Tsushima, Y.
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ON THE WEAKLY REGULAR p-BLOCKS WITH RESPECT TO $O_{p'}(G)$

YUKIO TSUSHIMA

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1. Introduction

We begin with a consequence of a result of Fong ([3] Theorem 1. F.). Let G be a finite group and p a fixed prime number. If D is a defect group of an element of $Irr(O_{p'}(G))$ (that is, D is an S_p-subgroup of the inertia group of an irreducible complex character of $O_{p'}(G)$), then it is also a defect group of a p-block of G. Furthermore, among those p-blocks that have defect group D, there exists a B which is weakly regular with respect to $O_{p'}(G)$. That is, there exists a conjugate class C of G satisfying (1) $C\subset O_{p'}(G)$ (2) C has a defect group D and (3) $\omega_B(\hat{C})\not\equiv 0 \ mod\ \mathfrak{p}$, where $\hat{C}=\sum_{x\in C}x$ (For the definition of the weak regularity, see Brauer [1]).

In this paper, we shall show if D is a defect group of an element of $O_{p'}(G)$, then it is also a defect group of a p-block of G, which is weakly regular with respect to $O_{p'}(G)$. As a corollary, we get if $O_{p'}(G)$ has an element of p-defect d in G, then G has an irreducible character whose degree is divisible by p^{e-d}, where p^e is the p-part of the order of G. As an application of this fact, we shall study those solvable groups all of whose irreducible characters are divisible by p at most to the first power.

NOTATION. p is a fixed prime number. G is a finite group of order $|G|=p^eg'$, $(p, g')=1$. G_p denotes an S_p-subgroup of G. $Irr(G)$ denotes the set of all irreducible characters of G. We fix a prime divisor $\mathfrak{p}$ of p in the ring of integers $\mathfrak{o}=Z[\varepsilon]$, where ε is a primitive $|G|$-th root of unity and we denote by k the residue class field $\mathfrak{o}/\mathfrak{p}$. If C is a conjugate class of G, then we denote by $\hat{C}$ the sum $\sum_{x\in C}x$ in the group ring of G over the field under consideration. Let $F(G)$ denote the Fitting subgroup of G. If G is solvable, we have the normal series,

$$G=F_n\supsetneqq F_{n-1}\supsetneqq\cdots\supsetneqq F_1\supsetneqq F_0=1, \qquad \text{where} \quad F_i/F_{i-1}=F(G/F_{i-1}).$$

The number n is called the nilpotent length of G, which will be denoted by $n(G)$. Some other notations and terminologies which will be used in this paper will be found in Curtis and Reiner [2] or Gorenstein [5].

2. Weakly regular p-blocks with respect to $O_{p'}(G)$

The main purpose of this section is to prove the following

Theorem 1. *Let D be a p-subgroup of G. Suppose there exist conjugate classes $C_1, C_2, \cdots C_r$ of G, which have defect group D in common and are contained in $O_{p'}(G)$. Then there exist p-blocks $B_1, B_2, \cdots B_r$ of G, satisfying*

(1) *Each B_i has defect group D.*

(2) *If χ_i is any irreducible character belonging to B_i with height 0, then $\chi_1, \chi_2, \cdots, \chi_r$ are linearly independent mod $\mathfrak{p}$ on $C_1, C_2, \cdots, C_r$ (Hence each B_i is weakly regular with respect to $O_{p'}(G)$.*

Proof. In case $D=1$, we have already proved the corresponding assertion in [9]. However, for the convenience of the reader, we give here an alternative proof, which is rather elementary. Hence assume first $D=1$. Let $B_1, B_2, \cdots B_s$ be the set of p-blocks of defect zero and assume $0 \leq s < r$. Then the matrix $M=[\bar{\omega}_i(\hat{C}_j)]$ has the rank smaller than r, where $\bar{\omega}_i=\bar{\omega}_{B_i}$ ("bar" indicates the image of the natural map $\mathfrak{o} \rightarrow k$). Hence there exists a non zero $x=\sum \lambda_j \hat{C}_j \in kG$, $(\lambda_j \in k)$, with $\bar{\omega}_i(x)=0$ for any i. Then it follows that $\bar{\omega}_B(x)=0$ for any p-block B of G, since $\bar{\omega}_B(\hat{C}_j)=0$ for any B of positive defect. Therefore x is contained in the radical of the group ring kG, a contradiction, since x is in $kO_{p'}(G)$, which is semisimple. Hence the rank of M is equal to r and so $s \geq r$. Hence we may assume, after a suitable change of indexes if necessary, there exist r blocks $B_1, B_2, \cdots B_r$ of defect zero such that *det* $[\bar{\omega}_i(\hat{C}_j)] \neq 0$. Then, using that $|C_j|/\chi_i(1)$ is a unit in $\mathfrak{o}_\mathfrak{p}$, we have $det[\chi_i(c_j)] \not\equiv 0$ *mod* $\mathfrak{p}$, where $\chi_i \in B_i$ and $c_j \in C_j$. Hence $\chi_1, \chi_2, \cdots \chi_r$ are linearly independent *mod* $\mathfrak{p}$ on $C_1, C_2, \cdots C_r$.

The proof of the general case is reduced to the above by virtue of the Brauer's Theorem (see *e.g.* [2] Theorem 88.8). Let $C_i'=C_i \cap C_G(D)$ and $\bar{C}_i'=\{\bar{x} \in \bar{N}=N/D \mid x \in C_i'\}$, where $N=N_G(D)$. Then $\bar{C}_1', \bar{C}_2', \cdots \bar{C}_r'$ are distict conjugate classes of $\bar{N}$ contained in $O_p(\bar{N})$. Furthermore, each of them has p-defect zero in $\bar{N}$, as is easily checked. Therefore, there exist r characters $\zeta_1, \zeta_2, \cdots \zeta_r$ of $\bar{N}$ of p-defect zero such that the associated linear functions $\omega_{\zeta_1}, \omega_{\zeta_2}, \cdots \omega_{\zeta_r}$ are linearly independent *mod* $\mathfrak{p}$ on those classes. In particular, for each i, there exist some j such that $\omega_{\zeta_i}(\bar{C}_j') \not\equiv 0$ *mod* $\mathfrak{p}$. Let b_i be the p-block of N which contains ζ_i. Then $\omega_{b_i}(\hat{C}_j') \equiv \omega_{\zeta_i}(\hat{C}_j') \not\equiv 0$ *mod* $\mathfrak{p}$, since $|C_j'|=|\bar{C}_j'|$. Hence D is a defect group of b_i, since $N \triangleright D$. Let B_i be the p-block of G which corresponds to b_i under the Brauer homomorphism. Then D is a defect group of B_i and $\omega_{\chi_i}(\hat{C}_j) \equiv \omega_{\zeta_i}(\hat{C}_j')$ *mod* $\mathfrak{p}$, where $\chi_i \in B_i$. Now the second assertion follows from this by the same way as that of the case $D=1$.

Corollary 2. *If $O_{p'}(G)$ contains an element of p-defect d in G, then G has an irreducible character whose degree is divisible by p^{e-d} exactly.*

We note here the follwing Corollary, the first of which has been proved by Fong [4]. But the second half seems not to have been noticed in even the p-solvable case.

Corollary 3. *Let G be a p-constrained group. Then the following conditions are equivalent.*

(1) *Every p-block of G is of full defect.*

(2) *Every element of $O_{p'}(G)$ is of full defect in G.*

If the aboves are satisfied, then we have $O_{p'p}(G)=O_{p'}(G)\times O_p(G)$. In particular we have $O_p(G)\neq 1$ (unless G is a p'-group).

The following Lemma is purely group theoritic and probably known. Our proof requires the modular representation theories.

Lemma 4. *Let G be any finite group. If every p-regular element of G is of full p-defect, then $G=G_p\times O_{p'}(G)$.*

Proof. By the assumption, the Cartan matrix of each p-block is necessary of the form (p^e) (see [2] §89). In particular, the principal block possesses only one modular irreducible character. Hence G has the normal p-complement K, $G=G_pK$. For any $x\in K$, there exists some $t\in K$ such that $C_G(x)\supset G_p{}^t$. Therefore, $K=\bigcup_{t\in K} C_K(G_p)^t$. Then as is well known, we have $K=C_K(G_p)$, which implies our assertion.

Proof of Corollary 3.

"(1)⇒(2)" is clear by Theorem 1. Now assume (2) and let $H=O_{p'p}(G)$. Then, every element of $O_p(G)$ has full defect in H and so we have $H=H_p\times O_{p'}(G)=O_p(G)\times O_{p'}(G)$ by the above Lemma. Let D be a defect group of a p-block of G. Then D contains $O_p(G)$. Furthermore, there exists a p-regular element x such that D is an S_p-subgroup of $C_G(x)$. Hence $x\in C_G(D)\subset C_G(O_p(G))\subset H$, since G is p-constrained. Then x is contained in $O_{p'}(G)$ and D is an S_p-subgroup of G.

3. Application

We let p a fixed prime as before. We consider the following condition

(*) *G is solvable and for any χ of $Irr(G)$, $p^2\not|\chi(1)$.*

As a corollary of results of Issacs [7], we have

Theorem A. *Under the condition* (*), *it holds that an S_p-subgroup of $G/O_p(G)$ is abelian. If $p>3$, then $O_p(G)$ coinsides with G_p or is abelian.*

First we show the following refinement of the above result.

Theorem 5. *Under the condition* (*), *it holds that an* S_p*-subgroup of* $G/O_p(G)$ *is elementary, whose rank does not exceed the nilpotent length of a p-complement of* G.

Proof. We proceed by the induction on the order of G. To prove the first, we may clearly assume $O_p(G)=1$ and $G=G_pK$, $K=O_{p'}(G)$, since the p-length of G is at most one by Theorem A. If the condition (*) holds, then by Corollary 2, the order of an S_p-subgroup of $C_G(x)$ is not smaller than p^{e-1} for any $x\in K$. Hence we have that $C_G(x)\supset\Phi(G_p)^g$ for some $g\in K$, where $\Phi(G_p)$ denotes the Frattini subgroup of G_p. Then it follows that $K=\bigcup_{g\in K} C_K(\Phi(G_p))^g$ and so $K=C_K(\Phi(G_p))$. However, since $O_p(G)=1$, we have $C_G(K)\subset K$. Therefore we have $\Phi(G_p)=1$, implying G_p is elementary.

To prove the second, we need the following, which is a special case of the Theorem 2 of Ito [8].

Theorem B. *Suppose* G *is a solvable group with an abelian* S_p*-subgroup. If the nilpotent length of* G *is at most* 2 *and* $O_p(G)=1$, *then* G *has an irreducible character of p-defect zero.*

Now let $n(G)$ denote the nilpotent length of G. Then as is easily shown, for a subgroup H of G, we have $n(G)\geq n(H)$ and also $n(G)\geq n(G/H)$, if $G\triangleright H$. Hence we may assume $O_p(G)=1$ and $G=G_pK$, $K=O_{p'}(G)$. If $O_p(G/F(K))=1$, then our assertion follows from the induction hypothesis on $G/F(K)$. Let $O_p(G/F)=QF/F$, where $F=F(K)$ and assume Q is a non-trivial p-subgroup of G. Since $G\triangleright QF$, we have $O_p(QF)=1$. Then by Theorem B, QF has an irreducible character whose degree is divisible by $|Q|$. By the Theorem of Clifford, also G does. Hence by the condition (*), we have $|Q|=p$. Let $\bar{G}=G/F$ and $r(G_p)$ denote the rank of G_p. Then $r(G_p)=r(\bar{G}_p)+1$ and $r(\bar{G}_p)\leq n(\bar{K})$ by the induction hypothesis. Since $n(\bar{K})=n(K)-1$, we have $r(G_p)\leq n(K)$, as desired.

Finally, we show

Theorem 6. *Suppose the condition* (*) *holds for every prime p, then for each p, and* S_p*-subgroup of* $G/F(G)$ *is elementary, whose rank is at most* 2.

Proof. Clearly the conclusion is equivalent to the following; *For any p, an* S_p*-subgroup of* $G/O_p(G)$ *is elementary, whose rank is at most* 2.

We prove this by the induction on the order of G. Let p be any prime and fixed. We may assume $O_p(G)=1$, and that G has no proper normal subgroup of index prime to p. Then it follows that $G=G_pK$, $K=O_{p'}(G)$, since G_p is abelian.

If there exists a non-trivial normal p'-subgroup V such that $O_p(G/V)=1$, then our assertion follows from the induction hypothesis on G/V. With these remarks in mind, we proceed our proof.

Step 1. *G has only one minimal normal subgroup.*

Proof. Suppose G has two distinct minimal normal subgroups V_1 and V_2. Let $O_p(G/V_i)=Q_iV_i/V_i$, where Q_i is a non-trivial p-subgroup of G. We may assume $Q_1, Q_2\subset G_p$. Let $Q=Q_1Q_2$ and $V=V_1V_2=V_1\times V_2$. Note that both Q and V are abelian. Since each Q_iV_i/V_i is central, we have $[G,Q_i]\subset V_i$ and so $G\triangleright QV$. Then by Theorem B, QV has an irreducible character whose degree is divisible by $|Q|$. By the Theorem of Clifford, also G does. Therefore we have $|Q|=p$ and hence $Q_1=Q_2=Q$. We then have $[K, Q]\subset V_1\cap V_2=1$, a contradiction, since $C_G(K)\subset K$.

Step 2. *$V=F(G)$, where V is the unique minimal normal subgroup of G.*

Proof. From the proof of Step 1, we see $O_p(G/V)$ is a central subgroup of order p. Let $O_p(G/V)=QV/V$ and $Q=\langle a\rangle, a^p=1$. Then $[G, a]\subset V$. We have also that $C_V(a)=1$ and $[V, a]=V$, since $G\triangleright QV$ and V is minimal. For $g\in C_G(V)$, we set $\tilde{a}(g)=g^{-1}g^a$. Then $\tilde{a}$ is a homomorphism from $C_G(V)$ into V, which is an epimorphism, as is remarked above. On the other hand, *ker* $\tilde{a}=C_G(a)\cap C_G(V)=C_G(QV)\triangleleft G$. If *ker* $\tilde{a}$ is not trivial, then it contains V, which contradicts $C_V(a)=1$. Hence $\tilde{a}$ is an isomorphism and we have $C_G(V)=V$. Then we have $V=F(G)$, since $C_G(V)\supset F(G)$. (see *e.g.* Huppert [7])

Step 3. *$F(G/V)$ is cyclic.*

Proof. Let $F(G/V)=W/V=\bar{W}$. Then $\bar{W}=\bar{Q}\times F(\bar{K})$. Since $|Q|=p$, it suffices to show $F(\bar{K})=T/V$ is cyclic. Since $V=F(G)$, we have $(|V|, |T/V)=1$. Let r be any prime dividing $|T/V|$. Then by Theorem B and the assumption (*), we see an S_p-subgroup of T/V is cyclic of prime order r. Hence T/V is cyclic, since it is nilpotent.

Step 4. $n(K)\leq 2$.

Proof. We have $G_G(F(\bar{G}))=F(\bar{G})$ from the above, since $C_G(F(\bar{G}))\subset F(\bar{G})$, where $\bar{G}=G/V$. Then G/W is isomorphic to a subgroup of $\mathrm{Aut}(W/V)$, which is abelian. Therefore $W\supset G'=K$ (since G has no proper normal subgroup index prime to p). In particular, K/V is abelian. Since $V=F(G)=F(K)$, we have $n(K)\leq 2$, completing the proof of Step 4.

Now the Theorem 6 follows at once from Theorem 5.

4. Correction

The final example in the previous paper [9] is not correct. The two dimensional affine group over the field $GF(3)$ *has* a character of 2-defect zero. The author expresses his gratitude to Proffesor B. Huppert for pointing out the error.

OSAKA CITY UNIVERSITY

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