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Osaka University
ON THE COEFFICIENT RING OF A TORUS EXTENSION

KEN-ICHI YOSHIDA

(Received June 20, 1979)

Introduction. S. Abhyankar, W. Heinzer and P. Eakin treated the following problem in [1]: if \( A[X] = B[Y] \), when is \( A \) isomorphic or identical to \( B \)? Replacing the polynomial ring by the torus extension we shall take up the following problem: if \( A[X, X^{-1}] = B[Y, Y^{-1}] \), when is \( A \) isomorphic or identical to \( B \)? We say that \( A \) is torus invariant (resp. strongly torus invariant) whenever \( A[X, X^{-1}] = B[Y, Y^{-1}] \) implies \( A \cong B \) (resp. \( A = B \)). The roles played by polynomial rings in [1] are played by the graded rings in our theory. A graded ring \( A = \sum A_i, i \in \mathbb{Z} \), with the property that \( A_i \neq 0 \) for each \( i \in \mathbb{Z} \), will be called a \( \mathbb{Z} \)-graded ring. Main results are the followings.

An affine domain \( A \) of dimension one over a field \( k \) is always torus invariant. Moreover \( A \) is not strongly torus invariant if and only if \( A \) has a graded ring structure. An affine domain of dimension two is not always torus invariant. We shall construct an affine domain of dimension two which is not torus invariant. Let \( A \) be an affine domain over \( k \) of dimension two. Assume that the field \( k \) contains all roots of "unity" and is of characteristic zero. If \( A \) is not torus invariant, then \( A \) is a \( \mathbb{Z} \)-graded ring such that there exist invertible elements of non-zero degree.

In Section 1 we study elementary properties of graded rings. Especially we are interested in \( \mathbb{Z} \)-graded rings with invertible elements of non-zero degree. In Section 2 we discuss some conditions for \( A \) to be torus invariant. In Section 3 we give several sufficient conditions for an integral domain to be strongly torus invariant. Some relevant results will be found in S. Iitaka and T. Fujita [2]. Section 4 is devoted to the proof of the main results mentioned above. In Section 5 we fix an integral domain \( D \) and we treat only \( D \)-algebras and \( D \)-isomorphisms there. We shall prove the following two results. When \( A \) is a \( D \)-algebra of \( \text{tr. deg}_D A = 1 \) and \( A \) is not \( D \)-torus invariant, \( A \) is a \( \mathbb{Z} \)-graded ring such that \( D \) is contained in \( A \). If \( A \) is a \( \mathbb{Z} \)-graded ring such as \( D = A_0 \), then the number of elements of the set of \( \{ D \text{-isomorphic classes of } D \text{-algebras } B \text{ such that } A[X, X^{-1}] = B[Y, Y^{-1}] \} \) is \( \Phi(d) \), where \( d \) is the smallest positive integer among the degrees of units in \( A \) and \( \Phi \) is the Euler function.

I'd like to express my sincere gratitude to the referee for his valuable advices.
1. Some properties of graded rings

Let \( R \) be commutative ring with identity. The ring \( R \) is said to be a graded ring if \( R \) is a graded module, \( R = \bigoplus R_i \), and \( R_i R_m \subseteq R_{i+m} \).

**Lemma 1.1.** Let \( R \) be a graded domain. Then we have the following.

1. The unity element of \( R \) is homogeneous.
2. If \( a \) is homogeneous and \( a = bc \), then \( b \) and \( c \) are both homogeneous. In particular every invertible element is homogeneous.
3. If \( R \) contains a field \( k \), then \( k \) is a subring of \( R_0 \).

**Proof.** (1) follows immediately from the relation \( 1^2 = 1 \). The proof of (2) is easy and will be omitted. To prove (3) we can assume \( k \) is different from \( F \) by (1). Let \( a \) be an element of \( k \) different from 1. Then \( 1 - a \) is homogeneous from (2). The unity 1 is homogeneous of degree 0 by (1). Hence \( a \) should be homogeneous of degree 0.

We call a graded ring \( R = \bigoplus R_i \) to be a \( \mathbb{Z} \)-graded ring if \( R_\geq 0 \) for some \( \mathbb{Z} \). Proof. Choose non-zero elements \( x \in R_i \), and \( y \in R_j \). Since \( R_0 \) is a field, \( 0 \neq xy \) is invertible, therefore \( x \) and \( y \) are units in \( R \), hence \( R = R_0[x, x^{-1}] \).

**Proposition 1.2.** Let \( R \) be a \( \mathbb{Z} \)-graded domain. Let \( S = \{ i \in \mathbb{Z} ; R_i \neq 0 \} \). Then \( S = n\mathbb{Z} \) for a certain integer \( n \).

**Proof.** Since \( R \) is a domain, \( S \) is a semi-group. Hence (1.2) is immediately seen by the following lemma.

**Lemma 1.3.** Let \( S \subseteq \mathbb{Z} \) be a semi-group. If \( S \cap \mathbb{Z}^+ \neq \emptyset \) and \( S \cap \mathbb{Z}^- \neq \emptyset \), then \( S \) is a subgroup of \( \mathbb{Z} \).

If \( R \) is a \( \mathbb{Z} \)-graded domain, then we may assume \( R_i \neq 0 \) for any \( i \in \mathbb{Z} \).

**Proposition 1.4.** Let \( R \) be a graded ring. If there is an invertible element \( x \) in \( R_i \), then \( R = R_0[x, x^{-1}] \).

**Corollary.** Let \( R \) be a \( \mathbb{Z} \)-graded domain. If \( R_0 \) is a field, so \( R = R_0[x, x^{-1}] \) for every \( x \in R_i \), \( x \neq 0 \).

**Proof.** Choose non-zero elements \( x \in R_i \), and \( y \in R_j \). Since \( R_0 \) is a field, \( 0 \neq xy \) is invertible, therefore \( x \) and \( y \) are units in \( R \), hence \( R = R_0[x, x^{-1}] \).

2. Torus invariant rings

A ring \( A \) is said to be torus invariant provided that \( A \) has the following property:

If there exist a ring \( B \), a variable \( Y \) over \( B \), and a variable \( X \) over \( A \) such that \( A[X, X^{-1}] \) is isomorphic to \( B[Y, Y^{-1}] \).
\[ \Phi: A[X, X^{-1}] \rightarrow B[Y, Y^{-1}] , \]

then \( A \) is always isomorphic to \( B \).

Especially if we have always \( \Phi(A)=B \) in such case, we say that the ring \( A \) is strongly torus invariant.

To show \( A \) is torus invariant (resp. strongly torus invariant) it suffices to prove that \( A \) is isomorphic to \( B \) (resp. \( A=B \)) under the assumption: \( A[X, X^{-1}]=B[Y, Y^{-1}] \).

(2.0) We begin with some elementary observations. Assume that

\[(1) \quad R = A[X, X^{-1}] = B[Y, Y^{-1}] . \]

Then \( X \) and \( Y \) are units of \( R \). It follows from (1.1) that we have

\[(2) \quad X = vY' \text{ and } Y = uX' , \quad v \in B \text{ and } u \in A , \]

or equivalently

\[(3) \quad v = u^{-1}X^{-1}Y' \text{ and } u = v^{-1}Y^{-1}X' . \]

In the rest of our paper we shall use the letters \( u \) and \( v \) to denote the elements of \( A \) and \( B \) respectively satisfying the relations (2) and (3) whenever we encounter the situation (1).

(2.1) The element \( u \) is in \( B \) if and only if \( ff'=1 \). In this case we have \( A[X, X^{-1}]=B[X, X^{-1}] \), thus we have \( A \approx B \).

Proof is easy and is omitted.

**Proposition 2.2.** Let \( k \) be a field and \( A \) be a \( k \)-algebra. If \( A^* \) (the set of all invertible elements in \( A \))=\( k^* \), then the ring \( A \) is torus invariant.

Proof. Let \( R=A[X, X^{-1}]=B[Y, Y^{-1}] \). By (1.1) the field \( k \) is contained in \( B \). Since \( A^*=k^* \), the unit element \( u \) of \( A \) is in \( k \), hence in \( B \). It follows from (2.1) that \( A \) is torus invariant.

**Proposition 2.3.** Let \( A=A_0[t_1, t_2, \ldots, t_n, (t_1t_2\cdots t_n)^{-1}] \) where \( t_i \)'s are independent variables over \( k \)-algebra \( A_0 \) and \( A^*_0=k^* \), then \( A \) is torus invariant.

Proof. Let \( R=A[X, X^{-1}]=B[Y, Y^{-1}] \). Then by the lemma (1.1) \( Y=uxX' \) and \( X=vY' \). Since \( u \) is invertible in \( A=A_0[t_1, t_2, \ldots, t_n, (t_1\cdots t_n)^{-1}] \), \( Y=rt_1\cdots t_n, r \in A^*_0=k^* \). We may assume that \( r=1 \), so \( Y=t_1\cdots t_nX' \).

On the other hand as \( t_i \) is invertible in \( R=B[Y, Y^{-1}] \), \( t_i=b_iY' \), \( b_i \in B^* \). Then we have that

\[ ff'+\sum e_if_i = 1 . \]

Therefore the following natural homomorphism is surjective.
Since $Z$ is P.I.D., we can construct a basis of $Z^{(n+1)}$ containing this vector $(f', e_1, \ldots, e_n)$. Put this basis

$$ e_0 = (f', e_1, \ldots, e_n) $$

$$ e_i = (f_i, f_{i1}, \ldots, f_{in}) $$

and put $u_i = t_1^{i_1} \cdots t_n^{i_n} X^{i_1}$.

$$ R = A_0[u_1, \ldots, u_n, (u_1 \cdots u_n)^{-1}][Y, Y^{-1}] = A[X, X^{-1}] = B[Y, Y^{-1}] $$

Therefore $A$ is isomorphic to $B$. Hence $A$ is torus invariant.

(2.4) Let $R = A[X, X^{-1}] = B[Y, Y^{-1}]$. An ideal $I$ of $R$ is said to be vertical relative to $A$ if there exists an ideal $J$ of $A$ such that $JR = I$. If $J$ is an ideal of $A$ such that $JR$ is vertical relative to $B$, then we will simply say that $J$ is vertical relative to $B$. If $A$ is a $k$-affine domain, the prime ideals defined by the singular locus of Spec $A$ are vertical relative to $B$.

**Proposition 2.5.** Let $R = A[X, X^{-1}] = B[Y, Y^{-1}]$. If there exists a maximal ideal of $A$ which is vertical relative to $B$, then $A[X, X^{-1}] = B[X, X^{-1}]$. In particular $A$ and $B$ are isomorphic.

Proof. Let $m$ be a maximal ideal of $A$ which is vertical relative to $B$. Then there exists an ideal $n$ of $B$ such that $mR = nR$. Therefore $R/mR = A/m[X, X^{-1}] = B/n[Y, Y^{-1}]$, where $X = vY$ and $Y = uX$. Since $m$ is a maximal ideal, $A/m$ is a field. Hence $u$ is in $B/n$ by (1.1). Therefore we obtain $f = \pm 1$ by (2.1). Thus $A$ is isomorphic to $B$.

**Corollary 2.6.** Let $A$ be a $k$-affine domain with isolated singular points, then $A$ is torus invariant.

3. **Strongly torus invariant rings**

In this section we investigate strongly torus invariant rings.

**Proposition 3.1.** Let $A[X, X^{-1}] = B[Y, Y^{-1}]$. If $Q(A) \subseteq Q(B)$, then $A = B$, where $Q(R)$ is the total quotient field of $R$.

Proof. Let $x$ be an element of $A$, then there exist two elements $b$ and $b'$ of $B$ such as $x = b/b'$. Hence $b = b'x$. In the graded ring $B[Y, Y^{-1}]$ the elements $b$ and $b'$ are homogeneous of degree zero, thus $x$ is also degree zero. Hence we have $A \subseteq B$. Let $b$ be an element of $B$. Then $b = \sum a_j X^j$, $a_j \in A$. By (2) of (2.0) we have that $b = \sum a_j v^j Y^{j'}. \text{ if } f = 0, \text{ then } X \in B$. Thus $A[X, X^{-1}] \subseteq B$,
it's a contradiction, hence \( f \neq 0 \). Since \( a_j\varepsilon B \) and \( Y \) is a variable over \( B \), \( b=a_0\varepsilon A \). Thus \( A=B \).

**Corollary 3.2.** Let \( \bar{A} \) denote the integral closure of \( A \). If \( \bar{A} \) is strongly torus invariant, then \( A \) is also so.

Proof. It is easily seen that if \( A[X, X^{-1}]=B[Y, Y^{-1}] \) then \( \bar{A}[X, X^{-1}]=\bar{B}[Y, Y^{-1}] \). Since \( \bar{A} \) is strongly torus invariant, \( \bar{A}=\bar{B} \). Hence \( Q(A)=Q(B) \), and we have that \( A=B \).

**Proposition 3.3.** Let \( A \) be a domain with \( J(A)\neq 0 \), where \( J(D) \) is the Jacobson radical of a ring \( D \). Then \( A \) is strongly torus invariant.

Proof. Let \( a \) be a non-zero element of \( J(A) \). Then \( 1+a \) is unit, so in the graded ring \( B[Y, Y^{-1}] \), \( 1+a \) is homogeneous. Since the "unity 1" is a homogeneous element of degree 0, the element \( a \) is also so. Thus the element \( a \) is contained in \( B \).

Let \( x \) be any element of \( A \). Since \( xa \) is contained in \( J(A) \), \( xa \) is in \( B \). Hence \( A \) is contained in \( Q(B) \). By (3.1), we have that \( A=B \).

**Corollary 3.4.** If \( A \) is a local domain, then \( A \) is strongly torus invariant.

**Proposition 3.5.** Let \( A \) be an affine ring over a field \( k \) and let \( A[X, X^{-1}]=B[Y, Y^{-1}] \). Then \( A=B \) if and only if every maximal ideals of \( A \) is vertical relative to \( B \).

Proof. It suffices to prove the "if" part of the (3.5). By (3.3) we may assume that \( J(A)=0 \). Let \( x \) be an element of \( B \) and let \( x=\sum_{j=1}^{t} a_j X^j \), where \( s<t \), \( a_j\varepsilon A \) and \( a_s\neq 0 \) and \( a_t\neq 0 \). For any maximal ideal \( m \) of \( A \) there exists a maximal ideal \( n \) of \( B \) such as \( mR=nR \), where \( R=A[X, X^{-1}] \). Let \( \bar{x} \) denote the residue class of \( x \) in \( B/n \). Then \( \bar{x} \) is algebraic over the coefficient field \( k \), hence there exist elements \( \lambda_0, \lambda_1, \ldots, \lambda_{n-1} \) in \( k \), such that \( \bar{f}(\bar{x})=x^n+\lambda_{n-1}x^{n-1}+\cdots+\lambda_0\varepsilon nR=mR \). If \( t\neq 0 \), then the highest degree term of \( f(x) \) with respect to \( X \) is \( a_0 X^s \varepsilon mR \), thus \( a_t \) is contained in \( m \) for every maximal ideal in \( A \). Since \( J(A)=0 \), \( a_t=0 \). It's a contradiction. Therefore \( t=0 \). By the same way, we have that \( s=0 \), hence \( x \) is in \( A \). Thus \( A=B \).

We denote the subring generated by all the units of \( A \) by \( A_u \).

**Proposition 3.6.** Let \( A \) be a \( k \)-affine domain with an isolated singular point. If \( A \) is algebraic over \( A_u \), then \( A \) is strongly torus invariant.

Proof. Let \( A[X, X^{-1}]=B[Y, Y^{-1}] \) and let \( m \) be the maximal ideal defined by the isolated singular point. Then there exists a maximal ideal \( n \) of \( B \) such as \( mR=nR \). Let \( a \) be a unit element of \( A \). In the graded ring \( B[Y, Y^{-1}] \), the
element $a$ is also invertible, so $a=bY^j$ for some invertible element $b$ in $B$ and a certain integer $j$. Since $A/m$ is algebraic over $k$, there exist elements $\lambda_0, \lambda_1, \ldots, \lambda_n \in k$ such that $\lambda_n a^n + \cdots + \lambda_0 a + \lambda_0 \in mR = nR$. If $j \neq 0$, $\lambda_n b^n$ is in $n$, hence $b$ is not invertible, it's a contradiction. Thus we have that $A \subseteq B$. By the following lemma our proof is over.

**Lemma 3.7.** Let $A[X, X^{-1}] = B[Y, Y^{-1}]$. If $A$ is algebraic over $A \cap B$, then $A=B$.

Proof. Since $A$ is algebraic over $A \cap B$, $A$ is also algebraic over $B$, but $B$ is algebraically closed in $B[Y, Y^{-1}]$, therefore $A$ is contained in $B$. Thus we have that $A=B$.

Let $A$ be an integral domain containing a field $k$. We denote the set of all automorphisms of $A$ over $k$ by $\text{Aut}_k(A)$.

**Proposition 3.8.** Let $A$ be an integral domain containing an infinite field $k$. If $\text{Aut}_k(A)$ is a finite set, then $A$ is strongly torus invariant.

Proof. Let $R = A[X, X^{-1}] = B[Y, Y^{-1}]$. Let $\Phi_{\lambda}, \lambda \in k^*$, be an automorphism of $R$ defined by $\Phi_{\lambda}(Y) = \lambda Y$ and $\Phi_{\lambda}(b) = b$ for $b \in B$. Following the notation of (2.0) we have $X = vY^j$, thus $\Phi_{\lambda}(X) = \lambda^jX$, therefore $R = \Phi_{\lambda}(A)[X, X^{-1}]$. Let $p$ be the projection $A[X, X^{-1}] \to A$ defined by $p(X) = 1$ and $i$ be the canonical injection $A \to A[X, X^{-1}]$. Define $\sigma_{\lambda} = q \circ \Phi_{\lambda} \circ i$. Then $\sigma_{\lambda}$ is an endomorphism of $A$. We shall show that $\sigma_{\lambda}$ is surjective. Let $x$ be an element of $A$. Since $R = \Phi_{\lambda}(A)[X, X^{-1}]$, there exist elements $a_i$'s of $A$ such as $x = \sum \Phi_{\lambda}(a_i)X^i$. Hence $x = \sum p(\Phi_{\lambda}(a_i))$. Let $x' = \sum a_i \in A$, then $\sigma_{\lambda}(x') = \sum p(\Phi_{\lambda}(a_i)) = x$. Thus $\sigma_{\lambda}$ is surjective. Next we shall show that $\sigma_{\lambda}$ is injective. Since $\Phi_{\lambda}(X - 1)R \cap \Phi_{\lambda}(A) = \Phi_{\lambda}(X - 1)R \cap A = (\lambda^jX - 1)R \cap A = 0$, we have $(X - 1)R \cap \Phi(A) = 0$, therefore $\sigma_{\lambda}$ is injective. Hence $\sigma_{\lambda}$ is an automorphism of $A$.

We shall prove that the set $\{\sigma_{\lambda}; \lambda \in k^*\}$ is infinite when $A \neq B$. Suppose $ff' = 0$. Suppose $ff' = 1$. Then we may assume that $R = A[X, X^{-1}] = B[X, X^{-1}]$. If $A \subseteq B$, then $A = B$, so there exists an element $x$ of $A$ not contained in $B$, say $x = \sum b_j X^j$, $t > s$. Since $\ker p = (X - 1)R$ and $(X - 1)R \cap B = 0$, $p(b_j) \neq 0$ for $b_j \neq 0$. Since $\sigma_{\lambda}(x) = \sum p(b_j)\lambda^j$ and $p(b_j) \neq 0$ for some $j \neq 0$, the set $\{\sigma_{\lambda}; \lambda \in k^*\}$ is infinite.

Next we shall give two cases of rings which are not strongly torus invariant. If $A$ has a non-trivial locally finite iterative higher derivation $\psi: A \to A[T]$, then $A[T] = B[T]$, where $B = \psi(A)$ and $A \neq B$, as is proved in [4]. Hence we have that $A[T, T^{-1}] = B[T, T^{-1}]$ and $A \neq B$. If $A$ is a graded ring, then $A$ is not strongly torus invariant. Indeed, let $X$ be a variable over $A$ and let
$B_i = \{a_i X^i; a_i \in A_i\}$. Then $B_i$ is an $A_\sigma$-module contained in $A[X, X^{-1}]$. Let $B = \sum B_i$. Then $B$ is a graded ring and we easily see that $A[X, X^{-1}] = B[X, X^{-1}]$. We shall show that $X$ is a variable over $B$. Assume that there exist elements $b_0, b_1, \ldots, b_n$ in $B$ such that $b_n \neq 0$ and $b_n X^n + \cdots + b_1 X + b_0 = 0$. By the definition of $B$ we denote $b_i = \sum a_{ij} X^j$, $a_{ij} \in A_j$. In the graded ring $A[X, X^{-1}]$ the homogeneous term of degree $t$ of this equation is that
\[(a_{n,t-n} + a_{n-1,t-n+1} + \cdots + a_{0,t}) X^t = 0 .\]
Since $A$ is a graded ring and $a_{ij}$ is a homogeneous element of degree $j$, we obtain $a_{ij} = 0$ for all index $i$ and $j$, hence $X$ is a variable over $B$.

By [4] we have that a $k$-algebra $A$ has a non-trivial locally finite iterative higher derivation if and only if $\text{Aut}_k(A)$ has a subgroup isomorphic to $G_a = \text{Spec } k[T]$. We easily see that $A$ is a non-trivial graded ring if and only if $\text{Aut}_k(A)$ has a subgroup isomorphic to $G_a = \text{Spec } (k[T, T^{-1}])$.

**Proposition 3.9.** A $k$-algebra $A$ is not strongly torus invariant, if $\text{Aut}_k(A)$ has a subgroup isomorphic to $G_a$ or $G_m$.

Assume that $\text{Aut}_k(A)$ is an infinite group. If $\text{Aut}_k(A)$ has an algebraic group structure, then there exists the following exact sequence:
\[0 \to T \to \text{Aut}_k(A) \to \theta \to 0\]
where $\text{Aut}_k(A)_0$ is the connected component containing the identity $I_A$, and $T$ is a maximal torus subgroup of $\text{Aut}_k(A)_0$ and $\theta$ is an abelian variety. Let $P$ be an arbitrary closed point of $\text{Spec}(A)$. If $T = 0$, then there exists a regular map
\[\Phi: \text{Aut}_k(A)_0 \to \text{Spec}(A)\]
\[\sigma \to \sigma(P) .\]
Since $\text{Im}(\Phi)$ is a projective variety contained in the affine variety $\text{Spec}(A)$, the set $\text{Im}(\Phi)$ consists of one point, it contradicts $\dim \text{Aut}_k(A)_0 > 0$. Hence we have that $T \neq 0$. Since $T \cong G_a$ or $G_m$, we have the following result:

**Proposition 3.10.** If $\text{Aut}_k(A)$ is not a finite set and has an algebraic group structure, then $A$ is not strongly torus invariant.

4. Affine domains of dimension $\leq 2$

Let $k$ be a field of characteristic zero which contains all roots of "unity". In this section let $A$ be an affine domain over $k$. We shall see that if $\dim A = 1$, then $A$ is always torus invariant. Moreover $A$ is not strongly torus invariant if and only if $\text{Aut}_k(A) \cong G_m$. Let $\dim A \geq 2$. Then $A$ is not always torus
invariant. But if an integrally closed domain $A$ is not a $\mathbb{Z}$-graded ring, then $A$ is torus invariant.

For the proof we need a lemma.

**Lemma 4.1.** Let $K$ be a finite separable algebraic field extension of a field $k$. If $A$ is a one-dimensional affine normal ring such that $k \subset A \subseteq K[X, X^{-1}]$, then $A$ is a polynomial ring or a torus ring over $k'$ where $k'$ is the algebraic closure of $k$ in $A$.

Proof. We may assume that $k = k'$. Following the similar device to the proof of (2.9) in [1, p 322], we have $Q(A) = k(\theta)$ for some element $\theta$ of $A$.

Since $k[\theta] \subseteq A \subset k(\theta)$, $A = k[\theta]$ or $A = k\left[\theta, \frac{1}{f(\theta)}\right]$ for some polynomial $f(\theta) \in k[\theta]$. Let $A = k\left[\theta, \frac{1}{f(\theta)}\right]$. Then we may assume that $f(\theta)$ has no multiple factors. The element $f(\theta)$ is invertible in $A$, so is also invertible in $K[X, X^{-1}]$. Thus we have $f(\theta) = \beta X^r$, $\beta \in K$, $\theta \in K[X, X^{-1}]$. We may assume that $r \geq 0$, if necessary, by replacing $X$ with $X^r$. Then we easily see that $\theta \in K[X]$. The uniqueness of the irreducible decomposition in a polynomial ring implies that $\deg f(\theta) = 1$, since the polynomial $f(\theta)$ has no multiple factors and $f(\theta) = \beta X^r$. Hence we may assume that $f(\theta) = \theta$ and we obtain $A = k\left[\theta, \frac{1}{\theta}\right]$.

Let $A$ be an integral domain. If $A$ is contained in $K[X, X^{-1}]$, then $A$ is a polynomial ring or a torus ring over $k'$.

**Proposition 4.2.** Let $A$ be a one-dimensional affine domain over a field $k$ of characteristics zero. Then we obtain that

(1) $A$ is torus invariant,

(2) $A$ is not strongly torus invariant if and only if $\text{Aut}_k(A)$ has a subgroup isomorphic to $G_m$. If $A$ is not strongly torus invariant and $A$ is integrally closed, then $A$ is a polynomial ring or a torus ring over the algebraic closure of $k$ in $A$.

Proof. At first we shall prove (2). The sufficiency follows from (3.9). Let $R = A[X, X^{-1}] = B[Y, Y^{-1}]$ in which $A \neq B$. If $mR \cap A \neq 0$ for any maximal ideal $m$ of $B$, then $m$ is vertical relative to $A$, and we have $A = B$ by (3.5). Hence there exists a maximal ideal $m$ such as $mR \cap A = 0$. Since $ch k = 0$, $B/m = K$ is a finite separable algebraic field over $k$. The residue mapping of $R$ to $R/mR$ yields (up to isomorphism) $k \subset A \subseteq K[Y, Y^{-1}]$ where $Y$ is algebraically independent over $K$. Therefore $A$ is a polynomial ring or a torus ring by the lemma (4.1). Thus the automorphism group $\text{Aut}_kA$ contains a subgroup isomorphic to $G_m$.

Assume that $A$ is not integrally closed. Then prime divisors in $A$ of the conductor $t(\overline{A}/A)$ are vertical relative to $B$. Hence we may assume $X = Y$ by
The above lemma (4.1) implies that \( \overline{A} = k'[t, t^{-1}] \) or \( \overline{A} = k'[t] \) where \( k' \) is the algebraic closure of \( k \) in \( \overline{A} \).

Firstly let \( \overline{A} = k'[t] \). Since \( \overline{A} \cong \overline{B} \), there exists an element \( s \) in \( \overline{B} \) such that \( \overline{B} = \overline{A}[X, X^{-1}] = \overline{A}[X, X^{-1}] \). We have \( k'[X, X^{-1}] \) \( [t] = k'[X, X^{-1}] \) \([s] \), hence we easily see that \( t = f_1(X)s + f(X) \) and \( s = g(X)t + g(X) \) where \( f_1(X) \) \( g(X) = 1 \) and \( f(X) \), \( g(X) \in k'[X, X^{-1}] \). We may assume that \( t = X^{-1} + f(X) \) and \( s = X^{-1}t + g(X) \). Let \( n \) be a prime divisor in \( \overline{A} \) of the conductor \( t(A/A) \).

Then there exists a maximal ideal \( m \) of \( \overline{B} \) such that \( n\overline{R} = m\overline{R} \). Since \( \overline{A}/n \) is algebraic over \( k \), there exist elements \( \lambda_0, \lambda_1, \ldots, \lambda_{d-1} \in \overline{B} \) such that \( t^n + \lambda_{d-1}X^{-1} + \cdots + \lambda_0 \in n\overline{R} = m\overline{R} \). Hence we have that \( (X^n + f(X))^d + \lambda_{d-1}(X^n + f(X))^{d-1} + \cdots + \lambda_0 \in n\overline{R} \). The constant term of this polynomial with respect to \( s \) is the following;

\[
f(X)^d + \lambda_{d-1}f(X)^{d-1} + \cdots + \lambda_0 \in n\overline{k}'[s][X, X^{-1}].
\]

Therefore \( f(X)^d + \lambda_{d-1}f(X)^{d-1} + \cdots + \lambda_0 \in n\overline{k}'[s][X, X^{-1}] \).

Proof of (1). If \( A \) is not integrally closed, then the prime divisors of the conductor \( t(A/A) \) are vertical relative to \( B \). Since non-zero prime ideals of \( A \) are maximal, the ring \( A \) is isomorphic to \( B \) by (2.5). If \( A \) is integrally closed and \( A \) is neither a polynomial ring nor a torus ring, then \( A \) is strongly torus invariant, hence \( A \) is torus invariant. If \( A \) is either a polynomial ring or a torus ring, \( A \) is torus invariant by (2.2) and (2.3).

Next we shall consider the case; the coefficient field \( k \) has all roots of "unity" and its characteristic is zero. Then we prove the following:

**Theorem 4.3.** Let \( A \) be an integrally closed \( k \)-affine domain of dimension two, where the field \( k \) has all roots of "unity" and \( \text{ch } k = 0 \). If \( A \) is not torus invariant, then \( A \) is a \( \mathbb{Z} \)-graded ring which contains units of non-zero degree.

Proof. Assume that \( A \) is not torus invariant. Then there exist a \( k \)-algebra \( B \) and independent variables \( X, Y \) such that \( A \) is not isomorphic to \( B \) and \( R = A[X, X^{-1}] = B[Y, Y^{-1}] \). By (2.0) and (2.1) we obtain \( ff' \neq 1 \). We shall show
that it follows from $ff'\neq 1$ that $A$ is a $\mathbb{Z}$-graded ring. We may only consider the case $1-ff'>0$. Let $x$ be a $(1-ff')$-th root of $u$ and let $y=x^{-ff}/X$. Then $y^{-ff'}=v$. Since $(y^{-ff'})^{-ff'}=u$, $x=\lambda y^{-ff'}$ for some $(1-ff')$-th root $\lambda$ of “unity”. From the relations; $y=x^{-ff}/X$ and $Y=uX'$, we have $\lambda=1$.

Since $y=x^{-ff}/X$ and $y=x^{-ff'}Y$ are invertible, we have $A[x][X, X^{-ff}]=B[y][Y, Y^{-1}]=A[x][y, y^{-1}]=B[y][x, x^{-1}]$. Define a surjective homomorphism $j: A[x][y, y^{-1}] \rightarrow A[x]$ by $j(y)=1$. Let $A_0=j(B[y]) \subseteq A[x]$. We shall show that $A[x]=A_0[x, x^{-1}]$. Let $a$ be an element of $A$. Then $a=\sum b_i x^i$, $b_i \in B$. Since $j(a)=a$ and $j(x)=x$, we have that $a=\sum j(b_i)x^i$, $j(b_i) \in A_0$. Thus $A[x]=A_0[x, x^{-1}]$ and $x$ is algebraically independent over $A_0$. By the same way $B[y]=B_0[y, y^{-1}]$.

Since the every $(1-ff')$-th roots of “unity” is contained in $k$ and $ch k=0$ and $A$ is normal, the extension $A[x]/A$ is a Galois extension with a cyclic group $G=\langle \sigma \rangle$ (cf. [3] p 214). Indeed when $|G|=n$, $n(1-ff')$ and there exists a primitive $n$-th root $\lambda$ of “unity” such that $\sigma(x)=\lambda x$ and the invariant subring $(A[x])^G=A$ and $A[x]=A+Ax+\cdots+Ax^{n-1}$ is a free $A$-module.

Since the element $u$ is a unit of $A$ and $ch(k)=0$, the extension $A[x]/A$ is étale. Since $A$ is a normal domain, $A[x]$, hence $A_0[x, x^{-1}]$, is also a normal domain. From this we see that $A_0$ is always normal.

We shall show that there exists a subring $A'_0$ in $A[x]$ such that $A[x]=A'_0[x, x^{-1}]$ and $\sigma(A_0)=A'_0$. If $A_0$ is strongly torus invariant, then $\sigma(A_0)=A'_0$; for $\sigma(A_0)[x, x^{-1}]=A_0[x, x^{-1}]$, therefore $A_0$ satisfies the conditions. If $A_0$ is not strongly torus invariant, then $A_0=\mathcal{K}'[t]$ or $=\mathcal{K}'[t, t^{-1}]$ by (4.2). Firstly let $A_0=\mathcal{K}'[t]$. Since $\mathcal{K}'[x, x^{-1}][t]=\mathcal{K}'[x, x^{-1}][\sigma(t)]$, we easily see that $\sigma(t)=\mu\lambda x^t+f(x)$, $\mu \in \mathcal{K}'^*$. The order of $\sigma$ is $n$, i.e. $\sigma^n=\text{Identity}$, so $\sigma^n(t)=t$. The other hand $\sigma^n(t)=\mu\lambda x^{t^n}+g(x)$, $g(x) \in \mathcal{K}'[x, x^{-1}]$, therefore we have that $i=0$, thus $\sigma(t)=\mu t+f(x)$ and $\mu^n=1$. Let $f(x)=\sum f_i x^i$ and define the set $\Delta=\{j \in \mathbb{Z}; \lambda^j \neq \mu \}$. Let $h(x)=\sum h_i x^i$, where $h_i=f_i(\mu-\lambda)x^i$, and put $s=t+h(x)$. Then $\sigma(s)=\mu s+\sum f_i x^i$, hence $\sigma^n(s)=\mu^n s+n \mu^{n-1}(\sum f_i x^i) s+n \mu^{n-1} \cdot \sum f_i x^i$. Since $\sigma^n(s)=s$, we have $\sigma(s)=\mu s$. We set $A'_0=\mathcal{K}'[s]$, then $A'_0$ satisfies the conditions.

Secondary let $A_0=\mathcal{K}'[t, t^{-1}]$. Since $\mathcal{K}'[x, x^{-1}][t, t^{-1}]=\mathcal{K}'[x, x^{-1}][\sigma(t), \sigma(t)^{-1}]$, we easily see that $\sigma(t)=\mu x^t$ or $\sigma(t)=\mu x^{t^{-1}}$, $\mu \in \mathcal{K}'^*$. 

Case (i); $\sigma(t)=\mu x^t$. Since $\sigma^n(t)=\mu^n \lambda^{(1+\cdots+n-1)} x^t$ and $\sigma^n(t)=t$, we have that $\sigma(t)=\mu t$, so $\sigma(A_0)=A_0$.

Case (ii); $\sigma(t)=\mu x^{t^{-1}}$. If $n$ is odd, say $n=2m+1$, then $\sigma^n(t)=\mu \lambda^{im} x^{t^{-1}}$, but this is impossible for $\sigma^n(t)=t$. Therefore $n$ is even, say $n=2m$. Then $\sigma^n(t)=\lambda^{im} t$. Since $\lambda$ is a primitive $n$-th root of “unity”, the integer $i$ is even, say $i=2j$. Let $s=x^{jt}$ and $A'_0=\mathcal{K}'[s, s^{-1}]$. Then $A'_0$ satisfies the conditions.
Next we shall show that $A$ has a $\mathbb{Z}$-graded ring structure. Let $a$ be an element of $A$. Since $a \in A[[x, x^{-1}], a = \sum a_i x^i$. Then $a = \sigma(a) = \sum \sigma(a_i) \lambda^i x_i$ and $\sigma(a_i) \in A_i$. Comparing the coefficient of each term in the equality; $\sum a_i x^i = \sum \sigma(a_i) \lambda^i x_i$, we have that $a_i = \sigma(a_i) \lambda^i$, then $\sigma(a x^i) = a_i x^i$. Thus $a x^i$ is an element of $A$. Therefore $A$ is a graded ring. Since there exists units of non-zero degree, $A$ has a $\mathbb{Z}$-graded ring structure.

REMARK. The converse of this theorem is false. Indeed we find by (2.3) that the ring $k[T][X, X^{-1}]$ is a $\mathbb{Z}$-graded ring with respect to $X$ which is torus invariant.

EXAMPLE. We shall construct an example of an affine dimension $A$ of dimension two which is not torus invariant.

Let $D$ be an integrally closed domain of dimension one over an algebraically closed field $k$ and $D^* = k^*$. Let $a$ be a non-unit of $D$ and $a^2 = a$, $a \in D$. Assume that $D$ is noetherian and $D[\alpha]$ is strongly torus invariant. Since an affine domain of dimension one whose totally quotient field has a positive genus is strongly torus invariant by (4.2), this assumption can be satisfied for a suitable choice of $D$. Let $T$ be a variable over $D$ and $A = D[\alpha T, T^5]$. Let $X$ be a variable over $A$ and $S = T^2 X$ and $Y = T^5 X^2$. Let $B = D[\alpha S^3, S^5, S^{-5}]$. Since $T = S^2 Y$ and $X = S^5 Y^{-2}$, we have that $A[X, X^{-1}] = B[Y, Y^{-1}]$. By (1.1) invertible elements in the graded ring $A$ are homogeneous. Since $D^* = k^*$, we obtain $A^* = \{\eta T^i; \eta \in k^* \text{ and } i \in \mathbb{Z}\}$. Hence the quotient $A^*/k^*$ is generated by $T^i$. Similary $B^*/k^*$ is generated by $S^5$. We shall show that $A$ is not isomorphic to $B$. We assume that there exists an isomorphism $\sigma$ of $A$ to $B$. Since $\sigma$ is a group-isomorphism of $A^*$ to $B^*$, we have $\sigma(T^i) = \mu S^5$ or $\sigma(T^i) = \mu S^{-5}$, $\mu \in k^*$. We shall only consider the case: $\sigma(T^i) = \mu S^5$, since the proof of the other case is the similar. Let $\sigma$ be an isomorphism of $A[T]$ to $B[S]$ defined by $\sigma = \sigma$ on $A$ and $\sigma(T) = \tau S$, $\tau^5 = \mu$. Then we have that $D[\alpha] [S, S^{-1}] = \sigma(D[\alpha]) [S, S^{-1}]$, therefore $\sigma(D[\alpha]) = D[\alpha]$; for $D[\alpha]$ is strongly torus invariant. Since $\sigma(D) \subseteq D[\alpha] \cap B = D$, we have $\sigma(D) = D$, therefore we easily see that $\sigma$ is an isomorphism as graded rings. Thus we have $\sigma((\alpha T) D) = (\alpha^2 S) D$, hence $\sigma(a) = a^2 D$. Since the element $a$ is not a unit, $a^2 D \subseteq a D$, thus $\sigma(a) D \subseteq a^2 D \subseteq a D$, so $a D \subseteq \sigma^{-1}(a) D$, hence we have a proper ascending chain $\{\sigma^{-j}(a) D\}$, but it contradicts the netherian assumption of $D$. Hence $A$ is not torus invariant.

(4.4) Now let $A = \sum A_i$ be an integrally closed $\mathbb{Z}$-graded domain which contains invertible elements of non-zero degree. Let $e$ be an invertible element of $A$ with the smallest positive degree $d$. Let $a$ be a unit of $A$, then $a$ is a homogeneous elements with deg $a = jd$ for some integer $j$, and there exists an element $\xi$ of $A_j^*$ such as $a = \xi^j$. Let $i$ be any positive integer and $x$ be one of the $ijd$-th roots of $a$, say $x^{ijd} = a$. Since $A[x]$ is a $\mathbb{Z}$-graded ring with the
invertible elements \( x \) of degree one, \( A[x] = A'_0[x, x^{-1}] \) by (1.4) where \( A'_0 \) contains \( A_0 \). Let \( f \) and \( f' \) be integers such as \( ff' + ij = 1 \) and let \( X \) be a variable over \( A \). Put \( y = x^{-f'}X \) and \( Y = ay^{-f}y' \). Therefore \( A'_0[x, x^{-1}] [X, X^{-1}] = A'_0[y, y^{-1}] [Y, Y^{-1}] \). Since the every \( n\)-th roots of "unity" is contained in \( k \) and \( ^4\sigma \) is integral closed, the extension \( A[x]/A \) is a Galois extension with a cyclic group \( G = \langle \sigma \rangle \). Indeed \( |G| = di \) and there exists a primitive \( di\)-th root of "unity" such as \( \sigma(x) = \lambda x \), and \( (A[x])^\sigma = A \). Since \( A'_0 \) is algebraic over \( A_0 \), \( \sigma(A'_0) \) is also so, hence \( \sigma(A'_0) \) is algebraic over \( A'_0 \), but \( A'_0 \) is algebraically closed in \( A'_0[x, x^{-1}] \), therefore \( \sigma(A'_0) = A'_0 \). Since \( \sigma(y) = \lambda^{-f}y \), \( \sigma \) is an automorphism of \( A'_0[y, y^{-1}] \). Let \( B = A'_0[y, y^{-1}]^\sigma \) and \( \sigma \) be an automorphism of \( A'_0[x, x^{-1}] [X, X^{-1}] \) defined by \( \sigma(X) = X \) and \( \sigma = \sigma \) over \( A'_0[x, x^{-1}] \). Since \( \sigma(Y) = Y \) and \( \sigma(X) = X \), we obtain \( B[Y, Y^{-1}] = A'_0[y, y^{-1}] [Y, Y^{-1}]^\sigma = A'_0[x, x^{-1}] [X, X^{-1}]^\sigma = A[X, X^{-1}] \).

**Proposition 4.5.** Let \( A \) be an integrally closed \( k \)-affine domain of dimension 2. If \( A[X, X^{-1}] = B[Y, Y^{-1}] \) and \( ff' = 1 \), then \( A \) has a \( \mathbb{Z} \)-graded ring structure and \( B \) is isomorphic to one of algebras constructed in (4.4).

Proof. The first statement is already mentioned in the proof of (4.3) and we obtained \( A'_0[x, x^{-1}] [X, X^{-1}] = A'_0[y, y^{-1}] [Y, Y^{-1}] \) and \( \sigma(A'_0) = A'_0 \). Let \( B' = A'_0[y, y^{-1}]^\sigma \). Then \( B' \) is one of algebras in (4.4). Since \( B'[Y, Y^{-1}] = B[Y, Y^{-1}] \), \( B \) is isomorphic to \( B' \).

5. D-torus invariant

Let \( D \) be an integral domain containing a field \( k \) of characteristic zero and \( A \) be a \( D \)-algebra. The ring \( A \) is called \( D \)-torus invariant; if \( A[X, X^{-1}] = B[Y, Y^{-1}] \) for a certain \( D \)-algebra \( B \) and independent variables \( X \) and \( Y \), then we have always \( A \cong \sigma B \). Then we have the following result:

**Proposition 5.1.** Let \( A \) be an integrally closed domain over \( D \) and \( \text{tr. deg}_D A = 1 \). If \( A \) is not \( D \)-torus invariant, then \( A \) is a \( \mathbb{Z} \)-graded ring containing units of non-zero degree.

Proof. Let \( A[X, X^{-1}] = B[Y, Y^{-1}] \), where \( B \) is a \( D \)-algebra and not \( D \)-isomorphic to \( A \). By (2.0) and (2.1) we easily see that \( ff' = 1 \). Then we may assume \( 1 - ff' > 0 \). Let \( x \) be a \((1 - ff')^{-th} \) root of \( u \) and \( y = x^{-f'}X \). Then we have that \( A[x] = A_0 [x, x^{-1}] \) and \( B[y] = B_0 [y, y^{-1}] \) as the proof of (4.3), where \( A_0 \) and \( B_0 \) are respectively subalgebras of \( A[x] \) and \( B[y] \) containing \( D \). Let \( \sigma \) be a generator of the cyclic Galois group of the extension \( A[x]/A \). We shall show that \( \sigma(A_0) = A_0 \). Since \( \text{tr. deg}_D A_0 [x, X^{-1}] = 1 \), \( A_0 \) is algebraic over \( D \), thus \( \sigma(A_0) \) is also so. Since \( A_0 \) is algebraically closed in \( A_0 [x, x^{-1}] \), we have that \( \sigma(A_0) = A_0 \). Following the similar devise to the proof of (4.3) we obtain
that $A$ is a $\mathbb{Z}$-graded ring, and $D$ is contained in $A$.

In the following we shall consider the case where $A$ is a $\mathbb{Z}$-graded ring and $A_0=D$. We consider only $D$-isomorphisms of $D$-algebras.

**Theorem 5.2.** Let $A$ be an integrally closed $\mathbb{Z}$-graded ring. Assume that the subring $A_0$ contains an algebraically closed field $k$ and that $A_0^d=k^*$. Let $d$ be the smallest positive integer among the set of degrees of units in $A$. Then the number of the isomorphic classes of $A_0$-algebra as $B$ such that $A[X, X^{-1}]=B[Y, Y^{-1}]$ equals to $\Phi(d)$, where $\Phi$ is the Euler function.

**Proof.** Let $i$ be an integer such as $1 \leq i < d$ and $(i, d) = 1$. Since $(i, d) = 1$, $ij+dh = 1$ for some integers $j$ and $h$. Moreover we may assume $h \geq 0$. Fix a unit $c$ of degree $d$. Let $x$ be one of the $d$-th roots of $c$. Then we have that $A^d[x]=A_0[x, x^{-1}]$ for a subring $A^d$ containing $A_0$ by (1.4). Let $\sigma$ be a generator of the cyclic Galois group of the extension $A^d[A]/A$. Then $\sigma(x) = \lambda x$, where $\lambda$ is a primitive $d$-th root of “unity”. Since $A_0$ is algebraic over $A_0$ and algebraically closed in $A_0[x, x^{-1}]$, we obtain $\sigma(A_0) = A_0$. Let $X$ be a variable over $A$ and let $y=x^{-1}X$ and $Y=e^{\theta}x^l$. Then we have that $A_0'[x, x^{-1}] [X, X^{-1}] = A_0'[y, y^{-1}] [Y, Y^{-1}]$. Define $B_i = A_0'[y, y^{-1}]$ and let $\sigma$ be an isomorphism of $A_0'[x, x^{-1}] [X, X^{-1}]$ defined by $\sigma(X) = X$ and $\sigma = \sigma$ on $A_0'[x, x^{-1}]$. Since $Y=e^{\theta}X^l$, $\sigma(Y) = Y$, therefore we obtain that $A[X, X^{-1}] = B_i[Y, Y^{-1}]$. We can easily see that $B_i$ is a $X$-graded ring and $(B_i)_0 = A_0$. Especially we have $B_i \cong A$.

Let $i$ and $j$ be integers such as $1 \leq i_1 < i_2 < d$ and $(i_1, d) = (i_2, d) = 1$. Let $B' = A_0'[y, y^{-1}]$ and $B'' = A_0'[x, z^{-1}]$ where $\sigma(y) = \lambda^{-i_1}y$ and $\sigma(z) = \lambda^{-i_2}z$, i.e., $B' = B_{i_1}$ and $B'' = B_{i_2}$. We shall show that $B'$ and $B''$ are not isomorphic. Assume that there exists an $A_0$-isomorphism $\psi$ of $B'$ to $B''$. Let $a$ be a unit in $B'$ of non-zero degree, say degree $a = n, n \neq 0$. Let $b$ be a homogeneous element of $B'$ and degree $b = t$. Then we have $b^n = ra^t$ for an element $r$ in the coefficient ring $A_0$, hence $\psi(b^n) = \psi(b)^n = r\psi(a^t)$. Since $r$ and $\psi(a^t)$ are homogeneous, $\psi(b)$ is also homogeneous by (1.1), therefore $\psi$ is an isomorphism as graded rings.

Let $c$ be a homogeneous element in $B'$ of degree one. Then $c = s_1y$ for an element $s_1$ in $A_0'. \psi(c) = c$ and $\sigma(y) = \lambda^{-i_1}y$, we have $\sigma(s_1) = \lambda^{i_1}s_1$, hence $s_1$ is in $B'$. Since $\psi(s_1y) = s_1z$ for an element $s_2$ in $A_0'$. Since $\sigma(s_2z) = s_2z$ and $\sigma(z) = \lambda^{i_2}z$, we have $\sigma(s_2) = \lambda^{i_2}s_2$, hence $s_2$ is in $B''$. By the relations, $s_1^l \psi(y^d) = \psi((s_1y)^d) = \psi(s_1y)^d = s_1^d s_2^d$, we obtain $s_2^d = \psi(y^d) s^{-d} s_1^d$. Since $\psi(y^d) s^{-d}$ is an invertible element in $B''$ and degree zero, we have $\zeta = \psi(y^d) s^{-d} \in A_0^d = k^*$, therefore we have $s_2 = \eta\varsigma$, for some $\eta \in k, \eta^d = \zeta$. Hence $\sigma(s_2) = \lambda^{i_2}s_2$, but it contradicts the fact that $\sigma(s_2) = \lambda^{i_1}s_2$ and $\lambda$ is a primitive $d$-th root of “unity”. Therefore $B' \cong B''$.

Finally we shall show that if $A[X, X^{-1}] = B[Y, Y^{-1}]$ then $B$ is isomorphic to $B_i$ for some $i$ satisfying $0 < i < d$ and $(i, d) = 1$. The invertible element $u$
in (2.0) is homogeneous. Let \( n \) be the degree of \( u \). If \( n=0 \), then \( A \) is isomorphic to \( B \) by (2.1), hence \( B \cong B_1 \). Assume \( n \neq 0 \). Let \( c \) be a non-zero homogeneous element of degree 1 and put \( \eta = c^* u^{-1} \). Then \( \eta \) is an element of \( A_0 \).

In the graded ring \( B[Y, Y^{-1}] \) the elements \( u \) and \( \eta \) are homogeneous, hence \( c \) is also homogeneous, thus we denote \( c = b Y^j \) for some element \( b \) in \( B \) and some integer \( j \). Then we obtain that \( c^* = \eta u^{-1} Y^{1-j} \). On the other hand we have \( c^* = \eta u = \eta v^{-j} Y^{1-j} \) by (2.0). Therefore we have \( 1 - \eta v = \eta j \).

By the minimality of \( d \) we obtain \( n = ld \) for some integer \( l \) and \( u = \xi e \), \( \xi \in A_0^* = k^* \). Since the field \( k \) is algebraically closed, we may assume \( \xi = 1 \), then the \( d \)-th root \( x \) of \( e \) is an \( n \)-th root of \( u \). Since the element \( \lambda \) is a primitive \( d \)-th root of "unity", there exists the unique integer \( i \) such that \( \lambda^{-i} = \lambda^{-i} \), \( 0 < i < d \), then \( i, d = 1 \) since \( (f, d) = 1 \). Let \( y' = x^{-j} X^j \) and \( B' = (A_0[y', y'^{-1}]) \). Then \( \sigma(y') = \lambda^{-i} y' = \lambda^{-i} y' \), hence \( B' = B \). We can easily show that \( x = y'^{-i} