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ON THE COEFFICIENT RING OF A TORUS EXTENSION

KEN-ICHI YOSHIDA

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Introduction. S. Abhyankar, W. Heinzer and P. Eakin treated the following problem in [1]; if $A[X]=B[Y]$, when is A isomorphic or identical to B? Replacing the polynomial ring by the torus extension we shall take up the following problem; if $A[X, X^{-1}]=B[Y, Y^{-1}]$, when is A isomorphic or identical to B? We say that A is torus invariant (resp. strongly torus invariant) whenever $A[X, X^{-1}]=B[Y, Y^{-1}]$ implies $A\simeq B$ (resp. $A=B$). The rolles played by polynomial rings in [1] are played by the graded rings in our theory. A graded ring $A=\sum A_i$, $i\in \boldsymbol{Z}$, with the property that $A_i\neq 0$ for each $i\in \boldsymbol{Z}$, will be called a $\boldsymbol{Z}$-graded ring. Main results are the followings.

An affine domain A of dimension one over a field $\boldsymbol{k}$ is always torus invariant. Moreover A is not strongly torus invariant if and only if A has a graded ring structure. An affine domain of dimension two is not always torus invariant. We shall construct an affine domain of dimension two which is not torus invariant. Let A be an affine domain over $\boldsymbol{k}$ of dimension two. Assume that the field $\boldsymbol{k}$ contains all roots of "unity" and is of characteristic zero. If A is not torus invariant, then A is a $\boldsymbol{Z}$-graded ring such that there exist invertible elements of non-zero degree.

In Section 1 we study elementary properties of graded rings. Especially we are interested in $\boldsymbol{Z}$-graded rings with invertible elements of non-zero degree. In Section 2 we discuss some conditions for A to be torus invariant. In Section 3 we give several sufficient conditions for an integral domain to be strongly torus invariant. Some relevant results will be found in S. Iitaka and T. Fujita [2]. Section 4 is devoted to the proof of the main results mentioned above. In Section 5 we fix an integral domain D and we treat only D-algebras and D-isomorphisms there. We shall prove the following two results. When A is a D-algebra of $tr.\ deg_D\ A=1$ and A is not D-torus invariant, A is a $\boldsymbol{Z}$-graded ring such that D is contained in A_0. If A is a $\boldsymbol{Z}$-graded ring such as $D=A_0$, then the number of elements of the set of $\{D$-isomorphic classes of D-algebras B such that $A[X, X^{-1}]=B[Y, Y^{-1}]\}$ is $\Phi(d)$, where d is the smallest positive integer among the degrees of units in A and Φ is the Euler function.

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1. Some properties of graded rings

Let R be commutative ring with indentity. The ring R is said to be a graded ring if R is a graded module, $R=\sum R_i$, and $R_nR_m\subseteqq R_{n+m}$.

Lemma 1.1. *Let R be a graded domain. Then we have the following.*

(1) *The unity element of R is homogeneous.*

(2) *If a is homogeneous and $a=bc$, then b and c are both homogeneous. In particular every invertible element is homogeneous.*

(3) *If R contains a field k, then k is a subring of R_0.*

Proof. (1) follows immediately from the relation $1^2=1$. The proof of (2) is easy and will be omitted. To prove (3) we can assume $\boldsymbol{k}$ is different from F_2 by (1). Let a be an element of $\boldsymbol{k}$ different from 1. Then $1-a$ is homogeneous from (2). The unity 1 is homogeneous of degree 0 by (1). Hence a should be homogeneous of degree 0.

We call a graded ring $R=\sum R_i$ to be a $\boldsymbol{Z}$-graded ring if $R_i\neq 0$, for some $i\in \boldsymbol{Z}^+$ and $\boldsymbol{Z}^-$.

Proposition 1.2. *Let R be a $\boldsymbol{Z}$-graded domain. Let $S=\{i\in Z;\ R_i\neq 0\}$. Then $S=n\boldsymbol{Z}$ for a certain integer n.*

Proof. Since R is a domain, S is a semi-group. Hence (1.2) is immediately seen by the following lemma.

Lemma 1.3. *Let $S\subseteqq \boldsymbol{Z}$ be a semi-group. If $S\cap \boldsymbol{Z}^+\neq 0$ and $S\cap \boldsymbol{Z}^-\neq 0$, then S is a subgroup of $\boldsymbol{Z}$.*

If R is a $\boldsymbol{Z}$-graded domain, then we may assume $R_i\neq 0$ for any $i\in \boldsymbol{Z}$.

Proposition 1.4. *Let R be a graded ring. If there is an invertible element x in R_1, then $R=R_0[x, x^{-1}]$.*

Proof. For any $r\in R_n$, $r=r(x^{-1}x)^n=rx^{-n}x^n$ and rx^{-n} is in R_0, therefore $r\in R_0x^n$. Hence $R=R_0[x, x^{-1}]$.

Corollary. *Let R be a $\boldsymbol{Z}$-graded domain. If R_0 is a field, so $R=R_0[x, x^{-1}]$ for every $x\in R_1$, $x\neq 0$.*

Proof. Choose non-zero elements $x\in R_1$, and $y\in R_1$. Since R_0 is a field, $0\neq xy$ is invertible, therefore x and y are units in R, hence $R=R_0[x, x^{-1}]$.

2. Torus invariant rings

A ring A is said to be torus invariant provided that A has the following property:

If there exist a ring B, a variable Y over B, and a variable X over A such that $A[X, X^{-1}]$ is isomorphic to $B[Y, Y^{-1}]$,

$$\Phi\colon A[X, X^{-1}] \to B[Y, Y^{-1}],$$

then A is always isomorphic to B.

Especially if we have always $\Phi(A)=B$ in such case, we say that the ring A is strongly torus invariant.

To show A is torus invariant (resp. strongly torus invariant) it suffices to prove that A is isomorphic to B (resp. $A=B$) under the assumption: $A[X, X^{-1}]=B[Y, Y^{-1}]$.

(2.0) We begin with some elementary observations. Assume that

$$(1) \qquad R = A[X, X^{-1}] = B[Y, Y^{-1}].$$

Then X and Y are units of R. It follows from (1.1) that we have

$$(2) \qquad X = vY^{f} \text{ and } Y = uX^{f'},\ v \in B \text{ and } u \in A,$$

or equivalently

$$(3) \qquad v = u^{-f}X^{1-ff'} \text{ and } u = v^{-f'}Y^{1-ff'}.$$

In the rest of our paper we shall use the letters u and v to denote the elements of A and B respectively satisfying the relations (2) and (3) whenever we encounter the situation (1).

