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ON THE COEFFICIENT RING OF A TORUS EXTENSION

KEN-ICHI YOSHIDA

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Introduction. S. Abhyankar, W. Heinzer and P. Eakin treated the following problem in [1]; if \( A[X]=B[Y] \), when is \( A \) isomorphic or identical to \( B \)? Replacing the polynomial ring by the torus extension we shall take up the following problem; if \( A[X, X^{-1}]=B[Y, Y^{-1}] \), when is \( A \) isomorphic or identical to \( B \)? We say that \( A \) is torus invariant (resp. strongly torus invariant) whenever \( A[X, X^{-1}]=B[Y, Y^{-1}] \) implies \( A=B \) (resp. \( A\approx B \)). The roles played by polynomial rings in [1] are played by the graded rings in our theory. A graded ring \( A=\sum A_i, i\in\mathbb{Z} \), with the property that \( A_i\neq 0 \) for each \( i\in\mathbb{Z} \), will be called a \( \mathbb{Z} \)-graded ring. Main results are the followings.

An affine domain \( A \) of dimension one over a field \( k \) is always torus invariant. Moreover \( A \) is not strongly torus invariant if and only if \( A \) has a graded ring structure. An affine domain of dimension two is not always torus invariant. We shall construct an affine domain of dimension two which is not torus invariant. Let \( A \) be an affine domain over \( k \) of dimension two. Assume that the field \( k \) contains all roots of "unity" and is of characteristic zero. If \( A \) is not torus invariant, then \( A \) is a \( \mathbb{Z} \)-graded ring such that there exist invertible elements of non-zero degree.

In Section 1 we study elementary properties of graded rings. Especially we are interested in \( \mathbb{Z} \)-graded rings with invertible elements of non-zero degree. In Section 2 we discuss some conditions for \( A \) to be torus invariant. In Section 3 we give several sufficient conditions for an integral domain to be strongly torus invariant. Some relevant results will be found in S. Iitaka and T. Fujita [2]. Section 4 is devoted to the proof of the main results mentioned above.

In Section 5 we fix an integral domain \( D \) and we treat only \( D \)-algebras and \( D \)-isomorphisms there. We shall prove the following two results. When \( A \) is a \( D \)-algebra of \( tr.\ deg_d A=1 \) and \( A \) is not \( D \)-torus invariant, \( A \) is a \( \mathbb{Z} \)-graded ring such that \( D \) is contained in \( A \). If \( A \) is a \( \mathbb{Z} \)-graded ring such as \( D=A_0 \), then the number of elements of the set of \( \{D \text{-isomorphic classes of } D \text{-algebras } B \text{ such that } A[X, X^{-1}]=B[Y, Y^{-1}] \} \) is \( \Phi(d) \), where \( d \) is the smallest positive integer among the degrees of units in \( A \) and \( \Phi \) is the Euler function.

I'd like to express my sincere gratitude to the referee for his valuable advices.
1. Some properties of graded rings

Let $R$ be commutative ring with indentity. The ring $R$ is said to be a graded ring if $R$ is a graded module, $R=\sum R_i$, and $R_nR_m=\sum R_{n+m}$.

**Lemma 1.1.** Let $R$ be a graded domain. Then we have the following.

1. The unity element of $R$ is homogeneous.
2. If $a$ is homogeneous and $a=bc$, then $b$ and $c$ are both homogeneous. In particular every invertible element is homogeneous.
3. If $R$ contains a field $k$, then $k$ is a subring of $R_0$.

Proof. (1) follows immediately from the relation $1^2=1$. The proof of (2) is easy and will be omitted. To prove (3) we can assume $k$ is different from $F$ by (1). Let $a$ be an element of $k$ different from 1. Then $1-a$ is homogeneous from (2). The unity 1 is homogeneous of degree 0 by (1). Hence $a$ should be homogeneous of degree 0.

We call a graded ring $R=\sum R_i$ to be a $\mathbb{Z}$-graded ring if $i\in \mathbb{Z}$ and $Z^+$.

**Proposition 1.2.** Let $R$ be a $\mathbb{Z}$-graded domain. Let $S=\{i \in \mathbb{Z}; R_i \neq 0\}$. Then $S=\sum n\mathbb{Z}$ for a certain integer $n$.

Proof. Since $R$ is a domain, $S$ is a semi-group. Hence (1.2) is immediately seen by the following lemma.

**Lemma 1.3.** Let $S \subseteq \mathbb{Z}$ be a semi-group. If $S \cap \mathbb{Z}^+ \neq 0$ and $S \cap \mathbb{Z}^- \neq 0$, then $S$ is a subgroup of $\mathbb{Z}$.

If $R$ is a $\mathbb{Z}$-graded domain, then we may assume $R_i \neq 0$ for any $i \in \mathbb{Z}$.

**Proposition 1.4.** Let $R$ be a graded ring. If there is an invertible element $x$ in $R_i$, then $R=R_0[x, x^{-1}]$.

Proof. For any $r \in R_{\mathbb{N}}$, $r=r(x^{-1})x=rx^{-1}x$ and $rx^{-n}$ is in $R_0$, therefore $r \in R_0x^n$. Hence $R=R_0[x, x^{-1}]$.

**Corollary.** Let $R$ be a $\mathbb{Z}$-graded domain. If $R_0$ is a field, so $R=R_0[x, x^{-1}]$ for every $x \in R_i$, $x \neq 0$.

Proof. Choose non-zero elements $x \in R_i$, and $y \in R_i$. Since $R_0$ is a field, $0 \neq xy$ is invertible, therefore $x$ and $y$ are units in $R$, hence $R=R_0[x, x^{-1}]$.

2. Torus invariant rings

A ring $A$ is said to be torus invariant provided that $A$ has the following property:

If there exist a ring $B$, a variable $Y$ over $B$, and a variable $X$ over $A$ such that $A[X, X^{-1}]$ is isomorphic to $B[Y, Y^{-1}]$,
then $A$ is always isomorphic to $B$.

Especially if we have always $\Phi(A)=B$ in such case, we say that the ring $A$ is strongly torus invariant.

To show $A$ is torus invariant (resp. strongly torus invariant) it suffices to prove that $A$ is isomorphic to $B$ (resp. $A=B$) under the assumption: $A[X, X^{-1}]=B[Y, Y^{-1}]$.

**2.0** We begin with some elementary observations. Assume that

$$ R = A[X, X^{-1}] = B[Y, Y^{-1}] $$

Then $X$ and $Y$ are units of $R$. It follows from (1.1) that we have

$$ X = vY^f \text{ and } Y = uX^{f'} \text{, } v \in B \text{ and } u \in A, $$

or equivalently

$$ v = u^{-f'}X^{1-f'} \text{ and } u = v^{-f'}Y^{1-f'}. $$

In the rest of our paper we shall use the letters $u$ and $v$ to denote the elements of $A$ and $B$ respectively satisfying the relations (2) and (3) whenever we encounter the situation (1).

