ON THE COEFFICIENT RING OF A TORUS EXTENSION

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Introduction. S. Abhyankar, W. Heinzer and P. Eakin treated the following problem in [1]: if $A[X]=B[Y]$, when is $A$ isomorphic or identical to $B$? Replacing the polynomial ring by the torus extension we shall take up the following problem: if $A[X, X^{-1}]=B[Y, Y^{-1}]$, when is $A$ isomorphic or identical to $B$? We say that $A$ is torus invariant (resp. strongly torus invariant) whenever $A[X, X^{-1}]=B[Y, Y^{-1}]$ implies $A\cong B$ (resp. $A=B$). The roles played by polynomial rings in [1] are played by the graded rings in our theory. A graded ring $A=\sum A_i$, $i\in\mathbb{Z}$, with the property that $A_i\neq 0$ for each $i\in\mathbb{Z}$, will be called a $\mathbb{Z}$-graded ring. Main results are the followings.

An affine domain $A$ of dimension one over a field $k$ is always torus invariant. Moreover $A$ is not strongly torus invariant if and only if $A$ has a graded ring structure. An affine domain of dimension two is not always torus invariant. We shall construct an affine domain of dimension two which is not torus invariant. Let $A$ be an affine domain over $k$ of dimension two. Assume that the field $k$ contains all roots of "unity" and is of characteristic zero. If $A$ is not torus invariant, then $A$ is a $\mathbb{Z}$-graded ring such that there exist invertible elements of non-zero degree.

In Section 1 we study elementary properties of graded rings. Especially we are interested in $\mathbb{Z}$-graded rings with invertible elements of non-zero degree. In Section 2 we discuss some conditions for $A$ to be torus invariant. In Section 3 we give several sufficient conditions for an integral domain to be strongly torus invariant. Some relevant results will be found in S. Iitaka and T. Fujita [2]. Section 4 is devoted to the proof of the main results mentioned above. In Section 5 we fix an integral domain $D$ and we treat only $D$-algebras and $D$-isomorphisms there. We shall prove the following two results. When $A$ is a $D$-algebra of $tr.\ deg_D A=1$ and $A$ is not $D$-torus invariant, $A$ is a $\mathbb{Z}$-graded ring such that $D$ is contained in $A$. If $A$ is a $\mathbb{Z}$-graded ring such as $D=A$, then the number of elements of the set of $\{D$-isomorphic classes of $D$-algebras $B$ such that $A[X, X^{-1}]=B[Y, Y^{-1}]\}$ is $\Phi(d)$, where $d$ is the smallest positive integer among the degrees of units in $A$ and $\Phi$ is the Euler function.

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1. Some properties of graded rings

Let \( R \) be commutative ring with identity. The ring \( R \) is said to be a graded ring if \( R \) is a graded module, \( R = \bigoplus R_i \), and \( R_n R_m = R_{n+m} \).

**Lemma 1.1.** Let \( R \) be a graded domain. Then we have the following.

1. The unity element of \( R \) is homogeneous.
2. If \( a \) is homogeneous and \( a = bc \), then \( b \) and \( c \) are both homogeneous. In particular every invertible element is homogeneous.
3. If \( R \) contains a field \( k \), then \( k \) is a subring of \( R_0 \).

**Proof.** (1) follows immediately from the relation \( 1^2 = 1 \). The proof of (2) is easy and will be omitted. To prove (3) we can assume \( k \) is different from \( F_2 \) by (1). Let \( a \) be an element of \( k \) different from 1. Then \( 1 - a \) is homogeneous from (2). The unity 1 is homogeneous of degree 0 by (1). Hence \( a \) should be homogeneous of degree 0.

We call a graded ring \( R = \bigoplus R_i \) to be a \( \mathbb{Z} \)-graded ring if \( R \neq \Phi 0 \) for some \( i \in \mathbb{Z} \).

**Proposition 1.2.** Let \( R \) be a \( \mathbb{Z} \)-graded domain. Let \( S = \{ i \in \mathbb{Z}; R_i \neq 0 \} \). Then \( S = n\mathbb{Z} \) for a certain integer \( n \).

**Proof.** Since \( R \) is a domain, \( S \) is a semi-group. Hence (1.2) is immediately seen by the following lemma.

**Lemma 1.3.** Let \( S \subseteq \mathbb{Z} \) be a semi-group. If \( S \cap \mathbb{Z}^+ \neq 0 \) and \( S \cap \mathbb{Z}^- \neq 0 \), then \( S \) is a subgroup of \( \mathbb{Z} \).

If \( R \) is a \( \mathbb{Z} \)-graded domain, then we may assume \( R_i \neq 0 \) for any \( i \in \mathbb{Z} \).

**Proposition 1.4.** Let \( R \) be a graded ring. If there is an invertible element \( x \) in \( R_i \), then \( R = R_0[x, x^{-1}] \).

**Proof.** For any \( r \in R_0, r = r(x^{-1}x)^* = rx^{-n}x^* \) and \( rx^{-n} \) is in \( R_0 \), therefore \( r = R_0[x, x^{-1}] \).

**Corollary.** Let \( R \) be a \( \mathbb{Z} \)-graded domain. If \( R_0 \) is a field, so \( R = R_0[x, x^{-1}] \) for every \( x \in R_i, x \neq 0 \).

**Proof.** Choose non-zero elements \( x \in R_i \), and \( y \in R_i \). Since \( R_0 \) is a field, \( 0 \neq xy \) is invertible, therefore \( x \) and \( y \) are units in \( R \), hence \( R = R_0[x, x^{-1}] \).

2. Torus invariant rings

A ring \( A \) is said to be torus invariant provided that \( A \) has the following property:

If there exist a ring \( B \), a variable \( Y \) over \( B \), and a variable \( X \) over \( A \) such that \( A[X, X^{-1}] \) is isomorphic to \( B[Y, Y^{-1}] \),
\[ \Phi : A[X, X^{-1}] \to B[Y, Y^{-1}], \]

then \( A \) is always isomorphic to \( B \).

Especially if we have always \( \Phi(A)=B \) in such case, we say that the ring \( A \) is strongly torus invariant.

To show \( A \) is torus invariant (resp. strongly torus invariant) it suffices to prove that \( A \) is isomorphic to \( B \) (resp. \( A=B \)) under the assumption: \( A[X, X^{-1}]=B[Y, Y^{-1}] \).

(2.0) We begin with some elementary observations. Assume that

\[ (1) \quad R = A[X, X^{-1}] = B[Y, Y^{-1}]. \]

Then \( X \) and \( Y \) are units of \( R \). It follows from (1.1) that we have

\[ (2) \quad X = vY' \text{ and } Y = uX', \quad v \in B \text{ and } u \in A, \]

or equivalently

\[ (3) \quad v = u^{-1}X^{-1}f' \text{ and } u = v^{-1}Y^{-1}f'. \]

In the rest of our paper we shall use the letters \( u \) and \( v \) to denote the elements of \( A \) and \( B \) respectively satisfying the relations (2) and (3) whenever we encounter the situation (1).

(2.1) The element \( u \) is in \( B \) if and only if \( ff'=1 \). In this case we have \( A[X, X^{-1}]=B[X, X^{-1}] \), thus we have \( A \approx B \).

Proof is easy and is omitted.

**Proposition 2.2.** Let \( k \) be a field and \( A \) be a \( k \)-algebra. If \( A^* \) (the set of all invertible elements in \( A \))=\( k^* \), then the ring \( A \) is torus invariant.

Proof. Let \( R = A[X, X^{-1}]=B[Y, Y^{-1}] \). By (1.1) the field \( k \) is contained in \( B \). Since \( A^* = k^* \), the unit element \( u \) of \( A \) is in \( k \), hence in \( B \). It follows from (2.1) that \( A \) is torus invariant.

**Proposition 2.3.** Let \( A = A_0[t_1, t_2, \ldots, t_n, (t_1t_2\cdots t_n)^{-1}] \) where \( t_i \)'s are independent variables over \( k \)-algebra \( A_0 \) and \( A^*_0=k^* \), then \( A \) is torus invariant.

Proof. Let \( R = A[X, X^{-1}]=B[Y, Y^{-1}] \). Then by the lemma (1.1) \( Y = uX' \) and \( X = vY' \). Since \( u \) is invertible in \( A = A_0[t_1, t_2, \ldots, t_n, (t_1\cdots t_n)^{-1}] \), \( Y = rt_1\cdots t_n, \quad r \in A^*_0= k^* \). We may assume that \( r=1 \), so \( Y = t_1\cdots t_nX' \).

On the other hand as \( t_i \) is invertible in \( R = B[Y, Y^{-1}] \), \( t_i = b_iY' \), \( b_i \in B^* \). Then we have that

\[ ff' + \sum e_if_i = 1. \]

Therefore the following natural homomorphism is surjective.
Since $Z$ is P.I.D., we can construct a basis of $Z^{(n+1)}$ containing this vector $(f', e_1, \ldots, e_n)$. Put this basis

$$e_0 = (f', e_1, \ldots, e_n)$$

and put $u_i = t_i^{(i_i)} x_i$. 

$$R = A_0[u_1, \ldots, u_n, (u_1 \cdots u_n)^{-1}] [Y, Y^{-1}] = A[X, X^{-1}] = B[Y, Y^{-1}]$$

Therefore $A$ is isomorphic to $B$. Hence $A$ is torus invariant.

(2.4) Let $R = A[X, X^{-1}] = B[Y, Y^{-1}]$. An ideal $I$ of $R$ is said to be vertical relative to $A$ if there exists an ideal $J$ of $A$ such that $JR = I$. If $J$ is an ideal of $A$ such that $JR$ is vertical relative to $B$, then we will simply say that $J$ is vertical relative to $B$. If $A$ is a $k$-affine domain, the prime ideals defined by the singular locus of Spec $A$ are vertical relative to $B$.

**Proposition 2.5.** Let $R = A[X, X^{-1}] = B[Y, Y^{-1}]$. If there exists a maximal ideal of $A$ which is vertical relative to $B$, then $A[X, X^{-1}] = B[X, X^{-1}]$. In particular $A$ and $B$ are isomorphic.

Proof. Let $m$ be a maximal ideal of $A$ which is vertical relative to $B$. Then there exists an ideal $n$ of $B$ such that $mR = nR$. Therefore $R/mR = A/m[X, X^{-1}] = R/nR = B/n[Y, Y^{-1}]$, where $X = vY$ and $Y = uX$. Since $m$ is a maximal ideal, $A/m$ is a field. Hence $u$ is in $B/n$ by (1.1). Therefore we obtain $f = \pm 1$ by (2.1). Thus $A$ is isomorphic to $B$.

**Corollary 2.6.** Let $A$ be a $k$-affine domain with isolated singular points, then $A$ is torus invariant.

3. Strongly torus invariant rings

In this section we investigate strongly torus invariant rings.

**Proposition 3.1.** Let $A[X, X^{-1}] = B[Y, Y^{-1}]$. If $Q(A) \subseteq Q(B)$, then $A = B$, where $Q(R)$ is the total quotient field of $R$.

Proof. Let $x$ be an element of $A$, then there exist two elements $b$ and $b'$ of $B$ such as $x = b/b'$. Hence $b = b'x$. In the graded ring $B[Y, Y^{-1}]$ the elements $b$ and $b'$ are homogeneous of degree zero, thus $x$ is also degree zero. Hence we have $A \subseteq B$. Let $b$ be an element of $B$. Then $b = \sum a_j X^{j}$, $a_j \in A$. By (2) of (2.0) we have that $b = \sum a_j v^j Y^{j f}$. If $f = 0$, then $X \in B$. Thus $A[X, X^{-1}] \subseteq B$, as desired.
it's a contradiction, hence \( f \neq 0 \). Since \( a_jv^i \in B \) and \( Y \) is a variable over \( B \), \( b = a_5 \in A \). Thus \( A = B \).

**Corollary 3.2.** Let \( \bar{A} \) denote the integral closure of \( A \). If \( \bar{A} \) is strongly torus invariant, then \( A \) is also so.

Proof. It is easily seen that if \( A[X, X^{-1}] = B[Y, Y^{-1}] \) then \( \bar{A}[X, X^{-1}] = \bar{B}[Y, Y^{-1}] \). Since \( \bar{A} \) is strongly torus invariant, \( \bar{A} = \bar{B} \). Hence \( Q(A) = Q(B) \), and we have that \( A = B \).

**Proposition 3.3.** Let \( A \) be a domain with \( J(A) \neq 0 \), where \( J(D) \) is the Jacobson radical of a ring \( D \). Then \( A \) is strongly torus invariant.

Proof. Let \( a \) be a non-zero element of \( J(A) \). Then \( 1 + a \) is unit, so in the graded ring \( B[Y, Y^{-1}] \), \( 1 + a \) is homogeneous. Since the "unity 1" is a homogeneous element of degree 0, the element \( a \) is also so. Thus the element \( a \) is contained in \( B \).

Let \( x \) be any element of \( A \). Since \( xa \) is contained in \( J(A) \), \( xa \) is in \( B \). Hence \( A \) is contained in \( Q(B) \). By (3.1), we have that \( A = B \).

**Corollary 3.4.** If \( A \) is a local domain, then \( A \) is strongly torus invariant.

**Proposition 3.5.** Let \( A \) be an affine ring over a field \( k \) and let \( A[X, X^{-1}] = B[Y, Y^{-1}] \). Then \( A = B \) if and only if every maximal ideals of \( A \) is vertical relative to \( B \).

Proof. It suffices to prove the "if" part of the (3.5). By (3.3) we may assume that \( J(A) = 0 \). Let \( x \) be an element of \( B \) and let \( x = \sum_{j=1}^{s} a_j X^j \), where \( s < t \), \( a_j \in A \) and \( a_s \neq 0 \) and \( a_t \neq 0 \). For any maximal ideal \( m \) of \( A \) there exists a maximal ideal \( n \) of \( B \) such as \( mR = nR \), where \( R = A[X, X^{-1}] \). Let \( x \) denote the residue class of \( x \) in \( B/n \). Then \( x \) is algebraic over the coefficient field \( k \), hence there exist elements \( \lambda_0, \lambda_1, \ldots, \lambda_{s-1} \) in \( k \), such that \( f(x) = x^s + \lambda_{s-1}x^{s-1} + \ldots + \lambda_0 \in R = mR \). If \( t \neq 0 \), then the highest degree term of \( f(x) \) with respect to \( X \) is \( a_t x^t \in mR \), thus \( a_t \) is contained in \( m \) for every maximal ideal in \( A \). Since \( J(A) = 0 \), \( a_t = 0 \). It's a contradiction. Therefore \( t = 0 \). By the same way, we have that \( s = 0 \), hence \( x \) is in \( A \). Thus \( A = B \).

We denote the subring generated by all the units of \( A \) by \( A_u \).

**Proposition 3.6.** Let \( A \) be a \( k \)-affine domain with an isolated singular point. If \( A \) is algebraic over \( A_u \), then \( A \) is strongly torus invariant.

Proof. Let \( A[X, X^{-1}] = B[Y, Y^{-1}] \) and let \( m \) be the maximal ideal defined by the isolated singular point. Then there exists a maximal ideal \( n \) of \( B \) such as \( mR = nR \). Let \( a \) be a unit element of \( A \). In the graded ring \( B[Y, Y^{-1}] \), the
element \(a\) is also invertible, so \(a = bY^j\), for some invertible element \(b\) in \(B\) and a certain integer \(j\). Since \(A/m\) is algebraic over \(k\), there exist elements \(\lambda_0, \lambda_1, \ldots, \lambda_n \in k\) such that \(\lambda_0a^n + \cdots + \lambda_n a + \lambda_0 \in mR = nR\). If \(j \neq 0\), \(\lambda_nb^n\) is in \(n\), hence \(b\) is not invertible, it's a contradiction. Thus we have that \(A \subseteq B\). By the following lemma our proof is over.

**Lemma 3.7.** Let \(A[X, X^{-1}] = B[Y, Y^{-1}]\). If \(A\) is algebraic over \(A \cap B\), then \(A = B\).

**Proof.** Since \(A\) is algebraic over \(A \cap B\), \(A\) is also algebraic over \(B\), but \(B\) is algebraically closed in \(B[Y, Y^{-1}]\), therefore \(A\) is contained in \(B\). Thus we have that \(A = B\).

Let \(A\) be an integral domain containing a field \(k\). We denote the set of all automorphisms of \(A\) over \(k\) by \(\text{Aut}_k(A)\).

**Proposition 3.8.** Let \(A\) be an integral domain containing an infinite field \(k\). If \(\text{Aut}_k(A)\) is a finite set, then \(A\) is strongly torus invariant.

**Proof.** Let \(R = A[X, X^{-1}] = B[Y, Y^{-1}]\). Let \(\Phi_\lambda, \lambda \in k^*\), be an automorphism of \(R\) defined by \(\Phi_\lambda(Y) = \lambda Y\) and \(\Phi_\lambda(b) = b\) for \(b \in B\). Following the notation of (2.0) we have \(X = vY^f\), thus \(\Phi_\lambda(X) = \lambda'X\), therefore \(R = \Phi_\lambda(A)[X, X^{-1}]\). Let \(p\) be the projection \(A[X, X^{-1}] \rightarrow A\) defined by \(p(X) = 1\) and \(\iota\) be the canonical injection \(A \rightarrow A[X, X^{-1}]\). Define \(\sigma_\lambda = q \circ \Phi_\lambda \circ \iota\). Then \(\sigma_\lambda\) is an endomorphism of \(A\). We shall show that \(\sigma_\lambda\) is surjective. Let \(x\) be an element of \(A\). Since \(R = \Phi_\lambda(A)[X, X^{-1}]\), there exist elements \(a_i\)'s of \(A\) such as \(x = \sum \Phi_\lambda(a_i)X^{i}\). Hence \(x = p(x) = \sum p\Phi_\lambda(a_i)\). Let \(x' = \sum a_i \in A\), then \(\sigma_\lambda(x') = \sum p\Phi_\lambda(a_i) = x\). Thus \(\sigma_\lambda\) is surjective. Next we shall show that \(\sigma_\lambda\) is injective. Since \(\Phi_\lambda^{-1}((X-1)R \cap \Phi_\lambda(A)) = \Phi_\lambda^{-1}(X-1)R \cap A = (\lambda'X-1)R \cap A = 0\), we have \((X-1)R \cap \Phi_\lambda(A) = 0\), therefore \(\sigma_\lambda\) is injective. Hence \(\sigma_\lambda\) is an automorphism of \(A\).

We shall prove that the set \(\{\sigma_\lambda; \lambda \in k^*\}\) is infinite when \(A \neq B\). Since \(u = v^{-f}Y^{1-f'}\), \(\sigma_\lambda(u) = \lambda^{1-f'}u\). Therefore our assertion is proved when \(1 - ff' \neq 0\). Suppose \(ff' = 1\). Then we may assume that \(R = A[X, X^{-1}] = B[X, X^{-1}]\). If \(A \subseteq B\), then \(A = B\), so there exists an element \(x\) of \(A\) not contained in \(B\), say \(x = \sum b_jX^j, \ t > s\). Since \(\ker p = (X-1)R\) and \((X-1)R \cap B = 0\), \(p(b_j) \neq 0\) for \(b_j \neq 0\). Since \(\sigma_\lambda(x) = \sum p(b_j)\lambda^j\) and \(p(b_j) \neq 0\) for some \(j \neq 0\), the set \(\{\sigma_\lambda; \lambda \in k^*\}\) is infinite.

Next we shall give two cases of rings which are not strongly torus invariant. If \(A\) has a non-trivial locally finite iterative higher derivation \(\psi: A \rightarrow A[T]\), then \(A[T] = B[T]\), where \(B = \psi(A)\) and \(A \neq B\), as is proved in [4]. Hence we have that \(A[T, T^{-1}] = B[T, T^{-1}]\) and \(A \neq B\). If \(A\) is a graded ring, then \(A\) is not strongly torus invariant. Indeed, let \(X\) be a variable over \(A\) and let
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$B_i = \{a_i X^i; \ a_i \in A_i\}$. Then $B_i$ is an $A_\circ$-module contained in $A[X, X^{-1}]$. Let $B = \sum B_i$. Then $B$ is a graded ring and we easily see that $A[X, X^{-1}] = B[X, X^{-1}]$. We shall show that $X$ is a variable over $B$. Assume that there exist elements $b_0, b_1, \ldots, b_n$ in $B$ such that $b_n X^n + \cdots + b_1 X + b_0 = 0$. By the definition of $B$ we denote $b_i = \sum a_{ij} X^j$, $a_{ij} \in A_j$. In the graded ring $A[X, X^{-1}]$ the homogeneous term of degree $t$ of this equation is that

$$(a_{n,t-n} + a_{n-1,t-n+1} + \cdots + a_{0,t})X^t = 0.$$ 

Since $A$ is a graded ring and $a_{ij}$ is a homogeneous element of degree $j$, we obtain $a_{ij} = 0$ for all index $i$ and $j$, hence $X$ is a variable over $B$.

By [4] we have that a $k$-algebra $A$ has a non-trivial locally finite iterative higher derivation if and only if $\text{Aut}_k(A)$ has a subgroup isomorphic to $G_a = \text{Spec} k[T]$. We easily see that $A$ is a non-trivial graded ring if and only if $\text{Aut}_k(A)$ has a subgroup isomorphic to $G_a = \text{Spec}(k[T, T^{-1}])$.

**Proposition 3.9.** A $k$-algebra $A$ is not strongly torus invariant, if $\text{Aut}_k(A)$ has a subgroup isomorphic to $G_a$ or $G_m$.

Assume that $\text{Aut}_k(A)$ is an infinite group. If $\text{Aut}_k(A)$ has an algebraic group structure, then there exists the following exact sequence;

$$0 \to T \to \text{Aut}_k(A) \to \theta \to 0$$

where $\text{Aut}_k(A)_0$ is the connected component containing the identity $I_A$, and $T$ is a maximal torus subgroup of $\text{Aut}_k(A)_0$ and $\theta$ is an abelian variety. Let $P$ be an arbitrary closed point of $\text{Spec}(A)$. If $T = 0$, then there exists a regular map

$$\Phi: \text{Aut}_k(A)_0 \to \text{Spec}(A)$$

$$\sigma \to \sigma(P).$$

Since $\text{Im}(\Phi)$ is a projective variety contained in the affine variety $\text{Spec}(A)$, the set $\text{Im}(\Phi)$ consists of one point, it contradicts $\dim \text{Aut}_k(A)_0 > 0$. Hence we have that $T \neq 0$. Since $T \cong G_a$ or $G_m$, we have the following result:

**Proposition 3.10.** If $\text{Aut}_k(A)$ is not a finite set and has an algebraic group structure, then $A$ is not strongly torus invariant.

4. **Affine domains of dimension $\leq 2$**

Let $k$ be a field of characteristic zero which contains all roots of "unity". In this section let $A$ be an affine domain over $k$. We shall see that if $\dim A = 1$, then $A$ is always torus invariant. Moreover $A$ is not strongly torus invariant if and only if $\text{Aut}_k(A) \cong G_m$. Let $\dim A \geq 2$. Then $A$ is not always torus
invariant. But if an integrally closed domain $A$ is not a $\mathbb{Z}$-graded ring, then $A$ is torus invariant.

For the proof we need a lemma.

**Lemma 4.1.** Let $K$ be a finite separable algebraic field extension of a field $k$. If $A$ is a one-dimensional affine normal ring such that $k \subseteq A \subseteq K[X, X^{-1}]$, then $A$ is a polynomial ring or a torus ring over $k'$ where $k'$ is the algebraic closure of $k$ in $A$.

Proof. We may assume that $k = k'$. Following the similar device to the proof of (2.9) in [1, p 322], we have $Q(A) = k(\theta)$ for some element $\theta$ of $A$.

Since $k[\theta] \subseteq A \subseteq k(\theta)$, $A = k[\theta]$ or $A = k[\theta, \frac{1}{f(\theta)}]$ for some polynomial $f(\theta) \in k[\theta]$. Let $A = k[\theta, \frac{1}{f(\theta)}]$. Then we may assume that $f(\theta)$ has no multiple factors. The element $f(\theta)$ is invertible in $A$, so is also invertible in $K[X, X^{-1}]$. Thus we have $f(\theta) = \beta X^r$, $\beta \in K$, $\theta \in K[X, X^{-1}]$. We may assume that $r \geq 0$, if necessary, by replacing $X$ with $X^r$. Then we easily see that $\theta \in K[X]$. The uniqueness of the irreducible decomposition in a polynomial ring implies that $\deg f(\theta) = 1$, since the polynomial $f(\theta)$ has no multiple factors and $f(\theta) = \beta X^r$. Hence we may assume that $f(\theta) = \theta$ and we obtain $A = k[\theta, \frac{1}{\theta}]$.

Let $A$ be an integral domain. If $A$ is contained in $K[X, X^{-1}]$, then $A$ is a polynomial ring or a torus ring over $k'$.

**Proposition 4.2.** Let $A$ be a one-dimensional affine domain over a field $k$ of characteristics zero. Then we obtain that

1. $A$ is torus invariant,
2. $A$ is not strongly torus invariant if and only if $\text{Aut}_k(A)$ has a subgroup isomorphic to $G_m$. If $A$ is not strongly torus invariant and $A$ is integrally closed, then $A$ is a polynomial ring or a torus ring over the algebraic closure of $k$ in $A$.

Proof. At first we shall prove (2). The sufficiency follows from (3.9). Let $R = A[X, X^{-1}] = B[Y, Y^{-1}]$ in which $A \neq B$. If $mR \cap A \neq 0$ for any maximal ideal $m$ of $B$, then $m$ is vertical relative to $A$, and we have $A = B$ by (3.5). Hence there exists a maximal ideal $m$ such as $mR \cap A = 0$. Since $\text{ch } k = 0$, $B/m = K$ is a finite separable algebraic field over $k$. The residue mapping of $R$ to $R/mR$ yields (up to isomorphism) $k \subseteq A \subseteq K[Y, Y^{-1}]$ where $Y$ is algebraically independent over $K$. Therefore $A$ is a polynomial ring or a torus ring by the lemma (4.1). Thus the automorphism group $\text{Aut}_k A$ contains a subgroup isomorphic to $G_m$.

Assume that $A$ is not integrally closed. Then prime divisors in $A$ of the conductor $t(A/A)$ are vertical relative to $B$. Hence we may assume $X = Y$ by
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(2.5). The above lemma (4.1) implies that $\bar{A} = k'[t, t^{-1}]$ or $\bar{A} = k'[t]$ where $k'$ is the algebraic closure of $k$ in $\bar{A}$.

Firstly let $\bar{A} = k'[t]$. Since $\bar{A} \cong \bar{B}$, there exists an element $s$ in $\bar{B}$ such that $\bar{B} = \bar{A}[X, X^{-1}] = \bar{B}[X, X^{-1}]$. We have $k'[X, X^{-1}]$ or $A = k'[t]$. Let $\bar{s} = g(X)t + g(X)$ where $g(X)$ and $f(X)$ are the constant terms of this polynomial with respect to $s$.

Therefore $f(X) = f^k$. Hence we may assume that $t = X\bar{s}$. We shall show that $A$ is a graded ring. Let $\alpha$ be an element of $A$. Since $\alpha$ is contained in $A = k'[t]$ and $t = X\bar{s}$, we have that $\alpha = \sum \lambda_i X^{i\bar{s}}$.

Secondary let $\bar{A} = k'[t, t^{-1}]$. Then $\bar{B} = k'[s, s^{-1}]$. Since $t$ and $s$ are invertible in $\bar{B}$, we may assume that $t = s^t X^s$ and $s = t^t X^s$, then $t = (t')^t X^s = t^i X^{im+s}$, therefore $ij = 1$. Hence we may assume $t = s X^s$. By the same method as in the case $\bar{A} = k'[t]$ we have that $A$ is a graded ring.

Proof of (1). If $A$ is not integrally closed, then the prime divisors of the conductor $t(\bar{A}/A)$ are vertical relative to $B$. Since non-zero prime ideals of $A$ are maximal, the ring $A$ is isomorphic to $B$ by (2.5). If $A$ is integrally closed and $A$ is neither a polynomial ring nor a torus ring, then $A$ is strongly torus invariant, hence $A$ is torus invariant. If $A$ is either a polynomial ring or a torus ring, $A$ is torus invariant by (2.2) and (2.3).

Next we shall consider the case; the coefficient field $k$ has all roots of "unity" and its characteristic is zero. Then we prove the following:

**Theorem 4.3.** Let $A$ be an integrally closed $k$-affine domain of dimension two, where the field $k$ has all roots of "unity" and $ch k = 0$. If $A$ is not torus invariant, then $A$ is a $Z$-graded ring which contains units of non-zero degree.

Proof. Assume that $A$ is not torus invariant. Then there exist a $k$-algebra $B$ and independent variables $X, Y$ such that $A$ is not isomorphic to $B$ and $R = A[X, X^{-1}] = B[Y, Y^{-1}]$. By (2.0) and (2.1) we obtain $ff' \neq 1$. We shall show...
that it follows from $ff' \neq 1$ that $A$ is a $\mathbb{Z}$-graded ring. We may only consider the case $1-ff' > 0$. Let $x$ be a $(1-ff')^{-th}$ root of $u$ and let $y=x^{-f}/X$. Then $y^{-f}f'=v$. Since $(y^{-f}y)^{-ff'}=u$, $x=\lambda y^{-f}Y$ for some $(1-ff')^{-th}$ root $\lambda$ of "unity". From the relations; $y=x^{-f}/X$ and $Y=uxf'$, we have $\lambda=1$.

Since $y=x^{-f}/X$ and $x=y^{-f}Y$ are invertible, we have $A[x][X, X^{-1}]=B[y][Y, Y^{-1}]=A[x][y, y^{-1}]=B[y][x, x^{-1}]$. Define a surjective homomorphism $j: A[x][y, y^{-1}] \to A[x]$ by $j(y)=1$. Let $A_0=j(B[y]) \subseteq A[x]$. We shall show that $A[x]=A_0[x, x^{-1}]$. Let a be an element of $A$. Then $a=\sum b_i x^i$, $b_i \in B$. Since $j(a)=a$ and $j(x)=x$, we have that $a=\sum j(b_i) x^i$, $j(b_i) \in A_0$. Thus $A[x]=A_0[x, x^{-1}]$ and $x$ is algebraically independent over $A_0$. By the same way $B[y]=B_0[y, y^{-1}]$.

Since the every $(1-ff')^{-th}$ root of "unity" is contained in $k$ and $ch k=0$ and $A$ is normal, the extension $A[x]/A$ is a Galois extension with a cyclic group $G=<\sigma>$ (cf. [3] p 214). Indeed when $|G|=n$, $1-ff'$ and there exists a primitive $n-th$ root $\lambda$ of "unity" such that $\sigma(x)=\lambda x$ and the invariant subring $(A[x])^\sigma=A$ and $A[x]=A_0[x, x^{-1}]$ and $x$ is algebraically independent over $A_0$. From this we see that $A_0$ is always normal.

We shall show that there exists a subring $A'_0$ in $A[x]$ such that $A[x]=A'_0[x, x^{-1}]$ and $\sigma(A'_0)=A'_0$. If $A_0$ is strongly torus invariant, then $\sigma(A_0)=A'_0$ for $\sigma(A_0)[x, x^{-1}]=A_0[x, x^{-1}]$, therefore $A_0$ satisfies the conditions. If $A_0$ is not strongly torus invariant, then $A_0=k'[t]$ or $k'[t, t^{-1}]$ by (4.2). Firstly let $A_0=k'[t]$. Since $k'[x, x^{-1}][t]=k'[x, x^{-1}][\sigma(t)]$, we easily see that $\sigma(t)=\mu x^i t+f(x)$, $\mu \in k^*$ and $f(x) \in k'[x, x^{-1}]$. The order of $\sigma$ is $n$, i.e. $\sigma^n=\text{Identity}$, so $\sigma^n(t)=t$, the other hand $\sigma^n(t)=\mu^n x^{(i+1-1)n} t^{n}+g(x)$, $g(x) \in k'[x, x^{-1}]$, therefore we have that $i=0$, thus $\sigma(t)=\mu t+f(x)$ and $\mu^n=1$. Let $f(x)=\sum f_i x^i$ and define the set $\Delta=\{j \in Z; \lambda^j \neq \mu \}$. Let $h(x)=\sum h_i x^i$, where $h_i=f_i(\mu-\lambda x)^i$, and put $s=\sum h_i x^i$. Then $\sigma(s)=\mu s+\sum f_i x^i$, hence $\sigma^n(s)=\mu^n s+n \sum f_i x^i=s+\sum f_i x^i$. Since $\sigma^n(s)=s$, we have $\sigma(s)=\mu s$. We set $A'_0=k'[s]$, then $A'_0$ satisfies the conditions.

Secondary let $A_0=k'[t, t^{-1}]$. Since $k'[x, x^{-1}][t, t^{-1}]=k'[x, x^{-1}][\sigma(t), \sigma(t)^{-1}]$, we easily see that $\sigma(t)=\mu x t$ or $\sigma(t)=\mu x^{-t}$, $\mu \in k^*$.  

Case (i); $\sigma(t)=\mu x t$. Since $\sigma^n(t)=\mu^n \lambda x^{(i+1-1)n} x^{n} t$ and $\sigma^n(t)=t$, we have that $\sigma(t)=\mu t$, so $\sigma(A_0)=A_0$.

Case (ii); $\sigma(t)=\mu x^{-t}$. If $n$ is odd, say $n=2m+1$, then $\sigma^n(t)=\mu \lambda^m x t^{-1}$, but this is impossible for $\sigma^n(t)=t$. Therefore $n$ is even, say $n=2m$. Then $\sigma^n(t)=\lambda^{im} t$. Since $\lambda$ is a primitive $n-th$ root of "unity", the integer $i$ is even, say $i=2j$. Let $s=xt^j$ and $A'_0=k'[s, s^{-1}]$. Then $A'_0$ satisfies the conditions.
Next we shall show that $A$ has a $\mathbb{Z}$-graded ring structure. Let $a$ be an element of $A$. Since $a \in A_0[x, x^{-1}]$, $a = \sum a_i x^i$. Then $a = a(a) = \sum \sigma(a_i) x^i$ and $\sigma(a) \in A_0$. Comparing the coefficient of each term in the equality; $\sum a_i x^i = \sum \sigma(a_i) x^i$, we have that $a_i = \sigma(a_i) x^i$, then $\sigma(a x^i) = a x^i$. Thus $a x^i$ is an element of $A$. Therefore $A$ is a graded ring. Since there exists units of non-zero degree, $A$ has a $\mathbb{Z}$-graded ring structure.

REMARK. The converse of this theorem is false. Indeed we find by (2.3) that the ring $k[T][X, X^{-1}]$ is a $\mathbb{Z}$-graded ring with respect to $X$ which is not torus invariant.

EXAMPLE. We shall construct an example of an affine dimension $A$ of dimension two which is not torus invariant.

Let $D$ be an integrally closed domain of dimension one over an algebraically closed field $k$ and $D^* = k^*$. Let $a$ be a non-unit of $D$ and $a^2 = a$, $a \in D$. Assume that $D$ is noetherian and $D[a]$ is strongly torus invariant. Since an affine domain of dimension one whose totally quotient field has a positive genus is strongly torus invariant by (4.2), this assumption can be satisfied for a suitable choice of $D$. Let $T$ be a variable over $D$ and $A = D[aT, T]^\times$. Let $X$ be a variable over $A$ and $S = T^2X$ and $Y = T^5X^2$. Since $T = S^2Y$ and $X = S^5Y^{-2}$, we have that $A[X, X^{-1}] = B[Y, Y^{-1}]$. By (1.1) invertible elements in the graded ring $A$ are homogeneous. Since $D^* = k^*$, we obtain $A^* = \{\eta T^\mu; \eta \in k^* \text{ and } \mu \in \mathbb{Z}\}$. Hence the quotient $A^*/k^*$ is generated by $T^\mu$. Similarly $B^*/k^*$ is generated by $S^\mu$. We shall show that $A$ is not isomorphic to $B$. We assume that there exists an isomorphism $\sigma$ of $A$ to $B$. Since $\sigma$ is a group-isomorphism of $A^*$ to $B^*$, we have $\sigma(T^\mu) = \mu S^\mu$ or $\sigma(T^\mu) = \mu S^{-\mu}$, $\mu \in k^*$. We shall only consider the case: $\sigma(T^\mu) = \mu S^\mu$, since the proof of the other case is the similar. Let $\sigma$ be an isomorphism of $A[T]$ to $B[S]$ defined by $\sigma = \sigma$ on $A$ and $\sigma(T) = \xi S$, $\xi^2 = \mu$. Then we have that $D[a] [S, S^{-1}] = \sigma(D[a]) [S, S^{-1}]$, therefore $\sigma(D[a]) = D[a]$; for $D[a]$ is strongly torus invariant. Since $\sigma(D) \subseteq D[a] \cap B = D$, we have $\sigma(D) = D$, therefore we easily see that $\sigma$ is an isomorphism as graded rings. Thus we have $\sigma((aT)D) = (a^2S)D$, hence $\sigma(a) \in a^2D$. Since the element $a$ is not a unit, $a^2D \subseteq aD$, thus $\sigma(a)D \subseteq a^2D \subseteq aD$, hence we have a proper ascending chain $\{\sigma^{-1}(a)D\}$, but it contradicts the netherian assumption of $D$. Hence $A$ is not torus invariant.

(4.4) Now let $A = \sum A_i$ be an integrally closed $\mathbb{Z}$-graded domain which contains invertible elements of non-zero degree. Let $e$ be an invertible element of $A$ with the smallest positive degree $d$. Let $a$ be a unit of $A$, then $a$ is a homogeneous elements with $\deg a = jd$ for some integer $j$, and there exists an element $\xi$ of $A^*_e$ such as $a = \xi e^j$. Let $i$ be any positive integer and $x$ be one of the $ijd$-th roots of $a$, say $x^{ijd} = a$. Since $A[x]$ is a $\mathbb{Z}$-graded ring with the
invertible elements $x$ of degree one, $A[x]=A_0[x, x^{-1}]$ by (1.4) where $A_0$ contains $A_0$. Let $f$ and $f'$ be integers such as $ff'+ijd=1$ and let $X$ be a variable over $A$. Put $y=x^{-f}X$ and $Y=ax^{f'}$. Then $x=y^{-f'}Y$ and $y=i^jdY'$. Therefore $A_0[x, x^{-1}][X, X^{-1}]=A_0[y, y^{-1}][Y, Y^{-1}]$. Since the every $n$-th roots of “unity” is contained in $k$ and $A$ is integral closed, the extension $A[x]/A$ is a Galois extension with a cyclic group $G=\langle \sigma \rangle$. Indeed $|G|=di$ and there exists a primitive $di$-th root $\lambda$ of “unity” such as $\sigma(x)=\lambda x$, and $(A[x])^\sigma=A$. Since $A_0$ is algebraic over $A_0$, $\sigma(A_0)$ is also so, hence $\sigma(A_0)$ is algebraic over $A_0$, but $A_0$ is algebraically closed in $A_0[x, x^{-1}]$, therefore $\sigma(A_0)=A_0$. Since $\sigma(y)=\lambda^{-f'}y$, $\sigma$ is an automorphism of $A_0[y, y^{-1}]$. Let $B=A_0[y, y^{-1}]^\sigma$ and $\sigma$ be an automorphism of $A_0[x, x^{-1}][X, X^{-1}]$ defined by $\sigma(X)=X$ and $\sigma=\sigma$ over $A_0[x, x^{-1}]$. Since $\sigma(Y)=Y$ and $\sigma(X)=X$, we obtain $B[Y, Y^{-1}]=A_0[y, y^{-1}][Y, Y^{-1}]^\sigma=A_0[x, x^{-1}][X, X^{-1}]^\sigma=A[X, X^{-1}]$.

**Proposition 4.5.** Let $A$ be an integrally closed $k$-affine domain of dimension 2. If $A[X, X^{-1}]=B[Y, Y^{-1}]$ and $ff'=1$, then $A$ has a $\mathbb{Z}$-graded ring structure and $B$ is isomorphic to one of algebras constructed in (4.4).

Proof. The first statement is already mentioned in the proof of (4.3) and we obtained $A_0[x, x^{-1}][X, X^{-1}]=A_0[y, y^{-1}][Y, Y^{-1}]$ and $\sigma(A_0)=A_0$. Let $B'=A_0[y, y^{-1}]^\sigma$. Then $B'$ is one of algebras in (4.4). Since $B'[Y, Y^{-1}]=B[Y, Y^{-1}], B$ is isomorphic to $B'$.

5. D-torus invariant

Let $D$ be an integral domain containing a field $k$ of characteristic zero and $A$ be a $D$-algebra. The ring $A$ is called $D$-torus invariant; if $A[X, X^{-1}]=B[Y, Y^{-1}]$ for a certain $D$-algebra $B$ and independent variables $X$ and $Y$, then we have always $A\cong B$. Then we have the following result:

**Proposition 5.1.** Let $A$ be an integrally closed domain over $D$ and $tr. deg_D A=1$. If $A$ is not $D$-torus invariant, then $A$ is a $\mathbb{Z}$-graded ring containing units of non-zero degree.

Proof. Let $A[X, X^{-1}]=B[Y, Y^{-1}]$, where $B$ is a $D$-algebra and not $D$-isomorphic to $A$. By (2.0) and (2.1) we easily see that $ff'=1$. Then we may assume $1-ff'>0$. Let $x$ be a $(1-ff')^{-th}$ root of $u$ and $y=x^{-f'}X$. Then we have that $A[x]=A_0[x, x^{-1}]$ and $B[y]=B_0[y, y^{-1}]$ as the proof of (4.3), where $A_0$ and $B_0$ are respectively subalgebras of $A[x]$ and $B[y]$ containing $D$. Let $\sigma$ be a generator of the cyclic Galois group of the extension $A[x]/A$. We shall show that $\sigma(A_0)=A_0$. Since $tr. deg_D A_0[x, x^{-1}]=1$, $A_0$ is algebraic over $D$, thus $\sigma(A_0)$ is also so. Since $A_0$ is algebraically closed in $A_0[x, x^{-1}]$, we have that $\sigma(A_0)=A_0$. Following the similar devise to the proof of (4.3) we obtain
that $A$ is a $\mathbb{Z}$-graded ring, and $D$ is contained in $A$.

In the following we shall consider the case where $A$ is a $\mathbb{Z}$-graded ring and $A_0 = D$. We consider only $D$-isomorphisms of $D$-algebras.

**Theorem 5.2.** Let $A$ be an integrally closed $\mathbb{Z}$-graded ring. Assume that the subring $A_0$ contains an algebraically closed field $k$ and that $A_0^\times = k^\times$. Let $d$ be the smallest positive integer among the set of degrees of units in $A$. Then the number of isomorphic classes of $A_r$-algebra as $B$ such that $A[X, X^{-1}] = B[Y, Y^{-1}]$ equals to $\Phi(d)$, where $\Phi$ is the Euler function.

Proof. Let $i$ be an integer such as $1 \leq i < d$ and $(i, d) = 1$. Since $(i, d) = 1$, $ij + dh = 1$ for some integers $j$ and $h$. Moreover we may assume $h \geq 0$. Fix a unit $e$ of degree $d$. Let $x$ be one of the $d$-th roots of $e$. Then we have that $A[x] = A_0[x, x^{-1}]$ for a subring $A_0$ containing $A_0$ by (1.4). Let $\sigma$ be a generator of the cyclic Galois group of the extension $A[x]/A$. Then $\sigma(x) = \lambda x$, where $\lambda$ is a primitive $d$-th root of “unity”. Since $A_0'$ is algebraic over $A_0$ and algebraically closed in $A_0[x, x^{-1}]$, we obtain $\sigma(A_0') = A_0'$. Let $X$ be a variable over $A$ and let $y = x^{-1}X$ and $Y = e^xX$. Then we have that $A_0'[x, x^{-1}] [X, X^{-1}] = A_0'[y, y^{-1}] [Y, Y^{-1}]$. Define $B_i = A_0'[y, y^{-1}]$ and let $\sigma$ be an isomorphism of $A_0'[x, x^{-1}] [X, X^{-1}]$ defined by $\sigma(X) = X$ and $\sigma = \sigma$ on $A_0'[x, x^{-1}]$. Since $Y = e^xX$, $\sigma(Y) = Y$, therefore we obtain that $A[X, X^{-1}] = B[Y, Y^{-1}]$. We can easily see that $B_i$ is a $X$-graded ring and $(B_i)_0 = A_0$. Especially we have $B_i \cong A$.

Let $i_1$ and $i_2$ be integers such as $1 \leq i_1 < i_2 < d$ and $(i_1, d) = (i_2, d) = 1$. Let $B' = A_0'[y, y^{-1}]$ and $B'' = A_0'[x, x^{-1}]$ where $\sigma(y) = \lambda^{-i_1}y$ and $\sigma(z) = \lambda^{-i_2}z$, i.e., $B' = B_{i_1}$ and $B'' = B_{i_2}$. We shall show that $B'$ and $B''$ are not isomorphic. Assume that there exists an $A_r$-isomorphism $\psi$ of $B'$ to $B''$. Let $a$ be a unit in $B'$ of non-zero degree, say degree $a = n$, $n \neq 0$. Let $b$ be a homogeneous element of $B'$ and degree $b = t$. Then we have $b^n = ra^n$ for an element $r$ in the coefficient ring $A_0$, hence $\psi(b^n) = \psi(b)^n = r\psi(a^n)$. Since $r$ and $\psi(a^n)$ are homogeneous, $\psi(b)$ is also homogeneous by (1.1), therefore $\psi$ is an isomorphism as graded rings.

Let $c$ be a homogeneous element in $B'$ of degree one. Then $c = s_1y$ for an element $s_1$ in $A_0$. Since $\sigma(c) = c$ and $\sigma(y) = \lambda^{-i_1}y$, we have $\sigma(s_1) = \lambda^{i_1}s_1$. Hence $s_1^d$ is in $B'$. Since $\psi(s_1y) = s_2z$ for an element $s_2$ in $A_0'$. Since $\sigma(s_2z) = s_2z$ and $\sigma(z) = \lambda^{-i_2}z$, we have $\sigma(s_2) = \lambda^{i_2}s_2$, hence $s_2^d$ is in $B''$. By the relations; $s_1^d\psi(y^d) = \psi((s_1y)^d) = \psi(s_1y)^d = s_2^dz^d$, we obtain $s_2^d = \psi(y^d)z^{-d}s_1^d$. Since $\psi(y^d)z^{-d}$ is an invertible element in $B''$ and degree zero, we have $\zeta = \psi(y^d)z^{-d} \in A_0^\times = k^\times$, therefore we have $s_2^d = \eta s_1^d$ for some $\eta \in k$, $\eta^d = \zeta$. Hence $\sigma(s_2) = \lambda^{i_2}s_2$, it contradicts the fact that $\sigma(s_2) = \lambda^{i_2}s_2$ and $\lambda$ is a primitive $d$-th root of “unity”. Therefore $B' \cong B''$.

Finally we shall show that if $A[X, X^{-1}] = B[Y, Y^{-1}]$ then $B$ is isomorphic to $B_i$ for some $i$ satisfying $0 < i < d$ and $(i, d) = 1$. The invertible element $u$
in (2.0) is homogeneous. Let $n$ be the degree of $u$. If $n=0$, then $A$ is isomorphic to $B$ by (2.1), hence $B \cong B_1$. Assume $n \neq 0$. Let $c$ be a non-zero homogeneous element of degree 1 and put $\eta = c^* u^{-1}$. Then $\eta$ is an element of $A_0$.

In the graded ring $B[Y, Y^{-1}]$ the elements $u$ and $\eta$ are homogeneous, hence $c$ is also homogeneous, thus we denote $c = bY^j$ for some element $b$ in $B$ and some integer $j$. Then we obtain that $c^* = \eta u^{-1} \eta^{-1} Y^{1-ff}$ by (2.0). Therefore we have $1-ff=\eta j$.

By the minimality of $d$ we obtain $n=ld$ for some integer $l$ and $u=\xi^e$, $\xi \in A^*_k=k^*$. Since the field $k$ is algebraically closed, we may assume $\xi = 1$, then the $d$-th root $x$ of $e$ is an $n$-th root of $u$. Since the element $\lambda$ is a primitive $d$-th root of "unity", there exists the unique integer $i$ such that $\lambda^{-i}=\lambda^i$, $0<i<d$, then $(i,d)=1$ since $(f,d)=1$. Let $y^j=x^{-j} X^j$ and $B'=A_0[y^j, y^{-j} y^j]$. Then $\sigma(y^j)=\lambda^{-i} y^j=\lambda^{-i} y$, hence $B'=B_1$. We can easily show that $x=y^{-j} Y^i$, therefore we obtain $A_0[x, x^{-1}] [X, X^{-1}]=A_0[y^j, y^{-j}] [Y, Y^{-1}]$. Since $\sigma(X)=X$ and $\sigma(Y)=Y$, we have $A[X, X^{-1}]=B_1[Y, Y^{-1}]$, hence $B[Y, Y^{-1}]=B_1[Y, Y^{-1}]$. Thus we have $B \cong B_1$.

References