(2.1) The element u is in B if and only if $ff'=1$. In this case we have $A[X, X^{-1}]=B[X, X^{-1}]$, thus we have $A \simeq B$.

Proof is easy and is omitted.

Proposition 2.2. *Let $\boldsymbol{k}$ be a field and A be a $\boldsymbol{k}$-algebra. If A^* (the set of all invertible elements in A)$=\boldsymbol{k}^*$, then the ring A is torus invariant.*

Proof. Let $R=A[X, X^{-1}]=B[Y, Y^{-1}]$. By (1.1) the field $\boldsymbol{k}$ is contained in B. Since $A^*=\boldsymbol{k}^*$, the unit element u of A is in $\boldsymbol{k}$, hence in B, It follows from (2.1) that A is torus invariant.

Proposition 2.3. *Let $A=A_0[t_1, t_2, \cdots, t_n, (t_1t_2\cdots t_n)^{-1}]$ where t_i's are independent variables over $\boldsymbol{k}$-algebra A_0 and $A_0^*=\boldsymbol{k}^*$, then A is torus invariant.*

Proof. Let $R=A[X, X^{-1}]=B[Y, Y^{-1}]$. Then by the lemma (1.1) $Y=uX^{f'}$ and $X=vY^{f}$. Since u is invertible in $A=A_0[t_1, t_2, \cdots, t_n, (t_1\cdots t_n)^{-1}]$, $Y=rt_1^{e_1}\cdots t_n^{e_n}$, $r \in A_0^*=\boldsymbol{k}^*$. We may assume that $r=1$, so $Y=t_1^{e_1}\cdots t_n^{e_n}X^{f'}$.

On the other hand as t_i is invertible in $R=B[Y, Y^{-1}]$, $t_i=b_iY^{f_i}$, $b_i \in B^*$. Then we have that

$$ff'+\sum e_if_i = 1.$$

Therefore the following natural homomorphism is surjective.

$$j: \boldsymbol{Z}^{(n+1)} = \boldsymbol{Z}\oplus\cdots\oplus\boldsymbol{Z} \to \boldsymbol{Z}$$
$$j(i_0, i_1, \cdots, i_n) = i_0 f' + \sum i_j e_j\,.$$

Since $\boldsymbol{Z}$ is P.I.D., we can construct a basis of $\boldsymbol{Z}^{(n+1)}$ containing this vector $(f', e_1, \cdots, e_n)$. Put this basis

$$e_0 = (f', e_1, \cdots, e_n)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$e_i = (f_i, f_{i1}, \cdots, f_{in})$$

and put $u_i = t_1^{f_{i1}} \cdots t_n^{f_{in}} X^{f_i}$.

$$R = A_0[u_1, \cdots, u_n, (u_1 \cdots u_n)^{-1}]\,[Y, Y^{-1}] = A[X, X^{-1}] = B[Y, Y^{-1}].$$

Therefore A is isomorphic to B. Hence A is torus invariant.

(2.4) Let $R=A[X, X^{-1}]=B[Y, Y^{-1}]$. An ideal I of R is said to be vertical relative to A if there exists an ideal J of A such that $JR=I$. If J is an ideal of A such that JR is vertical relative to B, then we will simply say that J is vertical relative to B. If A is a $\boldsymbol{k}$-affine domain, the prime ideals defined by the singular locus of Spec A are vertical relative to B.

Proposition 2.5. *Let $R=A[X, X^{-1}]=B[Y, Y^{-1}]$. If there exists a maximal ideal of A which is vertical relative to B, then $A[X, X^{-1}]=B[X, X^{-1}]$. In particular A and B are isomorphic.*

Proof. Let m be a maximal ideal of A which is vertical relative to B. Then there exists an ideal n of B such that $mR=nR$. Therefore $R/mR=A/m[X, X^{-1}]=R/nR=B/n[Y, Y^{-1}]$, where $X=\bar{v}Y^f$ and $Y=\bar{u}X^{f'}$. Since m is a maximal ideal, A/m is a field. Hence $\bar{u}$ is in B/n by (1.1). Therefore we obtain $f=\pm 1$ by (2.1). Thus A is isomorphic to B.

Corollary 2.6. *Let A be a k-affine domain with isolated singular points, then A is torus invariant.*

3. Strongly torus invariant rings

In this section we investigate strongly torus invariant rings.

Proposition 3.1. *Let $A[X, X^{-1}]=B[Y, Y^{-1}]$. If $Q(A)\subseteq Q(B)$, then $A=B$, where $Q(R)$ is the total quotient field of R.*

Proof. Let x be an element of A, then there exist two elements b and b' of B such as $x=b/b'$. Hence $b=b'x$. In the graded ring $B[Y, Y^{-1}]$ the elements b and b' are homogeneous of degree zero, thus x is also degree zero. Hence we have $A\subseteq B$. Let b be an element of B. Then $b=\sum a_j X^j$, $a_j \in A$. By (2) of (2.0) we have that $b=\sum a_j v^j Y^{jf}$. If $f=0$, then $X\in B$. Thus $A[X, X^{-1}]\subseteq B$, it's a contradiction, hence $f \neq 0$. Since $a_j v^j \in B$ and Y is a variable over B, $b = a_0 \in A$. Thus $A = B$.

Corollary 3.2. *Let $\bar{A}$ denote the integral closure of A. If $\bar{A}$ is strongly torus invariant, then A is also so.*

Proof. It is easily seen that if $A[X, X^{-1}] = B[Y, Y^{-1}]$ then $\bar{A}[X, X^{-1}] = \bar{B}[Y, Y^{-1}]$. Since $\bar{A}$ is strongly torus invariant, $\bar{A} = \bar{B}$. Hence $Q(A) = Q(B)$, and we have that $A = B$.

Proposition 3.3. *Let A be a domain with $J(A) \neq 0$, where $J(D)$ is the Jacobson radical of a ring D. Then A is strongly torus invariant.*

Proof. Let a be a non-zero element of $J(A)$. Then $1+a$ is unit, so in the graded ring $B[Y, Y^{-1}]$, $1+a$ is homogeneous. Since the "unity 1" is a homogeneous element of degree 0, the element a is also so. Thus the element a is contained in B.

Let x be any element of A. Since xa is contained in $J(A)$, xa is in B. Hence A is contained in $Q(B)$. By (3.1), we have that $A = B$.

Corollary 3.4. *If A is a local domain, then A is strongly torus invariant.*

Proposition 3.5. *Let A be an affine ring over a field $\boldsymbol{k}$ and let $A[X, X^{-1}] = B[Y, Y^{-1}]$. Then $A = B$ if and only if every maximal ideals of A is vertical relative to B.*

Proof. It suffices to prove the "if" part of the (3.5). By (3.3) we may assume that $J(A) = 0$. Let x be an element of B and let $x = \sum_{j=s}^{t} a_j X^j$, where $s < t$, $a_j \in A$ and $a_t \neq 0$ and $a_s \neq 0$. For any maximal ideal m of A there exists a maximal ideal n of B such as $mR = nR$, where $R = A[X, X^{-1}]$. Let $\bar{x}$ denote the residue class of x in B/n. Then $\bar{x}$ is algebraic over the coefficient field $\boldsymbol{k}$, hence there exist elements $\lambda_0, \lambda_1, \cdots \lambda_{n-1}$ in $\boldsymbol{k}$, such that $f(x) = x^n + \lambda_{n-1} x^{n-1} + \cdots + \lambda_0 \in nR = mR$. If $t \neq 0$, then the highest degree term of $f(x)$ with respect to X is $a_t^n x^{nt} \in mR$, thus a_t is contained in m for every maximal ideal in A. Since $J(A) = 0$, $a_t = 0$. It's a contradiction. Therefore $t = 0$. By the same way, we have that $s = 0$, hence x is in A. Thus $A = B$.

We denote the subring generated by all the units of A by A_u.

Proposition 3.6. *Let A be a $\boldsymbol{k}$-affine domain with an isolated singular point. If A is algebraic over A_u, then A is strongly torus invariant.*

Proof. Let $A[X, X^{-1}] = B[Y, Y^{-1}]$ and let m be the maximal ideal defined by the isolated singular point. Then there exists a maximal ideal n of B such as $mR = nR$. Let a be a unit element of A. In the graded ring $B[Y, Y^{-1}]$, the element a is also invertible, so $a=bY^j$, for some invertible element b in B and a certain integer j. Since A/m is algebraic over $\boldsymbol{k}$, there exist elements $\lambda_0, \lambda_1, \cdots, \lambda_n \in \boldsymbol{k}$ such that $\lambda_n a^n + \cdots + \lambda_1 a + \lambda_0 \in mR = nR$. If $j \neq 0$, $\lambda_n b^n$ is in n, hence b is not invertible, it's a contradiction. Thus we have that $A_u \subseteq B$. By the following lemma our proof is over.

Lemma 3.7. *Let $A[X, X^{-1}]=B[Y, Y^{-1}]$. If A is algebraic over $A \cap B$, then $A=B$.*

Proof. Since A is algebraic over $A \cap B$, A is also algebraic over B, but B is algebraically closed in $B[Y, Y^{-1}]$, therefore A is contained in B. Thus we have that $A=B$.

Let A be an integral domain containing a field $\boldsymbol{k}$. We denote the set of all automorphisms of A over $\boldsymbol{k}$ by $\mathrm{Aut}_k(A)$.

Proposition 3.8. *Let A be an integral domain containing an infinite field $\boldsymbol{k}$. If $Aut_k(A)$ is a finite set, then A is strongly torus invariant.*

Proof. Let $R=A[X, X^{-1}]=B[Y, Y^{-1}]$. Let Φ_λ, $\lambda \in \boldsymbol{k}^*$, be an automorphism of R defined by $\Phi_\lambda(Y)=\lambda Y$ and $\Phi_\lambda(b)=b$ for $b \in B$. Following the notation of (2.0) we have $X=vY^f$, thus $\Phi_\lambda(X)=\lambda^f X$, therefore $R=\Phi_\lambda(A)[X, X^{-1}]$. Let p be the projection $A[X, X^{-1}] \rightarrow A$ defined by $p(X)=1$ and i be the canonical injection $A \hookrightarrow A[X, X^{-1}]$. Define $\sigma_\lambda = q \circ \Phi_\lambda \circ i$. Then σ_λ is an endomorphism of A. We shall show that σ_λ is surjective. Let x be an element of A. Since $R=\Phi_\lambda(A)[X, X^{-1}]$, there exist elements a_j's of A such as $x=\sum \Phi_\lambda(a_j)X^j$. Hence $x=p(x)=\sum p\Phi_\lambda(a_j)$. Let $x'=\sum a_j \in A$, then $\sigma_\lambda(x')=\sum p\Phi_\lambda(a_j)=x$. Thus σ_λ is surjective. Next we shall show that σ_λ is injective. Since $\Phi_\lambda^{-1}((X-1)R \cap \Phi_\lambda(A))=\Phi_\lambda^{-1}(X-1)R \cap A=(\lambda^{-f}X-1)R \cap A=0$, we have $(X-1)R \cap \Phi(A)=0$, therefore σ_λ is injective. Hence σ_λ is an automorphism of A.

We shall prove that the set $\{\sigma_\lambda | \lambda \in \boldsymbol{k}^*\}$ is infinite when $A \neq B$. Since $u=v^{-f}Y^{1-ff'}$, $\sigma_\lambda(u)=\lambda^{1-ff'}u$. Therefore our assertion is proved when $1-ff' \neq 0$. Suppose $ff'=1$. Then we may assume that $R=A[X, X^{-1}]=B[X, X^{-1}]$. If $A \subseteq B$, then $A=B$, so there exists an element x of A not contained in B, say $x=\sum_{j=s}^{t} b_j X^j$, $t>s$. Since $\ker p=(X-1)R$ and $(X-1)R \cap B=0$, $p(b_j) \neq 0$ for $b_j \neq 0$. Since $\sigma_\lambda(x)=\sum p(b_j)\lambda^j$ and $p(b_j) \neq 0$ for some $j \neq 0$, the set $\{\sigma_\lambda;\, \lambda \in \boldsymbol{k}^*\}$ is infinite.

Next we shall give two cases of rings which are not strongly torus invariant. If A has a non-trivial locally finite iterative higher derivation ψ: $A \rightarrow A[T]$, then $A[T]=B[T]$, where $B=\psi(A)$ and $A \neq B$, as is proved in [4]. Hence we have that $A[T, T^{-1}]=B[T, T^{-1}]$ and $A \neq B$. If A is a graded ring, then A is not strongly torus invariant. Indeed, let X be a variable over A and let

$B_i=\{a_iX^i;\ a_i\in A_i\}$. Then B_i is an A_0-module contained in $A[X, X^{-1}]$. Let $B=\sum B_i$. Then B is a graded ring and we easily see that $A[X, X^{-1}]=B[X, X^{-1}]$. We shall show that X is a variable over B. Assume that there exist elements $b_0, b_1, \cdots, b_n$ in B such that $b_n\neq 0$ and $b_nX^n+\cdots+b_1X+b_0=0$. By the definition of B we denote $b_i=\sum a_{ij}X^j$, $a_{ij}\in A_j$. In the graded ring $A[X, X^{-1}]$ the homogeneous term of degree t of this equation is that

$$(a_{n,t-n}+a_{n-1,t-n+1}+\cdots+a_{0,t})X^t = 0\,.$$

Since A is a graded ring and a_{ij} is a homogeneous element of degree j, we obtain $a_{ij}=0$ for all index i and j, hence X is a variable over B.

By [4] we have that a $\boldsymbol{k}$-algebra A has a non-trivial locally finite iterative higher derivation if and only if $\mathrm{Aut}_k(A)$ has a subgroup isomorphic to $G_a=\mathrm{Spec}\ \boldsymbol{k}[T]$. We easily see that A is a non-trivial graded ring if and only if $\mathrm{Aut}_k(A)$ has a subgroup isomorphic to $G_m=Spec(\boldsymbol{k}[T, T^{-1}])$.

Proposition 3.9. *A k-algebra A is not strongly torus invariant, if $Aut_k(A)$ has a subgroup isomorphic to G_a or G_m.*

Assume that $\mathrm{Aut}_k(A)$ is an infinite group. If $\mathrm{Aut}_k(A)$ has an algebraic group structure, then there exists the following exact sequence;

$$0 \to T \to \mathrm{Aut}_k(A)_0 \to \theta \to 0$$

where $\mathrm{Aut}_k(A)_0$ is the connected component containing the identity I_A, and T is a maximal torus subgroup of $\mathrm{Aut}_k(A)_0$ and θ is an abelian variety. Let P be an arbitrary closed point of $Spec(A)$. If $T=0$, then there exists a regular map

$$\begin{aligned}\Phi\colon \mathrm{Aut}_k(A)_0 &\to \mathrm{Spec}(A)\\ \sigma &\to \sigma(P)\,.\end{aligned}$$

Since $Im(\Phi)$ is a projective variety contained in the affine variety $Spec(A)$, the set $Im(\Phi)$ consists of one point, it contradicts $\dim \mathrm{Aut}_k(A)_0>0$. Hence we have that $T\neq 0$. Since $T\supseteqq G_a$ or G_m, we have the following result:

Proposition 3.10. *If $Aut_k(A)$ is not a finite set and has an algebraic group structure, then A is not strongly torus invariant.*

4. Affine domains of dimension $\leqq 2$

Let $\boldsymbol{k}$ be a field of characteristic zero which contains all roots of "unity". In this section let A be an affine domain over $\boldsymbol{k}$. We shall see that if $\dim A=1$, then A is always torus invariant. Moreover A is not strongly torus invariant if and only if $\mathrm{Aut}_k(A)\supseteqq G_m$. Let $\dim A\geqq 2$. Then A is not always torus invariant. But if an integrally closed domain A is not a $\boldsymbol{Z}$-graded ring, then A is torus invariant.

For the proof we need a lemma.

Lemma 4.1. *Let K be a finite separable algebraic field extension of a field $\boldsymbol{k}$. If A is a one-dimensional affine normal ring such that $\boldsymbol{k}\subset A\subseteqq K[X, X^{-1}]$, then A is a polynomial ring or a torus ring over $\boldsymbol{k}'$ where $\boldsymbol{k}'$ is the algebraic closure of $\boldsymbol{k}$ in A.*

Proof. We may assume that $\boldsymbol{k}=\boldsymbol{k}'$. Following the similar device to the proof of (2.9) in [1, p 322], we have $Q(A)=\boldsymbol{k}(\theta)$ for some element θ of A. Since $\boldsymbol{k}[\theta]\subseteqq A\subset\boldsymbol{k}(\theta)$, $A=\boldsymbol{k}[\theta]$ or $A=\boldsymbol{k}\left[\theta, \frac{1}{f(\theta)}\right]$ for some polynomial $f(\theta)\in\boldsymbol{k}[\theta]$. Let $A=\boldsymbol{k}\left[\theta, \frac{1}{f(\theta)}\right]$. Then we may assume that $f(\theta)$ has no multiple factors. The element $f(\theta)$ is invertible in A, so is also invertible in $K[X, X^{-1}]$. Thus we have $f(\theta)=\beta X^r$, $\beta\in K$, $\theta\in K[X, X^{-1}]$. We may assume that $r\geqq 0$, if neccessary, by replacing X with X^{-1}. Then we easily see that $\theta\in K[X]$. The uniqueness of the irreducible decomposition in a polynomial ring implies that $\deg_\theta f(\theta)=1$, since the polynomial $f(\theta)$ has not multiple factors and $f(\theta)=\beta X^r$. Hence we may assume that $f(\theta)=\theta$ and we obtain $A=\boldsymbol{k}\left[\theta, \frac{1}{\theta}\right]$.

Let A be an integral domain. If A is contained in $K[X, X^{-1}]$, then $\bar{A}$ is a polynomial ring or a torus ring over $\boldsymbol{k}'$.

Proposition 4.2. *Let A be a one-dimensional affine domain over a field $\boldsymbol{k}$ of characteristics zero. Then we obtain that*

(1) *A is torus invariant,*

(2) *A is not strongly torus invariant if and only if $Aut_k(A)$ has a subgroup isomorphic to G_m. If A is not strongly torus invariant and A is integrally closed, then A is a polynomial ring or a torus ring over the algebraic closure of $\boldsymbol{k}$ in A.*

Proof. At first we shall prove (2). The sufficiency follows from (3.9). Let $R=A[X, X^{-1}]=B[Y, Y^{-1}]$ in which $A\neq B$. If $mR\cap A\neq 0$ for any maximal ideal m of B, then m is vertical relative to A, and we have $A=B$ by (3.5). Hence there exists a maximal ideal m such as $mR\cap A=0$. Since $ch\ \boldsymbol{k}=0$, $B/m=K$ is a finite separable algebraic field over $\boldsymbol{k}$. The residue mapping of R to R/mR yields (up to isomorphism) $\boldsymbol{k}\subset A\subseteqq K[Y, Y^{-1}]$ where Y is algebraiclly independent over K. Therefore A is a polynomial ring or a torus ring by the lemma (4.1). Thus the automorphism group $\mathrm{Aut}_k A$ contains a subgroup isomorphic to G_m.

Assume that A is not integrally closed. Then prime divisors in A of the conductor $\mathfrak{k}(\bar{A}/A)$ are vertical relative to B. Hence we may assume $X=Y$ by

(2.5). The above lemma (4.1) implies that $\bar{A}=\boldsymbol{k}'[t, t^{-1}]$ or $\bar{A}=\boldsymbol{k}'[t]$ where $\boldsymbol{k}'$ is the algebraic closure of $\boldsymbol{k}$ in $\bar{A}$.

