag{1.2}$$

Here $G_t(x) = (4\pi t)^{-1/2} \exp(-x^2/4t)$ is the fundamental solution of the heat equation. We call each solution $\{u_t(x): t \geq 0,\ x \in \mathbb{R}\}$ of (1.1) a random string process with values in $\mathbb{R}^d$, or simple a random string as in Mueller and Tribe (2002). Note that, whenever the initial conditions u_0 are deterministic, or are Gaussian fields independent of the driving noise, the random string processes are Gaussian.

Many authors have studied the properties of the solutions of (1.1). For example, Funaki (1983) investigated various properties of the solutions of semi-linear type stochastic partial differential equations which are more general than (1.1). In particular, his results imply that every solution $\{u_t(x): t \geq 0,\ x \in \mathbb{R}\}$ of (1.1) is Hölder continuous of any order less than $1/2$ in space and $1/4$ in time. The anisotropic property of the process $\{u_t(x): t \geq 0,\ x \in \mathbb{R}\}$ makes it a very interesting object to study. Mueller and Tribe (2002) found necessary and sufficient conditions for a random string in $\mathbb{R}^d$ to hit points or to have double points of various types. They also studied the question of recurrence and transience for $\{u_t(x): t \geq 0,\ x \in \mathbb{R}\}$. Recently Wu and Xiao (2006) have determined the dimensions of the range, graph and level sets of the random string process $\{u_t(x): t \geq 0,\ x \in \mathbb{R}\}$. Note that, in general, a random string may not be Gaussian, a powerful step in the proofs of Mueller and Tribe (2002) is to reduce the problems about a general random string process to those of the stationary pinned string $U = \{U_t(x) \colon t \geq 0,\ x \in \mathbb{R}\}$, obtained by taking the initial functions $U_0(\cdot)$ in (1.2) to be defined by

$$U_0(x) = \int_0^\infty \int (G_r(x-z) - G_r(z))\tilde{W}(dz\,dr), \tag{1.3}$$

where $\tilde{W}$ is a space-time white noise independent of the white noise $\dot{W}$. One can verify that $U_0 = \{U_0(x),\ x \in \mathbb{R}\}$ is a two-sided $\mathbb{R}^d$ valued Brownian motion satisfying $U_0(0) = 0$ and

$$\mathbb{E}[(U_0(x) - U_0(y))^2] = |x - y|.$$

We assume, by extending the probability space if needed, that U_0 is $\mathcal{F}_0$-measurable. As pointed out by Mueller and Tribe (2002), the solution to (1.1) driven by the noise $W(x, s)$ is then given by

$$\begin{aligned} U_t(x) &= \int G_t(x-z)U_0(z)\,dz + \int_0^t \int G_r(x-z)W(dz\,dr) \\ &= \int_0^\infty \int (G_{t+r}(x-z) - G_r(z))\tilde{W}(dz\,dr) + \int_0^t \int G_r(x-z)W(dz\,dr). \end{aligned} \tag{1.4}$$

A continuous version of the above solution is called *a stationary pinned string*. The components $\{U_t^j(x) \colon t \geq 0,\ x \in \mathbb{R}\}$ for $j = 1, \ldots, d$ are independent and identically distributed Gaussian processes. The stationary pinned string has following scaling property (or operator-self-similarity): For any constant $c > 0$,

$$\{c^{-1}U_{c^4 t}(c^2 x) \colon t \geq 0,\ x \in \mathbb{R}\} \stackrel{d}{=} \{U_t(x) \colon t \geq 0,\ x \in \mathbb{R}\}, \tag{1.5}$$

where $\stackrel{d}{=}$ means equality in finite dimensional distributions; see Corollary 1 in Mueller and Tribe (2002).

Now we recall briefly some basic theory of local times for the proof of the theorem in this paper. More information on local times can be found in Geman and Horowitz (1980), Ehm (1981) and Xiao and Zhang (2002).

Let $X(t)$ be a Borel vector field on $\mathbb{R}^N$ with values in $\mathbb{R}^d$. For any Borel set $T \subseteq \mathbb{R}^N$, the occupation measure of X on T is defined as the following measure on $\mathbb{R}^d$:

$$\mu_T(\bullet) = \lambda_N\{t \in T \colon X(t) \in \bullet\}.$$

If μ_T is absolutely continuous with respect to λ_d, one say that $X(t)$ has local times on T, and define its local times $l(\bullet, T)$ as the Radon-Nikodým derivative of μ_T with respect to λ_d, i.e.,

$$l(u, T) = \frac{d\mu_T}{d\lambda_d}(u), \quad u \in \mathbb{R}^d. \tag{1.6}$$

In the above, u is the so-called *space variable*, and T is the *time variable*. Sometimes, we write $l(u, T)$ in place of $l(u, [0, t])$. It is clear that if X have local times on T, then for every Borel set $E \subseteq T$, $l(u, E)$ also exists.

By standard martingale and monotone class arguments, one can deduce that the local times have a measurable modification that satisfies the following occupation density formula: for every Borel set $T \subseteq \mathbb{R}^N$, and for every measurable function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\int_T f(X(t))\,dt = \int_{\mathbb{R}^d} f(u)l(u, T)\,du. \tag{1.7}$$

For all $E, F \in \mathbb{R}_+^N$, define

$$X(E) - X(F) \mathrel{\hat{=}} \{X(s) - X(t) : s \in E,\ t \in F\}. \tag{1.8}$$

As usual, $X(E) - X(F)$ is said to the algebraic sum of the image sets on E and F for the random string process.

This paper is to study the fractal properties of algebraic sum of the image sets generated by the random string process. In Section 2, we determine the Hausdorff and packing dimensions of algebraic sum of the image sets $u([1, 2] \times [0, 1]) - u([3, 4] \times [0, 1])$. In Section 3, we consider the existence of the local times of the process $\{u_s(y) - u_t(x) : s,\ t \in [0, \infty),\ x, y \in \mathbb{R}\}$. We also obtain the Hausdorff and packing dimensions of the so-called level set $L_u = \{(s, t, x, y) : u_s(y) - u_t(x) = u,\ s, t \in [0, \infty)$ and $x, y \in \mathbb{R}\}$, where $u \in \mathbb{R}^d$.

We will use $c, c_1, c_2, \ldots,$ to denote unspecified positive finite constants whose precise values are not important and may be different in each appearance.

2. Dimension of algebraic sum of the image sets

In this section, we discuss the Hausdorff and packing dimensions of algebraic sum of the image sets $u([1, 2] \times [0, 1]) - u([3, 4] \times [0, 1])$. We refer to Falconer (1990) for the definitions and properties of Hausdorff dimension $\dim_{\mathrm{H}}(\cdot)$ and packing dimension $\dim_{\mathrm{P}}(\cdot)$.

For proving the results in this section, we need some lemmas. Lemma 1.1 below is Proposition 1 of Mueller and Tribe (2002).

Lemma 2.1. *The components $\{U_t^j(x) : t \geq 0,\ x \in \mathbb{R}\}$ of the stationary pinned string are mean zero Gaussian fields with the following covariance structure*: *for $t \geq 0$, $x, y \in \mathbb{R}$,*

$$\mathbb{E}[(U_t^j(x) - U_t^j(y))^2] = |x - y|, \tag{2.1}$$

and for all $0 \le s < t$, $x, y \in \mathbb{R}$,

$$\mathbb{E}[(U_t^j(x) - U_s^j(y))^2] = (t-s)^{1/2} F(|x-y|(t-s)^{-1/2}), \tag{2.2}$$

where

$$F(a) = \frac{1}{\sqrt{2\pi}} + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} G_1(a-z)G_1(a-z')(|z| + |z'| - |z-z'|)\, dz\, dz'.$$

$F(x)$ *is a smooth function, bounded below by* $(2\pi)^{-1/2}$, *and* $F(x)/|x| \to 1$ *as* $|x| \to \infty$. *Furthermore there exists a positive constant* $c_{2.1}$ *such that for all* $s, t \in [0, \infty)$ *and all* $x, y \in \mathbb{R}$,

$$c_{2.1}(|x-y| + |t-s|^{1/2}) \le \mathbb{E}[(U_t^j(x) - U_s^j(y))^2] \le 2(|x-y| + |t-s|^{1/2}). \tag{2.3}$$

Lemma 2.2. *Let* λ *and* L *be given positive constants with* $L < \lambda$. *Then there exist constants* $c_{2.2} > 0$ *and* $c_{2.3} > 0$, *depending on* L *and* λ *only, such that*

$$\begin{aligned} &c_{2.2}(|x_1 - x_2| + |y_1 - y_2| + |s_1 - s_2|^{1/2} + |t_1 - t_2|^{1/2}) \\ &\le \mathbb{E}[((U_{s_1}^j(y_1) - U_{t_1}^j(x_1)) - (U_{s_2}^j(y_2) - U_{t_2}^j(x_2)))^2] \\ &\le c_{2.3}(|x_1 - x_2| + |y_1 - y_2| + |s_1 - s_2|^{1/2} + |t_1 - t_2|^{1/2}), \end{aligned} \tag{2.4}$$

for all $(s_k, t_k, x_k, y_k) \in [0, \lambda] \times [0, \lambda] \times [-\lambda, \lambda] \times [-\lambda, \lambda]$ *such that* $|s_\ell - t_\flat| > L$, *where* $k, \ell, \flat \in \{1, 2\}$.