**2.1** The element $u$ is in $B$ if and only if $ff'=1$. In this case we have $A[X, X^{-1}]=B[X, X^{-1}]$, thus we have $A\cong B$.

Proof is easy and is omitted.

**Proposition 2.2.** Let $k$ be a field and $A$ be a $k$-algebra. If $A^*$ (the set of all invertible elements in $A$)$=k^*$, then the ring $A$ is torus invariant.

Proof. Let $R=A[X, X^{-1}]=B[Y, Y^{-1}]$. By (1.1) the field $k$ is contained in $B$. Since $A^*=k^*$, the unit element $u$ of $A$ is in $k$, hence in $B$. It follows from (2.1) that $A$ is torus invariant.

**Proposition 2.3.** Let $A=A_0[t_1, t_2, \ldots, t_n, (t_1t_2\cdots t_n)^{-1}]$ where $t_i$'s are independent variables over $k$-algebra $A_0$ and $A^*_0=k^*$, then $A$ is torus invariant.

Proof. Let $R=A[X, X^{-1}]=B[Y, Y^{-1}]$. Then by the lemma (1.1) $Y=ux^{f'}$ and $X=vy^{f'}$. Since $u$ is invertible in $A=A_0[t_1, t_2, \ldots, t_n, (t_1\cdots t_n)^{-1}]$, $Y=rt_1^{e_1}\cdots t_n^{e_n}$, $r \in A_0^*=k^*$. We may assume that $r=1$, so $Y=t_1^{e_1}\cdots t_n^{e_n}X^{f'}$.

On the other hand as $t_i$ is invertible in $R=B[Y, Y^{-1}]$, $t_i=b_iY^{f_i}$, $b_i \in B^*$. Then we have that

$$ ff' + \sum e_if_i = 1. $$

Therefore the following natural homomorphism is surjective.
Since \( Z \) is P.I.D., we can construct a basis of \( Z^{(n+1)} \) containing this vector \((f', e_1, \ldots, e_n)\). Put this basis

\[
e_0 = (f', e_1, \ldots, e_n) \\
e_i = (f', i, \ldots, i) \in B \]

and put \( u_i = t^i_{1 \ldots n} X^i \).

\[
R = A_0[u_1, \ldots, u_n, (u_1 \cdots u_n)^{-1}] [Y, Y^{-1}] = A[X, X^{-1}] = B[Y, Y^{-1}].
\]

Therefore \( A \) is isomorphic to \( B \). Hence \( A \) is torus invariant.

(2.4) Let \( R = A[X, X^{-1}] = B[Y, Y^{-1}] \). An ideal \( I \) of \( R \) is said to be vertical relative to \( A \) if there exists an ideal \( J \) of \( A \) such that \( Jr = I \). If \( J \) is an ideal of \( A \) such that \( Jr \) is vertical relative to \( B \), then we will simply say that \( J \) is vertical relative to \( B \). If \( A \) is a \( k \)-affine domain, the prime ideals defined by the singular locus of Spec \( A \) are vertical relative to \( B \).

**Proposition 2.5.** Let \( R = A[X, X^{-1}] = B[Y, Y^{-1}] \). If there exists a maximal ideal of \( A \) which is vertical relative to \( B \), then \( A[X, X^{-1}] = B[X, X^{-1}] \). In particular \( A \) and \( B \) are isomorphic.

**Proof.** Let \( m \) be a maximal ideal of \( A \) which is vertical relative to \( B \). Then there exists an ideal \( n \) of \( B \) such that \( mR = nR \). Therefore \( R/mR = A/m[X, X^{-1}] = R/nR = B/n(Y, Y^{-1}) \), where \( X = X'Y \) and \( Y = uX' \). Since \( m \) is a maximal ideal, \( A/m \) is a field. Hence \( \bar{u} \) is in \( B/n \) by (1.1). Therefore we obtain \( f = \pm 1 \) by (2.1). Thus \( A \) is isomorphic to \( B \).

**Corollary 2.6.** Let \( A \) be a \( k \)-affine domain with isolated singular points, then \( A \) is torus invariant.

### 3. Strongly torus invariant rings

In this section we investigate strongly torus invariant rings.

**Proposition 3.1.** Let \( A[X, X^{-1}] = B[Y, Y^{-1}] \). If \( Q(A) \subseteq Q(B) \), then \( A = B \), where \( Q(R) \) is the total quotient field of \( R \).

**Proof.** Let \( x \) be an element of \( A \), then there exist two elements \( b \) and \( b' \) of \( B \) such as \( x = b/b' \). Hence \( b = b'x \). In the graded ring \( B[Y, Y^{-1}] \) the elements \( b \) and \( b' \) are homogeneous of degree zero, thus \( x \) is also degree zero. Hence we have \( A \subseteq B \). Let \( b \) be an element of \( B \). Then \( b = \sum a_jX^j, \ a_j \in A \). By (2) of (2.0) we have that \( b = \sum a_j Y^{i'j} \). If \( f = 0 \), then \( X \in B \). Thus \( A[X, X^{-1}] \subseteq B \),
it's a contradiction, hence \( f \neq 0 \). Since \( a_jx^j \in B \) and \( Y \) is a variable over \( B \), \( b = a_0 \in A \). Thus \( A = B \).

**Corollary 3.2.** Let \( \overline{A} \) denote the integral closure of \( A \). If \( \overline{A} \) is strongly torus invariant, then \( A \) is also so.

Proof. It is easily seen that if \( A[X, X^{-1}] = B[Y, Y^{-1}] \) then \( \overline{A}[X, X^{-1}] = \overline{B}[Y, Y^{-1}] \). Since \( \overline{A} \) is strongly torus invariant, \( \overline{A} = \overline{B} \). Hence \( Q(A) = Q(B) \), and we have that \( A = B \).

**Proposition 3.3.** Let \( A \) be a domain with \( J(A) \neq 0 \), where \( J(D) \) is the Jacobson radical of a ring \( D \). Then \( A \) is strongly torus invariant.

Proof. Let \( a \) be a non-zero element of \( J(A) \). Then \( 1 + a \) is unit, so in the graded ring \( B[Y, Y^{-1}] \), \( 1 + a \) is homogeneous. Since the "unity 1" is a homogeneous element of degree 0, the element \( a \) is also so. Thus the element \( a \) is contained in \( B \).

Let \( x \) be any element of \( A \). Since \( xa \) is contained in \( J(A) \), \( xa \) is in \( B \). Hence \( A \) is contained in \( Q(B) \). By (3.1), we have that \( A = B \).

**Corollary 3.4.** If \( A \) is a local domain, then \( A \) is strongly torus invariant.

**Proposition 3.5.** Let \( A \) be an affine ring over a field \( k \) and let \( A[X, X^{-1}] = B[Y, Y^{-1}] \). Then \( A = B \) if and only if every maximal ideals of \( A \) is vertical relative to \( B \).

Proof. It suffices to prove the "if" part of the (3.5). By (3.3) we may assume that \( J(A) = 0 \). Let \( x \) be an element of \( B \) and let \( x = \sum_{j=1}^{t} a_jX^j \), where \( s < t \), \( a_j \in A \) and \( a_s \neq 0 \) and \( a_t \neq 0 \). For any maximal ideal \( m \) of \( A \) there exists a maximal ideal \( n \) of \( B \) such as \( mR = nR \), where \( R = A[X, X^{-1}] \). Let \( \overline{x} \) denote the residue class of \( x \) in \( B/n \). Then \( \overline{x} \) is algebraic over the coefficient field \( k \), hence there exist elements \( \lambda_0, \lambda_1, \ldots, \lambda_{n-1} \) in \( k \), such that \( f(x) = x^n + \lambda_{n-1}x^{n-1} + \cdots + \lambda_0 \in nR = mR \). If \( t \neq 0 \), then the highest degree term of \( f(x) \) with respect to \( X \) is \( a_tx^t \in mR \), thus \( a_t \) is contained in \( m \) for every maximal ideal in \( A \). Since \( J(A) = 0 \), \( a_t = 0 \). It's a contradiction. Therefore \( t = 0 \). By the same way, we have that \( s = 0 \), hence \( x \) is in \( A \). Thus \( A = B \).

We denote the subring generated by all the units of \( A \) by \( A_u \).

**Proposition 3.6.** Let \( A \) be a \( k \)-affine domain with an isolated singular point. If \( A \) is algebraic over \( A_u \), then \( A \) is strongly torus invariant.

Proof. Let \( A[X, X^{-1}] = B[Y, Y^{-1}] \) and let \( m \) be the maximal ideal defined by the isolated singular point. Then there exists a maximal ideal \( n \) of \( B \) such as \( mR = nR \). Let \( a \) be a unit element of \( A \). In the graded ring \( B[Y, Y^{-1}] \), the
element \(a\) is also invertible, so \(a = bY^j\), for some invertible element \(b\) in \(B\) and a certain integer \(j\). Since \(A/m\) is algebraic over \(k\), there exist elements \(\lambda_0, \lambda_1, \ldots, \lambda_n \in k\) such that \(\lambda_0 a^n + \cdots + \lambda_n a + \lambda_0 \in mR = nR\). If \(j \neq 0\), \(\lambda_n b^n\) is in \(n\), hence \(b\) is not invertible, it's a contradiction. Thus we have that \(A \subseteq B\). By the following lemma our proof is over.

**Lemma 3.7.** Let \(A[X, X^{-1}] = B[Y, Y^{-1}]\). If \(A\) is algebraic over \(A \cap B\), then \(A = B\).

Proof. Since \(A\) is algebraic over \(A \cap B\), \(A\) is also algebraic over \(B\), but \(B\) is algebraically closed in \(B[Y, Y^{-1}]\), therefore \(A\) is contained in \(B\). Thus we have that \(A = B\).

Let \(A\) be an integral domain containing a field \(k\). We denote the set of all automorphisms of \(A\) over \(k\) by \(\text{Aut}_k(A)\).

**Proposition 3.8.** Let \(A\) be an integral domain containing an infinite field \(k\). If \(\text{Aut}_k(A)\) is a finite set, then \(A\) is strongly torus invariant.

Proof. Let \(R = A[X, X^{-1}] = B[Y, Y^{-1}]\). Let \(\Phi_\lambda, \lambda \in k^*\), be an automorphism of \(R\) defined by \(\Phi_\lambda(Y) = \lambda Y\) and \(\Phi_\lambda(b) = b\) for \(b \in B\). Following the notation of (2.0) we have \(X = vY^i\), thus \(\Phi_\lambda(X) = \lambda^i X\), therefore \(R = \Phi_\lambda(A)[X, X^{-1}]\). Let \(p\) be the projection \(A[X, X^{-1}] \to A\) defined by \(p(X) = 1\) and \(i\) be the canonical injection \(A \to A[X, X^{-1}]\). Define \(\sigma_\lambda = q \circ \Phi_\lambda \circ i\). Then \(\sigma_\lambda\) is an endomorphism of \(A\).

We shall show that \(\sigma_\lambda\) is surjective. Let \(x\) be an element of \(A\). Since \(R = \Phi_\lambda(A)[X, X^{-1}]\), there exist elements \(a_i\) of \(A\) such as \(x = \sum \Phi_\lambda(a_i)X^i\). Hence \(x = p(x) = \sum p \Phi_\lambda(a_i)\). Let \(x = \sum a_i \in A\), then \(\sigma_\lambda(x') = \sum p \Phi_\lambda(a_i) = x\). Thus \(\sigma_\lambda\) is surjective. Next we shall show that \(\sigma_\lambda\) is injective. Since \(\Phi_\lambda((X-1)R \cap \Phi_\lambda(A)) = \Phi_\lambda((X-1)R \cap A = (\lambda^i X - 1)R \cap A = 0\), we have \((X-1)R \cap \Phi_\lambda(A) = 0\), therefore \(\sigma_\lambda\) is injective. Hence \(\sigma_\lambda\) is an automorphism of \(A\).

We shall prove that the set \(\{\sigma_\lambda ; \lambda \in k^*\}\) is infinite when \(A \neq B\). Since \(u = v^{-1} Y^{i-ff'}\), \(\sigma_\lambda(u) = \lambda^{i-ff'}u\). Therefore our assertion is proved when \(1 - ff' \neq 0\). Suppose \(ff' = 1\). Then we may assume that \(R = A[X, X^{-1}] = B[X, X^{-1}]\). If \(A \subseteq B\), then \(A = B\), so there exists an element \(x\) of \(A\) not contained in \(B\), say \(x = \sum b_j X^j, t > s\). Since \(\ker p = (X - 1)R\) and \((X - 1)R \cap B = 0\), \(p(b_j) \neq 0\) for \(b_j \neq 0\). Since \(\sigma_\lambda(x) = \sum p(b_j) \lambda^j\) and \(p(b_j) \neq 0\) for some \(j \neq 0\), the set \(\{\sigma_\lambda ; \lambda \in k^*\}\) is infinite.

Next we shall give two cases of rings which are not strongly torus invariant. If \(A\) has a non-trivial locally finite iterative higher derivation \(\psi : A \to A[T]\), then \(A[T] = B[T]\), where \(B = \psi(A)\) and \(A \neq B\), as is proved in [4]. Hence we have that \(A[T, T^{-1}] = B[T, T^{-1}]\) and \(A \neq B\). If \(A\) is a graded ring, then \(A\) is not strongly torus invariant. Indeed, let \(X\) be a variable over \(A\) and let
$B_i = \{a_i X^i; a_i \in A_i\}$. Then $B_i$ is an $A_\sigma$-module contained in $A[X, X^{-1}]$. Let $B = \sum B_i$. Then $B$ is a graded ring and we easily see that $A[X, X^{-1}] = B[X, X^{-1}]$. We shall show that $X$ is a variable over $B$. Assume that there exist elements $b_0, b_1, \ldots, b_n$ in $B$ such that $b_n \neq 0$ and $b_n X^n + \cdots + b_1 X + b_0 = 0$. By the definition of $B$ we denote $b_i = \sum a_{ij} X^i, a_{ij} \in A_j$. In the graded ring $A[X, X^{-1}]$ the homogeneous term of degree $t$ of this equation is that

$$
(a_{n,t-n} + a_{n-1, t-n+1} + \cdots + a_{0,t}) X^t = 0.
$$

Since $A$ is a graded ring and $a_{ij}$ is a homogeneous element of degree $j$, we obtain $a_{ij} = 0$ for all index $i$ and $j$, hence $X$ is a variable over $B$.

By [4] we have that a $k$-algebra $A$ has a non-trivial locally finite iterative higher derivation if and only if $\text{Aut}_k(A)$ has a subgroup isomorphic to $G_n = \text{Spec} k[T]$. We easily see that $A$ is a non-trivial graded ring if and only if $\text{Aut}_k(A)$ has a subgroup isomorphic to $G_n = \text{Spec}(k[T, T^{-1}])$.

**Proposition 3.9.** A $k$-algebra $A$ is not strongly torus invariant, if $\text{Aut}_k(A)$ has a subgroup isomorphic to $G_n$ or $G_m$.

Assume that $\text{Aut}_k(A)$ is an infinite group. If $\text{Aut}_k(A)$ has an algebraic group structure, then there exists the following exact sequence;

$$
0 \to T \to \text{Aut}_k(A)_0 \to \theta \to 0
$$

where $\text{Aut}_k(A)_0$ is the connected component containing the identity $I_A$, and $T$ is a maximal torus subgroup of $\text{Aut}_k(A)_0$ and $\theta$ is an abelian variety. Let $P$ be an arbitrary closed point of $\text{Spec}(A)$. If $T = 0$, then there exists a regular map

$$
\Phi: \text{Aut}_k(A)_0 \to \text{Spec}(A),
$$

$$
\sigma \mapsto \sigma(P).
$$

Since $\text{Im}(\Phi)$ is a projective variety contained in the affine variety $\text{Spec}(A)$, the set $\text{Im}(\Phi)$ consists of one point, it contradicts $\text{dim} \text{Aut}_k(A)_0 > 0$. Hence we have that $T \neq 0$. Since $T \supseteq G_n$ or $G_m$, we have the following result:

**Proposition 3.10.** If $\text{Aut}_k(A)$ is not a finite set and has an algebraic group structure, then $A$ is not strongly torus invariant.

4. **Affine domains of dimension $\leq 2$**

Let $k$ be a field of characteristic zero which contains all roots of "unity". In this section let $A$ be an affine domain over $k$. We shall see that if $\text{dim } A = 1$, then $A$ is always torus invariant. Moreover $A$ is not strongly torus invariant if and only if $\text{Aut}_k(A) \supseteq G_m$. Let $\text{dim } A \geq 2$. Then $A$ is not always torus invariant.
invariant. But if an integrally closed domain \( A \) is not a \( \mathbb{Z} \)-graded ring, then \( A \) is torus invariant.

For the proof we need a lemma.

**Lemma 4.1.** Let \( K \) be a finite separable algebraic field extension of a field \( k \). If \( A \) is a one-dimensional affine normal ring such that \( k \subset A \subset K[X, X^{-1}] \), then \( A \) is a polynomial ring or a torus ring over \( k' \) where \( k' \) is the algebraic closure of \( k \) in \( A \).

**Proof.** We may assume that \( k = k' \). Following the similar device to the proof of (2.9) in [1, p 322], we have \( Q(A) = k(\theta) \) for some element \( \theta \) of \( A \).

Since \( k[\theta] \subset A \subset k(\theta) \), \( A = k[\theta] \) or \( A = k[\theta, \frac{1}{f(\theta)}] \) for some polynomial \( f(\theta) \in k[\theta] \). Let \( A = k[\theta, \frac{1}{f(\theta)}] \). Then we may assume that \( f(\theta) \) has no multiple factors. The element \( f(\theta) \) is invertible in \( A \), so is also invertible in \( K[X, X^{-1}] \). Thus we have \( f(\theta) = \beta X^r \), \( \beta \in K \), \( \theta \in K[X, X^{-1}] \). We may assume that \( r \geq 0 \), if necessary, by replacing \( X \) with \( X^\alpha \). Then we easily see that \( \theta \in K[X] \). The uniqueness of the irreducible decomposition in a polynomial ring implies that \( \deg \theta = 1 \), since the polynomial \( f(\theta) \) has no multiple factors and \( f(\theta) = \beta X^r \). Hence we may assume that \( f(\theta) = \theta \) and we obtain \( A = k[\theta, \frac{1}{\theta}] \).

Let \( A \) be an integral domain. If \( A \) is contained in \( K[X, X^{-1}] \), then \( \bar{A} \) is a polynomial ring or a torus ring over \( k' \).

**Proposition 4.2.** Let \( A \) be a one-dimensional affine domain over a field \( k \) of characteristics zero. Then we obtain that

1. \( A \) is torus invariant,
2. \( A \) is not strongly torus invariant if and only if \( \text{Aut}_k(A) \) has a subgroup isomorphic to \( G_m \). If \( A \) is not strongly torus invariant and \( A \) is integrally closed, then \( A \) is a polynomial ring or a torus ring over the algebraic closure of \( k \) in \( A \).

**Proof.** At first we shall prove (2). The sufficiency follows from (3.9). Let \( R = A[X, X^{-1}] = B[Y, Y^{-1}] \) in which \( A \neq B \). If \( m \cap A \neq 0 \) for any maximal ideal \( m \) of \( B \), then \( m \) is vertical relative to \( A \), and we have \( A = B \) by (3.5). Hence there exists a maximal ideal \( m \) such as \( m \cap A = 0 \). Since \( ch k = 0 \), \( B/m = K \) is a finite separable algebraic field over \( k \). The residue mapping of \( R \) to \( R/mR \) yields (up to isomorphism) \( k \subset A \subset K[Y, Y^{-1}] \) where \( Y \) is algebraically independent over \( K \). Therefore \( A \) is a polynomial ring or a torus ring by the lemma (4.1). Thus the automorphism group \( \text{Aut}_k A \) contains a subgroup isomorphic to \( G_m \).

Assume that \( A \) is not integrally closed. Then prime divisors in \( A \) of the conductor \( t(\bar{A}/A) \) are vertical relative to \( B \). Hence we may assume \( X = Y \) by
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(2.5). The above lemma (4.1) implies that \( \bar{A} = k'[t, t^{-1}] \) or \( \bar{A} = k'[t] \) where \( k' \) is the algebraic closure of \( k \) in \( \bar{A} \).

Firstly let \( A = k'[t] \). Since \( A \cong \bar{B} \), there exists an element \( s \) in \( \bar{B} \) such as \( \bar{B} = k'[s] \). Since \( R = \bar{A}[X, X^{-1}] = \bar{B}[X, X^{-1}] \), we have \( k'[X, X^{-1}] [t] = k'[X, X^{-1}] [s] \), hence we easily see that \( t = f_1(X)s + f(X) \) and \( s = g_1(X)t + g(X) \) where \( f_1(X) = 1 \) and \( f(X), g(X) \in k'[X, X^{-1}] \). We may assume that \( t = X^s + f(X) \) and \( s = X^{-t} + g(X) \). Let \( \bar{n} \) be a prime divisor in \( \bar{A} \) of the conductor \( t(A/A) \).

Then there exists a maximal ideal \( \bar{m} \) of \( \bar{B} \) such as \( \bar{n} \bar{R} = \bar{m} \bar{R} \). Since \( A/\bar{n} \) is algebraic over \( k \), there exist elements \( \lambda_0, \lambda_1, \ldots, \lambda_{d-1} \in k \) such that \( t^{d-1} + \lambda_{d-1} + \cdots + \lambda_0 \in \bar{m} \bar{R} = \bar{n} \bar{R} \). Hence we have that \( (X^s + f(X))^d + \lambda_{d-1}(X^s + f(X))^{d-1} + \cdots + \lambda_0 \in \bar{n} \bar{R} \). The constant term of this polynomial with respect to \( s \) is the following;

\[
f(X)^d + \lambda_{d-1}f(X)^{d-1} + \cdots + \lambda_0 \in \bar{n}k'[s][X, X^{-1}].
\]

Therefore \( f(X) = f \in k' \). Hence we may assume that \( t = X^s \). We shall show that \( A \) is a graded ring. Let \( a \) be an element of \( A \). Since \( a \) is contained in \( A = k'[t] \) and \( t = X^s \), we have that \( a = \sum \lambda_i t^i = \sum \lambda_i s^i X^{jn} \), \( \lambda_i, s^i \in \overline{B} \). On the other hand, as the element \( a \) is contained in \( B[X, X^{-1}] \), \( a = \sum b_i X^i \), \( b_i \in B \).

Comparing the coefficient of each term in the following;

\[
\sum \lambda_i s^i X^{jn} = \sum b_i X^i,
\]

we have \( b_i = \lambda_i s^i (i = jn) \) and \( b_i = 0 \) \((i \notin n\mathbb{Z})\). If \( b_i \neq 0 \), then \( b_i X^i = \lambda_i s^i X^{jn} = \lambda_i t^i \in B[X, X^{-1}] \cap \bar{A} = A[X, X^{-1}] \cap \bar{A} = A \). Therefore \( A \) has a graded ring structure.

Secondary let \( \bar{A} = k'[t, t^{-1}] \). Then \( \bar{B} = k'[s, s^{-1}] \). Since \( t \) and \( s \) are invertible in \( \bar{R} \), we may assume that \( t = s^i X^s \) and \( s = t^j X^s \), then \( t = (t^j X^s)^i X^s = t^i X^{jn+s} \), therefore \( ij = 1 \). Hence we may assume \( t = s X^s \). By the same method as in the case \( \bar{A} = k'[t] \) we have that \( A \) is a graded ring.

Proof of (1). If \( A \) is not integrally closed, then the prime divisors of the conductor \( t(\bar{A}/A) \) are vertical relative to \( B \). Since non-zero prime ideals of \( A \) are maximal, the ring \( A \) is isomorphic to \( B \) by (2.5). If \( A \) is integrally closed and \( A \) is neither a polynomial ring nor a torus ring, then \( A \) is strongly torus invariant, hence \( A \) is torus invariant. If \( A \) is either a polynomial ring or a torus ring, \( A \) is torus invariant by (2.2) and (2.3).

Next we shall consider the case; the coefficient field \( k \) has all roots of "unity" and its characteristic is zero. Then we prove the following:

**Theorem 4.3.** Let \( A \) be an integrally closed \( k \)-affine domain of dimension two, where the field \( k \) has all roots of "unity" and \( \text{ch } k = 0 \). If \( A \) is not torus invariant, then \( A \) is a \( \mathbb{Z} \)-graded ring which contains units of non-zero degree.

Proof. Assume that \( A \) is not torus invariant. Then there exist a \( k \)-algebra \( B \) and independent variables \( X, Y \) such that \( A \) is not isomorphic to \( B \) and \( R = A[X, X^{-1}] = B[Y, Y^{-1}] \). By (2.0) and (2.1) we obtain \( ff' \neq 1 \). We shall show
that it follows from \( ff' > 0 \) that \( A \) is a \( \mathbb{Z} \)-graded ring. We may only consider the case \( 1-ff' > 0 \). Let \( x \) be a \( (1-ff')-th \) root of \( u \) and let \( y = x^{-ff'} \). Then \( x^{-ff'} = v \). Since \( y^{-ff'} = u \), \( x = \lambda y^{-ff'} \) for some \((1-ff')-th \) root \( \lambda \) of “unity”. From the relations; \( y = x^{-ff'} \) and \( Y = uX^{ff'} \), we have \( \lambda = 1 \).

Since \( y = x^{-ff'} \) and \( x = y^{-ff'} \) are invertible, we have \( A[x, X^{-1}] = B[y, Y^{-1}] \). Define a surjective homomorphism \( j: A[x, y^{-ff}] \to A[x] \) by \( j(y) = 1 \). Let \( A_0 = j(B[y]) \subseteq A[x] \). We shall show that \( A[x] \sim A_0[x, x^{-ff}] \). Let a be an element of \( A \). Then \( a = \sum b_i x^i \), \( b_i \in B \). Since \( j(a) = a \) and \( j(x) = x \), we have that \( a = \sum b_i x^i \), \( b_i \in B_0 \). Thus \( A[x] = A_0[x, x^{-ff}] \) and \( x \) is algebraically independent over \( A_0 \). By the same way \( B[y] = B_0[y, y^{-ff}] \).

Since the every \((1-ff')-th \) roots of “unity” is contained in \( k \) and \( ch k = 0 \) and \( A \) is normal, the extension \( A[x]/A \) is a Galois extension with a cyclic group \( G = \langle \sigma \rangle \) (cf. [3] p 214). Indeed when \( |G| = n, n \mid (1-ff') \) and there exists a primitive \( n- \)th root \( \lambda \) of “unity” such that \( \sigma(x) = \lambda x \) and the invariant subring \( (A[x])^\sigma = A \) and \( A[x] = A + Ax + \cdots + Ax^{n-1} \) is a free \( A \)-module.

Since the element \( u \) is a unit of \( A \) and \( ch(k) = 0 \), the extension \( A[x]/A \) is étale. Since \( A \) is a normal domain, \( A[x] \), hence \( A_0[x, x^{-1}] \), is also a normal domain. From this we see that \( A_0 \) is always normal.

We shall show that there exists a subring \( A'_0 \) in \( A[x] \) such that \( A[x] = A'_0[x, x^{-1}] \) and \( \sigma(A_0) = A'_0 \). If \( A_0 \) is strongly torus invariant, then \( \sigma(A_0) = A'_0 \); for \( \sigma(A_0) = A_0[\mu, x^{-1}] = A_0[x, x^{-1}] \), therefore \( A_0 \) satisfies the conditions. If \( A_0 \) is not strongly torus invariant, then \( A_0 = k'[t] \) or \( A'_0 = k'[t, t^{-1}] \) by (4.2). Firstly let \( A_0 = k'[t] \). Since \( k'[x, x^{-1}] = k'[x, x^{-1}] \), \( \sigma(t) = \mu x^i t + f(x) \), \( \mu \in k^* \) and \( f(x) \in k'[x, x^{-1}] \). The order of \( \sigma \) is \( n \), i.e. \( \sigma^n \) is identity, so \( \sigma^n(t) = t \). The other hand \( \sigma^n(t) = \mu \lambda x^n t + f(x) \), \( \mu \in k^* \) and \( f(x) \in k'[x, x^{-1}] \), therefore we have that \( i = 0 \), thus \( \sigma(t) = \mu t + f(x) \) and \( \mu^n = 1 \). Let \( f(x) = \sum f_j x^j \) and define the set \( \Delta = \{ j \in Z; \lambda^j \neq \mu \} \). Let \( h(x) = \sum h_j x^j \), where \( h_j = f_j (\mu - \lambda)^{-j} \), and put \( s = t + h(x) \). Then \( \sigma(s) = \mu s + \sum f_j x^j \), hence \( \sigma^n(s) = \mu^n s + \mu^n + (\sum f_j x^j) = s + \mu^n + (\sum f_j x^j) \). Since \( \sigma^n(s) = s \), we have \( \sigma(s) = \mu s \). We set \( A'_0 = k'[s] \), then \( A'_0 \) satisfies the conditions.

Secondary let \( A_0 = k'[t, t^{-1}] \). Since \( k'[x, x^{-1}] = k'[x, x^{-1}] \), we easily see that \( \sigma(t) = \mu x^t \) or \( \sigma(t) = \mu x^{-t} \), \( \mu \in k^* \).

Case (i); \( \sigma(t) = \mu x^t \). Since \( \sigma^n(t) = \mu^n \lambda^{(n+1)} x^{nt} \) and \( \sigma^n(t) = t \), we have that \( \sigma(t) \neq \mu t \), so \( \sigma(A_0) = A_0 \).

Case (ii); \( \sigma(t) = \mu x^{-t} \). If \( n \) is odd, say \( n = 2m + 1 \), then \( \sigma^n(t) = \mu \lambda^{im} x^{-t} \), but this is impossible for \( \sigma^n(t) = t \). Therefore \( n \) is even, say \( n = 2m \). Then \( \sigma^n(t) = \lambda^{im} t \). Since \( \lambda \) is a primitive \( n- \)th root of “unity”, the integer \( i \) is even, say \( i = 2j \). Let \( s = x^j t \) and \( A'_0 = k'[s, s^{-1}] \). Then \( A'_0 \) satisfies the conditions.
Next we shall show that $A$ has a $\mathbb{Z}$-graded ring structure. Let $a$ be an element of $A$. Since $a \in A[[x, x^{-1}]]$, $a = \sum a_i x^i$. Then $a = \sigma(a) = \sum \sigma(a_i) x^i$ and $\sigma(a_i) \in A_i$. Comparing the coefficient of each term in the equality, $\sum a_i x^i = \sum \sigma(a_i) x^i$, we have that $a_i = \sigma(a_i) x^i$, then $\sigma(a x^i) = a x^i$. Thus $a x^i$ is an element of $A$. Therefore $A$ is a graded ring. Since there exists units of non-zero degree, $A$ has a $\mathbb{Z}$-graded ring structure.

**Remark.** The converse of this theorem is false. Indeed we find by (2.3) that the ring $k[T][X, X^{-1}]$ is a $\mathbb{Z}$-graded ring with respect to $X$ which is torus invariant.

**Example.** We shall construct an example of an affine dimension $A$ of dimension two which is not torus invariant.

Let $D$ be an integrally closed domain of dimension one over an algebraically closed field $k$ and $D^* = k^*$. Let $a$ be a non-unit of $D$ and $a^2 = a, \alpha \in D$. Assume that $D$ is noetherian and $D[\alpha]$ is strongly torus invariant. Since an affine domain of dimension one whose totally quotient field has a positive genus is strongly torus invariant by (4.2), this assumption can be satisfied for a suitable choice of $D$. Let $T$ be a variable over $D$ and $A = D[\alpha T, T^5 T^{-5}]$. Let $X$ be a variable over $A$ and $S = T^2 X$ and $Y = T^5 X^2$. Let $B = D[\alpha S^4, S^5, S^{-5}]$. Since $T = S^{-2} Y$ and $X = S^5 Y^{-2}$, we have that $A[X, X^{-1}] = B[Y, Y^{-1}]$. By (1.1) invertible elements in the graded ring $A$ are homogeneous. Since $D^* = k^*$, we obtain $A^* = \{ \eta T^{3i}; \eta \in k^* \text{ and } i \in \mathbb{Z} \}$. Hence the quotient $A^*/k^*$ is generated by $T^5$. Similary $B^*/k^*$ is generated by $S^5$. We shall show that $A$ is not isomorphic to $B$. We assume that there exists an isomorphism $\sigma$ of $A$ to $B$. Since $\sigma$ is a group-isomorphism of $A^*$ to $B^*$, we have $\sigma(T^5) = \mu S^5$ or $\sigma(T^5) = \mu S^{-5}$, $\mu \in k^*$. We shall only consider the case: $\sigma(T^5) = \mu S^5$, since the proof of the other case is the similar. Let $\sigma$ be an isomorphism of $A[T]$ to $B[S]$ defined by $\sigma = \sigma$ on $A$ and $\sigma(T) = \xi S$, $\xi^5 = \mu$. Then we have that $D[\alpha] [S, S^{-1}] = \sigma(D[\alpha]) [S, S^{-1}]$, therefore $\sigma(D[\alpha]) = D[\alpha]$; for $D[\alpha]$ is strongly torus invariant. Since $\sigma(D) \subseteq D[\alpha] \cap B = D$, we have $\sigma(D) = D$, therefore we easily see that $\sigma$ is an isomorphism as graded rings. Thus we have $\sigma((\alpha T) D) = (\alpha^2 S) D$, hence $a D = a^2 D$. Since the element $a$ is not a unit, $a^2 D \subseteq a D$, thus $\sigma(a) a D \subseteq a^2 D \subseteq a D$, hence we have a proper ascending chain $\{ \sigma^{-i}(a) D \}$, but it contradicts the netherian assumption of $D$. Hence $A$ is not torus invariant.

(4.4) Now let $A = \sum A_i$ be an integrally closed $\mathbb{Z}$-graded domain which contains invertible elements of non-zero degree. Let $e$ be an invertible element of $A$ with the smallest positive degree $d$. Let $a$ be a unit of $A$, then $a$ is a homogeneous elements with $\deg a = jd$ for some integer $j$, and there exists an element $\xi$ of $A_0^*$ such as $a = \xi^j e$. Let $i$ be any positive integer and $x$ be one of the $ijd$-th roots of $a$, say $x^{ijd} = a$. Since $A[x]$ is a $\mathbb{Z}$-graded ring with the
invertible elements \( x \) of degree one, \( A[x]=A'_0[x, x^{-1}] \) by (1.4) where \( A'_0 \) contains \( A_0 \). Let \( f \) and \( f' \) be integers such as \( f f'+ijd=1 \) and let \( X \) be a variable over \( A \). Put \( y=x^{-f}X \) and \( Y=aX^{f'} \). Therefore \( A'_0[x, x^{-1}][X, X'] = A'_0[y, y^{-1}][Y, Y'] \). Since the every \( n \)-th roots of "unity" is contained in \( k \) and \( A \) is integral closed, the extension \( A[x]/A \) is a Galois extension with a cyclic group \( G=\langle \sigma \rangle \). Indeed \( |G|=di \) and there exists a primitive \( di \)-th root \( \lambda \) of "unity" such as \( \sigma(x)=\lambda x \), and \( (A[x])^\lambda =A \). Since \( A'_0 \) is algebraic over \( A_0 \), \( \sigma(A'_0) \) is also so, hence \( \sigma(A'_0) \) is algebraic over \( A_0 \), but \( A'_0 \) is algebraically closed in \( A'_0[x, x^{-1}] \), therefore \( \sigma(A'_0)=A'_0 \). Since \( \sigma(y)=\lambda^{-j}y \), \( \sigma \) is an automorphism of \( A'_0[y, y^{-1}] \). Let \( B=A'_0[y, y^{-1}]^\sigma \) and \( \sigma \) be an automorphism of \( A'_0[x, x^{-1}][X, X'] \) defined by \( \sigma(X)=X \) and \( \sigma(y)=\sigma \) over \( A'_0[x, x^{-1}] \). Since \( \sigma(Y)=Y \) and \( \sigma(X)=X \), we obtain \( B[Y, Y^{-1}]=A'_0[y, y^{-1}][Y, Y^{-1}]^\sigma =A'_0[x, x^{-1}][X, X^{-1}]^\sigma =A[X, X^{-1}] \).

**Proposition 4.5.** Let \( A \) be an integrally closed \( k \)-affine domain of dimension 2. If \( A[X, X^{-1}]=B[Y, Y^{-1}] \) and \( ff'=1 \), then \( A \) has a \( \mathbb{Z} \)-graded ring structure and \( B \) is isomorphic to one of algebras constructed in (4.4).

Proof. The first statement is already mentioned in the proof of (4.3) and we obtained \( A'_0[x, x^{-1}][X, X'] = A'_0[y, y^{-1}][Y, Y^{-1}] \) and \( \sigma(A'_0)=A'_0 \). Let \( B'=A'_0[y, y^{-1}]^\sigma \). Then \( B' \) is one of algebras in (4.4). Since \( B'[Y, Y^{-1}]=B[Y, Y^{-1}] \), \( B \) is isomorphic to \( B' \).

5. **D-torus invariant**

Let \( D \) be an integral domain containing a field \( k \) of characteristic zero and \( A \) be a \( D \)-algebra. The ring \( A \) is called \( D \)-torus invariant; if \( A[X, X^{-1}]=B[Y, Y^{-1}] \) for a certain \( D \)-algebra \( B \) and independent variables \( X \) and \( Y \), then we have always \( A \cong B \). Then we have the following result:

**Proposition 5.1.** Let \( A \) be an integrally closed domain over \( D \) and \( \text{tr. deg}_D A=1 \). If \( A \) is not \( D \)-torus invariant, then \( A \) is a \( \mathbb{Z} \)-graded ring containing units of non-zero degree.

Proof. Let \( A[X, X^{-1}]=B[Y, Y^{-1}] \), where \( B \) is a \( D \)-algebra and not \( D \)-isomorphic to \( A \). By (2.0) and (2.1) we easily see that \( ff'=1 \). Then we may assume \( 1-ff'>0 \). Let \( x \) be a \((1-ff')-th \) root of \( u \) and \( y=x^{-f}X \). Then we have that \( A[x]=A_0[x, x^{-1}] \) and \( B[y]=B_0[y, y^{-1}] \) as the proof of (4.3), where \( A_0 \) and \( B_0 \) are respectively subalgebras of \( A[x] \) and \( B[y] \) containing \( D \). Let \( \sigma \) be a generator of the cyclic Galois group of the extension \( A[x]/A \). We shall show that \( \sigma(A_0)=A_0 \). Since \( \text{tr. deg}_D A_0[x, X^{-1}]=1 \), \( A_0 \) is algebraic over \( D \), thus \( \sigma(A_0) \) is also so. Since \( A_0 \) is algebraically closed in \( A_0[x, x^{-1}] \), we have that \( \sigma(A_0)=A_0 \). Following the similar devise to the proof of (4.3) we obtain
that \( A \) is a \( \mathbb{Z} \)-graded ring, and \( D \) is contained in \( A \).

In the following we shall consider the case where \( A \) is a \( \mathbb{Z} \)-graded ring and \( A_0 = D \). We consider only \( D \)-isomorphisms of \( D \)-algebras.

**Theorem 5.2.** Let \( A \) be an integrally closed \( \mathbb{Z} \)-graded ring. Assume that the subring \( A_0 \) contains an algebraically closed field \( k \) and that \( A_0^k = k^* \). Let \( d \) be the smallest positive integer among the set of degrees of units in \( A \). Then the number of the isomorphic classes of \( A \)-algebra as \( B \) such that \( A[X, X^{-1}] = B[Y, Y^{-1}] \) equals to \( \Phi(d) \), where \( \Phi \) is the Euler function.

**Proof.** Let \( i \) be an integer such as \( 1 \leq i < d \) and \( (i, d) = 1 \). Since \( (i, d) = 1 \), \( ij + dh = 1 \) for some integers \( j \) and \( h \). Moreover we may assume \( h \geq 0 \). Fix a unit \( e \) of degree \( d \). Let \( x \) be one of the \( d \)-th roots of \( e \). Then we have that \( A[x] = A_0[x, x^{-1}] \) for a subring \( A_0 \) containing \( A_0 \) by (1.4). Let \( \sigma \) be a generator of the cyclic Galois group of the extension \( A[x] / A \). Then \( \sigma(x) = \lambda x \), where \( \lambda \) is a primitive \( d \)-th root of “unity”. Since \( A_0 \) is algebraic over \( A_0 \) and algebraically closed in \( A_0[x, x^{-1}] \), we obtain \( \sigma(A_0) = A_0 \).

Let \( X \) be a variable over \( A \) and let \( y = x^{-1} X \) and \( Y = e^y x^i \). Then we have that

\[
A_0[x, x^{-1}] = A_0[y, y^{-1}] [X, X^{-1}] = A_0[y, y^{-1}] [Y, Y^{-1}].
\]

Define \( B_1 = A_0[y, y^{-1}] \) and let \( \sigma \) be an isomorphism of \( A_0[x, x^{-1}] [X, X^{-1}] \) defined by \( \sigma(X) = X \) and \( \sigma = \sigma \). Since \( Y = e^y X^i \), we can easily see that \( B_1 \) is a \( X \)-graded ring and \( B_1 \). Especially we have \( B_1 = A_0 \).

Let \( i_1 \) and \( i_2 \) be integers such as \( 1 \leq i_1 < i_2 < d \) and \( (i_1, d) = (i_2, d) = 1 \). Let

\[
B' = A_0[y, y^{-1}] \quad \text{and} \quad B'' = A_0[z, z^{-1}] \quad \text{where} \quad \sigma(y) = \lambda^{-i_1} y \quad \text{and} \quad \sigma(z) = \lambda^{-i_2} z.\]

We shall show that \( B' \) and \( B'' \) are not isomorphic. Assume that there exists an \( A_0 \)-isomorphism \( \psi \) of \( B' \) to \( B'' \). Let \( a \) be a unit in \( B' \) of non-zero degree, say degree \( a = n \), \( n \neq 0 \). Let \( b \) be a homogeneous element of \( B' \) and degree \( b = t \). Then we have \( b^n = rb^t \) for an element \( r \) in the coefficient ring \( A_0 \), hence \( \psi(b^n) = \psi(b)^n = r \psi(b)^t \).

Since \( r \) and \( \psi(b) \) are homogeneous, \( \psi(b) \) is also homogeneous by (1.1), therefore \( \psi \) is an isomorphism as graded rings.

Let \( c \) be a homogeneous element in \( B' \) of degree one. Then \( c = s_1 y \) for an element \( s_1 \) in \( A_0 \). Since \( \sigma(c) = c \) and \( \sigma(y) = \lambda^{-i_1} y \), we have \( \sigma(s_1) = \lambda^{i_1} s_1 \), hence \( s_1' \) is in \( B' \). Since \( \psi(s_1 y) = s_2 z \) for an element \( s_2 \) in \( A_0 \). Since \( \sigma(s_2 z) = s_2 z \) and \( \sigma(z) = \lambda^{-i_2} z \), we have \( \sigma(s_2) = \lambda^{i_2} s_2 \) hence \( s_2' \) is in \( B'' \). By the relations;

\[
\psi(y^d) = \psi(s_1 y)^d = \psi(s_1)^d = s_2^d z^d.\]

Since \( \psi(y^d) z^{-d} \) is an invertible element in \( B'' \) and degree zero, we have \( \xi = \psi(y^d) z^{-d} h \in A_0^n \), therefore we have \( s_2 = \eta s_1 \), for some \( \eta \in k \), \( \eta^{-d} = \xi \). Hence \( \sigma(s_2) = \lambda^{i_1} s_2 \), but it contradicts the fact that \( \sigma(s_2) = \lambda^{i_2} s_2 \) and \( \lambda \) is a primitive \( d \)-th root of “unity”. Therefore \( B' \not \cong B'' \).

Finally we shall show that if \( A[X, X^{-1}] = B[Y, Y^{-1}] \) then \( B \) is isomorphic to \( B_i \) for some \( i \) satisfying \( 0 < i < d \) and \( (i, d) = 1 \). The invertible element \( u \)
in (2.0) is homogeneous. Let $n$ be the degree of $u$. If $n=0$, then $A$ is isomorphic to $B$ by (2.1), hence $B \cong B_1$. Assume $n \neq 0$. Let $c$ be a non-zero homogeneous element of degree 1 and put $\eta = c u^{-1}$. Then $\eta$ is an element of $A_0$.

In the graded ring $B[Y, Y^{-1}]$ the elements $u$ and $\eta$ are homogeneous, hence $c$ is also homogeneous, thus we denote $c = b Y^j$ for some element $b$ in $B$ and some integer $j$. Then we obtain that $c^* = \eta u = \eta u^{-1} Y^{1-j}$ by (2.0). Therefore we have $1 - ff' = nj$.

By the minimality of $d$ we obtain $n = ld$ for some integer $l$ and $u = \xi e$, $\xi \in A^*_g = k^*$. Since the field $k$ is algebraically closed, we may assume $\xi = 1$, then the $d$-th root $x$ of $e$ is an $n$-th root of $u$. Since the element $\lambda$ is a primitive $d$-th root of "unity", there exists the unique integer $i$ such that $\lambda^{-i} = \lambda^{-i}$, $0 < i < d$, then $(i, d) = 1$ since $(f, d) = 1$. Let $y' = x^{-j} X^j$ and $B' = (A_0[y', y'^{-1}])^g$. Then $\sigma(y') = \lambda^{-j} y' = \lambda^{-i} y'$, hence $B' = B_1$. We can easily show that $x = y'^{-j} Y^j$, therefore we obtain $A_0[x, x^{-1}] [X, X^{-1}] = A_0[y', y'^{-1}] [Y, Y^{-1}]$. Since $\sigma(X) = X$ and $\sigma(Y) = Y$, we have $A[X, X^{-1}] = B_1[Y, Y^{-1}]$, hence $B[Y, Y^{-1}] = B_1[Y, Y^{-1}]$. Thus we have $B \cong B_1$.

References