Firstly let $\bar{A}=\boldsymbol{k}'[t]$. Since $\bar{A}\cong\bar{B}$, there exists an element s in $\bar{B}$ such as $\bar{B}=\boldsymbol{k}'[s]$. Since $\bar{R}=\bar{A}[X, X^{-1}]=\bar{B}[X, X^{-1}]$, we have $\boldsymbol{k}'[X, X^{-1}]\,[t]=\boldsymbol{k}'[X, X^{-1}]\,[s]$, hence we easily see that $t=f_1(X)s+f(X)$ and $s=g_1(X)t+g(X)$ where $f_1(X)\,g_1(X)=1$ and $f(X)$, $g(X)\in\boldsymbol{k}'[X, X^{-1}]$. We may assume that $t=X^n s+f(X)$ and $s=X^{-n}t+g(X)$. Let $\bar{n}$ be a prime divisor in $\bar{A}$ of the conductor $\mathfrak{k}(\bar{A}/A)$.

Then there exists a maximal ideal $\bar{m}$ of $\bar{B}$ such as $\bar{n}\bar{R}=\bar{m}\bar{R}$. Since $\bar{A}/\bar{n}$ is algebraic over $\boldsymbol{k}$, there exist elements $\lambda_0, \lambda_1, \cdots, \lambda_{d-1}\in\boldsymbol{k}$ such that $t^{d-1}+\lambda_{d-1}t^{d-1}+\cdots+\lambda_0\in\bar{m}\bar{R}=\bar{n}\bar{R}$. Hence we have that $(X^n s+f(X))^d+\lambda_{d-1}(X^n s+f(X))^{d-1}+\cdots+\lambda_0\in\bar{n}\bar{R}$. The constant term of this polynomial with respect to s is the following;

$$f(X)^d+\lambda_{d-1}f(X)^{d-1}+\cdots+\lambda_0\in\bar{n}\boldsymbol{k}'[s]\,[X, X^{-1}]\,.$$

Therefore $f(X)=f\in\boldsymbol{k}'$. Hence we may assume that $t=X^n s$. We shall show that A is a graded ring. Let a be an element of A. Since a is contained in $\bar{A}=\boldsymbol{k}'[t]$ and $t=X^n s$, we have that $a=\sum\lambda_j t^j=\sum\lambda_j s^j X^{jn}$, $\lambda_j s^j\in\bar{B}$. On the other hand, as the element a is contained in $B[X, X^{-1}]$, $a=\sum b_i X^i$, $b_i\in B$. Comparing the coefficient of the each term in the following; $\sum\lambda_j s^j X^{jn}=\sum b_i X^i$, we have $b_i=\lambda_j s^j$ $(i=jn)$ and $b_i=0$ $(i\notin n\boldsymbol{Z})$. If $b_i\neq 0$, then $b_i X^i=\lambda_j s^j X^{jn}=\lambda_j t^j\in B[X, X^{-1}]\cap\bar{A}=A[X, X^{-1}]\cap\bar{A}=A$. Therefore A has a graded ring structure.

Secondary let $\bar{A}=\boldsymbol{k}'[t, t^{-1}]$. Then $\bar{B}=\boldsymbol{k}'[s, s^{-1}]$. Since t and s are invertible in $\bar{R}$, we may assume that $t=s^i X^n$ and $s=t^j X^m$, then $t=(t^j X^m)^i X^n=t^{ij}X^{im+n}$, therefore $ij=1$. Hence we may assume $t=sX^n$. By the same method as in the case $\bar{A}=\boldsymbol{k}'[t]$ we have that A is a graded ring.

Proof of (1). If A is not integrally closed, then the prime divisors of the conductor $\mathfrak{k}(\bar{A}/A)$ are vertical relative to B. Since non-zero prime ideals of A are maximal, the ring A is isomorphic to B by (2.5). If A is integrally closed and A is neither a polynomial ring nor a torus ring, then A is strongly torus invariant, hence A is torus invariant. If A is either a polynomial ring or a torus ring, A is torus invariant by (2.2) and (2.3).

Next we shall consider the case; the coefficient field k has all roots of "unity" and its characteristic is zero. Then we prove the following:

Theorem 4.3. *Let A be an integrally closed $\boldsymbol{k}$-affine domain of dimension two, where the field $\boldsymbol{k}$ has all roots of "unity" and ch $\boldsymbol{k}=0$. If A is not torus invariant, then A is a $\boldsymbol{Z}$-graded ring which contains units of non-zero degree.*

Proof. Assume that A is not torus invariant. Then there exist a $\boldsymbol{k}$-algebra B and independent variables X, Y such that A is not isomorphic to B and $R=A[X, X^{-1}]=B[Y, Y^{-1}]$. By (2.0) and (2,1) we obtain $ff'\neq 1$. We shall show that it follows from $ff'\neq 1$ that A is a $\boldsymbol{Z}$-graded ring. We may only consider the case $1-ff'>0$. Let x be a $(1-ff')-th$ root of u and let $y=x^{-f}/X$. Then $y^{1-ff'}=v$. Since $(y^{-f'}y)^{1-ff'}=u$, $x=\lambda y^{-f'}Y$ for some $(1-ff')-th$ root λ of "unity". From the relations; $y=x^{-f}X$ and $Y=uX^{f'}$, we have $\lambda=1$.

Since $y=x^{-f}X$ and $x=y^{-f'}Y$ are invertible, we have $A[x]\,[X, X^{-1}]=B[y]\,[Y, Y^{-1}]=A[x]\,[y, y^{-1}]=B[y]\,[x, x^{-1}]$. Define a surjective homomorphism $j\colon A[x]\,[y, y^{-1}]\to A[x]$ by $j(y)=1$. Let $A_0=j(B[y])\subseteq A[x]$. We shall show that $A[x]=A_0[x, x^{-1}]$. Let a be an element of A. Then $a=\sum b_ix^i$, $b_i\in B$. Since $j(a)=a$ and $j(x)=x$, we have that $a=\sum j(b_i)x^i$, $j(b_j)\in A_0$. Thus $A[x]=A_0[x, x^{-1}]$ and x is algebraically independent over A_0. By the same way $B[y]=B_0[y, y^{-1}]$.

Since the every $(1-ff')-th$ roots of "unity" is contained in k and $ch\ k=0$ and A is normal, the extension $A[x]/A$ is a Galois extension with a cyclic group $G=\langle\sigma\rangle$ (cf. [3] p 214). Indeed when $|G|=n$, $n\,|\,1-ff'$ and there exists a primitive $n-th$ root λ of "unity" such that $\sigma(x)=\lambda x$ and the invariant subring $(A[x])^{\sigma}=A$ and $A[x]=A+Ax+\cdots+Ax^{n-1}$ is a free A-module.

Since the element u is a unit of A and $ch(\boldsymbol{k})=0$, the extension $A[x]/A$ is étale. Since A is a normal domain, $A[x]$, hence $A_0[x, x^{-1}]$, is also a normal domain. From this we see that A_0 is always normal.

We shall show that there exists a subring A_0' in $A[x]$ such that $A[x]=A_0'[x, x^{-1}]$ and $\sigma(A_0')=A_0'$. If A_0 is strongly torus invariant, then $\sigma(A_0)=A_0$; for $\sigma(A_0)\,[x, x^{-1}]=A_0[x, x^{-1}]$, therefore A_0 satisfies the conditions. If A_0 is not strongly torus invariant, then $A_0=\boldsymbol{k}'[t]$ or $=\boldsymbol{k}'[t, t^{-1}]$ by (4.2). Firstly let $A_0=\boldsymbol{k}'[t]$. Since $\boldsymbol{k}'[x, x^{-1}]\,[t]=\boldsymbol{k}'[x, x^{-1}]\,[\sigma(t)]$, we easily see that $\sigma(t)=\mu x^it+f(x)$, $\mu\in k^*$ and $f(x)\in\boldsymbol{k}'[x, x^{-1}]$. The order of σ is n, i.e. $\sigma^n=$Identity, so $\sigma^n(t)=t$, the other hand $\sigma^n(t)=\mu^n\lambda^{(1+\cdots n-1)i}x^{in}t+g(x)$, $g(x)\in\boldsymbol{k}'[x, x^{-1}]$, therefore we have that $i=0$, thus $\sigma(t)=\mu t+f(x)$ and $\mu^n=1$. Let $f(x)=\sum f_ix^i$ and define the set $\Delta=\{j\in Z;\ \lambda^j\neq\mu\}$. Let $h(x)=\sum_{j\in\Delta}h_jx^j$, where $h_j=f_j(\mu-\lambda^j)^{-1}$, and put $s=t+h(x)$. Then $\sigma(s)=\mu s+\sum_{i\notin\Delta}f_ix^i$, hence $\sigma^n(s)=\mu^n s+n\mu^{n-1}(\sum_{i\notin\Delta}f_ix^i)=s+n\mu^{n-1}(\sum_{i\notin\Delta}f_ix^i)$. Since $\sigma^n(s)=s$, we have $\sigma(s)=\mu s$. We set $A_0'=\boldsymbol{k}'[s]$, then A_0' satisfies the conditions.

Secondary let $A_0=\boldsymbol{k}'[t, t^{-1}]$. Since $\boldsymbol{k}'[x, x^{-1}]\,[t, t^{-1}]=\boldsymbol{k}'[x, x^{-1}]\,[\sigma(t), \sigma(t)^{-1}]$, we easily see that $\sigma(t)=\mu x^it$ or $\sigma(t)=\mu x^it^{-1}$, $\mu\in\boldsymbol{k}'^*$.

Case (i); $\sigma(t)=\mu x^it$. Since $\sigma^n(t)=\mu^n\lambda^{(1+\cdots+n-1)i}x^{ni}t$ and $\sigma^n(t)=t$, we have that $\sigma(t)=\mu t$, so $\sigma(A_0)=A_0$.

Case (ii); $\sigma(t)=\mu x^it^{-1}$. If n is odd, say $n=2m+1$, then $\sigma^n(t)=\mu\lambda^{im}x^it^{-1}$, but this is imposible for $\sigma^n(t)=t$. Therefore n is even, say $n=2m$. Then $\sigma^n(t)=\lambda^{im}t$. Since λ is a primitive $n-th$ root of "unity", the integer i is even, say $i=2j$. Let $s=x^{-j}t$ and $A_0'=\boldsymbol{k}'[s, s^{-1}]$. Then A_0' satisfies the conditions.

Next we shall show that A has a $\boldsymbol{Z}$-graded ring structure. Let a be an element of A. Since $a\in A_0'[x, x^{-1}]$, $a=\sum a_i x^i$. Then $a=\sigma(a)=\sum\sigma(a_i)\lambda^i x^i$ and $\sigma(a_i)\in A_0'$. Comparing the coefficient of each term in the equality; $\sum a_i x^i=\sum\sigma(a_i)\lambda^i x^i$, we have that $a_i=\sigma(a_i)\lambda^i$, then $\sigma(a_i x^i)=a_i x^i$. Thus $a_i x^i$ is an element of A. Therefore A is a graded ring. Since there exists units of non-zero degree, A has a $\boldsymbol{Z}$-graded ring structure.

REMARK. The converse of this theorem is false. Indeed we find by (2.3) that the ring $\boldsymbol{k}[T][X, X^{-1}]$ is a $\boldsymbol{Z}$-graded ring with respect to X which is torus invariant.

EXAMPLE. We shall construct an example of an affine dimension A of dimension two which is not torus invariant.

Let D be an integrally closed domain of dimension one over an algebraically closed field $\boldsymbol{k}$ and $D^*=\boldsymbol{k}^*$. Let a be a non-unit of D and $\alpha^5=a$, $\alpha\in D$. Assume that D is noetherian and $D[\alpha]$ is strongly torus invariant. Since an affine domain of dimension one whose totally quotient field has a positive genus is strongly torus invariant by (4.2), this assumption can be satisfied foı a suitable choice of D. Let T be a variable over D and $A=D[\alpha T, T^5, T^{-5}]$. Let X be a variable over A and $S=T^2X$ and $Y=T^5X^2$. Let $B=D[\alpha S^3, S^5, S^{-5}]$. Since $T=S^{-2}Y$ and $X=S^5Y^{-2}$, we have that $A[X, X^{-1}]=B[Y, Y^{-1}]$. By (1.1) invertible elements in the graded ring A are homogeneous. Since $D^*=\boldsymbol{k}^*$, we obtain $A^*=\{\eta T^{5i};\ \eta\in\boldsymbol{k}^* \text{ and } i\in\boldsymbol{Z}\}$. Hence the quotient $A^*/\boldsymbol{k}^*$ is generated by T^5. Similary $B^*/\boldsymbol{k}^*$ ıs generated by S^5. We shall show that A is not isomorphic to B. We assume that there exists an isomorphism σ of A to B. Since σ is a group-isomorphism of A^* to B^*, we have $\sigma(T^5)=\mu S^5$ or $\sigma(T^5)=\mu S^{-5}$, $\mu\in\boldsymbol{k}^*$. We shall only consider the case: $\sigma(T^5)=\mu S^5$, since the proof of the other case is the similar. Let $\bar{\sigma}$ be an isomorphism of $A[T]$ to $B[S]$ defined by $\bar{\sigma}=\sigma$ on A and $\bar{\sigma}(T)=\zeta S$, $\zeta^5=\mu$. Then we have that $D[\alpha][S, S^{-1}]=\bar{\sigma}(D[\alpha])[S, S^{-1}]$, therefore $\bar{\sigma}(D[\alpha])=D[\alpha]$; for $D[\alpha]$ is strongly torus invariant. Since $\sigma(D)\subseteq D[\alpha]\cap B=D$, we have $\sigma(D)=D$, therefore we easily see that σ is an isomorphism as graded rings. Thus we have $\sigma((\alpha T)D)=(\alpha^2 S)D$, hence $\sigma(a)\in a^2D$. Since the element a is not a unit, $a^2D\subsetneqq aD$, thus $\sigma(a)D\subseteqq a^2D\subsetneqq aD$, so $aD\subsetneqq\sigma^{-1}(a)D$, hence we have a proper ascending chain $\{\sigma^{-n}(a)D\}$, but it contradicts the netherian assumption of D. Hence A is not torus invariant.

(4.4) Now let $A=\sum A_i$ be an integrally closed $\boldsymbol{Z}$-graded domain which contains invertible elements of non-zero degree. Let e be an invertible element of A with the smallest positive degree d. Let a be a unit of A, then a is a homogeneous elements with deg $a=jd$ for some integer j, and there exists an element ξ of A_0^* such as $a=\xi e^j$. Let i be any positive integer and x be one of the *ijd-th* roots of a, say $x^{ijd}=a$. Since $A[x]$ is a $\boldsymbol{Z}$-graded ring with the invertible elements x of degree one, $A[x]=A_0'[x, x^{-1}]$ by (1.4) where A_0' contains A_0. Let f and f' be integers such as $ff'+ijd=1$ and let X be a variable over A. Put $y=x^{-f}X$ and $Y=aX^{f'}$. Then $x=y^{-f'}Y$ and $X=y^{ijd}Y^f$. Therefoιe $A_0'[x, x^{-1}][X, X^{-1}]=A_0'[y, y^{-1}][Y, Y^{-1}]$. Since the every *n-th* roots of "unity" is contained in $\boldsymbol{k}$ and A is integral closed, the extension $A[x]/A$ is a Galois extension with a cyclic group $G=\langle\sigma\rangle$. Indeed $|G|=di$ and there exists a primitive *di-th* root λ of "unity" such as $\sigma(x)=\lambda x$, and $(A[x])^\sigma=A$. Since A_0' is algebraic over A_0, $\sigma(A_0')$ is also so, hence $\sigma(A_0')$ is algebraic over A_0', but A_0' is algebraically closed in $A_0'[x, x^{-1}]$, therefore $\sigma(A_0')=A_0'$. Since $\sigma(y)=\lambda^{-f}y$, σ is an automorphism of $A_0'[y, y^{-1}]$. Let $B=A_0'[y, y^{-1}]^\sigma$ and $\bar{\sigma}$ be an automorphism of $A_0'[x, x^{-1}][X, X^{-1}]$ defined by $\bar{\sigma}(X)=X$ and $\bar{\sigma}=\sigma$ over $A_0'[x, x^{-1}]$. Since $\bar{\sigma}(Y)=Y$ and $\bar{\sigma}(X)=X$, we obtain $B[Y, Y^{-1}]=A_0'[y, y^{-1}][Y, Y^{-1}]^\sigma$ $=A_0'[x, x^{-1}][X, X^{-1}]^\sigma=A[X, X^{-1}]$.

Proposition 4.5. *Let A be an integrally closed $\boldsymbol{k}$-affine domain of dimension* 2. *If $A[X, X^{-1}]=B[Y, Y^{-1}]$ and $ff'\neq 1$, then A has a $\boldsymbol{Z}$-graded ring structure and B is isomorphic to one of algebras constructed in* (4.4).

Proof. The first statement is already mentioned in the proof of (4.3) and we obtained $A_0'[x, x^{-1}][X, X^{-1}]=A_0'[y, y^{-1}][Y, Y^{-1}]$ and $\sigma(A_0')=A_0'$. Let $B'=A_0'[y, y^{-1}]^\sigma$. Then B' is one of algebras in (4.4). Since $B'[Y, Y^{-1}]=B[Y, Y^{-1}]$, B is isomorphic to B'.

5. D-torus invariant

Let D be an integral domain contining a field $\boldsymbol{k}$ of characteristic zero and A be a D-algebra. The ring A is called D-torus invariant; if $A[X, X^{-1}]=B[Y, Y^{-1}]$ for a certain D-algebra B and independent variables X and Y, then we have always $A\cong_D B$. Then we have the following result:

Proposition 5.1. *Let A be an integrally closed domain over D and tr. $\deg_D A=1$. If A is not D-torus invariant, then A is a $\boldsymbol{Z}$-graded ring containing units of non-zero degree.*

Proof. Let $A[X, X^{-1}]=B[Y, Y^{-1}]$, where B is a D-algebra and not D-isomorphic to A. By (2.0) and (2.1) we easily see that $ff'=1$. Then we may assume $1-ff'>0$. Let x be a $(1-ff')-th$ root of u and $y=x^{-f}X$. Then we have that $A[x]=A_0[x, x^{-1}]$ and $B[y]=B_0[y, y^{-1}]$ as the proof of (4.3), where A_0 and B_0 are respectively subalgebras of $A[x]$ and $B[y]$ containing D. Let σ be a generator of the cyclic Galois group of the extension $A[x]/A$. We shall show that $\sigma(A_0)=A_0$. Since *tr.* $\deg_D A_0[x, X^{-1}]=1$, A_0 is algebraic over D, thus $\sigma(A_0)$ is also so. Since A_0 is algebraically closed in $A_0[x, x^{-1}]$, we have that $\sigma(A_0)=A_0$. Following the similar devise to the proof of (4.3) we obtain that A is a $\boldsymbol{Z}$-graded ring, and D is contained in A.

In the following we shall consider the case where A is a $\boldsymbol{Z}$-graded ring and $A_0=D$. We consider only D-isomorphisms of D-algebras.

