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Osaka University
Introduction. S. Abhyankar, W. Heinzer and P. Eakin treated the following problem in [1]; if \(A[X]=B[Y]\), when is \(A\) isomorphic or identical to \(B\)? Replacing the polynomial ring by the torus extension we shall take up the following problem; if \(A[X, X^{-1}]=B[Y, Y^{-1}]\), when is \(A\) isomorphic or identical to \(B\)? We say that \(A\) is torus invariant (resp. strongly torus invariant) whenever \(A[X, X^{-1}]=B[Y, Y^{-1}]\) implies \(A\approx B\) (resp. \(A=B\)). The roles played by polynomial rings in [1] are played by the graded rings in our theory. A graded ring \(A=\sum A_i, i\in\mathbb{Z}\), with the property that \(A_i\neq 0\) for each \(i\in\mathbb{Z}\), will be called a \(\mathbb{Z}\)-graded ring. Main results are the followings.

An affine domain \(A\) of dimension one over a field \(k\) is always torus invariant. Moreover \(A\) is not strongly torus invariant if and only if \(A\) has a graded ring structure. An affine domain of dimension two is not always torus invariant. We shall construct an affine domain of dimension two which is not torus invariant. Let \(A\) be an affine domain over \(k\) of dimension two. Assume that the field \(k\) contains all roots of “unity” and is of characteristic zero. If \(A\) is not torus invariant, then \(A\) is a \(\mathbb{Z}\)-graded ring such that there exist invertible elements of non-zero degree.

In Section 1 we study elementary properties of graded rings. Especially we are interested in \(\mathbb{Z}\)-graded rings with invertible elements of non-zero degree. In Section 2 we discuss some conditions for \(A\) to be torus invariant. In Section 3 we give several sufficient conditions for an integral domain to be strongly torus invariant. Some relevant results will be found in S. Iitaka and T. Fujita [2]. Section 4 is devoted to the proof of the main results mentioned above. In Section 5 we fix an integral domain \(D\) and we treat only \(D\)-algebras and \(D\)-isomorphisms there. We shall prove the following two results. When \(A\) is a \(D\)-algebra of \(tr.\ deg_D A=1\) and \(A\) is not \(D\)-torus invariant, \(A\) is a \(\mathbb{Z}\)-graded ring such that \(D\) is contained in \(A_0\). If \(A\) is a \(\mathbb{Z}\)-graded ring such as \(D=A_0\), then the number of elements of the set of \(D\)-isomorphic classes of \(D\)-algebras \(B\) such that \(A[X, X^{-1}]=B[Y, Y^{-1}]\) is \(\Phi(d)\), where \(d\) is the smallest positive integer among the degrees of units in \(A\) and \(\Phi\) is the Euler function.

I’d like to express my sincere gratitude to the referee for his valuable advices.
1. Some properties of graded rings

Let $R$ be commutative ring with indentity. The ring $R$ is said to be a graded ring if $R$ is a graded module, $R = \bigoplus R_i$, and $R_nR_m \subseteq R_{n+m}$.

**Lemma 1.1.** Let $R$ be a graded domain. Then we have the following.

1. The unity element of $R$ is homogeneous.
2. If $a$ is homogeneous and $a = bc$, then $b$ and $c$ are both homogeneous. In particular every invertible element is homogeneous.
3. If $R$ contains a field $k$, then $k$ is a subring of $R_0$.

Proof. (1) follows immediately from the relation $1^2 = 1$. The proof of (2) is easy and will be omitted. To prove (3) we can assume $k$ is different from $F_2$ by (1). Let $a$ be an element of $k$ different from 1. Then $1 - a$ is homogeneous from (2). The unity 1 is homogeneous of degree 0 by (1). Hence $a$ should be homogeneous of degree 0.

We call a graded ring $R = \bigoplus R_i$ to be a $\mathbb{Z}$-graded ring if $i? \Phi 0$, for some $i \in \mathbb{Z}$.

**Proposition 1.2.** Let $R$ be a $\mathbb{Z}$-graded domain. Let $S = \{i \in \mathbb{Z}; R_i \neq 0\}$. Then $S = n\mathbb{Z}$ for a certain integer $n$.

Proof. Since $R$ is a domain, $S$ is a semi-group. Hence (1.2) is immediately seen by the following lemma.

**Lemma 1.3.** Let $S \subseteq \mathbb{Z}$ be a semi-group. If $S \cap \mathbb{Z}^+ \neq 0$ and $S \cap \mathbb{Z}^- \neq 0$, then $S$ is a subgroup of $\mathbb{Z}$.

If $R$ is a $\mathbb{Z}$-graded domain, then we may assume $R_i \neq 0$ for any $i \in \mathbb{Z}$.

**Proposition 1.4.** Let $R$ be a graded ring. If there is an invertible element $x$ in $R_i$, then $R = R_0[x, x^{-1}]$.

Proof. For any $r \in R_n$, $r = r(x^{-1}x)^s = rx^{-s+n}$ and $rx^{-s}$ is in $R_0$, therefore $r \in R_0x^n$. Hence $R = R_0[x, x^{-1}]$.

**Corollary.** Let $R$ be a $\mathbb{Z}$-graded domain. If $R_0$ is a field, so $R = R_0[x, x^{-1}]$ for every $x \in R_i$, $x \neq 0$.

Proof. Choose non-zero elements $x \in R_i$, and $y \in R_j$. Since $R_0$ is a field, $0 \neq xy$ is invertible, therefore $x$ and $y$ are units in $R$, hence $R = R_0[x, x^{-1}]$.

2. Torus invariant rings

A ring $A$ is said to be torus invariant provided that $A$ has the following property:

If there exist a ring $B$, a variable $Y$ over $B$, and a variable $X$ over $A$ such that $A[X, X^{-1}]$ is isomorphic to $B[Y, Y^{-1}]$. 


\( \Phi: A[X, X^{-1}] \to B[Y, Y^{-1}] \),

then \( A \) is always isomorphic to \( B \).

Especially if we have always \( \Phi(A) = B \) in such case, we say that the ring \( A \) is strongly torus invariant.

To show \( A \) is torus invariant (resp. strongly torus invariant) it suffices to prove that \( A \) is isomorphic to \( B \) (resp. \( A = B \)) under the assumption: \( A[X, X^{-1}] = B[Y, Y^{-1}] \).

(2.0) We begin with some elementary observations. Assume that

\( R = A[X, X^{-1}] = B[Y, Y^{-1}] \).

Then \( X \) and \( Y \) are units of \( R \). It follows from (1.1) that we have

\( X = vY' \) and \( Y = uX', v \in B \) and \( u \in A \),
or equivalently

\( v = u^{-1}X^{1-f_f} \) and \( u = v^{-1}Y^{1-f_f} \).

In the rest of our paper we shall use the letters \( u \) and \( v \) to denote the elements of \( A \) and \( B \) respectively satisfying the relations (2) and (3) whenever we encounter the situation (1).

(2.1) The element \( u \) is in \( B \) if and only if \( ff' = 1 \). In this case we have \( A[X, X^{-1}] = B[X, X^{-1}] \), thus we have \( A \cong B \).

Proof is easy and is omitted.

Proposition 2.2. Let \( k \) be a field and \( A \) be a \( k \)-algebra. If \( A^* \) (the set of all invertible elements in \( A \)) = \( k^* \), then the ring \( A \) is torus invariant.

Proof. Let \( R = A[X, X^{-1}] = B[Y, Y^{-1}] \). By (1.1) the field \( k \) is contained in \( B \). Since \( A^* = k^* \), the unit element \( u \) of \( A \) is in \( k \), hence in \( B \). It follows from (2.1) that \( A \) is torus invariant.

Proposition 2.3. Let \( A = A_0[t_1, t_2, \ldots, t_n] \) (where \( t_i \)'s are independent variables over \( k \)-algebra \( A_0 \) and \( A^*_0 = k^* \), then \( A \) is torus invariant.

Proof. Let \( R = A[X, X^{-1}] = B[Y, Y^{-1}] \). Then by the lemma (1.1) \( Y = uX' \) and \( X = vY' \). Since \( u \) is invertible in \( A = A_0[t_1, t_2, \ldots, t_n] \), \( Y = rt_1^{\xi_1} \cdots t_n^{\xi_n} \), \( r \in A^*_0 = k^* \). We may assume that \( r = 1 \), so \( Y = t_1^{\xi_1} \cdots t_n^{\xi_n}X' \).

On the other hand as \( t_i \) is invertible in \( R = B[Y, Y^{-1}] \), \( t_i = b_iY' \), \( b_i \in B^* \). Then we have that

\[ ff' + \sum e_if_i = 1. \]

Therefore the following natural homomorphism is surjective.
$j: \mathbb{Z}^{(n+1)} = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \to \mathbb{Z}$

$j(i_0, i_1, \ldots, i_n) = i_0 f' + \sum i_j e_j$.

Since $\mathbb{Z}$ is P.I.D., we can construct a basis of $\mathbb{Z}^{(n+1)}$ containing this vector $(f', e_1, \ldots, e_n)$. Put this basis

$e_0 = (f', e_1, \ldots, e_n)$

$e_i = (f_i, f_{i1}, \ldots, f_{in})$

and put $u_i = t_i^{i_1} \cdots t_i^{i_n} X^i_i$.

$R = A_0[u_1, \ldots, u_n, (u_1 \cdots u_n)^{-1}] [Y, Y^{-1}] = A[X, X^{-1}] = B[Y, Y^{-1}]$.

Therefore $A$ is isomorphic to $B$. Hence $A$ is torus invariant.

(2.4) Let $R = A[X, X^{-1}] = B[Y, Y^{-1}]$. An ideal $I$ of $R$ is said to be vertical relative to $A$ if there exists an ideal $J$ of $A$ such that $JR = I$. If $J$ is an ideal of $A$ such that $JR$ is vertical relative to $B$, then we will simply say that $J$ is vertical relative to $B$. If $A$ is a $k$-affine domain, the prime ideals defined by the singular locus of Spec $A$ are vertical relative to $B$.

**Proposition 2.5.** Let $R = A[X, X^{-1}] = B[Y, Y^{-1}]$. If there exists a maximal ideal of $A$ which is vertical relative to $B$, then $A[X, X^{-1}] = B[X, X^{-1}]$. In particular $A$ and $B$ are isomorphic.

**Proof.** Let $m$ be a maximal ideal of $A$ which is vertical relative to $B$. Then there exists an ideal $n$ of $B$ such that $mR = nR$. Therefore $R/mR = A/m[X, X^{-1}] = R/nR = B/n[Y, Y^{-1}]$, where $X = vY'$ and $Y = uX'$. Since $m$ is a maximal ideal, $A/m$ is a field. Hence $u$ is in $B/n$ by (1.1). Therefore we obtain $f = \pm 1$ by (2.1). Thus $A$ is isomorphic to $B$.

**Corollary 2.6.** Let $A$ be a $k$-affine domain with isolated singular points, then $A$ is torus invariant.

3. **Strongly torus invariant rings**

In this section we investigate strongly torus invariant rings.

**Proposition 3.1.** Let $A[X, X^{-1}] = B[Y, Y^{-1}]$. If $Q(A) \subseteq Q(B)$, then $A = B$, where $Q(R)$ is the total quotient field of $R$.

**Proof.** Let $x$ be an element of $A$, then there exist two elements $b$ and $b'$ of $B$ such as $x = b/b'$. Hence $b = b'x$. In the graded ring $B[Y, Y^{-1}]$ the elements $b$ and $b'$ are homogeneous of degree zero, thus $x$ is also degree zero. Hence we have $A \subseteq B$. Let $b$ be an element of $B$. Then $b = \sum a_j X^j$, $a_j \in A$. By (2) of (2.0) we have that $b = \sum a_j v^j Y^{j'}$. If $f = 0$, then $X \in B$. Thus $A[X, X^{-1}] \subseteq B$,.
it's a contradiction, hence $f \neq 0$. Since $a_jv^i \in B$ and $Y$ is a variable over $B$, $b=a_0 \in A$. Thus $A=B$.

**Corollary 3.2.** Let $\tilde{A}$ denote the integral closure of $A$. If $\tilde{A}$ is strongly torus invariant, then $A$ is also so.

Proof. It is easily seen that if $A[X, X^{-1}]=B[Y, Y^{-1}]$ then $\tilde{A}[X, X^{-1}]=\tilde{B}[Y, Y^{-1}]$. Since $\tilde{A}$ is strongly torus invariant, $\tilde{A}=\tilde{B}$. Hence $Q(A)=Q(B)$, and we have that $A=B$.

**Proposition 3.3.** Let $A$ be a domain with $J(A) \neq 0$, where $J(D)$ is the Jacobson radical of a ring $D$. Then $A$ is strongly torus invariant.

Proof. Let $a$ be a non-zero element of $J(A)$. Then $1+a$ is unit, so in the graded ring $B[Y, Y^{-1}]$, $1+a$ is homogeneous. Since the "unity 1" is a homogeneous element of degree 0, the element $a$ is also so. Thus the element $a$ is contained in $B$.

Let $x$ be any element of $A$. Since $xa$ is contained in $J(A)$, $xa$ is in $B$. Hence $A$ is contained in $Q(B)$. By (3.1), we have that $A=B$.

**Corollary 3.4.** If $A$ is a local domain, then $A$ is strongly torus invariant.

**Proposition 3.5.** Let $A$ be an affine ring over a field $k$ and let $A[X, X^{-1}]=B[Y, Y^{-1}]$. Then $A=B$ if and only if every maximal ideals of $A$ is vertical relative to $B$.

Proof. It suffices to prove the "if" part of the (3.5). By (3.3) we may assume that $J(A)=0$. Let $x$ be an element of $B$ and let $x=\sum_{j=1}^{t} a_jX^j$, where $s<t, a_j \in A$ and $a_s \neq 0$ and $a_t \neq 0$. For any maximal ideal $m$ of $A$ there exists a maximal ideal $n$ of $B$ such as $mR=nR$, where $R=A[X, X^{-1}]$. Let $\bar{x}$ denote the residue class of $x$ in $B/n$. Then $\bar{x}$ is algebraic over the coefficient field $k$, hence there exist elements $\lambda_0, \lambda_1, \ldots, \lambda_{s-1}$ in $k$, such that $f(x)=x^s+\lambda_{s-1}x^{s-1}+\ldots+\lambda_0 \in nR=mR$. If $t \neq 0$, then the highest degree term of $f(x)$ with respect to $X$ is $a_t x^t \in mR$, thus $a_t$ is contained in $m$ for every maximal ideal in $A$. Since $J(A)=0$, $a_t=0$. It's a contradiction. Therefore $t=0$. By the same way, we have that $s=0$, hence $x$ is in $A$. Thus $A=B$.

We denote the subring generated by all the units of $A$ by $A_u$.

**Proposition 3.6.** Let $A$ be a $k$-affine domain with an isolated singular point. If $A$ is algebraic over $A_u$, then $A$ is strongly torus invariant.

Proof. Let $A[X, X^{-1}]=B[Y, Y^{-1}]$ and let $m$ be the maximal ideal defined by the isolated singular point. Then there exists a maximal ideal $n$ of $B$ such as $mR=nR$. Let $a$ be a unit element of $A$. In the graded ring $B[Y, Y^{-1}]$, the
element $a$ is also invertible, so $a = bY^j$, for some invertible element $b$ in $B$ and a certain integer $j$. Since $A[m]$ is algebraic over $k$, there exist elements $\lambda_0, \lambda_1, \ldots, \lambda_n \in k$ such that $\lambda_0 a^n + \cdots + \lambda_n a + \lambda_0 \in mR = nR$. If $j \neq 0$, $\lambda_i b^n$ is in $n$, hence $b$ is not invertible, it's a contradiction. Thus we have that $A \subseteq B$. By the following lemma our proof is over.

**Lemma 3.7.** Let $A[X, X^{-1}] = B[Y, Y^{-1}]$. If $A$ is algebraic over $A \cap B$, then $A = B$.

**Proof.** Since $A$ is algebraic over $A \cap B$, $A$ is also algebraic over $B$, but $B$ is algebraically closed in $B[Y, Y^{-1}]$, therefore $A$ is contained in $B$. Thus we have that $A = B$.

Let $A$ be an integral domain containing a field $k$. We denote the set of all automorphisms of $A$ over $k$ by $\text{Aut}_k(A)$.

**Proposition 3.8.** Let $A$ be an integral domain containing an infinite field $k$. If $\text{Aut}_k(A)$ is a finite set, then $A$ is strongly torus invariant.

**Proof.** Let $R = A[X, X^{-1}] = B[Y, Y^{-1}]$. Let $\Phi_\lambda, \lambda \in k^*$, be an automorphism of $R$ defined by $\Phi_\lambda(Y) = \lambda Y$ and $\Phi_\lambda(b) = b$ for $b \in B$. Following the notation of (2.0) we have $X = vY^j$, thus $\Phi_\lambda(X) = \lambda'X$, therefore $R = \Phi_\lambda(A)[X, X^{-1}]$. Let $p$ be the projection $A[X, X^{-1}] \rightarrow A$ defined by $p(X) = 1$ and $i$ be the canonical injection $A \hookrightarrow A[X, X^{-1}]$. Define $\sigma_\lambda = q \Phi_\lambda o i$. Then $\sigma_\lambda$ is an endomorphism of $A$. We shall show that $\sigma_\lambda$ is surjective. Let $x$ be an element of $A$. Since $R = \Phi_\lambda(A)[X, X^{-1}]$, there exist elements $a_j$ of $A$ such as $x = \sum \Phi_\lambda(a_j)X^j$. Hence $x = p(x) = \sum p\Phi_\lambda(a_j)$. Let $x' = \sum a_j \in A$, then $\sigma_\lambda(x') = \sum p\Phi_\lambda(a_j) = x$. Thus $\sigma_\lambda$ is surjective. Next we shall show that $\sigma_\lambda$ is injective. Since $\Phi_\lambda((X-1)R \cap \Phi_\lambda(A)) = \Phi_\lambda((X-1)R \cap A) = (\lambda'X-1)R \cap A = 0$, we have $(X-1)R \cap \Phi_\lambda(A) = 0$, therefore $\sigma_\lambda$ is injective. Hence $\sigma_\lambda$ is an automorphism of $A$.

We shall prove that the set $\{\sigma_\lambda; \lambda \in k^*\}$ is infinite when $A \neq B$. Since $u = v^{-1}Y^{1-f'f}$, $\sigma_\lambda(u) = \lambda^{1-f'f}u$. Therefore our assertion is proved when $1-f'f \neq 0$. Suppose $f'f = 1$. Then we may assume that $R = A[X, X^{-1}] = B[X, X^{-1}]$. If $A \subseteq B$, then $A = B$, so there exists an element $x$ of $A$ not contained in $B$, say $x = \sum b_jX^j, \ t > s$. Since $\ker p = (X-1)R$ and $(X-1)R \cap B = 0$, $p(b_j) = 0$ for $b_j = 0$. Since $\sigma_\lambda(x) = \sum p(b_j)\lambda^j$ and $p(b_j) = 0$ for some $j \not= 0$, the set $\{\sigma_\lambda; \lambda \in k^*\}$ is infinite.

Next we shall give two cases of rings which are not strongly torus invariant. If $A$ has a non-trivial locally finite iterative higher derivation $\psi: A \rightarrow A[T]$, then $A[T] = B[T]$, where $B = \psi(A)$ and $A \neq B$, as is proved in [4]. Hence we have that $A[T, T^{-1}] = B[T, T^{-1}]$ and $A \neq B$. If $A$ is a graded ring, then $A$ is not strongly torus invariant. Indeed, let $X$ be a variable over $A$ and let
$B_i = \{ a_i X^i ; a_i \in A_i \}$. Then $B_i$ is an $A_0$-module contained in $A[X, X^{-1}]$. Let $B=\sum B_i$. Then $B$ is a graded ring and we easily see that $A[X, X^{-1}]=B[X, X^{-1}]$. We shall show that $X$ is a variable over $B$. Assume that there exist elements $b_0, b_1, \ldots, b_n$ in $B$ such that $b_n \neq 0$ and $b_n X^n + \cdots + b_1 X + b_0 = 0$. By the definition of $B$ we denote $b_i=\sum a_{ij} X^i$, $a_{ij} \in A_j$. In the graded ring $A[X, X^{-1}]$ the homogeneous term of degree $t$ of this equation is that

$$(a_{n,t-n} X^n a_{n-1,t-n-1} + \cdots + a_{0,t}) X^t = 0.$$ 

Since $A$ is a graded ring and $a_{ij}$ is a homogeneous element of degree $j$, we obtain $a_{ij}=0$ for all index $i$ and $j$, hence $X$ is a variable over $B$.

By [4] we have that a $k$-algebra $A$ has a non-trivial locally finite iterative higher derivation if and only if $Aut_k(A)$ has a subgroup isomorphic to $G_a = \text{Spec } k[T]$. We easily see that $A$ is a non-trivial graded ring if and only if $Aut_k(A)$ has a subgroup isomorphic to $G_m = \text{Spec } (k[T, T^{-1}])$.

\textbf{Proposition 3.9.} A $k$-algebra $A$ is not strongly torus invariant, if $Aut_k(A)$ has a subgroup isomorphic to $G_a$ or $G_m$.

Assume that $Aut_k(A)$ is an infinite group. If $Aut_k(A)$ has an algebraic group structure, then there exists the following exact sequence:

$$0 \to T \to Aut_k(A)_0 \to \theta \to 0$$

where $Aut_k(A)_0$ is the connected component containing the identity $I_A$, and $T$ is a maximal torus subgroup of $Aut_k(A)_0$ and $\theta$ is an abelian variety. Let $P$ be an arbitrary closed point of $\text{Spec}(A)$. If $T=0$, then there exists a regular map

$$\Phi : Aut_k(A)_0 \to \text{Spec}(A)$$

$$\sigma \to \sigma(P).$$

Since $\text{Im}(\Phi)$ is a projective variety contained in the affine variety $\text{Spec}(A)$, the set $\text{Im}(\Phi)$ consists of one point, it contradicts $\text{dim } Aut_k(A)_0 > 0$. Hence we have that $\text{dim } T=0$. Since $T \cong G_a$ or $G_m$, we have the following result:

\textbf{Proposition 3.10.} If $Aut_k(A)$ is not a finite set and has an algebraic group structure, then $A$ is not strongly torus invariant.

4. Affine domains of dimension $\leq 2$

Let $k$ be a field of characteristic zero which contains all roots of “unity”. In this section let $A$ be an affine domain over $k$. We shall see that if $\text{dim } A=1$, then $A$ is always torus invariant. Moreover $A$ is not strongly torus invariant if and only if $Aut_k(A) \cong G_m$. Let $\text{dim } A \geq 2$. Then $A$ is not always torus
invariant. But if an integrally closed domain $A$ is not a $\mathbb{Z}$-graded ring, then $A$ is torus invariant.

For the proof we need a lemma.

**Lemma 4.1.** Let $K$ be a finite separable algebraic field extension of a field $k$. If $A$ is a one-dimensional affine normal ring such that $k\subset A\subseteq K[X, X^{-1}]$, then $A$ is a polynomial ring or a torus ring over $k'$ where $k'$ is the algebraic closure of $k$ in $A$.

Proof. We may assume that $k=k'$. Following the similar device to the proof of (2.9) in [1, p 322], we have $Q(A)=k(\theta)$ for some element $\theta$ of $A$.

Since $k[\theta]\subseteq A\subseteq k(\theta)$, $A=k[\theta]$ or $A=k\left[\theta, \frac{1}{f(\theta)}\right]$ for some polynomial $f(\theta)\in k[\theta]$. Let $A=k\left[\theta, \frac{1}{f(\theta)}\right]$. Then we may assume that $f(\theta)$ has no multiple factors. The element $f(\theta)$ is invertible in $A$, so is also invertible in $K[X, X^{-1}]$. Thus we have $f(\theta)=\beta X^r$, $\beta\in K$, $\theta\in K[X, X^{-1}]$. We may assume that $r\geq 0$, if necessary, by replacing $X$ with $X^\prime$. Then we easily see that $\theta\in K[X]$. The uniqueness of the irreducible decomposition in a polynomial ring implies that deg$_A f(\theta)=1$, since the polynomial $f(\theta)$ has not multiple factors and $f(\theta)=\beta X^r$. Hence we may assume that $f(\theta)=\theta$ and we obtain $A=k\left[\theta, \frac{1}{\theta}\right]$.

Let $A$ be an integral domain. If $A$ is contained in $K[X, X^{-1}]$, then $\bar{A}$ is a polynomial ring or a torus ring over $k$.

**Proposition 4.2.** Let $A$ be a one-dimensional affine domain over a field $k$ of characteristics zero. Then we obtain that

1. $A$ is torus invariant,

2. $A$ is not strongly torus invariant if and only if Aut$_k(A)$ has a subgroup isomorphic to $G_m$. If $A$ is not strongly torus invariant and $A$ is integrally closed, then $A$ is a polynomial ring or a torus ring over the algebraic closure of $k$ in $A$.

Proof. At first we shall prove (2). The sufficiency follows from (3.9). Let $R=A[X, X^{-1}]=B[Y, Y^{-1}]$ in which $A\not\equiv B$. If $mR\cap A\neq 0$ for any maximal ideal $m$ of $B$, then $m$ is vertical relative to $A$, and we have $A=B$ by (3.5). Hence there exists a maximal ideal $m$ such as $mR\cap A=0$. Since $ch k=0$, $B/m=K$ is a finite separable algebraic field over $k$. The residue mapping of $R$ to $R/mR$ yields (up to isomorphism) $k\subset A\subseteq K[Y, Y^{-1}]$ where $Y$ is algebraically independent over $K$. Therefore $A$ is a polynomial ring or a torus ring by the lemma (4.1). Thus the automorphism group Aut$_kA$ contains a subgroup isomorphic to $G_m$.

Assume that $A$ is not integrally closed. Then prime divisors in $A$ of the conductor $t(\bar{A}/A)$ are vertical relative to $B$. Hence we may assume $X\equiv Y$ by
The above lemma (4.1) implies that $\overline{A}=k'[t, t^{-1}]$ or $\overline{A}=k'[t]$ where $k'$ is the algebraic closure of $k$ in $\overline{A}$.

Firstly let $\overline{A}=k'[t]$. Since $\overline{A}\cong \overline{B}$, there exists an element $s$ in $\overline{B}$ such as $\overline{B}=\overline{A}[X, X^{-1}]=\overline{B}[X, X^{-1}]$, we have $k'[X, X^{-1}][t]=k'[X, X^{-1}][s]$, hence we easily see that $t=f_1(X)s+f(X)$ and $s=g_1(X)t+g(X)$ where $f_1(X)$ and $g_1(X)$ are invertible in $k'[X, X^{-1}]$. We may assume that $t=X^ns+f(X)$ and $s=X^{-n}t+g(X)$. Let $\bar{n}$ be a prime divisor in $\overline{A}$ of the conductor $t(A/A)$.

Then $\overline{A}/\overline{n}$ is algebraic over $k'$, there exist elements $\lambda_0, \lambda_1, \ldots, \lambda_{d-1} \in k'$ such that $t^d+\lambda_{d-1}t^{d-1}+\cdots+\lambda_0 \in \overline{m}\overline{R}=\bar{n}\overline{R}$. Hence we have that $(X^ns+f(X))^d+\lambda_{d-1}(X^ns+f(X))^d-1+\cdots+\lambda_0 \in \overline{n}\overline{R}$. The constant term of this polynomial with respect to $s$ is the following:

$$f(X)^d+\lambda_{d-1}f(X)^{d-1}+\cdots+\lambda_0 \in \overline{n}k'[s][X, X^{-1}].$$

Therefore $f(X)=f \in k'$. Hence we may assume that $t=X^ns$. We shall show that $A$ is a graded ring. Let $a$ be an element of $A$. Since $a$ is contained in $\overline{A}=k'[t]$ and $t=X^ns$, we have that $a=\sum \lambda_it^i=\sum \lambda_is^iX^{in}$, $\lambda_is^i \in B$. On the other hand, as the element $a$ is contained in $B[X, X^{-1}]$, $a=\sum b_iX^i$, $b_i \in B$. Comparing the coefficient of each term in the following, we have $b_i=\sum \lambda_is^iX^i=\lambda_is^i \in B[X, X^{-1}] \cap \overline{A}=A[X, X^{-1}] \cap \overline{A}=A$. Therefore $A$ has a graded ring structure.

Secondary let $\overline{A}=k'[t, t^{-1}]$. Then $\overline{B}=k'[s, s^{-1}]$. Since $t$ and $s$ are invertible in $\overline{R}$, we may assume that $t=s^{-1}X^s$ and $s=t^{-1}X^s$, then $t=(t^{-1}X^s)s=t^{-1}X^{in+s}$, therefore $ij=1$. Hence we may assume $t=sX^s$. By the same method as in the case $\overline{A}=k'[t]$ we have that $A$ is a graded ring.

Proof of (1). If $A$ is not integrally closed, then the prime divisors of the conductor $t(\overline{A}/A)$ are vertical relative to $B$. Since non-zero prime ideals of $A$ are maximal, the ring $A$ is isomorphic to $B$ by (2.5). If $A$ is integrally closed and $A$ is neither a polynomial ring nor a torus ring, then $A$ is strongly torus invariant, hence $A$ is torus invariant. If $A$ is either a polynomial ring or a torus ring, $A$ is torus invariant by (2.2) and (2.3).

Next we shall consider the case; the coefficient field $k$ has all roots of "unity" and its characteristic is zero. Then we prove the following:

**Theorem 4.3.** Let $A$ be an integrally closed $k$-affine domain of dimension two, where the field $k$ has all roots of "unity" and $\text{ch } k=0$. If $A$ is not torus invariant, then $A$ is a $\mathbb{Z}$-graded ring which contains units of non-zero degree.

Proof. Assume that $A$ is not torus invariant. Then there exist a $k$-algebra $B$ and independent variables $X, Y$ such that $A$ is not isomorphic to $B$ and $R=A[X, X^{-1}]=B[Y, Y^{-1}]$. By (2.0) and (2.1) we obtain $ff' \neq 1$. We shall show
that it follows from $ff'\neq 1$ that $A$ is a $\mathbb{Z}$-graded ring. We may only consider the case $1-ff'>0$. Let $x$ be a $(1-ff')^{-th}$ root of $u$ and let $y=x^{-f}/X$. Then $y^{1-ff'}=v$. Since $(y^{-f'\cdot}y)^{1-ff'}=u$, $x=\lambda y^{-f'}Y$ for some $(1-ff')^{-th}$ root $\lambda$ of "unity". From the relations; $y=x^{-f}/X$ and $Y=ux'$, we have $\lambda=1$.

Since $y=x^{-f}/X$ and $x=y^{-f'}Y$ are invertible, we have $A[x][X, X^{-1}]=B[y][Y, Y^{-1}]=A[x][y, y^{-1}]=B[y][x, x^{-1}]$. Define a surjective homomorphism $j:A[x][y, y^{-1}]	o A[x]$ by $j(y)=1$. Let $A_0=j(B[y])\subseteq A[x]$. We shall show that $A[x]=A_0[x, x^{-1}]$. Let $a$ be an element of $A$. Then $a=\sum b_i x^i$, $b_i\in B$. Since $j(a)=a$ and $j(x)=x$, we have that $a=\sum j(b_i)x^i$, $j(b_i)\in A_0$. Then $A[x]=A_0[x, x^{-1}]$ and $x$ is algebraically independent over $A_0$. By the same way $B[y]=B_0[y, y^{-1}]$.

Since the every $(1-ff')^{-th}$ roots of "unity" is contained in $k$ and $ch\, k=0$ and $A$ is normal, the extension $A[x]/A$ is a Galois extension with a cyclic group $G=\langle \sigma \rangle$ (cf. [3] p 214). Indeed when $|G|=n$, $n\,(1-ff')$ and there exists a primitive $n-th$ root $\lambda$ of "unity" such that $\sigma(x)=\lambda x$ and the invariant subring $(A[x])^\sigma=A$ and $A[x]=A+Ax+\cdots+Ax^{n-1}$ is a free $A$-module.

Since the element $u$ is a unit of $A$ and $ch(k)=0$, the extension $A[x]/A$ is étale. Since $A$ is a normal domain, $A[x]$; hence $A_0[x, x^{-1}]$, is also a normal domain. From this we see that $A_0$ is always normal.

We shall show that there exists a subring $A'_0$ in $A[x]$ such that $A[x]=A'_0[x, x^{-1}]$ and $\sigma(A'_0)=A'_0$. If $A_0$ is strongly torus invariant, then $\sigma(A_0)=A'_0$; for $\sigma(A_0)[x, x^{-1}]=A_0[x, x^{-1}]$, therefore $A_0$ satisfies the conditions. If $A_0$ is not strongly torus invariant, then $A_0=k'[t]$ or $k'[t, t^{-1}]$ by (4.2). Firstly let $A_0=k'[t]$. Since $k'[x, x^{-1}][t]=k'[x, x^{-1}][\sigma(t)]$, we easily see that $\sigma(t)=\mu x^i t + f(x)$, $\mu\in k^*$ and $f(x)\in k'[x, x^{-1}]$. The order of $\sigma$ is $n$, i.e. $\sigma^n=Identity$, so $\sigma^n(t)=t$, the other hand $\sigma^n(t)=\mu x^i t + f(x)$, $\mu\in k^*$ and $f(x)\in k'[x, x^{-1}]$, therefore we have that $i=0$, thus $\sigma(t)=\mu x^i t + f(x)$ and $\mu^{n}=1$. Let $f(x)=\sum f_i x^i$ and define the set $\Delta=\{i\in \mathbb{Z}; \lambda^i \neq \mu^i \}$. Let $h(x)=\sum h_i x^i$, where $h_i=f_i (\mu-\lambda)^i$, and put $s=t+h(x)$. Then $\sigma(s)=\mu s+\sum f_i x^i$, hence $\sigma^n(s)=\mu^n s+n\mu^{n-1}(\sum f_i x^i)=s+n\mu^{n-1}(\sum f_i x^i)$. Since $\sigma^n(s)=s$, we have $\sigma(s)=\mu s$. We set $A_0'=k'[s, s^{-1}]$, then $A'_0$ satisfies the conditions.

Secondary let $A_0=k'[t, t^{-1}]$. Since $k'[x, x^{-1}][t, t^{-1}]=k'[x, x^{-1}][\sigma(t), \sigma(t)^{-1}]$, we easily see that $\sigma(t)=\mu x^i t$ or $\sigma(t)=\mu x^i t^{-1}$, $\mu\in k^*$.  

Case (i); $\sigma(t)=\mu x^i t$. Since $\sigma^n(t)=\mu x^{(1+n-n)} x^i t$ and $\sigma^n(t)=t$, we have that $\sigma(t)=\mu t$, so $\sigma(A_0)=A_0$.

Case (ii); $\sigma(t)=\mu x^i t^{-1}$. If $n$ is odd, 'say $n=2m+1$, then $\sigma^n(t)=\mu x^{imx^i} t^{-1}$, but this is impossible for $\sigma^n(t)=t$. Therefore $n$ is even, say $n=2m$. Then $\sigma^n(t)=\lambda^{imt}$. Since $\lambda$ is a primitive $n-th$ root of "unity", the integer $i$ is even, say $i=2j$. Let $s=x^{-j} t$ and $A_0'=k'[s, s^{-1}]$. Then $A'_0$ satisfies the conditions.
Next we shall show that \( A \) has a \( \mathbb{Z} \)-graded ring structure. Let \( a \) be an element of \( A \). Since \( a \in A[[x, x^{-1}]] \), \( a = \sum a_i x^i \). Then \( a = \sigma(a) = \sum \sigma(a_i) x^i \) and \( \sigma(a_i) \in A_i \). Comparing the coefficient of each term in the equality; \( \sum a_i x^i = \sum \sigma(a_i) x^i \), we have that \( a_i = \sigma(a_i) x^i \), then \( \sigma(a x^i) = a_i x^i \). Thus \( a x^i \) is an element of \( A \). Therefore \( A \) is a graded ring. Since there exists units of non-zero degree, \( A \) has a \( \mathbb{Z} \)-graded ring structure.

**Remark.** The converse of this theorem is false. Indeed we find by (2.3) that the ring \( k[T][X, X^{-1}] \) is a \( \mathbb{Z} \)-graded ring with respect to \( X \) which is torus invariant.

**Example.** We shall construct an example of an affine dimension \( A \) of dimension two which is not torus invariant.

Let \( D \) be an integrally closed domain of dimension one over an algebraically closed field \( k \) and \( D^* = k^* \). Let \( a \) be a non-unit of \( D \) and \( a^2 = a, \alpha \in D \). Assume that \( D \) is noetherian and \( D[\alpha] \) is strongly torus invariant. Since an affine domain of dimension one whose totally quotient field has a positive genus is strongly torus invariant by (4.2), this assumption can be satisfied for a suitable choice of \( D \). Let \( T \) be a variable over \( A \) and \( S = T^2 X \) and \( Y = T^5 X^2 \). Let \( B = D[\alpha S^3, S^5, S^{-3}] \). Since \( T = S^{-2} Y \) and \( X = S^5 Y^{-2} \), we have that \( A[X, X^{-1}] = B[Y, Y^{-1}] \). By (1.1) invertible elements in the graded ring \( A \) are homogeneous. Since \( D^* = k^* \), we obtain \( A^* = \{ \eta T^i; \eta \in k^* \text{ and } i \in \mathbb{Z} \} \). Hence the quotient \( A^*/k^* \) is generated by \( T^5 \). Similarly \( B^*/k^* \) is generated by \( S^5 \). We shall show that \( A \) is not isomorphic to \( B \). We assume that there exists an isomorphism \( \sigma \) of \( A \) to \( B \). Since \( \sigma \) is a group-isomorphism of \( A^* \) to \( B^* \), we have \( \sigma(T^5) = \mu S^5 \) or \( \sigma(T^5) = \mu S^{-5} \), \( \mu \in k^* \). We shall only consider the case: \( \sigma(T^5) = \mu S^5 \), since the proof of the other case is the similar. Let \( \sigma \) be an isomorphism of \( A[T] \) to \( B[S] \) defined by \( \sigma = \sigma \) on \( A \) and \( \sigma(T) = T \xi S, \xi \xi = \mu \). Then we have that \( D[\alpha][S, S^{-1}] = \sigma(D[\alpha]) [S, S^{-1}] \), therefore \( \sigma(D[\alpha]) = D[\alpha] \); for \( D[\alpha] \) is strongly torus invariant. Since \( \sigma(D) \subseteq D[\alpha] \cap B = D \), we have \( \sigma(D) = D \); therefore we easily see that \( \sigma \) is an isomorphism as graded rings. Thus we have \( \sigma((\alpha T)D) = (\alpha^2 S) D \), hence \( \sigma(a) \in a^2 D \). Since the element \( a \) is not a unit, \( a^2 D \subseteq a D \), thus \( \sigma(a) D \subseteq a^2 D \subseteq a D \), so \( a D \subseteq a^{-1} \sigma(a) D \), hence we have a proper ascending chain \( \langle \sigma^{-1}(a) D \rangle \), but it contradicts the netherian assumption of \( D \). Hence \( A \) is not torus invariant.

(4.4) Now let \( A = \sum A_i \) be an integrally closed \( \mathbb{Z} \)-graded domain which contains invertible elements of non-zero degree. Let \( e \) be an invertible element of \( A \) with the smallest positive degree \( d \). Let \( a \) be a unit of \( A \), then \( a \) is a homogeneous elements with \( \deg a = jd \) for some integer \( j \), and there exists an element \( \xi \) of \( A^*_d \) such as \( a = \xi^d \). Let \( i \) be any positive integer and \( x \) be one of the \( ijd \)-th roots of \( a \), say \( x^{ijd} = a \). Since \( A[x] \) is a \( \mathbb{Z} \)-graded ring with the
invertible elements \( x \) of degree one, \( A[x]=A_0[[x, x^{-1}]] \) by (1.4) where \( A_0 \) contains \( A_0 \).

Let \( f \) and \( f' \) be integers such as \( ff'+ijd=1 \) and let \( X \) be a variable over \( A \).

Put \( y=x^{-f}X \) and \( Y=ax^{f'} \). Then \( x=y^{-f'}Y \) and \( Y=x^{ff}Y' \). Therefore \( A_0[x, x^{-1}][X, X^{-1}]=A_0[y, y^{-1}][Y, Y^{-1}] \). Since the every \( n \)-th roots of "unity" is contained in \( k \) and \( A \) is integral closed, the extension \( A[x]/A \) is a Galois extension with a cyclic group \( G=\langle \sigma \rangle \). Indeed \( |G|=di \) and there exists a primitive \( di \)-th root of "unity" such as \( \sigma(x)=\lambda x \), and \( (A[x])^*=A \). Since \( A_0 \) is algebraic over \( A_0 \), \( \sigma(A_0) \) is also so, hence \( \sigma(A_0) \) is algebraic over \( A_0 \), but \( A_0 \) is algebraically closed in \( A_0[x, x^{-1}] \), therefore \( \sigma(A_0)=A_0 \). Since \( \sigma(y)=\lambda^{-f}y \), \( \sigma \) is an automorphism of \( A_0[y, y^{-1}] \). Let \( B=A_0[y, y^{-1}]^* \) and \( \sigma \) be an automorphism of \( A_0[y, y^{-1}][X, X^{-1}] \) defined by \( \sigma(X)=X \) and \( \sigma=\sigma \) over \( A_0[x, x^{-1}] \). Since \( \sigma(Y)=Y \) and \( \sigma(X)=X \), we obtain \( B[Y, Y^{-1}]=A_0[y, y^{-1}][Y, Y^{-1}]^* =A_0[x, x^{-1}][X, X^{-1}]^*=A[X, X^{-1}] \).

**Proposition 4.5.** Let \( A \) be an integrally closed \( k \)-affine domain of dimension 2. If \( A[X, X^{-1}]=B[Y, Y^{-1}] \) and \( ff'=1 \), then \( A \) has a \( \mathbb{Z} \)-graded ring structure and \( B \) is isomorphic to one of algebras constructed in (4.4).

Proof. The first statement is already mentioned in the proof of (4.3) and we obtained \( A_0[x, x^{-1}][X, X^{-1}]=A_0[y, y^{-1}][Y, Y^{-1}] \) and \( \sigma(A_0)=A_0 \). Let \( B'=A_0[y, y^{-1}]^* \). Then \( B' \) is one of algebras in (4.4). Since \( B'[Y, Y^{-1}]=B[Y, Y^{-1}] \), \( B \) is isomorphic to \( B' \).

5. D-torus invariant

Let \( D \) be an integral domain containing a field \( k \) of characteristic zero and \( A \) be a \( D \)-algebra. The ring \( A \) is called \( D \)-torus invariant; if \( A[X, X^{-1}]=B[Y, Y^{-1}] \) for a certain \( D \)-algebra \( B \) and independent variables \( X \) and \( Y \), then we have always \( A \cong B \). Then we have the following result:

**Proposition 5.1.** Let \( A \) be an integrally closed domain over \( D \) and \( \text{tr. deg}_D A=1 \). If \( A \) is not \( D \)-torus invariant, then \( A \) is a \( \mathbb{Z} \)-graded ring containing units of non-zero degree.

Proof. Let \( A[X, X^{-1}]=B[Y, Y^{-1}] \), where \( B \) is a \( D \)-algebra and not \( D \)-isomorphic to \( A \). By (2.0) and (2.1) we easily see that \( ff'=1 \). Then we may assume \( 1-ff'>0 \). Let \( u \) be a \((1-ff')^{-th} \) root of \( u \) and \( y=x^{-f}X \). Then we have that \( A[x]=A_0[x, x^{-1}] \) and \( B[y]=B_0[y, y^{-1}] \) as the proof of (4.3), where \( A_0 \) and \( B_0 \) are respectively subalgebras of \( A[x] \) and \( B[y] \) containing \( D \). Let \( \sigma \) be a generator of the cyclic Galois group of the extension \( A[x]/A \). We shall show that \( \sigma(A_0)=A_0 \). Since \( \text{tr. deg}_D A_0[x, X^{-1}]=1 \), \( A_0 \) is algebraic over \( D \), thus \( \sigma(A_0) \) is also so. Since \( A_0 \) is algebraically closed in \( A_0[x, x^{-1}] \), we have that \( \sigma(A_0)=A_0 \). Following the similar devise to the proof of (4.3) we obtain
that $A$ is a $\mathbb{Z}$-graded ring, and $D$ is contained in $A$.

In the following we shall consider the case where $A$ is a $\mathbb{Z}$-graded ring and $A_0=D$. We consider only $D$-isomorphisms of $D$-algebras.

**Theorem 5.2.** Let $A$ be an integrally closed $\mathbb{Z}$-graded ring. Assume that the subring $A_0$ contains an algebraically closed field $k$ and that $A_0^* = k^*$. Let $d$ be the smallest positive integer among the set of degrees of units in $A$. Then the number of the isomorphic classes of $A_0$-algebra as $B$ such that $A[X, X^{-1}] = B[Y, Y^{-1}]$ equals to $\Phi(d)$, where $\Phi$ is the Euler function.

**Proof.** Let $i$ be an integer such as $1 \leq i < d$ and $(i, d) = 1$. Since $(i, d) = 1$, $ij + dh = 1$ for some integers $j$ and $h$. Moreover we may assume $h \geq 0$. Fix a unit $e$ of degree $d$. Let $x$ be one of the $d$-th roots of $e$. Then we have that $A'[x] = A_0'[x, x^{-1}]$ for a subring $A'_0$ containing $A_0$ by (1.4). Let $\sigma$ be a generator of the cyclic Galois group of the extension $A[x]/A$. Then $\sigma(x) = \lambda x$, where $\lambda$ is a primitive $d$-th root of "unity". Since $A'_0$ is algebraic over $A_0$ and algebraically closed in $A'_0[x, x^{-1}]$, we obtain $\sigma(A'_0) = A'_0$. Let $X$ be a variable over $A$ and let $y = x^{-1}X$ and $Y = e^X$. Then we have that $A'_0[x, x^{-1}] [X, X^{-1}] = A'_0[y, y^{-1}] [Y, Y^{-1}]$. Define $B_i = A'_0[y, y^{-1}]^\sigma$ and let $\sigma$ be an isomorphism of $A'_0[x, x^{-1}] [X, X^{-1}]$ defined by $\sigma(X) = X$ and $\sigma = \sigma$ on $A'_0[x, x^{-1}]$. Since $Y = e^X$, $\sigma(Y) = Y$, therefore we obtain that $A[X, X^{-1}] = B_i[Y, Y^{-1}]$. We can easily see that $B_i$ is a $X$-graded ring and $(B_i)_0 = A_0$. Especially we have $B_i \simeq A$.

Let $i_1$ and $i_2$ be integers such as $1 \leq i_1 < i_2 < d$ and $(i_1, d) = (i_2, d) = 1$. Let $B' = A'_0[y, y^{-1}]^\sigma$ and $B'' = A'_0[z, z^{-1}]^\sigma$ where $\sigma(y) = \lambda^{-i_1}y$ and $\sigma(z) = \lambda^{-i_2}z$ i.e., $B' = B_{i_1}$ and $B'' = B_{i_2}$. We shall show that $B'$ and $B''$ are not isomorphic. Assume that there exists an $A_0$-isomorphism $\psi$ of $B'$ to $B''$. Let a $b$ be a unit in $B'$ of non-zero degree, say degree $a = n$, $n \neq 0$. Let $b$ be a homogeneous element of $B'$ and degree $b = t$. Then we have $b^n = ra^t$ for an element $r$ in the coefficient ring $A_0$, hence $\psi(b^n) = \psi(b)^n = r\psi(a^t)$. Since $r$ and $\psi(a^t)$ are homogeneous, $\psi(b)$ is also homogeneous by (1.1), therefore $\psi$ is an isomorphism as graded rings.

Let $c$ be a homogeneous element in $B'$ of degree one. Then $c = s_1y$ for an element $s_1$ in $A'_0$. Since $\sigma(c) = c$ and $\sigma(y) = \lambda^{-i_1}y$, we have $\sigma(s_1) = \lambda^{i_2}s_1$ hence $s_1^d$ is in $B'$. Since $\psi(s_1y) = s_2z$ for an element $s_2$ in $A'_0$. Since $\sigma(s_2z) = s_2\zeta$ and $\sigma(z) = \lambda^{-iz}z$, we have $\sigma(s_2) = \lambda^{i_2}s_2$, hence $s_2^d$ is in $B''$. By the relations, $\psi(y) = \psi(s_1y)^\sigma = \psi(s_2)^d = s_2^dz^d = s_2^dz^d$, we obtain $s_2^d = \psi(y)^d z^{-d} s_2^d$. Since $\psi(y)^d z^{-d}$ is an invertible element in $B''$ and degree zero, we have $\zeta = \psi(y)^d z^{-d} \in A_0^* = k^*$, therefore we have $s_2 = \eta s_1$ for some $\eta \in k$, $\eta^d = \xi$. Hence $\sigma(s_2) = \lambda^{i_2}s_2$, but it contradicts the fact that $\sigma(s_2) = \lambda^{i_2}s_2$ and $\lambda$ is a primitive $d$-th root of "unity". Therefore $B' \ncong B''$.

Finally we shall show that if $A[X, X^{-1}] = B[Y, Y^{-1}]$ then $B$ is isomorphic to $B_i$ for some $i$ satisfying $0 < i < d$ and $(i, d) = 1$. The invertible element $u$
in (2.0) is homogeneous. Let $n$ be the degree of $u$. If $n=0$, then $A$ is isomorphic to $B$ by (2.1), hence $B \cong B_i$. Assume $n \neq 0$. Let $c$ be a non-zero homogeneous element of degree 1 and put $\eta = c^*u^{-1}$. Then $\eta$ is an element of $A_0$. In the graded ring $B[Y, Y^{-1}]$ the elements $u$ and $\eta$ are homogeneous, hence $c$ is also homogeneous, thus we denote $c = bY^j$ for some element $b$ in $B$ and some integer $j$. Then we obtain that $c^* = \eta u = \eta v^{-j'} Y^{1-j'}$ by (2.0). Therefore we have $1 - jj' = nj$.

By the minimality of $d$ we obtain $n = ld$ for some integer $l$ and $u = \xi e$, $\xi \in A_d^* = k^*$. Since the field $k$ is algebraically closed, we may assume $\xi = 1$, then the $d$-th root $x$ of $e$ is an $n$-th root of $u$. Since the element $\lambda$ is a primitive $d$-th root of “unity”, there exists the unique integer $i$ such that $\lambda^{-i} = \lambda^{-i} < i < d$, then $(i, d) = 1$ since $(f, d) = 1$. Let $y' = x^{-j}X^j$ and $B' = (A_d[y', y'^{-1}])^\sigma$. Then $\sigma(y') = \lambda^{-j'}y' = \lambda^{-i}y'$, hence $B' = B_i$. We can easily show that $x = y'^{-j'}Y^j$, therefore we obtain $A_d[x, x^{-1}][X, X^{-1}] = A_d[y', y'^{-1}][Y, Y^{-1}]$. Since $\sigma(X) = X$ and $\sigma(Y) = Y$, we have $A[X, X^{-1}] = B_i[Y, Y^{-1}]$, hence $B[Y, Y^{-1}] = B_i[Y, Y^{-1}]$. Thus we have $B \cong B_i$.

References


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