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ON THE WEAKLY REGULAR p -BLOCKS WITH RESPECT TO $O_{p'}(G)$

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1. Introduction

We begin with a consequence of a result of Fong ([3] Theorem 1. F.). Let G be a finite group and p a fixed prime number. If D is a defect group of an element of $\text{Irr}(O_{p'}(G))$ (that is, D is an S_p -subgroup of the inertia group of an irreducible complex character of $O_{p'}(G)$), then it is also a defect group of a p -block of G . Furthermore, among those p -blocks that have defect group D , there exists a B which is weakly regular with respect to $O_{p'}(G)$. That is, there exists a conjugate class C of G satisfying (1) $C \subset O_{p'}(G)$ (2) C has a defect group D and (3) $\omega_B(\hat{C}) \not\equiv 0 \pmod{p}$, where $\hat{C} = \sum_{x \in C} x$ (For the definition of the weak regularity, see Brauer [1]).

In this paper, we shall show if D is a defect group of an element of $O_{p'}(G)$, then it is also a defect group of a p -block of G , which is weakly regular with respect to $O_{p'}(G)$. As a corollary, we get if $O_{p'}(G)$ has an element of p -defect d in G , then G has an irreducible character whose degree is divisible by $p^{\epsilon-d}$, where p^ϵ is the p -part of the order of G . As an application of this fact, we shall study those solvable groups all of whose irreducible characters are divisible by p at most to the first power.

NOTATION. p is a fixed prime number. G is a finite group of order $|G| = p^e g'$, $(p, g') = 1$. G_p denotes an S_p -subgroup of G . $\text{Irr}(G)$ denotes the set of all irreducible characters of G . We fix a prime divisor \mathfrak{p} of p in the ring of integers $\mathfrak{o} = \mathbb{Z}[\epsilon]$, where ϵ is a primitive $|G|$ -th root of unity and we denote by k the residue class field $\mathfrak{o}/\mathfrak{p}$. If C is a conjugate class of G , then we denote by \hat{C} the sum $\sum_{x \in C} x$ in the group ring of G over the field under consideration. Let $F(G)$ denote the Fitting subgroup of G . If G is solvable, we have the normal series,

$$G = F_n \supseteq F_{n-1} \supseteq \cdots \supseteq F_1 \supseteq F_0 = 1, \quad \text{where } F_i/F_{i-1} = F(G/F_{i-1}).$$

The number n is called the nilpotent length of G , which will be denoted by $n(G)$. Some other notations and terminologies which will be used in this paper will be found in Curtis and Reiner [2] or Gorenstein [5].

2. Weakly regular p -blocks with respect to $O_{p'}(G)$

The main purpose of this section is to prove the following

Theorem 1. *Let D be a p -subgroup of G . Suppose there exist conjugate classes C_1, C_2, \dots, C_r of G , which have defect group D in common and are contained in $O_{p'}(G)$. Then there exist p -blocks B_1, B_2, \dots, B_r of G , satisfying*

- (1) *Each B_i has defect group D .*
- (2) *If χ_i is any irreducible character belonging to B_i with height 0, then $\chi_1, \chi_2, \dots, \chi_r$ are linearly independent mod \mathfrak{p} on C_1, C_2, \dots, C_r (Hence each B_i is weakly regular with respect to $O_{p'}(G)$).*

Proof. In case $D=1$, we have already proved the corresponding assertion in [9]. However, for the convenience of the reader, we give here an alternative proof, which is rather elementary. Hence assume first $D=1$. Let B_1, B_2, \dots, B_s be the set of p -blocks of defect zero and assume $0 \leq s < r$. Then the matrix $M = [\bar{\omega}_i(\hat{C}_j)]$ has the rank smaller than r , where $\bar{\omega}_i = \bar{\omega}_{B_i}$ ("bar" indicates the image of the natural map $\mathfrak{o} \rightarrow k$). Hence there exists a non zero $x = \sum \lambda_j \hat{C}_j \in kG$, ($\lambda_j \in k$), with $\bar{\omega}_i(x) = 0$ for any i . Then it follows that $\bar{\omega}_B(x) = 0$ for any p -block B of G , since $\bar{\omega}_B(\hat{C}_j) = 0$ for any B of positive defect. Therefore x is contained in the radical of the group ring kG , a contradiction, since x is in $kO_{p'}(G)$, which is semisimple. Hence the rank of M is equal to r and so $s \geq r$. Hence we may assume, after a suitable change of indexes if necessary, there exist r blocks B_1, B_2, \dots, B_r of defect zero such that $\det [\bar{\omega}_i(\hat{C}_j)] \neq 0$. Then, using that $|C_j|/\chi_j(1)$ is a unit in \mathfrak{o}_p , we have $\det[\chi_i(c_j)] \not\equiv 0 \pmod{\mathfrak{p}}$, where $\chi_i \in B_i$ and $c_j \in C_j$. Hence $\chi_1, \chi_2, \dots, \chi_r$ are linearly independent mod \mathfrak{p} on C_1, C_2, \dots, C_r .

The proof of the general case is reduced to the above by virtue of the Brauer's Theorem (see e.g. [2] Theorem 88.8). Let $C_i' = C_i \cap C_G(D)$ and $\bar{C}_i' = \{\bar{x} \in \bar{N} = N/D \mid x \in C_i'\}$, where $N = N_G(D)$. Then $\bar{C}_1', \bar{C}_2', \dots, \bar{C}_r'$ are distinct conjugate classes of \bar{N} contained in $O_p(\bar{N})$. Furthermore, each of them has p -defect zero in \bar{N} , as is easily checked. Therefore, there exist r characters $\zeta_1, \zeta_2, \dots, \zeta_r$ of \bar{N} of p -defect zero such that the associated linear functions $\omega_{\zeta_1}, \omega_{\zeta_2}, \dots, \omega_{\zeta_r}$ are linearly independent mod \mathfrak{p} on those classes. In particular, for each i , there exist some j such that $\omega_{\zeta_i}(\bar{C}_j') \not\equiv 0 \pmod{\mathfrak{p}}$. Let b_i be the p -block of N which contains ζ_i . Then $\omega_{b_i}(\hat{C}_j') \equiv \omega_{\zeta_i}(\hat{C}_j') \not\equiv 0 \pmod{\mathfrak{p}}$, since $|C_j'| = |\bar{C}_j'|$. Hence D is a defect group of b_i , since $N \triangleright D$. Let B_i be the p -block of G which corresponds to b_i under the Brauer homomorphism. Then D is a defect group of B_i and $\omega_{\chi_i}(\hat{C}_j) \equiv \omega_{\zeta_i}(\hat{C}_j') \pmod{\mathfrak{p}}$, where $\chi_i \in B_i$. Now the second assertion follows from this by the same way as that of the case $D=1$.

Corollary 2. *If $O_{p'}(G)$ contains an element of p -defect d in G , then G has an irreducible character whose degree is divisible by p^{e-d} exactly.*

We note here the following Corollary, the first of which has been proved by Fong [4]. But the second half seems not to have been noticed in even the p -solvable case.

Corollary 3. *Let G be a p -constrained group. Then the following conditions are equivalent.*

- (1) *Every p -block of G is of full defect.*
- (2) *Every element of $O_{p'}(G)$ is of full defect in G .*

If the above are satisfied, then we have $O_{p'p}(G) = O_{p'}(G) \times O_p(G)$. In particular we have $O_p(G) \neq 1$ (unless G is a p' -group).

The following Lemma is purely group theoretic and probably known. Our proof requires the modular representation theories.

Lemma 4. *Let G be any finite group. If every p -regular element of G is of full p -defect, then $G = G_p \times O_{p'}(G)$.*

Proof. By the assumption, the Cartan matrix of each p -block is necessary of the form (p^e) (see [2] §89). In particular, the principal block possesses only one modular irreducible character. Hence G has the normal p -complement K , $G = G_p K$. For any $x \in K$, there exists some $t \in K$ such that $C_G(x) \supset G_p^t$. Therefore, $K = \bigcup_{t \in K} C_K(G_p)^t$. Then as is well known, we have $K = C_K(G_p)$, which implies our assertion.

Proof of Corollary 3.

“(1) \Rightarrow (2)” is clear by Theorem 1. Now assume (2) and let $H = O_{p'p}(G)$. Then, every element of $O_p(G)$ has full defect in H and so we have $H = H_p \times O_{p'}(G) = O_p(G) \times O_{p'}(G)$ by the above Lemma. Let D be a defect group of a p -block of G . Then D contains $O_p(G)$. Furthermore, there exists a p -regular element x such that D is an S_p -subgroup of $C_G(x)$. Hence $x \in C_G(D) \subset C_G(O_p(G)) \subset H$, since G is p -constrained. Then x is contained in $O_{p'}(G)$ and D is an S_p -subgroup of G .

3. Application

We let p a fixed prime as before. We consider the following condition

$$(*) \quad G \text{ is solvable and for any } \chi \text{ of } \text{Irr}(G), p^2 \nmid \chi(1).$$

As a corollary of results of Issacs [7], we have

Theorem A. *Under the condition (*), it holds that an S_p -subgroup of $G/O_p(G)$ is abelian. If $p > 3$, then $O_p(G)$ coincides with G_p or is abelian.*

First we show the following refinement of the above result.

Theorem 5. *Under the condition (*), it holds that an S_p -subgroup of $G/O_p(G)$ is elementary, whose rank does not exceed the nilpotent length of a p -complement of G .*

Proof. We proceed by the induction on the order of G . To prove the first, we may clearly assume $O_p(G)=1$ and $G=G_pK$, $K=O_{p'}(G)$, since the p -length of G is at most one by Theorem A. If the condition (*) holds, then by Corollary 2, the order of an S_p -subgroup of $C_G(x)$ is not smaller than p^{e-1} for any $x \in K$. Hence we have that $C_G(x) \supset \Phi(G_p)^g$ for some $g \in K$, where $\Phi(G_p)$ denotes the Frattini subgroup of G_p . Then it follows that $K = \bigcup_{g \in K} C_K(\Phi(G_p))^g$ and so $K = C_K(\Phi(G_p))$. However, since $O_p(G)=1$, we have $C_G(K) \subset K$. Therefore we have $\Phi(G_p)=1$, implying G_p is elementary.

To prove the second, we need the following, which is a special case of the Theorem 2 of Ito [8].

Theorem B. *Suppose G is a solvable group with an abelian S_p -subgroup. If the nilpotent length of G is at most 2 and $O_p(G)=1$, then G has an irreducible character of p -defect zero.*

Now let $n(G)$ denote the nilpotent length of G . Then as is easily shown, for a subgroup H of G , we have $n(G) \geq n(H)$ and also $n(G) \geq n(G/H)$, if $G \triangleright H$. Hence we may assume $O_p(G)=1$ and $G=G_pK$, $K=O_{p'}(G)$. If $O_p(G/F(K))=1$, then our assertion follows from the induction hypothesis on $G/F(K)$. Let $O_p(G/F)=QF/F$, where $F=F(K)$ and assume Q is a non-trivial p -subgroup of G . Since $G \triangleright QF$, we have $O_p(QF)=1$. Then by Theorem B, QF has an irreducible character whose degree is divisible by $|Q|$. By the Theorem of Clifford, also G does. Hence by the condition (*), we have $|Q|=p$. Let $\bar{G}=G/F$ and $r(G_p)$ denote the rank of G_p . Then $r(G_p)=r(\bar{G}_p)+1$ and $r(\bar{G}_p) \leq n(\bar{K})$ by the induction hypothesis. Since $n(\bar{K})=n(K)-1$, we have $r(G_p) \leq n(K)$, as desired.

Finally, we show

Theorem 6. *Suppose the condition (*) holds for every prime p , then for each p , and S_p -subgroup of $G/F(G)$ is elementary, whose rank is at most 2.*

Proof. Clearly the conclusion is equivalent to the following; For any p , an S_p -subgroup of $G/O_p(G)$ is elementary, whose rank is at most 2.

We prove this by the induction on the order of G . Let p be any prime and fixed. We may assume $O_p(G)=1$, and that G has no proper normal subgroup of index prime to p . Then it follows that $G=G_pK$, $K=O_{p'}(G)$, since G_p is abelian.

If there exists a non-trivial normal p' -subgroup V such that $O_p(G/V)=1$, then our assertion follows from the induction hypothesis on G/V . With these remarks in mind, we proceed our proof.

Step 1. G has only one minimal normal subgroup.

Proof. Suppose G has two distinct minimal normal subgroups V_1 and V_2 . Let $O_p(G/V_i) = Q_i V_i / V_i$, where Q_i is a non-trivial p -subgroup of G . We may assume $Q_1, Q_2 \subset G_p$. Let $Q = Q_1 Q_2$ and $V = V_1 V_2 = V_1 \times V_2$. Note that both Q and V are abelian. Since each $Q_i V_i / V_i$ is central, we have $[G, Q_i] \subset V_i$ and so $G \triangleright QV$. Then by Theorem B, QV has an irreducible character whose degree is divisible by $|Q|$. By the Theorem of Clifford, also G does. Therefore we have $|Q| = p$ and hence $Q_1 = Q_2 = Q$. We then have $[K, Q] \subset V_1 \cap V_2 = 1$, a contradiction, since $C_G(K) \subset K$.

Step 2. $V = F(G)$, where V is the unique minimal normal subgroup of G .

Proof. From the proof of Step 1, we see $O_p(G/V)$ is a central subgroup of order p . Let $O_p(G/V) = QV/V$ and $Q = \langle a \rangle, a^p = 1$. Then $[G, a] \subset V$. We have also that $C_V(a) = 1$ and $[V, a] = V$, since $G \triangleright QV$ and V is minimal. For $g \in C_G(V)$, we set $\bar{a}(g) = g^{-1}g^a$. Then \bar{a} is a homomorphism from $C_G(V)$ into V , which is an epimorphism, as is remarked above. On the other hand, $\ker \bar{a} = C_G(a) \cap C_G(V) = C_G(QV) \triangleleft G$. If $\ker \bar{a}$ is not trivial, then it contains V , which contradicts $C_V(a) = 1$. Hence \bar{a} is an isomorphism and we have $C_G(V) = V$. Then we have $V = F(G)$, since $C_G(V) \supset F(G)$. (see e.g. Huppert [7])

Step 3. $F(G/V)$ is cyclic.

Proof. Let $F(G/V) = W/V = \bar{W}$. Then $\bar{W} = \bar{Q} \times F(\bar{K})$. Since $|Q| = p$, it suffices to show $F(\bar{K}) = T/V$ is cyclic. Since $V = F(G)$, we have $(|V|, |T/V|) = 1$. Let r be any prime dividing $|T/V|$. Then by Theorem B and the assumption (*), we see an S_p -subgroup of T/V is cyclic of prime order r . Hence T/V is cyclic, since it is nilpotent.

Step 4. $n(K) \leq 2$.

Proof. We have $G_c(F(\bar{G})) = F(\bar{G})$ from the above, since $C_G(F(\bar{G})) \subset F(\bar{G})$, where $\bar{G} = G/V$. Then G/W is isomorphic to a subgroup of $\text{Aut}(W/V)$, which is abelian. Therefore $W \supset G' = K$ (since G has no proper normal subgroup index prime to p). In particular, K/V is abelian. Since $V = F(G) = F(K)$, we have $n(K) \leq 2$, completing the proof of Step 4.

Now the Theorem 6 follows at once from Theorem 5.

4. Correction

The final example in the previous paper [9] is not correct. The two dimensional affine group over the field $GF(3)$ has a character of 2-defect zero. The author expresses his gratitude to Professor B. Huppert for pointing out the error.

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