Theorem 5.2. *Let A be an integrally closed $\boldsymbol{Z}$-graded ring. Assume that the subring A_0 contains an algebraically closed field $\boldsymbol{k}$ and that $A_0^*=\boldsymbol{k}^*$. Let d be the smallest positive integer among the set of degrees of units in A. Then the number of the isomorphic classes of A_0-algebra as B such that $A[X, X^{-1}]=B[Y, Y^{-1}]$ equals to $\Phi(d)$, where Φ is the Euler function.*

Proof. Let i be an integer such as $1\leqq i<d$ and $(i, d)=1$. Since $(i, d)=1$, $ij+dh=1$ for some integers j and h. Moreover we may assume $h\geqq 0$. Fix a unit e of degree d. Let x be one of the *d-th* roots of e. Then we have that $A[x]=A_0'[x, x^{-1}]$ for a subring A_0' containing A_0 by (1.4). Let σ be a generator of the cyclic Galois group of the extension $A[x]/A$. Then $\sigma(x)=\lambda x$, where λ is a primitive *d-th* root of "unity". Since A_0' is algebraic over A_0 and algebraically closed in $A_0'[x, x^{-1}]$, we obtain $\sigma(A_0')=A_0'$. Let X be a variable over A and let $y=x^{-i}X$ and $Y=e^hx^j$. Then we have that $A_0'[x, x^{-1}]$ $[X, X^{-1}]=A_0'[y, y^{-1}]$ $[Y, Y^{-1}]$. Define $B_i=A_0'[y, y^{-1}]^\sigma$ and let $\bar{\sigma}$ be an isomorphism of $A_0'[x, x^{-1}]$ $[X, X^{-1}]$ defined by $\bar{\sigma}(X)=X$ and $\bar{\sigma}=\sigma$ on $A_0'[x, x^{-1}]$. Since $Y=e^hX^j$, $\bar{\sigma}(Y)=Y$, therefore we obtain that $A[X, X^{-1}]=B_i[Y, Y^{-1}]$. We can easily see that B_i is a $\boldsymbol{X}$-graded ring and $(B_i)_0=A_0$. Especially we have $B_1\simeq A$.

Let i_1 and i_2 be integers such as $1\leqq i_1<i_2<d$ and $(i_1, d)=(i_2, d)=1$. Let $B'=A_0'[y, y^{-1}]^\sigma$ and $B''=A_0'[z, z^{-1}]^\sigma$ where $\sigma(y)=\lambda^{-i_1}y$ and $\sigma(z)=\lambda^{-i_2}$ *i.e.*, $B'=B_{i_1}$ and $B''=B_{i_2}$. We shall show that B' and B'' are not isomorphic. Assume that there exists an A_0-isomorphism ψ of B' to B''. Let a be a unit in B' of non-zero degree, say degree $a=n$, $n\neq 0$. Let b be a homogeneous element of B' and degree $b=t$. Then we have $b^n=ra^t$ for an element r in the coefficient ring A_0, hence $\psi(b^n)=\psi(b)^n=r\psi(a^t)$. Since r and $\psi(a^t)$ are homogeneous, $\psi(b)$ is also homogeneous by (1.1), therefore ψ is an isomorphism as graded rings.

Let c be a homogeneous element in B' of degree one. Then $c=s_1y$ for an element s_1 in A_0'. Since $\sigma(c)=c$ and $\sigma(y)=\lambda^{-i_1}y$, we have $\sigma(s_1)=\lambda^{i_1}s_1$ hence s_1^d is in B'. Since $\psi(s_1y)$ is a homogeneous element of degree one, we obtain $\psi(s_1y)=s_2z$ for an element s_2 in A_0'. Since $\sigma(s_2z)=s_2z$ and $\sigma(z)=\lambda^{-i_2}z$, we have $\sigma(s_2)=\lambda^{i_2}s_2$, hence s_2^d is in B''. By the relations; $s_1^d\psi(y^d)=\psi((s_1y)^d)=\psi(s_1y)^d=s_2^dz^d$, we obtain $s_2^d=\psi(y^d)z^{-d}s_1^d$. Since $\psi(y^d)z^{-d}$ is an invertible element in B'' and degree zero, we have $\zeta=\psi(y^d)z^{-d}\in A_0^*=\boldsymbol{k}^*$, therefore we have $s_2=\eta s_1$ for some $\eta\in k$, $\eta^d=\xi$. Hence $\sigma(s_2)=\lambda^{i_1}s_2$, but it contadicts the fact that $\sigma(s_2)=\lambda^{i_2}s_2$ and λ is a primitive *d-th* root of "unity". Therefore $B'\not\cong B''$.

Finally we shall show that if $A[X, X^{-1}]=B[Y, Y^{-1}]$ then B is isomorphic to B_i for some i satisfying $0<i<d$ and $(i, d)=1$. The invertible element u

in (2.0) is homogeneous. Let n be the degree of u. If $n=0$, then A is isomorphic to B by (2.1), hence $B\cong B_1$. Assume $n\neq 0$. Let c be a non-zero homogeneous element of degree 1 and put $\eta=c^n u^{-1}$. Then η is an element of A_0. In the graded ring $B[Y, Y^{-1}]$ the elements u and η are homogeneous, hence c is also homogeneous, thus we denote $c=bY^j$ for some element b in B and some integer j. Then we obtain that $c^n=b^n Y^{nj}$. On the other hand we have $c^n=\eta u=\eta v^{-f'}Y^{1-ff'}$ by (2.0). Therefore we have $1-ff'=nj$.

By the minimality of d we obtain $n=ld$ for some integer l and $u=\xi e$, $\xi\in A_0^*=\boldsymbol{k}^*$. Since the field $\boldsymbol{k}$ is algebraically closed, we may assume $\xi=1$, then the d-th root x of e is an n-th root of u. Since the element λ is a primitive d-th root of "unity", there exists the unique integer i such that $\lambda^{-f}=\lambda^{-i}$, $0<i<d$, then $(i, d)=1$ since $(f, d)=1$. Let $y'=x^{-f}X^j$ and $B'=(A_0'[y', y'^{-1}])^{\bar{\sigma}}$. Then $\sigma(y')=\lambda^{-f}y'=\lambda^{-i}y'$, hence $B'=B_i$. We can easily show that $x=y'^{-f'}Y^j$, therefore we obtain $A_0'[x, x^{-1}]\,[X, X^{-1}]=A_0'[y', y'^{-1}]\,[Y, Y^{-1}]$. Since $\sigma(X)=X$ and $\sigma(Y)=Y$, we have $A[X, X^{-1}]=B_i[Y, Y^{-1}]$, hence $B[Y, Y^{-1}]=B_i[Y, Y^{-1}]$. Thus we have $B\cong B_i$.

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Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan