ON THE COEFFICIENT RING OF A TORUS EXTENSION

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Introduction. S. Abhyankar, W. Heinzer and P. Eakin treated the following problem in [1]; if $A[X]=B[Y]$, when is $A$ isomorphic or identical to $B$? Replacing the polynomial ring by the torus extension we shall take up the following problem; if $A[X, X^{-1}]=B[Y, Y^{-1}]$, when is $A$ isomorphic or identical to $B$? We say that $A$ is torus invariant (resp. strongly torus invariant) whenever $A[X, X^{-1}]=B[Y, Y^{-1}]$ implies $A=B$ (resp. $A=B$). The roles played by polynomial rings in [1] are played by the graded rings in our theory. A graded ring $A=\sum A_i$, $i\in \mathbb{Z}$, with the property that $A_i\neq 0$ for each $i\in \mathbb{Z}$, will be called a $\mathbb{Z}$-graded ring. Main results are the followings.

An affine domain $A$ of dimension one over a field $k$ is always torus invariant. Moreover $A$ is not strongly torus invariant if and only if $A$ has a graded ring structure. An affine domain of dimension two is not always torus invariant. We shall construct an affine domain of dimension two which is not torus invariant. Let $A$ be an affine domain over $k$ of dimension two. Assume that the field $k$ contains all roots of "unity" and is of characteristic zero. If $A$ is not torus invariant, then $A$ is a $\mathbb{Z}$-graded ring such that there exist invertible elements of non-zero degree.

In Section 1 we study elementary properties of graded rings. Especially we are interested in $\mathbb{Z}$-graded rings with invertible elements of non-zero degree. In Section 2 we discuss some conditions for $A$ to be torus invariant. In Section 3 we give several sufficient conditions for an integral domain to be strongly torus invariant. Some relevant results will be found in S. Iitaka and T. Fujita [2]. Section 4 is devoted to the proof of the main results mentioned above. In Section 5 we fix an integral domain $D$ and we treat only $D$-algebras and $D$-isomorphisms there. We shall prove the following two results. When $A$ is a $D$-algebra of $tr.\ deg_a A=1$ and $A$ is not $D$-torus invariant, $A$ is a $\mathbb{Z}$-graded ring such that $D$ is contained in $A_a$. If $A$ is a $\mathbb{Z}$-graded ring such as $D=A_a$, then the number of elements of the set of $\{D$-isomorphic classes of $D$-algebras $B$ such that $A[X, X^{-1}]=B[Y, Y^{-1}]\}$ is $\Phi(d)$, where $d$ is the smallest positive integer among the degrees of units in $A$ and $\Phi$ is the Euler function.

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1. Some properties of graded rings

Let $R$ be commutative ring with indentity. The ring $R$ is said to be a graded ring if $R$ is a graded module, $R=\sum R_i$, and $R_nR_m\subseteq R_{n+m}$.

**Lemma 1.1.** Let $R$ be a graded domain. Then we have the following.

1. The unity element of $R$ is homogeneous.
2. If $a$ is homogeneous and $a=bc$, then $b$ and $c$ are both homogeneous. In particular every invertible element is homogeneous.
3. If $R$ contains a field $k$, then $k$ is a subring of $R_0$.

Proof. (1) follows immediately from the relation $1^2=1$. The proof of (2) is easy and will be omitted. To prove (3) we can assume $k$ is different from $F_2$ by (1). Let $a$ be an element of $k$ different from 1. Then $1-a$ is homogeneous from (2). The unity 1 is homogeneous of degree 0 by (1). Hence $a$ should be homogeneous of degree 0.

We call a graded ring $R=\sum R_i$ to be a $\mathbb{Z}$-graded ring if $i\in\mathbb{Z}$ for some $i\in\mathbb{Z}^+$ and $\mathbb{Z}^-$. 

**Proposition 1.2.** Let $R$ be a $\mathbb{Z}$-graded domain. Let $S=\{i\in\mathbb{Z}; R_i\neq 0\}$. Then $S=n\mathbb{Z}$ for a certain integer $n$.

Proof. Since $R$ is a domain, $S$ is a semi-group. Hence (1.2) is immediately seen by the following lemma.

**Lemma 1.3.** Let $S\subseteq\mathbb{Z}$ be a semi-group. If $S\cap\mathbb{Z}^+\neq 0$ and $S\cap\mathbb{Z}^-\neq 0$, then $S$ is a subgroup of $\mathbb{Z}$.

If $R$ is a $\mathbb{Z}$-graded domain, then we may assume $R_i\neq 0$ for any $i\in\mathbb{Z}$.

**Proposition 1.4.** Let $R$ be a graded ring. If there is an invertible element $x$ in $R_i$, then $R=R_0[x, x^{-1}]$.

Proof. For any $r\in R_n$, $r=r(x^{-1}x)^n=rx^{-n}x^n$ and $rx^{-n}$ is in $R_0$, therefore $r\in R_0x^n$. Hence $R=R_0[x, x^{-1}]$.

**Corollary.** Let $R$ be a $\mathbb{Z}$-graded domain. If $R_0$ is a field, so $R=R_0[x, x^{-1}]$ for every $x\in R_i$, $x\neq 0$.

Proof. Choose non-zero elements $x\in R_i$, and $y\in R_i$. Since $R_0$ is a field, $0\neq xy$ is invertible, therefore $x$ and $y$ are units in $R$, hence $R=R_0[x, x^{-1}]$.

2. Torus invariant rings

A ring $A$ is said to be torus invariant provided that $A$ has the following property:

If there exist a ring $B$, a variable $Y$ over $B$, and a variable $X$ over $A$ such that $A[X, X^{-1}]$ is isomorphic to $B[Y, Y^{-1}]$, 

\[ \Phi: A[X, X^{-1}] \rightarrow B[Y, Y^{-1}] , \]
then \( A \) is always isomorphic to \( B \).

Especially if we have always \( \Phi(A)=B \) in such case, we say that the ring \( A \) is strongly torus invariant.

To show \( A \) is torus invariant (resp. strongly torus invariant) it suffices to prove that \( A \) is isomorphic to \( B \) (resp. \( A=B \)) under the assumption: \( A[X, X^{-1}]=B[Y, Y^{-1}] \).

(2.0) We begin with some elementary observations. Assume that

\[ (1) \quad R = A[X, X^{-1}] = B[Y, Y^{-1}] . \]

Then \( X \) and \( Y \) are units of \( R \). It follows from (1.1) that we have

\[ (2) \quad X = vY' \text{ and } Y = uX' , \quad v \in B \text{ and } u \in A , \]
or equivalently

\[ (3) \quad v = u^{-1}X^{-1}f' \text{ and } u = v^{-1}Y^{-1}f' . \]

In the rest of our paper we shall use the letters \( u \) and \( v \) to denote the elements of \( A \) and \( B \) respectively satisfying the relations (2) and (3) whenever we encounter the situation (1).

(2.1) The element \( u \) is in \( B \) if and only if \( ff'=1 \). In this case we have \( A[X, X^{-1}]=B[X, X^{-1}] \), thus we have \( A \cong B \).

Proof is easy and is omitted.

**Proposition 2.2.** Let \( k \) be a field and \( A \) be a \( k \)-algebra. If \( A^* \) (the set of all invertible elements in \( A \))=\( k^* \), then the ring \( A \) is torus invariant.

Proof. Let \( R=A[X, X^{-1}]=B[Y, Y^{-1}] \). By (1.1) the field \( k \) is contained in \( B \). Since \( A^*\cong k^* \), the unit element \( u \) of \( A \) is in \( k \), hence in \( B \). It follows from (2.1) that \( A \) is torus invariant.

**Proposition 2.3.** Let \( A=A_0[t_1, t_2, \ldots, t_n, (t_1t_2\cdots t_n)^{-1}] \) where \( t_i \)'s are independent variables over \( k \)-algebra \( A_0 \) and \( A^*_0=k^* \), then \( A \) is torus invariant.

Proof. Let \( R=A[X, X^{-1}]=B[Y, Y^{-1}] \). Then by the lemma (1.1) \( Y=uX' \) and \( X=vY' \). Since \( u \) is invertible in \( A=A_0[t_1, t_2, \ldots, t_n, (t_1\cdots t_n)^{-1}] \), \( Y=rt_1^i\cdots t_n^i, r \in A^*_0=k^* \). We may assume that \( r=1 \), so \( Y=t_1^i\cdots t_n^iX' \).

On the other hand as \( t_i \) is invertible in \( R=B[Y, Y^{-1}] \), \( t_i=b_iY'i, b_i \in B^* \). Then we have that

\[ ff'+\sum e_if_i = 1 . \]

Therefore the following natural homomorphism is surjective.
Since $Z$ is P.I.D., we can construct a basis of $Z^{(n+1)}$ containing this vector $(f', e_1, \ldots, e_n)$. Put this basis

$$
e_0 = (f', e_1, \ldots, e_n)$$

and put $u_i = t_i' n X_i$, $i = 0, 1, \ldots, n$.

$$R = A_0[u_1, \ldots, u_n, (u_1 \cdot u_n)^{-1}] [Y, Y^{-1}] = A[X, X^{-1}] = B[Y, Y^{-1}]$$

Therefore $A$ is isomorphic to $B$. Hence $A$ is torus invariant.

(2.4) Let $R = A[X, X^{-1}] = B[Y, Y^{-1}]$. An ideal $I$ of $R$ is said to be vertical relative to $A$ if there exists an ideal $J$ of $A$ such that $JR = I$. If $J$ is an ideal of $A$ such that $JR$ is vertical relative to $B$, then we will simply say that $J$ is vertical relative to $B$. If $A$ is a $k$-affine domain, the prime ideals defined by the singular locus of Spec $A$ are vertical relative to $B$.

**Proposition 2.5.** Let $R = A[X, X^{-1}] = B[Y, Y^{-1}]$. If there exists a maximal ideal of $A$ which is vertical relative to $B$, then $A[X, X^{-1}] = B[X, X^{-1}]$. In particular $A$ and $B$ are isomorphic.

Proof. Let $m$ be a maximal ideal of $A$ which is vertical relative to $B$. Then there exists an ideal $n$ of $B$ such that $mR = nR$. Therefore $R/mR = A/m[X, X^{-1}] = B/n[Y, Y^{-1}]$, where $X = uX'$ and $Y = uX'$. Since $m$ is a maximal ideal, $A/m$ is a field. Hence $u$ is in $B/n$ by (1.1). Therefore we obtain $f = \pm 1$ by (2.1). Thus $A$ is isomorphic to $B$.

**Corollary 2.6.** Let $A$ be a $k$-affine domain with isolated singular points, then $A$ is torus invariant.

3. Strongly torus invariant rings

In this section we investigate strongly torus invariant rings.

**Proposition 3.1.** Let $A[X, X^{-1}] = B[Y, Y^{-1}]$. If $Q(A) \subseteq Q(B)$, then $A = B$, where $Q(R)$ is the total quotient field of $R$.

Proof. Let $x$ be an element of $A$, then there exist two elements $b$ and $b'$ of $B$ such as $x = b/b'$. Hence $b = b'x$. In the graded ring $B[Y, Y^{-1}]$ the elements $b$ and $b'$ are homogeneous of degree zero, thus $x$ is also degree zero. Hence we have $A \subseteq B$. Let $b$ be an element of $B$. Then $b = \sum a_j X^j$, $a_j \in A$. By (2) of (2.0) we have that $b = \sum a_j v^j Y^j$. If $f = 0$, then $X \in B$. Thus $A[X, X^{-1}] \subseteq B$. 


it's a contradiction, hence \( f \neq 0 \). Since \( a_jv^j \in B \) and \( Y \) is a variable over \( B \), \( b = a_0 \in A \). Thus \( A = B \).

**Corollary 3.2.** Let \( A \) denote the integral closure of \( A \). If \( A \) is strongly torus invariant, then \( A \) is also so.

Proof. It is easily seen that if \( A[X, X^{-1}] = B[Y, Y^{-1}] \) then \( A[X, X^{-1}] = B[Y, Y^{-1}] \). Since \( A \) is strongly torus invariant, \( A = B \). Hence \( Q(A) = Q(B) \), and we have that \( A = B \).

**Proposition 3.3.** Let \( A \) be a domain with \( J(A) = 0 \), where \( J(D) \) is the Jacobson radical of a ring \( D \). Then \( A \) is strongly torus invariant.

Proof. Let \( a \) be a non-zero element of \( J(A) \). Then \( 1 + a \) is unit, so in the graded ring \( B[Y, Y^{-1}] \), \( 1 + a \) is homogeneous. Since the "unity 1" is a homogeneous element of degree 0, the element \( a \) is also so. Thus the element \( a \) is contained in \( B \).

Let \( x \) be any element of \( A \). Since \( xa \) is contained in \( J(A) \), \( xa \) is in \( B \). Hence \( A \) is contained in \( Q(B) \). By (3.1), we have that \( A = B \).

**Corollary 3.4.** If \( A \) is a local domain, then \( A \) is strongly torus invariant.

**Proposition 3.5.** Let \( A \) be an affine ring over a field \( k \) and let \( A[X, X^{-1}] = B[Y, Y^{-1}] \). Then \( A = B \) if and only if every maximal ideals of \( A \) is vertical relative to \( B \).

Proof. It suffices to prove the "if" part of the (3.5). By (3.3) we may assume that \( J(A) = 0 \). Let \( x \) be an element of \( B \) and let \( x = \sum_{j=1}^{t} a_jx^j \), where \( s < t \), \( a_j \in A \) and \( a_s \neq 0 \) and \( a_t \neq 0 \). For any maximal ideal \( m \) of \( A \) there exists a maximal ideal \( n \) of \( B \) such as \( mR = nR \), where \( R = A[X, X^{-1}] \). Let \( x \) denote the residue class of \( x \) in \( B/n \). Then \( x \) is algebraic over the coefficient field \( k \), hence there exist elements \( \lambda_0, \lambda_1, \ldots, \lambda_{s-1} \) in \( k \), such that \( f(x) = x^s + \lambda_{s-1}x^{s-1} + \cdots + \lambda_0 \in nR = mR \). If \( t \neq 0 \), then the highest degree term of \( f(x) \) with respect to \( X \) is \( a_t x^t \in mR \), thus \( a_t \) is contained in \( m \) for every maximal ideal in \( A \). Since \( J(A) = 0 \), \( a_t = 0 \). It's a contradiction. Therefore \( t = 0 \). By the same way, we have that \( s = 0 \), hence \( x \) is in \( A \). Thus \( A = B \).

We denote the subring generated by all the units of \( A \) by \( A_u \).

**Proposition 3.6.** Let \( A \) be a \( k \)-affine domain with an isolated singular point. If \( A \) is algebraic over \( A_u \), then \( A \) is strongly torus invariant.

Proof. Let \( A[X, X^{-1}] = B[Y, Y^{-1}] \) and let \( m \) be the maximal ideal defined by the isolated singular point. Then there exists a maximal ideal \( n \) of \( B \) such as \( mR = nR \). Let \( a \) be a unit element of \( A \). In the graded ring \( B[Y, Y^{-1}] \), the
element $a$ is also invertible, so $a = b Y^i$ for some invertible element $b$ in $B$ and a certain integer $j$. Since $A[m]$ is algebraic over $k$, there exist elements $\lambda_0, \lambda_1, \ldots, \lambda_n \in k$ such that $\lambda_n a^n + \cdots + \lambda_1 a + \lambda_0 \in mR = nR$. If $j \neq 0$, $\lambda_n b^n$ is in $n$, hence $b$ is not invertible, it's a contradiction. Thus we have that $A \subseteq B$. By the following lemma our proof is over.

**Lemma 3.7.** Let $A[X, X^{-1}] = B[Y, Y^{-1}]$. If $A$ is algebraic over $A \cap B$, then $A = B$.

Proof. Since $A$ is algebraic over $A \cap B$, $A$ is also algebraic over $B$, but $B$ is algebraically closed in $B[Y, Y^{-1}]$, therefore $A$ is contained in $B$. Thus we have that $A = B$.

Let $A$ be an integral domain containing a field $k$. We denote the set of all automorphisms of $A$ over $k$ by $\text{Aut}_k(A)$.

**Proposition 3.8.** Let $A$ be an integral domain containing an infinite field $k$. If $\text{Aut}_k(A)$ is a finite set, then $A$ is strongly torus invariant.

Proof. Let $R = A[X, X^{-1}] = B[Y, Y^{-1}]$. Let $\Phi_\lambda, \lambda \in k^*$, be an automorphism of $R$ defined by $\Phi_\lambda(Y) = \lambda Y$ and $\Phi_\lambda(b) = b$ for $b \in B$. Following the notation of (2.0) we have $X = vY^i$, thus $\Phi_\lambda(X) = \lambda' X$, therefore $R = \Phi_\lambda(A)[X, X^{-1}]$. Let $p$ be the projection $A[X, X^{-1}] \to A$ defined by $p(X) = 1$ and $i$ be the canonical injection $A \hookrightarrow A[X, X^{-1}]$. Define $\sigma_\lambda = q \circ \Phi_\lambda \circ i$. Then $\sigma_\lambda$ is an endomorphism of $A$. We shall show that $\sigma_\lambda$ is surjective. Let $x$ be an element of $A$. Since $R = \Phi_\lambda(A)[X, X^{-1}]$, there exist elements $a_j$'s of $A$ such as $x = \sum \Phi_\lambda(a_j)X^j$. Hence $x = p(x) = \sum p\Phi_\lambda(a_j)$. Let $x' = \sum a_j \in A$, then $\sigma_\lambda(x') = \sum p\Phi_\lambda(a_j) = x$. Thus $\sigma_\lambda$ is surjective. Next we shall show that $\sigma_\lambda$ is injective. Since $\Phi_\lambda^{-1}(X - 1)R \cap \Phi_\lambda(A) = \Phi_\lambda^{-1}(X - 1)R \cap A = \lambda' X - 1)R \cap A = 0$, we have $(X - 1)R \cap \Phi(A) = 0$, therefore $\sigma_\lambda$ is injective. Hence $\sigma_\lambda$ is an automorphism of $A$.

We shall prove that the set $\{\sigma_{\lambda}; \lambda \in k^*\}$ is infinite when $A \neq B$. Since $u = v \cdot Y^{1 - f'}, \sigma_\lambda(u) = \lambda^{1 - f'} u$. Therefore our assertion is proved when $1 - ff' \neq 0$. Suppose $ff' = 1$. Then we may assume that $R = A[X, X^{-1}] = B[X, X^{-1}]$. If $A \subseteq B$, then $A = B$, so there exists an element $x$ of $A$ not contained in $B$, say $x = \sum b_j X^j, t > s$. Since $\ker p = (X - 1)R$ and $(X - 1)R \cap B = 0$, $p(b_j) \neq 0$ for $b_j \neq 0$. Since $\sigma_\lambda(x) = \sum p(b_j)\lambda^j$ and $p(b_j) \neq 0$ for some $j \neq 0$, the set $\{\sigma_\lambda; \lambda \in k^*\}$ is infinite.

Next we shall give two cases of rings which are not strongly torus invariant. If $A$ has a non-trivial locally finite iterative higher derivation $\psi: A \to A[T]$, then $A[T] = B[T]$, where $B = \psi(A)$ and $A \neq B$, as is proved in [4]. Hence we have that $A[T, T^{-1}] = B[T, T^{-1}]$ and $A \neq B$. If $A$ is a graded ring, then $A$ is not strongly torus invariant. Indeed, let $X$ be a variable over $A$ and let
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Let \( B_i = \{ a_i X^i; a_i \in A_i \} \). Then \( B_i \) is an \( A_\sigma \)-module contained in \( A[X, X^{-1}] \). Let \( B = \sum B_i \). Then \( B \) is a graded ring and we easily see that \( A[X, X^{-1}] = B[X, X^{-1}] \). We shall show that \( X \) is a variable over \( B \). Assume that there exist elements \( b_0, b_1, \ldots, b_n \) in \( B \) such that \( b_n \neq 0 \) and \( b_n X^n + \cdots + b_1 X + b_0 = 0 \). By the definition of \( B \) we denote \( b_i = \sum a_i X^i, a_i \in A_i \). In the graded ring \( A[X, X^{-1}] \) the homogeneous term of degree \( t \) of this equation is that

\[
(a_{n,t-n} + a_{n-1,t-n+1} + \cdots + a_{0,t})X^t = 0.
\]

Since \( A \) is a graded ring and \( a_{ij} \) is a homogeneous element of degree \( j \), we obtain \( a_{ij} = 0 \) for all index \( i \) and \( j \), hence \( X \) is a variable over \( B \).

By [4] we have that a \( k \)-algebra \( A \) has a non-trivial locally finite iterative higher derivation if and only if \( \text{Aut}_k(A) \) has a subgroup isomorphic to \( G_a = \text{Spec} k[T] \). We easily see that \( A \) is a non-trivial graded ring if and only if \( \text{Aut}_k(A) \) has a subgroup isomorphic to \( G_a = \text{Spec} (k[T, T^{-1}]) \). 

**Proposition 3.9.** A \( k \)-algebra \( A \) is not strongly torus invariant, if \( \text{Aut}_k(A) \) has a subgroup isomorphic to \( G_a \) or \( G_m \).

Assume that \( \text{Aut}_k(A) \) is an infinite group. If \( \text{Aut}_k(A) \) has an algebraic group structure, then there exists the following exact sequence;

\[
0 \rightarrow T \rightarrow \text{Aut}_k(A)_0 \rightarrow \theta \rightarrow 0
\]

where \( \text{Aut}_k(A)_0 \) is the connected component containing the identity \( I_A \), and \( T \) is a maximal torus subgroup of \( \text{Aut}_k(A)_0 \) and \( \theta \) is an abelian variety. Let \( P \) be an arbitrary closed point of \( \text{Spec}(A) \). If \( T = 0 \), then there exists a regular map

\[
\Phi: \text{Aut}_k(A)_0 \rightarrow \text{Spec}(A), \sigma \mapsto \sigma(P).
\]

Since \( \text{Im}(\Phi) \) is a projective variety contained in the affine variety \( \text{Spec}(A) \), the set \( \text{Im}(\Phi) \) consists of one point, it contradicts \( \dim \text{Aut}_k(A)_0 > 0 \). Hence we have that \( T \neq 0 \). Since \( T \supseteq G_a \) or \( G_m \), we have the following result:

**Proposition 3.10.** If \( \text{Aut}_k(A) \) is not a finite set and has an algebraic group structure, then \( A \) is not strongly torus invariant.

4. **Affine domains of dimension \( \leq 2 \)**

Let \( k \) be a field of characteristic zero which contains all roots of “unity”. In this section let \( A \) be an affine domain over \( k \). We shall see that if \( \dim A = 1 \), then \( A \) is always torus invariant. Moreover \( A \) is not strongly torus invariant if and only if \( \text{Aut}_k(A) \supseteq G_m \). Let \( \dim A \geq 2 \). Then \( A \) is not always torus
invariant. But if an integrally closed domain \( A \) is not a \( \mathbb{Z} \)-graded ring, then \( A \) is torus invariant.

For the proof we need a lemma.

**Lemma 4.1.** Let \( K \) be a finite separable algebraic field extension of a field \( k \). If \( A \) is a one-dimensional affine normal ring such that \( k \subset A \subset K[X, X^{-1}] \), then \( A \) is a polynomial ring or a torus ring over \( k' \) where \( k' \) is the algebraic closure of \( k \) in \( A \).

Proof. We may assume that \( k=k' \). Following the similar device to the proof of (2.9) in [1, p 322], we have \( Q(A)=k(\theta) \) for some element \( \theta \) of \( A \).

Since \( k[\theta] \subset A \subset k(\theta) \), \( A=k[\theta] \) or \( A=k\left[ \theta, \frac{1}{f(\theta)} \right] \) for some polynomial \( f(\theta) \in k[\theta] \). Let \( A=k\left[ \theta, \frac{1}{f(\theta)} \right] \). Then we may assume that \( f(\theta) \) has no multiple factors. The element \( f(\theta) \) is invertible in \( A \), so is also invertible in \( K[X, X^{-1}] \). Thus we have \( f(\theta)=\beta X^r \), \( \beta \in K \), \( \theta \in K[X, X^{-1}] \). We may assume that \( r \geq 0 \), if necessary, by replacing \( X \) with \( X^r \). Then we easily see that \( \theta \in K[X] \). The uniqueness of the irreducible decomposition in a polynomial ring implies that \( \deg f(\theta) = 1 \), since the polynomial \( f(\theta) \) has not multiple factors and \( f(\theta)=\beta X^r \). Hence we may assume that \( f(\theta)=\theta \) and we obtain \( A=k\left[ \theta, \frac{1}{\theta} \right] \).

Let \( A \) be an integral domain. If \( A \) is contained in \( K[X, X^{-1}] \), then \( A \) is a polynomial ring or a torus ring over \( k' \).

**Proposition 4.2.** Let \( A \) be a one-dimensional affine domain over a field \( k \) of characteristics zero. Then we obtain that

1. \( A \) is torus invariant,
2. \( A \) is not strongly torus invariant if and only if \( \text{Aut}_k(A) \) has a subgroup isomorphic to \( G_m \). If \( A \) is not strongly torus invariant and \( A \) is integrally closed, then \( A \) is a polynomial ring or a torus ring over the algebraic closure of \( k \) in \( A \).

Proof. At first we shall prove (2). The sufficiency follows from (3.9). Let \( R=A[X, X^{-1}]=B[Y, Y^{-1}] \) in which \( A \supset B \). If \( mR \cap A \neq 0 \) for any maximal ideal \( m \) of \( B \), then \( m \) is vertical relative to \( A \), and we have \( A=B \) by (3.5). Hence there exists a maximal ideal \( m \) such as \( mR \cap A=0 \). Since \( ch k=0 \), \( B/m=k \) is a finite separable algebraic field over \( k \). The residue mapping of \( R \) to \( R/mR \) yields (up to isomorphism) \( k \subset A \subset K[Y, Y^{-1}] \) where \( Y \) is algebraically independent over \( K \). Therefore \( A \) is a polynomial ring or a torus ring by the lemma (4.1). Thus the automorphism group \( \text{Aut}_k A \) contains a subgroup isomorphic to \( G_m \).

Assume that \( A \) is not integrally closed. Then prime divisors in \( A \) of the conductor \( \tau(A/A) \) are vertical relative to \( B \). Hence we may assume \( X=Y \) by
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The above lemma (4.1) implies that \( \overline{A} = k'[t, t^{-1}] \) or \( \overline{A} = k'[t] \) where \( k' \) is the algebraic closure of \( k \) in \( \overline{A} \).

Firstly let \( \overline{A} = k'[t] \). Since \( \overline{A} \cong \overline{B} \), there exists an element \( s \) in \( \overline{B} \) such as \( \overline{B} = \overline{A}[X, X^{-1}] = \overline{A}[X, X^{-1}] \). We have \( k'[X, X^{-1}] [t] = k'[X, X^{-1}] [s] \), hence we easily see that \( t = f_1(X)s + f(X) \) and \( s = g_1(X)t + g(X) \) where \( f(X) \) and \( g(X) \) are both in \( k'[X, X^{-1}] \). We may assume that \( t = X^n s + f(X) \) and \( s = X^{-n} t + g(X) \). Let \( n \) be a prime divisor in \( \overline{A} \) of the conductor \( t(A/\overline{A}) \).

Then there exists a maximal ideal \( \overline{m} \) of \( \overline{B} \) such as \( \overline{n} \overline{R} = \overline{m} \overline{R} \). Since \( \overline{A}/\overline{n} \overline{A} \) is algebraic over \( k > \overline{k} \), there exist elements \( \lambda_0, \lambda_1, \ldots, \lambda_{d-1} \) such that \( \overline{m} \overline{R} = \overline{n} \overline{R} \overline{R} \). Hence, we have that \( (X^n s + f(X))^d + \lambda_{d-1}(X^n s + f(X))^{d-1} + \cdots + \lambda_0 \in \overline{n} \overline{R} \). The constant term of this polynomial with respect to \( s \) is the following:

\[ f(X)^d + \lambda_{d-1} f(X)^{d-1} + \cdots + \lambda_0 \in \overline{n} k'[s] [X, X^{-1}] \]

Therefore \( f(X) = f \in k' \). Hence we may assume that \( t = X^n s \). We shall show that \( A \) is a graded ring. Let \( a \) be an element of \( A \). Since \( a \) is contained in \( \overline{A} = k'[t] \) and \( t = X^n s \), we have that \( a = \sum \lambda_i t^i = \sum \lambda_i s^i X^i, \lambda_i \in \overline{B} \). On the other hand, as the element \( a \) is contained in \( B[X, X^{-1}] \), \( a = \sum b_i X^i, b_i \in \overline{B} \). Comparing the coefficient of the each term in the following:

\[ \sum \lambda_i s^i X^i = \sum b_i X^i \]

we have \( b_i = \lambda_i s^i (i = \overline{m}) \) and \( b_0 = 0 \) (i.e. the terms of degree zero). If \( b_i \neq 0 \), then \( b_i = \lambda_i s^i X^i = \lambda_i t^i \in B[X, X^{-1}] \subset A[A, X^{-1}] \cap \overline{A} = A[X, X^{-1}] \cap \overline{A} = A \). Therefore \( A \) has a graded ring structure.

Secondary let \( \overline{A} = k'[t, t^{-1}] \). Then \( \overline{B} = k'[s, s^{-1}] \). Since \( t \) and \( s \) are invertible in \( \overline{R} \), we may assume that \( t = s^i X^i \) and \( s = s^j X^j \), then \( t = (t^i X^i)^s = t^i X^{m+s} \), therefore \( ij = 1 \). Hence, we may assume \( t = X^s X \). By the same method as in the case \( \overline{A} = k'[t] \) we have that \( A \) is a graded ring.

Proof of (1). If \( A \) is not integrally closed, then the prime divisors of the conductor \( t(A/\overline{A}) \) are vertical relative to \( B \). Since non-zero prime ideals of \( A \) are maximal, the ring \( A \) is isomorphic to \( B \) by (2.5). If \( A \) is integrally closed and \( A \) is neither a polynomial ring nor a torus ring, then \( A \) is strongly torus invariant. If \( A \) is a polynomial ring or a torus ring, \( A \) is torus invariant by (2.2) and (2.3).

Next we shall consider the case; the coefficient field \( k \) has all roots of "unity" and its characteristic is zero. Then we prove the following:

**Theorem 4.3.** Let \( A \) be an integrally closed \( k \)-affine domain of dimension two, where the field \( k \) has all roots of "unity" and \( \text{ch } k = 0 \). If \( A \) is not torus invariant, then \( A \) is a \( \mathbb{Z} \)-graded ring which contains units of non-zero degree.

Proof. Assume that \( A \) is not torus invariant. Then there exist a \( k \)-algebra \( B \) and independent variables \( X, Y \) such that \( A \) is not isomorphic to \( B \) and \( \overline{R} = \overline{A}[X, X^{-1}] = \overline{B}[Y, Y^{-1}] \). By (2.0) and (2.1) we obtain \( f f' + 1 \). We shall show
that it follows from \( ff' \neq 1 \) that \( A \) is a \( \mathbf{Z} \)-graded ring. We may only consider the case \( 1-ff' > 0 \). Let \( x \) be a \((1-ff')-th\) root of \( u \) and let \( y=x^{-1}/X \). Then \( y^{i-ff'}=v \). Since \( (y^{-ff'})^{-1}=u \), \( x=\lambda y^{-ff'}Y \) for some \((1-ff')-th\) root \( \lambda \) of “unity”. From the relations; \( y=x^{-1}/X \) and \( Y=ux' \), we have \( \lambda=1 \).

Since \( y=x^{-1}/X \) and \( x=y^{-ff'}Y \) are invertible, we have \( A[x][Y, X^{-1}]=B[y][Y, Y^{-1}]=A[x][y, y^{-1}]=B[y][x, x^{-1}] \). Define a surjective homomorphism \( j: A[x][y, y^{-1}] \rightarrow A[x] \) by \( j(y)=1 \). Let \( A_0=j(B[y]) \subseteq A[x] \). We shall show that \( A[x]=A_0[x, x^{-1}] \), let \( a \) be an element of \( A_0 \). Then \( a=\sum b_ix^i, b_i \in B \). Since \( j(a)-a \) and \( j(x)=x \), we have that \( a=\sum j(b_i)x^i, j(b_i) \in A_0 \). Thus \( A[x]=A_0[x, x^{-1}] \) and \( x \) is algebraically independent over \( A_0 \). By the same way \( B[y]=B_0[y, y^{-1}] \).

Since the every \((1-ff')-th\) roots of “unity” is contained in \( k \) and \( ch k=0 \) and \( A \) is normal, the extension \( A[x]/A \) is a Galois extension with a cyclic group \( G=\langle \sigma \rangle \) (cf. [3] p 214). Indeed when \(|G|=n, n \neq ff' \) and there exists a primitive \( n-th \) root \( \lambda \) of “unity” such that \( \sigma(x)=\lambda x \) and the invariant subring \( \sigma(A_0[x, x^{-1}])=A \) and \( A[x]=A+Ax+\cdots+Ax^{n-1} \) is a free \( \mathbf{A} \)-module.

Since the element \( u \) is a unit of \( A \) and \( ch(k)=0 \), the extension \( A[x]/A \) is étale. Since \( A \) is a normal domain, \( A[x], \) hence \( A_0[x, x^{-1}] \), is also a normal domain. From this we see that \( A_0 \) is always normal.

We shall show that there exists a subring \( A'_0 \) in \( A[x] \) such that \( A[x]=A'_0[x, x^{-1}] \) and \( \sigma(A_0)=A'_0 \). If \( A_0 \) is strongly torus invariant, then \( \sigma(A_0)=A'_0 \) for \( \sigma(A_0)[x, x^{-1}]=A_0[x, x^{-1}] \), therefore \( A_0 \) satisfies the conditions. If \( A_0 \) is not strongly torus invariant, then \( A_0=k'[t] \) or \( k'[t, t^{-1}] \). Firstly let \( A_0=k'[t] \). Since \( k'[x, x^{-1}][t]=k'[x, x^{-1}][\sigma(t)] \), we easily see that \( \sigma(t)=\mu x^t + f(x), \mu \in k^* \) and \( f(x) \in k'[x, x^{-1}] \). The order of \( \sigma \) is \( 1 \), i.e. \( \sigma=\text{identity} \), so \( \sigma^n(t)=t \), the other hand \( \sigma^n(t)=\mu x^{(i+\cdots+i)x^n}+g(x), g(x) \in k'[x, x^{-1}] \), therefore we have that \( i=0 \), thus \( \sigma(t)=\mu t+f(x) \) and \( \mu^s=1 \). Let \( f(x)=\sum f_i x^i \) and define the set \( \Delta=\{ j \in \mathbf{Z}; \lambda^j \neq \mu \} \). Let \( h(x)=\sum h_i x^i \), where \( h_i=f_j(\mu-\lambda x)\). Then \( s=t+h(x) \). Then \( \sigma(s)=\mu s+\sum f_i x^i \), hence \( \sigma^n(s)=\mu^n s+\mu^n n \sum f_i x^i =s+n \mu^{n-1} \sum f_i x^i \). Since \( \sigma^n(s)=s \), we have \( \sigma(s)=\mu s \). We set \( A_0=k'[s] \), then \( A_0 \) satisfies the conditions.

Secondary let \( A_0=k'[t, t^{-1}] \). Since \( k'[x, x^{-1}][t, t^{-1}]=k'[x, x^{-1}][\sigma(t), \sigma(t)^{-1}] \), we easily see that \( \sigma(t)=\mu x^t \) or \( \sigma(t)=\mu x^{-t}, \mu \in k^* \).

Case (i); \( \sigma(t)=\mu x^t \). Since \( \sigma^n(t)=\mu x^{(1+\cdots+i)x^n t} \) and \( \sigma^n(t)=t \), we have that \( \sigma(t)=\mu t \), so \( \sigma(A_0)=A_0 \).

Case (ii); \( \sigma(t)=\mu x^{t-1} \). If \( n \) is even, say \( n=2m+1 \), then \( \sigma^n(t)=\mu x^{(m+1)x^n t} \), but this is impossible for \( \sigma^n(t)=t \). Therefore \( n \) is even, say \( n=2m \). Then \( \sigma^n(t)=\lambda^{im} t \). Since \( \lambda \) is a primitive \( n-th \) root of “unity”, the integer \( i \) is even, say \( i=2j \). Let \( s=x^{-t}t \) and \( A_0=k'[s, s^{-1}] \). Then \( A_0 \) satisfies the conditions.
Next we shall show that $A$ has a $\mathbb{Z}$-graded ring structure. Let $a$ be an element of $A$. Since $a \in A[[x, x^{-1}], a = \sum a_i x^i$. Then $a = \sigma(a) = \sum \sigma(a_i)x^i$ and $\sigma(a_i) \in A_0$. Comparing the coefficient of each term in the equality; $\sum a_i x^i = \sum \sigma(a_i)x^i$, we have that $a_i = \sigma(a_i)x^i$, then $\sigma(a^x_i) = a_i x^i$. Thus $a_i x^i$ is an element of $A$. Therefore $A$ is a graded ring. Since there exists units of non-zero degree, $A$ has a $\mathbb{Z}$-graded ring structure.

Remark. The converse of this theorem is false. Indeed we find by (2.3) that the ring $k[T][X, X^{-1}]$ is a $\mathbb{Z}$-graded ring with respect to $X$ which is torus invariant.

Example. We shall construct an example of an affine dimension $A$ of dimension two which is not torus invariant.

Let $D$ be an integrally closed domain of dimension one over an algebraically closed field $k$ and $D^* = k^*$. Let $a$ be a non-unit of $D$ and $a^2 = a$, $a \in D$. Assume that $D$ is noetherian and $D[a]$ is strongly torus invariant. Since an affine domain of dimension one whose totally quotient field has a positive genus is strongly torus invariant by (4.2), this assumption can be satisfied for a suitable choice of $D$. Let $T$ be a variable over $D$ and $A = D[aT, T^3T^{-5}]$. Let $X$ be a variable over $A$ and $S = T^2X$ and $Y = T^5X^2$. Let $B = D[\alpha S^3, S^5, S^{-7}]$. Since $T = S^{-2}Y$ and $X = S^5Y^{-5}$, we have that $A[X, X^{-1}] = B[Y, Y^{-1}]$. By (1.1) invertible elements in the graded ring $A$ are homogeneous. Since $D^* = k^*$, we obtain $A^* = \{\eta T^i; \eta \in k^* \text{ and } i \in \mathbb{Z} \}$. Hence the quotient $A^*/k^*$ is generated by $T^5$. Similarly $B^*/k^*$ is generated by $S^5$. We shall show that $A$ is not isomorphic to $B$. We assume that there exists an isomorphism $\sigma$ of $A$ to $B$. Since $\sigma$ is a group-isomorphism of $A^*$ to $B^*$, we have $\sigma(T^5) = \mu S^5$ or $\sigma(T^5) = \mu S^{-5}$, $\mu \in k^*$. We shall only consider the case: $\sigma(T^5) = \mu S^5$, since the proof of the other case is the similar. Let $\sigma$ be an isomorphism of $A[T]$ to $B[S]$ defined by $\sigma = \sigma$ on $A$ and $\sigma(T) = \xi S$, $\xi^5 = \mu$. Then we have that $D[\alpha][S, S^{-1}] = \sigma(D[\alpha]) [S, S^{-1}]$, therefore $\sigma(D[\alpha]) = D[\alpha]$; for $D[\alpha]$ is strongly torus invariant. Since $\sigma(D) \subseteq D[\alpha] \cap B = D$, we have $\sigma(D) = D$, therefore we easily see that $\sigma$ is an isomorphism as graded rings. Thus we have $\sigma((\alpha T)D) = (a^2 S)D$, hence $\sigma(a) \in a^2 D$. Since the element $a$ is not a unit, $a^2 D \subseteq a D$, thus $\sigma(a) D \subseteq a^2 D \subseteq a D$, so $aD \subseteq \sigma^{-1}(a)D$, hence we have a proper ascending chain $\{\sigma^{-n}(a)D\}$, but it contradicts the netherian assumption of $D$. Hence $A$ is not torus invariant.

(4.4) Now let $A = \sum A_i$ be an integrally closed $\mathbb{Z}$-graded domain which contains invertible elements of non-zero degree. Let $e$ be an invertible element of $A$ with the smallest positive degree $d$. Let $a$ be a unit of $A$, then $a$ is a homogeneous elements with $\deg a = jd$ for some integer $j$, and there exists an element $\xi$ of $A^*_e$ such as $a = \xi^d e$. Let $i$ be any positive integer and $x$ be one of the $ijd$-th roots of $a$, say $x^{ijd} = a$. Since $A[x]$ is a $\mathbb{Z}$-graded ring with the
invertible elements $x$ of degree one, $A[x]=A'_0[x, x^{-1}]$ by (1.4) where $A'_0$ contains $A_0$. Let $f$ and $f'$ be integers such as $ff'+ijd=1$ and let $X$ be a variable over $A$. Put $y=x^{-f}X$ and $X=ay^{jd}Y$. Therefore $A'_0[x, x^{-1}] [X, X^{-1}]=A'_0[y, y^{-1}] [Y, Y^{-1}]$. Since the every $n$th roots of “unity” is contained in $k$ and $A$ is integral closed, the extension $A[x]/A$ is a Galois extension with a cyclic group $G=\langle \sigma \rangle$. Indeed $|G|=di$ and there exists a primitive $di$-th root of “unity” such as $\sigma(x)=\lambda x$, and $(A[x])^\sigma=A$. Since $A'_0$ is algebraic over $A_0$, $\sigma(A'_0)$ is also so, hence $\sigma(A'_0)$ is algebraic over $A'_0$, but $A'_0$ is algebraically closed in $A'_0[x, x^{-1}]$, therefore $\sigma(A'_0)=A'_0$. Since $\sigma(y)=\lambda^{-f}y$, $\sigma$ is an automorphism of $A'_0[y, y^{-1}]$. Let $B=A'_0[y, y^{-1}]$ and $\sigma$ be an automorphism of $A'_0[x, x^{-1}] [X, X^{-1}]$ defined by $\sigma(X)=X$ and $\sigma=\sigma$ over $A'_0[x, x^{-1}]$. Since $\sigma(Y)=Y$ and $\sigma(X)=X$, we obtain $B[Y, Y^{-1}]=A'_0[y, y^{-1}] [Y, Y^{-1}]^\sigma=A'_0[x, x^{-1}] [X, X^{-1}]^\sigma=A[X, X^{-1}]$.

**Proposition 4.5.** Let $A$ be an integrally closed $k$-affine domain of dimension 2. If $A[X, X^{-1}]=B[Y, Y^{-1}]$ and $ff'+1$, then $A$ has a $Z$-graded ring structure and $B$ is isomorphic to one of algebras constructed in (4.4).

Proof. The first statement is already mentioned in the proof of (4.3) and we obtained $A'_0[x, x^{-1}] [X, X^{-1}]=A'_0[y, y^{-1}] [Y, Y^{-1}]$ and $\sigma(A'_0)=A'_0$. Let $B'=A'_0[y, y^{-1}]$. Then $B'$ is one of algebras in (4.4). Since $B'[Y, Y^{-1}]=B[Y, Y^{-1}]$, $B$ is isomorphic to $B'$.

5. D-torus invariant

Let $D$ be an integral domain containing a field $k$ of characteristic zero and $A$ be a $D$-algebra. The ring $A$ is called $D$-torus invariant; if $A[X, X^{-1}]=B[Y, Y^{-1}]$ for a certain $D$-algebra $B$ and independent variables $X$ and $Y$, then we have always $A\cong B$. Then we have the following result:

**Proposition 5.1.** Let $A$ be an integrally closed domain over $D$ and $tr.\ deg_{D} A=1$. If $A$ is not $D$-torus invariant, then $A$ is a $Z$-graded ring containing units of non-zero degree.

Proof. Let $A[X, X^{-1}]=B[Y, Y^{-1}]$, where $B$ is a $D$-algebra and not $D$-isomorphic to $A$. By (2.0) and (2.1) we easily see that $ff'=1$. Then we may assume $1-ff'>0$. Let $x$ be a $(1-ff')^{-th}$ root of $u$ and $y=x^{-f}X$. Then we have that $A[x]=A_0[x, x^{-1}]$ and $B[y]=B_0[y, y^{-1}]$ as the proof of (4.3), where $A_0$ and $B_0$ are respectively subalgebras of $A[x]$ and $B[y]$ containing $D$. Let $\sigma$ be a generator of the cyclic Galois group of the extension $A[x]/A$. We shall show that $\sigma(A_0)=A_0$. Since $tr.\ deg_{D} A_0[x, X^{-1}]=1$, $A_0$ is algebraic over $D$, thus $\sigma(A_0)$ is also so. Since $A_0$ is algebraically closed in $A_0[x, x^{-1}]$, we have that $\sigma(A_0)=A_0$. Following the similar devise to the proof of (4.3) we obtain
that $A$ is a $\mathbb{Z}$-graded ring, and $D$ is contained in $A$.

In the following we shall consider the case where $A$ is a $\mathbb{Z}$-graded ring and $A_0=D$. We consider only $D$-isomorphisms of $D$-algebras.

**Theorem 5.2.** Let $A$ be an integrally closed $\mathbb{Z}$-graded ring. Assume that the subring $A_0$ contains an algebraically closed field $k$ and that $A_0^*\cong k^*$. Let $d$ be the smallest positive integer among the set of degrees of units in $A$. Then the number of the isomorphic classes of $A_0$-algebra as $B$ such that $A[X, X^{-1}]=B[Y, Y^{-1}]$ equals to $\Phi(d)$, where $\Phi$ is the Euler function.

**Proof.** Let $i$ be an integer such as $1\leq i<d$ and $(i, d)=1$. Since $(i, d)=1$, $ij+dh=1$ for some integers $j$ and $h$. Moreover we may assume $h\geq 0$. Fix a unit $e$ of degree $d$. Let $x$ be one of the $d$-th roots of $e$. Then we have that $A[x]=A_0[x, x^{-1}]$ for a subring $A_0$ by (1.4). Let $\sigma$ be a generator of the cyclic Galois group of the extension $A[x]/A$. Then $\sigma(x)=\lambda x$, where $\lambda$ is a primitive $d$-th root of "unity". Since $A_0$ is algebraic over $A_0$ and algebraically closed in $A_0[x, x^{-1}]$, we obtain $\sigma(A_0)=A_0$. Let $X$ be a variable over $A$ and let $y=x^{-1}X$ and $Y=e^yX$. Then we have that $A_0[x, x^{-1}]=A_0[y, y^{-1}]$. Define $B_1=A_0[y, y^{-1}]^\sigma$ and let $\sigma$ be an isomorphism of $A_0[x, x^{-1}]$ defined by $\sigma(X)=X$ and $\sigma=\sigma$ on $A_0[x, x^{-1}]$. Since $Y=e^yX$, $\sigma(Y)=Y$, therefore we obtain that $A[X, X^{-1}]=B_1[Y, Y^{-1}]$. We can easily see that $B_1$ is a $X$-graded ring and $(B_1)_0=A_0$. Especially we have $B_1\cong A$.

Let $i_1$ and $i_2$ be integers such as $1\leq i_1<i_2<d$ and $(i_1, d)=(i_2, d)=1$. Let $B'=A_0[y, y^{-1}]^\sigma$ and $B''=A_0[x, x^{-1}]^\sigma$ where $\sigma(y)=\lambda^{-i_1}y$ and $\sigma(z)=\lambda^{-i_2}$ i.e., $B'=B_{i_1}$ and $B''=B_{i_2}$. We shall show that $B'$ and $B''$ are not isomorphic. Assume that there exists an $A_0$-isomorphism $\psi$ of $B'$ to $B''$. Let $a$ be a unit in $B'$ of non-zero degree, say degree $a=n$, $n\neq 0$. Let $b$ be a homogeneous element of $B'$ and degree $b=t$. Then we have $b^n=ra^t$ for an element $r$ in the coefficient ring $A_0$, hence $\psi(b^n)=\psi(b)^t=\sigma^{-t}(b^t)$. Since $r$ and $\psi(a^t)$ are homogeneous, $\psi(b)$ is also homogeneous by (1.1), therefore $\psi$ is an isomorphism as graded rings.

Let $c$ be a homogeneous element in $B'$ of degree one. Then $c=s_1y$ for an element $s_1$ in $A_0$. Since $\sigma(c)=c$ and $\sigma(y)=\lambda^{-i_1}y$, we have $\sigma(s_1)=\lambda^{i_1}s_1$ hence $s_1'$ is in $B'$. Since $\psi(s_1y)=s_2z$ for an element $s_2$ in $A_0$. Since $\sigma(s_2z)=s_2z$ and $\sigma(z)=\lambda^{-i_2}z$, we have $\sigma(s_2)=\lambda^{i_2}s_2$, hence $s_2'$ is in $B''$. By the relations; $s_1'\psi(y^d)=\psi((s_1y)^d)=\psi(s_1y)^d=s_2'z^d$, we obtain $s_2'=\psi(y^d)z^{-d}$. Since $\psi(y^d)z^{-d}$ is an invertible element in $B''$ and degree zero, we have $\zeta=\psi(y^d)z^{-d}\in A_0^*\cong k^*$, therefore we have $s_2'=\eta s_1$, for some $\eta\in k$, $\eta^d=\zeta$. Hence $\sigma(s_2)=\lambda^{i_2}s_2$, but it contradicts the fact that $\sigma(s_2)=\lambda^{i_1}s_1$ and $\lambda$ is a primitive $d$-th root of "unity". Therefore $B'\cong B''$.

Finally we shall show that if $A[X, X^{-1}]=B[Y, Y^{-1}]$ then $B$ is isomorphic to $B_1$ for some $i$ satisfying $0<i<d$ and $(i, d)=1$. The invertible element $u$
in (2.0) is homogeneous. Let $n$ be the degree of $u$. If $n=0$, then $A$ is isomorphic to $B$ by (2.1), hence $B \cong B_1$. Assume $n \neq 0$. Let $c$ be a non-zero homogeneous element of degree 1 and put $\eta = c^* u^{-1}$. Then $\eta$ is an element of $A_0$.

In the graded ring $B[Y, Y^{-1}]$ the elements $u$ and $\eta$ are homogeneous, hence $c$ is also homogeneous, thus we denote $c = b Y^j$ for some element $b$ in $B$ and some integer $j$. Then we obtain that $c^* = \eta u = \eta \overline{v} Y^{1/\overline{r}}$ by (2.0). Therefore we have $1 - \overline{f} = nj$.

By the minimality of $d$ we obtain $n = ld$ for some integer $l$ and $u = \xi e$, $\xi \in A_d^* = k^*$. Since the field $k$ is algebraically closed, we may assume $\xi = 1$, then the $d$-th root $x$ of $e$ is an $n$-th root of $u$. Since the element $\lambda$ is a primitive $d$-th root of “unity”, there exists the unique integer $i$ such that $\lambda^{-i} = \lambda^{-i}$, $0 < i < d$, then $(i, d) = 1$ since $(f, d) = 1$. Let $y = x^{-j} x^j$ and $B' = (A_0[y', y'^{-1}])^\overline{r}$. Then $\sigma(y') = \lambda^{-i} y' = \lambda^{-i} y'$, hence $B' = B_1$. We can easily show that $x = y^{-j} y^i$, therefore we obtain $A_0[x, x^i] = A_0[y', y'^{-1}] [Y, Y^{-1}]$. Since $\sigma(X) = X$ and $\sigma(Y) = Y$, we have $A[X, X^{-1}] = B_1[Y, Y^{-1}]$, hence $B[Y, Y^{-1}] = B_1[Y, Y^{-1}]$. Thus we have $B \cong B_1$.

References


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