Proof. We first prove the upper bound in (2.4). Let

$$(X, Y) = (U_{s_1}^j(y_1) - U_{t_1}^j(x_1), U_{s_2}^j(y_2) - U_{t_2}^j(x_2)).$$

By Lemma 2.1, we have

$$\begin{aligned} \rho_{X,Y}^2 &\mathrel{\hat=} \mathbb{E}[((U_{s_1}^j(y_1) - U_{t_1}^j(x_1)) - (U_{s_2}^j(y_2) - U_{t_2}^j(x_2)))^2] \\ &= \mathbb{E}[(U_{s_1}^j(y_1) - U_{s_2}^j(y_2))^2] + \mathbb{E}[(U_{t_1}^j(x_1) - U_{t_2}^j(x_2))^2] \\ &\quad - 2\mathbb{E}[(U_{s_1}^j(y_1) - U_{s_2}^j(y_2))(U_{t_1}^j(x_1) - U_{t_2}^j(x_2))] \\ &\le 2\mathbb{E}[(U_{s_1}^j(y_1) - U_{s_2}^j(y_2))^2] + 2\mathbb{E}[(U_{t_1}^j(x_1) - U_{t_2}^j(x_2))^2] \\ &\le c_{2.3}(|x_1 - x_2| + |y_1 - y_2| + |s_1 - s_2|^{1/2} + |t_1 - t_2|^{1/2}). \end{aligned} \tag{2.5}$$

Using the method similar to that in Mueller and Tribe (2002), we can give a proof for the lower bound in (2.4). By using the identity

$$(a-b-c+d)^2 = (a-b)^2 + (c-d)^2 + (a-c)^2 + (b-d)^2 - (a-d)^2 - (b-c)^2$$

and (2.2), we have

$$\begin{aligned}\rho_{X,Y}^2 \mathrel{\hat{=}} \mathbb{E}[(U_{s_1}^j(y_1) - U_{s_2}^j(y_2))^2] + \mathbb{E}[(U_{t_1}^j(x_1) - U_{t_2}^j(x_2))^2] \\ + H_{s_1-t_1}(x_1-y_1) + H_{s_2-t_2}(x_2-y_2) \\ - H_{s_1-t_2}(x_2-y_1) - H_{s_2-t_1}(x_1-y_2),\end{aligned} \tag{2.6}$$

where $H_r(z) = |r|^{1/2} F(|z|\,|r|^{-1/2})$.

Note that, under the conditions of our lemma, $|s_\ell - t_\flat| > L$, where $\ell, \flat \in \{1, 2\}$. The function $H_r(z)$ is smooth for $r \in [-\lambda, -L] \cup [L, \lambda]$, $z \in [-2\lambda, 2\lambda]$. The last four terms on the right hand side of (2.6) are differences of H at the four vertices of a parallelogram. Using the mean value theorem twice, these can be expressed as a double integral of second derivatives of H over the parallelogram. Hence the algebraic sum of the last terms is bounded by the size of the second derivatives and the area of the parallelogram. Denote the algebraic sum of the last terms by S and we can deduce that there exists a constant C such that

$$S \le C(|x_1 - x_2|^2 + |y_1 - y_2|^2 + |s_1 - s_2|^2 + |t_1 - t_2|^2). \tag{2.7}$$

Using (2.3), we have

$$\begin{aligned}&\mathbb{E}[(U_{s_1}^j(y_1) - U_{s_2}^j(y_2))^2] + \mathbb{E}[(U_{t_1}^j(x_1) - U_{t_2}^j(x_2))^2] \\ &\quad \ge c_{2.1}(|x_1 - x_2| + |y_1 - y_2| + |s_1 - s_2|^{1/2} + |t_1 - t_2|^{1/2}).\end{aligned} \tag{2.8}$$

Combining (2.7) and (2.8), we find there exists $\varepsilon > 0$ such that

$$\rho_{X,Y}^2 \ge \frac{c_{2.1}}{2}(|x_1 - x_2| + |y_1 - y_2| + |s_1 - s_2|^{1/2} + |t_1 - t_2|^{1/2}), \tag{2.9}$$

whenever $(s_k, t_k, x_k, y_k) \in [0, \lambda] \times [0, \lambda] \times [-\lambda, \lambda] \times [-\lambda, \lambda]$, $k \in \{1, 2\}$ and $|x_1 - x_2| + |y_1 - y_2| + |s_1 - s_2| + |t_1 - t_2| \le \varepsilon$. Because $\rho_{X,Y}^2$ is a continuous function of $(s_k, t_k, x_k, y_k) \in [0, \lambda] \times [0, \lambda] \times [-\lambda, \lambda] \times [-\lambda, \lambda]$, $k \in \{1, 2\}$, it vanishes in this region only on $x_1 = x_2$, $y_1 = y_2$, $s_1 = s_2$, $t_1 = t_2$. Therefore, $\rho_{X,Y}^2$ is bounded below when $|x_1 - x_2| + |y_1 - y_2| + |s_1 - s_2| + |t_1 - t_2| \ge \varepsilon$. Changing the constant C if necessary, the lower bound of $\rho_{X,Y}^2$ holds without the restriction $|x_1 - x_2| + |y_1 - y_2| + |s_1 - s_2| + |t_1 - t_2| \le \varepsilon$. This completes the proof of Lemma 2.2. $\square$

Lemma 2.3. *For any constants* $0 < \gamma_1 < 1/4$, $0 < \gamma_2 < 1/4$, $0 < \gamma_3 < 1/2$ *and* $0 < \gamma_4 < 1/2$, *there exist a random variable* $A > 0$ *of finite moments of all orders and an event* Ω_1 *of probability* 1 *such that for all* $\omega \in \Omega_1$,

(2.10)
$$\sup_{(s_1,t_1,x_1,y_1),(s_2,t_2,x_2,y_2)\in R} \frac{|(U_{s_1}(y_1,\omega)-U_{t_1}(x_1,\omega))-(U_{s_2}(y_2,\omega)-U_{t_2}(x_2,\omega))|}{|s_1-s_2|^{\gamma_1}+|t_1-t_2|^{\gamma_2}+|x_1-x_2|^{\gamma_3}+|y_1-y_2|^{\gamma_4}} \le A(\omega),$$

where $R = [1,2]\times[3,4]\times[0,1]\times[0,1]$.

Proof. Because of Lemma 2.2, we can use the standard entropy for estimating the tail probabilities of the supremum of a Gaussian process to establish the modulus of continuity of the process $Z(s,t;x,y) \hat{=} \{U_s(y)-U_t(x)\colon (s,t,x,y)\in[1,2]\times[3,4]\times[0,1]\times[0,1]\}$. Hence, one can apply the method similar to that in Kôno (1975) to prove the inequality in (2.10). $\square$

In the following theorem, we obtain the Hausdorff dimension of algebraic sum of the image sets $u([1,2]\times[0,1]) - u([3,4]\times[0,1])$.

Theorem 2.4. *Let* $\{u_t(x)\colon t\ge 0,\ x\in\mathbb{R}\}$ *be a random string process taking values in* $\mathbb{R}^d$. *Then with probability* 1

$$\dim_{\mathrm{H}}(u([1,2]\times[0,1]) - u([3,4]\times[0,1])) = \min\{d; 12\}. \tag{2.11}$$

Proof. Corollary 2 of Mueller and Tribe (2002) states that the distributions of $\{u_t(x)\colon t>0,\ x\in\mathbb{R}\}$ and the stationary pinned string $U=\{U_t(x)\colon t>0,\ x\in\mathbb{R}\}$ are mutually absolutely continuous. We only need to prove (2.11) for the stationary pinned string $U=\{U_t(x)\colon t>0,\ x\in\mathbb{R}\}$. For the upper bound in (2.11), we note that clearly

$$\dim_{\mathrm{H}}(U([1,2]\times[0,1]) - U([3,4]\times[0,1])) \le d \quad \text{a.s.} \tag{2.12}$$

Hence, it is enough for us to prove the almost sure upper bound

$$\dim_{\mathrm{H}}(U([1,2]\times[0,1]) - U([3,4]\times[0,1])) \le 12. \tag{2.13}$$

Let $\omega\in\Omega_1$ be fixed and then suppressed. For any integer $n\ge 2$, we divide $[1,2]\times[3,4]\times[0,1]\times[0,1]$ into n^{12} sub-rectangles $R_{n,i}$ with sides parallel to the axes and side-lengths n^{-4}, n^{-4}, n^{-2} and n^{-2}, respectively. Then $Z(R) = U([1,2]\times[0,1]) - U([3,4]\times[0,1])$ can be covered by the sets $Z(R_{n,i})$ $(1\le i\le n^{12})$. For any constants $0<\gamma_1'<\gamma_1<1/4$, $0<\gamma_2'<\gamma_2<1/4$, $0<\gamma_3'<\gamma_3<1/2$ and $0<\gamma_4'<\gamma_4<1/2$, we use (2.10) to deduce that the diameter of the image $Z(R_{n,i})$ satisfies

$$\operatorname{diam} Z(R_{n,i}) \le c_{2.4} n^{-1+\delta}, \tag{2.14}$$

where $\delta = \max\{1-4\gamma_1,\ 1-4\gamma_2,\ 1-2\gamma_3,\ 1-2\gamma_4\}$. We choose $\gamma_1 \in (\gamma_1', 1/4)$, $\gamma_2 \in (\gamma_2', 1/4)$, $\gamma_3 \in (\gamma_3', 1/2)$ and $\gamma_4 \in (\gamma_4', 1/2)$ such that

$$(1-\delta)\left(\frac{1}{\gamma_1'} + \frac{1}{\gamma_2'} + \frac{1}{\gamma_3'} + \frac{1}{\gamma_4'}\right) > 12.$$

Hence, for $\gamma = 1/\gamma_1' + 1/\gamma_2' + 1/\gamma_3' + 1/\gamma_4'$, it follows from (2.14) that

$$\sum_{i=1}^{n^{12}} [\operatorname{diam} Z(R_{n,i})]^\gamma \le c_{2.5} n^{12} n^{-(1-\delta)\gamma} \to 0 \tag{2.15}$$

as $n \to \infty$. This implies that $\dim_{\mathrm{H}}(U([1,2]\times[0,1]) - U([3,4]\times[0,1])) \le \gamma$ a.s. By letting $\gamma_1' \uparrow 1/4$, $\gamma_2' \uparrow 1/4$, $\gamma_3' \uparrow 1/2$ and $\gamma_4' \uparrow 1/2$ along rational numbers, respectively, we derive (2.13).

To prove the lower bound in (2.11), by Frostman's theorem it is sufficient to show that for any $0 < \gamma < \min\{d, 12\}$,

$$\begin{aligned}\varepsilon_\gamma &= \int_R \int_R \mathbb{E}\left[\frac{1}{|U_{s_1}(y_1) - U_{t_1}(x_1) - U_{s_2}(y_2) + U_{t_2}(x_2)|^\gamma}\right] ds_1\, dy_1\, ds_2\, dy_2\, dt_1\, dx_1\, dt_2\, dx_2 \\ &< \infty,\end{aligned} \tag{2.16}$$

where $R = [1,2]\times[3,4]\times[0,1]\times[0,1]$. See, e.g. Kahane (1985, Chapter 10). Since $0 < \gamma < d$, we have

$$\mathbb{E}(|\Xi|^{-\gamma}) < \infty,$$

where Ξ is a standard d-dimensional normal vector. Because the components of the process $\{u_s(y) - u_t(x)\colon s, t \in [0,\infty),\ x, y \in \mathbb{R}\}$ is i.i.d., we have

$$\begin{aligned}&\mathbb{E}\left[\frac{1}{|U_{s_1}(y_1) - U_{t_1}(x_1) - U_{s_2}(y_2) + U_{t_2}(x_2)|^\gamma}\right] \\ &= \mathbb{E}\left[\sum_{j=1}^d (U_{s_1}^j(y_1) - U_{t_1}^j(x_1) - U_{s_2}^j(y_2) + U_{t_2}^j(x_2))^2\right]^{-\gamma/2} \\ &= \rho_{X,Y}^{-\gamma}\mathbb{E}\left[\sum_{j=1}^d \left(\frac{U_{s_1}^j(y_1) - U_{t_1}^j(x_1) - U_{s_2}^j(y_2) + U_{t_2}^j(x_2)}{\rho_{X,Y}}\right)^2\right]^{-\gamma/2} \\ &= \rho_{X,Y}^{-\gamma}\mathbb{E}(|\Xi|^{-\gamma}).\end{aligned} \tag{2.17}$$

Combing (2.4), (2.16) and (2.17) with a change of variables, we have

$$\begin{aligned}\varepsilon_\gamma \leq c_{2.6} \int_0^1 ds_1 \int_0^1 ds_2 \int_0^1 dt_1 \int_0^1 dt_2 \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dy_1 \\ \times \int_0^1 \frac{1}{(|x_1-x_2|+|y_1-y_2|+|s_1-s_2|^{1/2}+|t_1-t_2|^{1/2})^{\gamma/2}}\, dy_2.\end{aligned} \tag{2.18}$$

Recall the weighted arithmetic-mean and geometric-mean inequality: for all integer $n \geq 2$ and $x_i \geq 0$, $\beta_i > 0$ ($i = 1, 2, \ldots, n$) such that $\sum_{i=1}^n \beta_i = 1$, we have

$$\prod_{i=1}^n x_i^{\beta_i} \leq \sum_{i=1}^n \beta_i x_i. \tag{2.19}$$

Applying (2.19) with $n = 4$, $\beta_1 = \beta_2 = 1/6$ and $\beta_3 = \beta_4 = 2/6$, we have

$$\begin{aligned}&|x_1-x_2|+|y_1-y_2|+|s_1-s_2|^{1/2}+|t_1-t_2|^{1/2} \\ &\geq \frac{1}{6}|x_1-x_2|+\frac{1}{6}|y_1-y_2|+\frac{2}{6}|s_1-s_2|^{1/2}+\frac{2}{6}|t_1-t_2|^{1/2} \\ &\geq |x_1-x_2|^{1/6}|y_1-y_2|^{1/6}|s_1-s_2|^{1/6}|t_1-t_2|^{1/6}.\end{aligned} \tag{2.20}$$

Since $0 < \gamma < 12$, we obtain

$$\begin{aligned}\varepsilon_\gamma \leq c_{2.6} \int_0^1 ds_1 \int_0^1 \frac{1}{|s_1-s_2|^{\gamma/12}}\, ds_2 \int_0^1 dt_1 \int_0^1 \frac{1}{|t_1-t_2|^{\gamma/12}}\, dt_2 \\ \times \int_0^1 dx_1 \int_0^1 \frac{1}{|x_1-x_2|^{\gamma/12}}\, dx_2 \int_0^1 dy_1 \int_0^1 \frac{1}{|y_1-y_2|^{\gamma/12}}\, dy_2 < \infty.\end{aligned} \tag{2.21}$$

This completes the proof of Theorem 2.4. □

By using the relationships among the Hausdorff dimension, the packing dimension and the box dimension in Falconer (1990), we determine the packing dimension of algebraic sum of the image sets $u([1,2]\times[0,1]) - u([3,4]\times[0,1])$ in the following theorem.

Theorem 2.5. *Let $\{u_t(x)\colon t \geq 0,\ x \in \mathbb{R}\}$ be a random string process taking values in $\mathbb{R}^d$. Then with probability* 1

$$\dim_{\mathrm{P}}(u([1,2]\times[0,1]) - u([3,4]\times[0,1])) = \min\{d; 12\}. \tag{2.22}$$

Proof. Corollary 2 of Mueller and Tribe (2002) states that the distributions of $\{u_t(x)\colon t > 0,\ x \in \mathbb{R}\}$ and the stationary pinned string $U = \{U_t(x)\colon t > 0,\ x \in \mathbb{R}\}$ are mutually absolutely continuous. We only need to prove (2.22) for the stationary pinned string $U = \{U_t(x)\colon t > 0,\ x \in \mathbb{R}\}$.

Using the relationship between the Hausdorff dimension and the packing dimension, by Theorem 2.4 we have

$$\begin{aligned}(2.23)\qquad &\dim_{\mathrm{P}}(u([1,2]\times[0,1]) - u([3,4]\times[0,1]))\\ &\geq \dim_{\mathrm{H}}(u([1,2]\times[0,1]) - u([3,4]\times[0,1]))\\ &= \min\{d;12\}\quad \text{a.s.}\end{aligned}$$

To prove the upper bound in (2.22), it suffices to prove that

$$\dim_{\mathrm{P}}(u([1,2]\times[0,1]) - u([3,4]\times[0,1])) \leq \min\{d;12\}\quad \text{a.s.}$$

Note that clearly $\dim_{\mathrm{P}}(u([1,2]\times[0,1]) - u([3,4]\times[0,1])) \leq d$ a.s., so we only need to prove the following inequality:

$$(2.24)\qquad \dim_{\mathrm{P}}(u([1,2]\times[0,1]) - u([3,4]\times[0,1])) \leq 12\quad \text{a.s.}$$

Let $\omega \in \Omega_1$ be fixed and then suppressed. For any integer $0 < \varepsilon < 1$, we divide $[1,2]\times[3,4]\times[0,1]\times[0,1]$ into ε^{-12} sub-rectangles $R_{\varepsilon,i}$ with sides parallel to the axes and side-lengths ε^4, ε^4, ε^2 and ε^2, respectively. Then $Z(R) = U([1,2]\times[0,1]) - U([3,4]\times[0,1])$ can be covered by the sets $Z(R_{\varepsilon,i})$ $(1 \leq i \leq \varepsilon^{-12})$. For any constants $0 < \gamma_1 < 1/4$, $0 < \gamma_2 < 1/4$, $0 < \gamma_3 < 1/2$ and $0 < \gamma_4 < 1/2$, we use (2.10) to deduce that the diameter of the image $Z(R_{\varepsilon,i})$ satisfies

$$(2.25)\qquad \operatorname{diam} Z(R_{\varepsilon,i}) \leq c_{2.7}\varepsilon^{1-\delta},$$

where $\delta = \max\{1-4\gamma_1,\ 1-4\gamma_2,\ 1-2\gamma_3,\ 1-2\gamma_4\}$.

For $R = [1,2]\times[3,4]\times[0,1]\times[0,1]$, let $N(R,\varepsilon)$ denote smallest number of balls of diameter ε needed to cover R. By (2.25),

$$\begin{aligned}(2.26)\qquad N(Z(R),\varepsilon) &\leq \sum_{i=1}^{\varepsilon^{-12}} N(Z(R_{\varepsilon,i}),\varepsilon)\\ &\leq \varepsilon^{-12}\left(\frac{c_{2.7}\varepsilon^{1-\delta}}{\varepsilon}\right)^d\\ &= c_{2.8}\varepsilon^{-12-\delta d}.\end{aligned}$$

Therefore,

$$\begin{aligned}(2.27)\qquad \triangle(Z(R)) &= \limsup_{\varepsilon\to 0}\frac{\log N(\varepsilon, Z(R))}{-\log\varepsilon}\\ &\leq \limsup_{\varepsilon\to 0}\frac{\log(c_{2.8}\varepsilon^{-12-\delta d})}{-\log\varepsilon}\\ &= 12+\delta d,\end{aligned}$$

where $\triangle(Z(R))$ denotes the Kolmogorov's upper index of $Z(R)$. By letting $\gamma_1 \uparrow 1/4$, $\gamma_2 \uparrow 1/4$, $\gamma_3 \uparrow 1/2$ and $\gamma_4 \uparrow 1/2$ along rational numbers, respectively, we can obtain (2.24). So we complete the proof of Theorem 2.5. $\square$

3. Existence of local times and dimension of the level sets

In this section, we will first consider the existence of the local times of the process $\{u_s(y) - u_t(x)\colon s, t \in [0, \infty),\ x, y \in \mathbb{R}\}$. Then, we discuss the Hausdorff and packing dimensions for the so-called level set $L_u = \{(s, t, x, y) \in [0, \infty) \times [0, \infty) \times \mathbb{R} \times \mathbb{R}\colon u_s(y) - u_t(x) = u\}$, where $u \in \mathbb{R}^d$ is fixed.

For proving the results in this section, we need the following lemmas. Lemma 3.1 below is implied by the proof of Lemma 4 in Mueller and Tribe (2002, p.21).

Lemma 3.1. *For any given constants* $0 < \lambda < 1$ *and* $L > 0$, *there exists a constant* $c_{3.1} > 0$, *depending on* L *and* λ *only, such that*

$$\begin{aligned} &\mathrm{Var}(U^j_{s_1}(y_1) - U^j_{t_1}(x_1) \mid U^j_{s_2}(y_2) - U^j_{t_2}(x_2)) \\ &\geq c_{3.1}(|x_1 - x_2| + |y_1 - y_2| + |s_1 - s_2|^{1/2} + |t_1 - t_2|^{1/2}) \end{aligned} \tag{3.1}$$

for all $(s_k, t_k, x_k, y_k) \in [\lambda, \lambda^{-1}] \times [\lambda, \lambda^{-1}] \times [-\lambda^{-1}, \lambda^{-1}] \times [-\lambda^{-1}, \lambda^{-1}]$ *such that* $|s_\ell - t_\flat| > L$, *where* $k, \ell, \flat \in \{1, 2\}$.

Note that in Lemma 3.1, the pairs s_ℓ and $t_\flat$, where $\ell, \flat \in \{1, 2\}$, are well separated. The following lemma is concerned with the case when $s_1 = t_1$, $s_2 = t_2$. By the same method as in proving Lemma 4 in Mueller and Tribe (2002, p.21), we can obtain the following lemma.

Lemma 3.2. *For any given constants* $0 < \lambda < 1$ *and* $L > 0$, *there exist constants* $h_0 \in (0, L/2)$ *and* $c_{3.2} > 0$, *depending on* L *and* λ *only, such that*

$$\begin{aligned} &\mathrm{Var}(U^j_t(x_2) - U^j_t(x_1) \mid U^j_s(y_2) - U^j_s(y_1)) \\ &\geq c_{3.2}(|x_1 - y_1| + |x_2 - y_2| + |t - s|^{1/2}) \end{aligned} \tag{3.2}$$

for all $s, t \in [\lambda, \lambda^{-1}]$ *with* $|s - t| \leq h_0$ *and all* $(x_k, y_k) \in [-\lambda^{-1}, \lambda^{-1}]$, *where* $k \in \{1, 2\}$, *such that* $|x_2 - x_1| \geq L$, $|y_2 - y_1| \geq L$ *and* $|x_k - y_k| \leq L/2$ *for* $k = 1, 2$.

The lemma below will be used to derive a lower bound in the proof of Theorem 3.6.

Lemma 3.3. *Let* α, β, η *and* $b > 0$ *be positive constants. For* $A > 0$ *and* $B > 0$, *let*

$$J \triangleq \int_0^b \frac{dt}{(A + t^\alpha)^\beta (B + t)^\eta}. \tag{3.3}$$

Then there exist positive and finite constants $c_{3.3}$ *and* $c_{3.4}$, *depending on* α, β, η *and* b *only, such that the following hold for all reals* A, $B > 0$ *satisfying* $A^{1/\alpha} \le c_{3.3} B$:

(i) *if* $\alpha\beta > 1$, *then*

$$J \le c_{3.4} \frac{1}{A^{\beta-\alpha^{-1}} B^{\eta}}; \tag{3.4}$$

(ii) *if* $\alpha\beta = 1$, *then*

$$J \le c_{3.4} \frac{1}{B^{\eta}} \log(1 + BA^{-1/\alpha}); \tag{3.5}$$

(iii) *if* $0 < \alpha\beta < 1$ *and* $\alpha\beta + \eta \ne 1$, *then*

$$J \le c_{3.4} \left(\frac{1}{B^{\alpha\beta+\eta-1}} + 1 \right). \tag{3.6}$$

Proof. If $b \le 1$, by using (3.3) and Lemma 10 in [2, p.430], we can prove (3.4)–(3.6). If $b > 1$, then we can split the integral in (3.3) such that

$$J = \int_0^1 \frac{dt}{(A + t^{\alpha})^{\beta} (B + t)^{\eta}} + \int_1^b \frac{dt}{(A + t^{\alpha})^{\beta} (B + t)^{\eta}}. \tag{3.7}$$

By changing the variable of the second term with $s = t/b$ in (3.7) and using again Lemma 10 in [2, p.430], we get

$$\begin{aligned} \int_1^b \frac{dt}{(A + t^{\alpha})^{\beta} (B + t)^{\eta}} &= b \int_{1/b}^1 \frac{ds}{(A + (bs)^{\alpha})^{\beta} (B + bs)^{\eta}} \\ &\le b^{1-\alpha\beta-\eta} \int_0^1 \frac{ds}{(Ab^{-\alpha} + s^{\alpha})^{\beta} (Bb^{-1} + s)^{\eta}} \\ &\le c_{3.5} b^{2(1-\alpha\beta-\eta)} \frac{1}{A^{\beta-\alpha^{-1}} B^{\eta}}. \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), we finish the proof of (3.4).

By using Lemma 10 in [2, p.430] and a similar argument as in the proof of (3.4), we can also prove (3.5) and (3.6). □

Lemma 3.4. *For any* $b > 0$, $\gamma > 0$ *and* $1 \le d < 12$, *let*

$$\Lambda(b, \gamma, d) = \int_0^b dx \int_0^b dy \int_0^b ds \int_0^b \frac{1}{(x + y + s^{1/2} + t^{1/2})^{d/2} (x + y + s + t)^{\gamma}} \, dt. \tag{3.9}$$

Then there exist positive and finite constants $c_{3.6}$, $c_{3.7}$, *depending on* b, γ *and* d *only, and* $\delta_0 > 0$ *small enough, such that the following hold for any* $\delta \in (0, \delta_0)$:

(i) *if* $1 \le d < 8$ *and* $\gamma = 4 - (1/4)(1+\delta)d$, *then* $\Lambda(b, \gamma, d) \le c_{3.6}$,
(ii) *if* $8 \le d < 12$ *and* $\gamma = 6 - (1/2)(1+\delta)d$, *then* $\Lambda(b, \gamma, d) \le c_{3.7}$.

Proof. In order to prove the above results, we need consider five cases: $1 \le d < 4$, $d = 4$, $4 < d < 8$, $d = 8$ and $8 < d < 12$, respectively.
(1) If $1 \le d < 4$, applying (3.6) of Lemma 3.3 with $\alpha = 1/2$, $\beta = d/2$, $\eta = \gamma$, $A = x + y + s^{1/2}$ and $B = x + y + s$, we can choose $\delta > 0$ small enough such that $0 < \alpha\beta < 1$ and $\alpha\beta + \eta = 4 - (1/4)\delta d \neq 1$. We integrate $[dt]$ first to get

$$\Lambda(b, \gamma, d) \le c_{3.8} \int_0^b dx \int_0^b dy \int_0^b \frac{1}{(x+y+s)^{d/4+\gamma-1}}\, ds \triangleq c_{3.6}, \tag{3.10}$$

since $d/4 + \gamma - 1 < 3$.
(2) If $d = 4$, applying Lemma 3.3 with $\alpha = 1/2$, $\beta = 2$, $\eta = \gamma$, $A = x + y + s^{1/2}$ and $B = x + y + s$, we have $\alpha\beta = 1$. We integrate $[dt]$ and use (3.5) to get

$$\begin{aligned} \Lambda(b, \gamma, d) &\le c_{3.8} \int_0^b dx \int_0^b dy \int_0^b \frac{1}{(x+y+s)^{\gamma}} \log\left(1 + \frac{x+y+s}{(x+y+s^{1/2})^2}\right) ds \\ &\le c_{3.8} \int_0^b dx \int_0^b dy \int_0^b \frac{1}{(x+y+s)^{\gamma}} \log\left(1 + \frac{1}{x+y+s}\right) ds \\ &\triangleq c_{3.6}, \end{aligned} \tag{3.11}$$

since $\gamma = 4 - (1/4)(1+\delta)d = 3 - \delta < 3$.
(3) If $4 < d < 8$, we integrate $[dt]$ first. Since $\alpha\beta = d/4 > 1$, then we can use (3.4) to get

$$\Lambda(b, \gamma, d) \le c_{3.8} \int_0^b dx \int_0^b dy \int_0^b \frac{1}{(x+y+s)^{\gamma}(x+y+s^{1/2})^{d/2-2}}\, ds. \tag{3.12}$$

Note that $0 < \alpha\beta = (1/2) \times (d/2 - 2) < 1$ and $\alpha\beta + \gamma - 1 = 2 - (1/4)\delta d \neq 0$ in (3.12), then we can use (3.6) again to deduce that

$$\Lambda(\delta, \gamma, d) \le c_{3.9} \int_0^b dx \int_0^b \frac{1}{(x+y)^{2-\delta d/4}}\, dy \triangleq c_{3.6}, \tag{3.13}$$

since $2 - \delta d/4 < 2$.
(4) If $d = 8$, then we apply (3.4) of Lemma 3.3 with $\alpha\beta = d/4 = 2$ and $\gamma = 6 - (1/2)(1+\delta)d$ to get

$$\Lambda(b, \gamma, d) \le c_{3.10} \int_0^b dx \int_0^b dy \int_0^b \frac{1}{(x+y+s)^{\gamma}(x+y+s^{1/2})^{2}}\, ds. \tag{3.14}$$

Note that $\alpha\beta = 1$ in (3.14), then we can use (3.5) again to deduce that

$$\Lambda(\delta, \gamma, d) \le c_{3.11} \int_0^b dx \int_0^b \frac{1}{(x+y)^\gamma} \log\left(1 + \frac{1}{x+y}\right) dy \mathrel{\hat{=}} c_{3.7}, \tag{3.15}$$

since $\gamma = 2 - 4\delta < 2$.

(5) If $8 < d < 12$, we integrate $[dt]$ first. Since $\alpha\beta > 2$ in (3.9), then we can use (3.4) to get

$$\Lambda(b, \gamma, d) \le c_{3.12} \int_0^b dx \int_0^b dy \int_0^b \frac{1}{(x+y+s)^\gamma (x+y+s^{1/2})^{d/2-2}}\, ds. \tag{3.16}$$

Note that $\alpha\beta = d/4 - 1 > 1$ in (3.16), then we can use (3.4) again to deduce that

$$\Lambda(\delta, \gamma, d) \le c_{3.13} \int_0^b dx \int_0^b \frac{1}{(x+y)^{d/2+\gamma-4}}\, dy \mathrel{\hat{=}} c_{3.7}, \tag{3.17}$$

since $d/2 + \gamma - 4 = 2 - (1/2)\delta d < 2$. Combining (3.10) through (3.17), we finish the proof of Lemma 3.4. $\square$

For any constants $0 < a_1 < a_2$ and $b_1 < b_2$, we choose $h > 0$ small enough, say,

$$0 < h < \frac{1}{3}(a_2 - a_1) \equiv L.$$

Let $I = [a_1, a_1 + h] \times [a_2, a_2 + h] \times [b_1, b_1 + h] \times [b_2, b_2 + h] \subset (0, \infty)^2 \times \mathbb{R}^2$ denote the corresponding hypercube. We denote the collection of the hypercube having the above properties by $\mathcal{A}$. The following theorem is concerned with the existence of the local times of the process $\{u_t(x) - u_s(y) \colon s,\ t \in [0, \infty),\ x, y \in \mathbb{R}\}$ on any hypercube $I \in \mathcal{A}$.

Theorem 3.5. *Let $\{u_t(x) \colon t \ge 0,\ x \in \mathbb{R}\}$ be a random string process in $\mathbb{R}^d$. If $d < 12$, then for every $I \in \mathcal{A}$, the process $\{u_s(y) - u_t(x) \colon s, t \in [0, \infty),\ x, y \in \mathbb{R}\}$ has local times $\{l(u, I),\ u \in \mathbb{R}^d\}$ on any hypercube I, and $l(u, I)$ admits the following $\mathbb{L}^2$ representation*:

$$l(u, I) = (2\pi)^{-d} \int_{\mathbb{R}^d} \exp(-i\langle v, u\rangle) \int_I \exp(i\langle v, u_s(y) - u_t(x)\rangle)\, ds\, dt\, dx\, dy\, dv, \quad \forall u \in \mathbb{R}^d, \tag{3.18}$$

where $l(u, I)$ is defined in (1.6).

Proof. By Corollary 2 of Mueller and Tribe (2002), we only need to prove that $l(u, I)$ admits the above $\mathbb{L}^2$ representation in (3.18) for the stationary pinned string $U = \{U_t(x) \colon t \ge 0,\ x \in \mathbb{R}\}$.

Let $I \in \mathcal{A}$ be fixed. Without loss of generality, we may assume $I = [a_1, a_1 + h] \times [a_2, a_2 + h] \times [b_1, b_1 + h] \times [b_2, b_2 + h]$. By (2.13) in Geman and Horowitz (1980) and using the characteristic functions of Gaussian random variables, it suffices to prove

$$\begin{aligned}\mathcal{J}(I) &\hat{=} \int_I ds_1\, dt_1\, dx_1\, dy_1 \int_I ds_2\, dt_2\, dx_2\, dy_2 \int_{\mathbb{R}^d} du \\ &\quad \times \int_{\mathbb{R}^d} |\mathbb{E} \exp(i\langle u, U_{s_1}(y_1) - U_{t_1}(x_1)\rangle + i\langle v, U_{s_2}(y_2) - U_{t_2}(x_2)\rangle|\, dv \\ &< \infty.\end{aligned} \tag{3.19}$$

Since the components of U are i.i.d., it is easy to deduce that

$$\begin{aligned}\mathcal{J}(I) &= (2\pi)^d \int_I ds_1 dt_1 dx_1 dy_1 \\ &\quad \times \int_I [\det \operatorname{Cov}(U_{s_1}(y_1) - U_{t_1}(x_1), U_{s_2}(y_2) - U_{t_2}(x_2))]^{-d/2}\, ds_2\, dt_2\, dx_2\, dy_2.\end{aligned} \tag{3.20}$$

For any $(s_k, t_k, x_k, y_k) \in I = [a_1, a_1 + h] \times [a_2, a_2 + h] \times [b_1, b_1 + h] \times [b_2, b_2 + h]$ $(k = 1, 2)$, we have $|s_\ell - t_\flat| > L$, $\ell, \flat \in \{1, 2\}$. By Lemma 2.1 and Lemma 3.1, we obtain

$$\begin{aligned}&\det \operatorname{Cov}(U_{s_1}(y_1) - U_{t_1}(x_1), U_{s_2}(y_2) - U_{t_2}(x_2)) \\ &= \operatorname{Var}(U_{s_1}(y_1) - U_{t_1}(x_1)) \operatorname{Var}(U_{s_2}(y_2) - U_{t_2}(x_2) \mid U_{s_1}(y_1) - U_{t_1}(x_1)) \\ &\geq c_{3.1} L^{1/2} (2\pi)^{-1/2} (|x_1 - x_2| + |y_1 - y_2| + |s_1 - s_2|^{1/2} + |t_1 - t_2|^{1/2}).\end{aligned} \tag{3.21}$$

Applying (2.19) with $n = 4$, $\beta_1 = \beta_2 = 1/6$ and $\beta_3 = \beta_4 = 2/6$, we have

$$\begin{aligned}&|x_1 - x_2| + |y_1 - y_2| + |s_1 - s_2|^{1/2} + |t_1 - t_2|^{1/2} \\ &\geq \frac{1}{6}|x_1 - x_2| + \frac{1}{6}|y_1 - y_2| + \frac{2}{6}|s_1 - s_2|^{1/2} + \frac{2}{6}|t_1 - t_2|^{1/2} \\ &\geq |x_1 - x_2|^{1/6} |y_1 - y_2|^{1/6} |s_1 - s_2|^{1/6} |t_1 - t_2|^{1/6}.\end{aligned} \tag{3.22}$$

Combining (3.20), (3.21) and (3.22), we obtain

$$\begin{aligned}\mathcal{J}(I) &\leq c_{3.14} \int_{a_1}^{a_1+h} ds_1 \int_{a_1}^{a_1+h} \frac{1}{|s_1 - s_2|^{d/12}}\, ds_2 \int_{a_2}^{a_2+h} dt_1 \int_{a_2}^{a_2+h} \frac{1}{|t_1 - t_2|^{d/12}}\, dt_2 \\ &\quad \times \int_{b_1}^{b_1+h} dx_1 \int_{b_1}^{b_1+h} \frac{1}{|x_1 - x_2|^{d/12}}\, dx_2 \int_{b_2}^{b_2+h} dy_1 \int_{b_2}^{b_2+h} \frac{1}{|y_1 - y_2|^{d/12}}\, dy_2 \\ &< \infty,\end{aligned} \tag{3.23}$$

since $d < 12$. This completes the proof of Theorem 3.5. □

Mueller and Tribe (2002) proved that for every $u \in \mathbb{R}^d$,

$$\mathbb{P}\{u_t(x) = u \text{ for some } (t, x) \in [0, \infty) \times \mathbb{R}\} > 0 \tag{3.24}$$

if and only if $d < 6$. Some related results for certain Gaussian random fields be found in Xiao (1999) and Wu and Xiao (2006, 2007).

Now we consider the Hausdorff and packing dimensions for the so-called level set $L_u = \{(s, t, x, y) \in [0, \infty) \times [0, \infty) \times \mathbb{R} \times \mathbb{R} : u_s(y) - u_t(x) = u\}$.

Theorem 3.6. *Let $\{u_t(x) : t \geq 0, x \in \mathbb{R}\}$ be a random string process in $\mathbb{R}^d$ with $d < 12$. Then for every $u \in \mathbb{R}^d$, with positive probability,*

$$\dim_{\mathrm{H}}(L_u \cap R) = \dim_{\mathrm{P}}(L_u \cap R) = \begin{cases} 4 - \dfrac{1}{4}d, & \text{if } 1 \leq d < 8, \\ 6 - \dfrac{1}{2}d, & \text{if } 8 \leq d < 12, \end{cases} \tag{3.25}$$

where $R = [0, 1] \times [2, 3] \times [0, 1] \times [0, 1]$.

Proof. By Corollary 2 of Mueller and Tribe (2002), we only need to prove (3.25) for the stationary pinned string $U = \{U_t(x) : t > 0, x \in \mathbb{R}\}$. By the σ-stability of $\dim_{\mathrm{P}}$, it is sufficient to show (3.25) holds for $L_u \cap R_\varepsilon \triangleq L_u \cap [\varepsilon, 1] \times [2 + \varepsilon, 3] \times [\varepsilon, 1] \times [\varepsilon, 1]$ for every $\varepsilon \in (0, 1)$. We first prove the almost sure upper bound

$$\dim_{\mathrm{P}}(L_u \cap R_\varepsilon) \leq \begin{cases} 4 - \dfrac{1}{4}d, & \text{if } 1 \leq d < 8, \\ 6 - \dfrac{1}{2}d, & \text{if } 8 \leq d < 12. \end{cases} \tag{3.26}$$

For this purpose, we construct coverings of $L_u \cap R_\varepsilon$ by cubes of the same side length.

For any integer $n \geq 2$, we divide R_ε into n^{12} sub-domain $T_{n,\ell} = R^1_{n,\ell} \times R^2_{n,\ell}$, where $R^1_{n,\ell}, R^2_{n,\ell} \subset (0, \infty) \times \mathbb{R}$ are rectangles of side lengths $n^{-4}(1 - \varepsilon)$ and $n^{-2}(1 - \varepsilon)$, respectively. Let $0 < \delta < 1$ be fixed and let $\tau^k_{n,\ell}$ be the lower-left vertex of $R^k_{n,\ell}$ $(k = 1, 2)$. Then the probability $\mathbb{P}\{u \in Z(T_{n,\ell})\}$ is at most

$$\begin{aligned}
&\mathbb{P}\Big\{\max_{(s_1,t_1,x_1,y_1)\in T_{n,\ell},(s_2,t_2,x_2,y_2)\in T_{n,\ell}} |Z(s_1,t_1,x_1,y_1) - Z(s_2,t_2,x_2,y_2)| \leq n^{-(1-\delta)}; u \in Z(T_{n,\ell})\Big\} \\
&+ \mathbb{P}\Big\{\max_{(s_1,t_1,x_1,y_1)\in T_{n,\ell},(s_2,t_2,x_2,y_2)\in T_{n,\ell}} |Z(s_1,t_1,x_1,y_1) - Z(s_2,t_2,x_2,y_2)| > n^{-(1-\delta)}\Big\} \\
&\leq \mathbb{P}\{|Z(\tau^1_{n,\ell}; \tau^2_{n,\ell}) - u| \leq n^{-(1-\delta)}\} \\
&\quad + \mathbb{P}\Big\{\max_{(s_1,t_1,x_1,y_1)\in T_{n,\ell},(s_2,t_2,x_2,y_2)\in T_{n,\ell}} |Z(s_1,t_1,x_1,y_1) - Z(s_2,t_2,x_2,y_2)| > n^{-(1-\delta)}\Big\},
\end{aligned} \tag{3.27}$$

where $Z(s_i, t_i; x_i, y_i) \hat{=} \{U_{s_i}(y_i) - U_{t_i}(x_i)$: $(s_i, t_i, x_i, y_i) \in T_{n,\ell},\ i = 1, 2\}$. For any $(s_k, t_k, x_k, y_k) \in R_\varepsilon = [\varepsilon, 1] \times [2+\varepsilon, 3] \times [\varepsilon, 1] \times [\varepsilon, 1]$ $(k = 1, 2)$, we have $|s_\ell - t_\flat| > 1 + \varepsilon$, $\ell, \flat \in \{1, 2\}$. By Lemma 2.1 and Lemma 3.1, we see that $Z(\tau^1_{n,\ell}; \tau^2_{n,\ell})$ is the Gaussian random variable with mean 0 and variance at least $c(1+\varepsilon)^{1/2}$. Hence,

$$\mathbb{P}\{|Z(\tau^1_{n,\ell}; \tau^2_{n,\ell}) - u| \leq n^{-(1-\delta)}\} \leq c_{3.15} n^{-(1-\delta)d}. \tag{3.28}$$

On the other hand, since

$$|Z(s_1, t_1, x_1, y_1) - Z(s_2, t_2, x_2, y_2)| \leq |U_{s_1}(y_1) - U_{s_2}(y_2)| + |U_{t_1}(x_1) - U_{t_2}(x_2)|,$$

we have

$$\begin{aligned}
&\mathbb{P}\left\{\max_{(s_1,t_1,x_1,y_1)\in T_{n,\ell},(s_2,t_2,x_2,y_2)\in T_{n,\ell}} |Z(s_1, t_1, x_1, y_1) - Z(s_2, t_2, x_2, y_2)| > n^{-(1-\delta)}\right\} \\
&\leq \mathbb{P}\left\{\max_{(s_1,y_1),(s_2,y_2)\in R^1_{n,\ell}} |U_{s_1}(y_1) - U_{s_2}(y_2)| > \frac{n^{-(1-\delta)}}{2}\right\} \\
&\quad + \mathbb{P}\left\{\max_{(t_1,x_1),(t_2,x_2)\in R^2_{n,\ell}} |U_{t_1}(x_1) - U_{t_2}(x_2)| > \frac{n^{-(1-\delta)}}{2}\right\} \\
&\leq \exp(-c_{3.16} n^{2\delta}),
\end{aligned} \tag{3.29}$$

where the last inequality follows from Lemma 2.1 and the Gaussian isoperimetric inequality of Lemma 2.1 in Talagrand (1995).

By (3.28) and (3.29), we have

$$\begin{aligned}
\mathbb{P}\{u \in Z(T_{n,\ell})\} &\leq c_{3.15} n^{-(1-\delta)d} + \exp(-c_{3.16} n^{2\delta}) \\
&\leq c_{3.17} n^{-(1-\delta)d}.
\end{aligned} \tag{3.30}$$

Define a covering $\{T'_{n,\ell}\}$ of $L_u \cap R$ by $T'_{n,\ell} = T_{n,\ell}$ if $u \in Z(T_{n,\ell})$ and $T'_{n,\ell} = \emptyset$ otherwise.

- Note that each $T'_{n,\ell}$ can be covered by n^4 cubes of side length $n^{-4}(1-\varepsilon)$. Therefore, for every $n \geq 2$, we have obtained a covering of the set $L_u \cap R$ by cubes of side length $n^{-4}(1-\varepsilon)$. Consider the sequence of integers $n = 2^k$ $(k \geq 1)$ and let N_k denote the minimum number of cubes of side length $2^{-4k}(1-\varepsilon)$ that are needed to cover $L_u \cap R$. It follows from (3.30) that

$$\mathbb{E}(N_k) \leq c_{3.17} 2^{12k} 2^{4k} 2^{-k(1-\delta)d} = c_{3.17} 2^{k(16-(1-\delta)d)}. \tag{3.31}$$

By (3.31), Markov's inequality and the Borel-Cantelli lemma, we deduce that for any $\delta' \in (0, \delta)$, almost surely for k large enough,

$$N_k \leq c_{3.17} 2^{k(16-(1-\delta')d)}. \tag{3.32}$$

• Observe that each $T'_{n,\ell}$ can also be covered by 1 cubes of side length $n^{-2}(1-\varepsilon)$. Therefore, for every $n \geq 2$, we also obtain a covering of the set $L_u \cap R$ by cubes of side length $n^{-2}(1-\varepsilon)$. Consider the sequence of integers $n = 2^k$ ($k \geq 1$) and let N_k denote the minimum number of cubes of side length $2^{-2k}(1-\varepsilon)$ that are needed to cover $L_u \cap R$. It follows from (3.30) that

$$\mathbb{E}(N_k) \leq c_{3.17} 2^{12k} 2^{-k(1-\delta)d} = c_{3.17} 2^{k(12-(1-\delta)d)}. \tag{3.33}$$

By (3.33), Markov's inequality and the Borel-Cantelli lemma, we also deduce that for any $\delta' \in (0, \delta)$, almost surely for k large enough,

$$N_k \leq c_{3.17} 2^{k(12-(1-\delta')d)}. \tag{3.34}$$

By using the relationship between the packing dimension and the box dimension, (3.32) and (3.34) imply that

$$\dim_{\mathrm{P}}(L_u \cap R_\varepsilon) \leq \min\left\{4 - \frac{1}{4}d,\, 6 - \frac{1}{2}d\right\} = \begin{cases} 4 - \dfrac{1}{4}d, & \text{if } \quad 1 \leq d < 8, \\ 6 - \dfrac{1}{2}d, & \text{if } \quad 8 \leq d < 12, \end{cases} \quad \text{a.s.} \tag{3.35}$$

Since $\varepsilon > 0$ is arbitrary, we obtain the desired upper bound for $\dim_{\mathrm{P}}(L_u \cap R)$.

Because of the fact that $\dim_{\mathrm{H}}(E) \leq \dim_{\mathrm{P}}(E)$ for all Borel set $E \subset \mathbb{R}^4$, it remains to show the following lower bound: for any $\varepsilon \in (0, 1)$, with positive probability

$$\dim_{\mathrm{H}}(L_u \cap R_\varepsilon) \geq \begin{cases} 4 - \dfrac{1}{4}d, & \text{if } \quad 1 \leq d < 8, \\ 6 - \dfrac{1}{2}d, & \text{if } \quad 8 \leq d < 12. \end{cases} \tag{3.36}$$

We only prove (3.36) for the case $1 \leq d < 8$. The other case $8 \leq d < 12$ is similar and is omitted. Let $\delta > 0$ such that

$$\gamma \mathrel{\hat{=}} 4 - \frac{1}{4}(1+\delta)d > 2.$$

Note that if we can prove that there exists a constant $c_{3.18} > 0$ such that

$$\mathbb{P}\{\dim_{\mathrm{H}}(L_u \cap R_\varepsilon) \geq \gamma\} \geq c_{3.18}, \tag{3.37}$$

then the lower bound in (3.36) will follow by letting $\delta \downarrow 0$. Our proof of (3.36) is based on the capacity argument due to Kahane (1985).

Let $\mathcal{M}_\gamma^+$ be the space of all non-negative measures on R with γ-energy. It is known due to Adler (1981) that $\mathcal{M}_\gamma^+$ is a complete metric space under the metric

$$\|\mu\|_\gamma = \int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \frac{\mu(ds_1\, dy_1\, dt_1\, dx_1)\mu(ds_2\, dy_2\, dt_2\, dx_2)}{(|s_1-s_2|^2 + |y_1-y_2|^2 + |t_1-t_2|^2 + |x_1-x_2|^2)^{\gamma/2}}. \tag{3.38}$$

We define a sequence of random positive measure μ_n on the Borel set R_ε by

(3.39)
$$\begin{aligned}\mu_n(C) &= \int_C (2\pi n)^{d/2} \exp\left(-\frac{n|(U_s(y)-U_t(x))-u|^2}{2}\right) ds\, dt\, dx\, dy\\ &= \int_C \int_{\mathbb{R}^d} \exp\left(-\frac{|\xi|^2}{2n} + i\langle \xi, U_s(y)-U_t(x)-u\rangle\right) ds\, dt\, dx\, dy, \quad \forall C \in \mathcal{B}(R_\varepsilon).\end{aligned}$$

It follows from Kahane (1985) or Testard (1986) that if there are positive constants $c_{3.19}$ and $c_{3.20}$, which depend on u, such that

$$\mathbb{E}(\|\mu_n\|) \ge c_{3.19}, \quad \mathbb{E}(\|\mu_n\|^2) \le c_{3.20}, \quad \mathbb{E}(\|\mu_n\|_\gamma) < \infty, \tag{3.40}$$

where $\|\mu_n\| = \mu_n(R_\varepsilon)$, then there is a subsequence of $\{\mu_n\}$, say, $\{\mu_{n_k}\}$, such that $\mu_{n_k} \to \mu$ in $\mathcal{M}_\gamma^+$ and μ is strictly positive with probability $\ge c_{3.19}^2/(2c_{3.20})$. It follows from (3.39) and the continuity of the process $\{U_s(y)-U_t(x)\colon s, t \in [0,\infty),\ x, y \in \mathbb{R}\}$ that μ has its support in $L_u \cap R_\varepsilon$ almost surely. Hence Frostman's theorem yields (3.37). We start the proof with the first inequality in (3.40). By Fubini's theorem and Lemma 2.1, we have

$$\begin{aligned}\mathbb{E}(\|\mu_n\|) &= \int_{R_\varepsilon}\int_{\mathbb{R}^d} \exp\left(-\frac{|\xi|^2}{2n}\right)\mathbb{E}[\exp(i\langle \xi, (U_s(y)-U_t(x))-u\rangle)]\, d\xi\, ds\, dt\, dx\, dy\\ &= \int_{R_\varepsilon}\int_{\mathbb{R}^d} \exp(-i\langle \xi, u\rangle)\\ &\qquad \times \exp\left(-\frac{1}{2}\xi(n^{-1}I_d + \mathrm{Cov}(U_s(y)-U_t(x)))\xi'\right) d\xi\, ds\, dt\, dx\, dy\\ &= \int_{R_\varepsilon} \left(\frac{2\pi}{n^{-1}+\mathrm{Var}(U_s^1(y)-U_t^1(x))}\right)^{d/2}\\ &\qquad \times \exp\left(-\frac{|u|^2}{2(n^{-1}+\mathrm{Var}(U_s^1(y)-U_t^1(x)))}\right) ds\, dt\, dx\, dy\\ &\ge \int_{R_\varepsilon} \left(\frac{2\pi}{1+\mathrm{Var}(U_s^1(y)-U_t^1(x))}\right)^{d/2}\\ &\qquad \times \exp\left(-\frac{|u|^2}{2\,\mathrm{Var}(U_s^1(y)-U_t^1(x))}\right) ds\, dt\, dx\, dy\\ &\mathrel{\hat=} c_{3.19}.\end{aligned}$$

Denote by $\mathrm{Cov}(U_{s_1}(y_1)-U_{t_1}(x_1), U_{s_2}(y_2)-U_{t_2}(x_2))$ the covariance matrix of the Gaussian vector $(U_{s_1}(y_1)-U_{t_1}(x_1), U_{s_2}(y_2)-U_{t_2}(x_2))$ and by I_{2d} the identity matrix of order $2d$. Let

$$\Gamma = n^{-1}I_{2d} + \mathrm{Cov}(U_{s_1}(y_1)-U_{t_1}(x_1), U_{s_2}(y_2)-U_{t_2}(x_2)).$$

Then by the definition of R_ε and (3.1) in Lemma 3.1, we have

$$\begin{aligned}
&\det \mathrm{Cov}(U_{s_1}(y_1) - U_{t_1}(x_1), U_{s_2}(y_2) - U_{t_2}(x_2)) \\
&= \mathrm{Var}(U_{s_1}(y_1) - U_{t_1}(x_1)) \mathrm{Var}(U_{s_2}(y_2) - U_{t_2}(x_2) \mid U_{s_1}(y_1) - U_{t_1}(x_1)) \qquad (3.41)\\
&\geq c_{3.21}(|x_1 - x_2| + |y_1 - y_2| + |s_1 - s_2|^{1/2} + |t_1 - t_2|^{1/2}).
\end{aligned}$$

By (3.41), we have

$$\begin{aligned}
\mathbb{E}(\|\mu_n\|^2) &= \int_{R_\varepsilon}\int_{R_\varepsilon}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \exp(-i\langle \xi + \eta, u\rangle) \\
&\qquad \times \exp\left(-\frac{1}{2}(\xi, \eta)\Gamma(\xi, \eta)'\right) d\xi\, d\eta\, ds_1\, dt_1\, dx_1\, dy_1\, ds_2\, dt_2\, dx_2\, dy_2 \\
&= \int_{R_\varepsilon}\int_{R_\varepsilon} \frac{(2\pi)^d}{\sqrt{\det \Gamma}} \exp\left(-\frac{1}{2}(u, u)\Gamma^{-1}(u, u)'\right) ds_1\, dt_1\, dx_1\, dy_1\, ds_2\, dt_2\, dx_2\, dy_2 \\
&\leq (2\pi)^d \int_{R_\varepsilon}\int_{R_\varepsilon} \frac{ds_1\, dt_1\, dx_1\, dy_1\, ds_2\, dt_2\, dx_2\, dy_2}{(\det \mathrm{Cov}(U^1_{s_1}(y_1) - U^1_{t_1}(x_1), U^1_{s_2}(y_2) - U^1_{t_2}(x_2))^{d/2}} \\
&\leq c_{3.22} \int_{R_\varepsilon}\int_{R_\varepsilon} \frac{ds_1\, dt_1\, dx_1\, dy_1\, ds_2\, dt_2\, dx_2\, dy_2}{(|x_1 - x_2| + |y_1 - y_2| + |s_1 - s_2|^{1/2} + |t_1 - t_2|^{1/2})^{d/2}} \\
&\leq c_{3.23} \int_{R_\varepsilon}\int_{R_\varepsilon} \frac{ds_1\, dt_1\, dx_1\, dy_1\, ds_2\, dt_2\, dx_2\, dy_2}{(|x_1 - x_2|\, |y_1 - y_2|\, |s_1 - s_2|\, |t_1 - t_2|)^{d/12}} \\
&\hat{=}\, c_{3.24} < \infty,
\end{aligned}$$

where the last inequality follows from $d < 12$. We have also applied (2.19) with $n = 4$, $\beta_1 = \beta_2 = 1/6$ and $\beta_3 = \beta_4 = 2/6$ in the above inequality.

Similar to the proof of the above inequality, we have

$$\begin{aligned}
\mathbb{E}(\|\mu_n\|_\gamma) &= \int_{R_\varepsilon}\int_{R_\varepsilon} \frac{ds_1\, dt_1\, dx_1\, dy_1\, ds_2\, dt_2\, dx_2\, dy_2}{(|x_1 - x_2|^2 + |y_1 - y_2|^2 + |s_1 - s_2|^2 + |t_1 - t_2|^2)^{\gamma/2}} \\
&\qquad \times \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \exp(-i\langle \xi + \eta, u\rangle) \exp\left(-\frac{1}{2}(\xi, \eta)\Gamma(\xi, \eta)'\right) d\xi\, d\eta \\
&\leq \int_{R_\varepsilon}\int_{R_\varepsilon} \frac{c_{3.25}}{(|x_1 - x_2| + |y_1 - y_2| + |s_1 - s_2|^{1/2} + |t_1 - t_2|^{1/2})^{d/2}} \\
&\qquad \times \frac{ds_1\, dt_1\, dx_1\, dy_1\, ds_2\, dt_2\, dx_2\, dy_2}{(|x_1 - x_2| + |y_1 - y_2| + |s_1 - s_2| + |t_1 - t_2|)^{\gamma}}.
\end{aligned}$$

By a change of variable, we can deduce that

$$\begin{aligned}
\mathbb{E}(\|\mu_n\|_\gamma) &\leq \int_0^{1-\varepsilon} dx \int_0^{1-\varepsilon} dy \int_0^{1-\varepsilon} ds \int_0^{1-\varepsilon} \frac{c_{3.26}}{(x + y + s^{1/2} + t^{1/2})^{d/2}(x + y + s + t)^{\gamma}}\, dt \qquad (3.42)\\
&< \infty,
\end{aligned}$$

where the last inequality follows from Lemma 3.4. This proves (3.40) and thus the proof of Theorem 3.6 is finished. □

Now we consider the Hausdorff and packing dimensions for the so-called level set $L_u = \{(t, x, y) \in [0, \infty) \times \mathbb{R} \times \mathbb{R} : u_t(x) - u_t(y) = u\}$. By using Lemma 3.2 and a similar argument as in the proof of Theorem 3.6, we can obtain the following dimension result.

Theorem 3.7. *Let $\{u_t(x) : t \geq 0,\ x \in \mathbb{R}\}$ be a random string process in $\mathbb{R}^d$ with $d < 8$. Then for every $u \in \mathbb{R}^d$, with positive probability,*

$$\dim_{\mathrm{H}}(L_u \cap J) = \dim_{\mathrm{P}}(L_u \cap J) = \begin{cases} 3 - \dfrac{1}{4}d, & \textit{if} \quad 1 \leq d < 4, \\ 4 - \dfrac{1}{2}d, & \textit{if} \quad 4 \leq d < 8, \end{cases} \tag{3.43}$$

where $J = [0, 1] \times [0, 1] \times [2, 3] \subset [0, \infty) \times \mathbb{R} \times \mathbb{R}$.

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College of Statistics and Mathematics
Zhejiang Gongshang University
Hangzhou 310018
P.R. China
e-mail: zlchenv@163.com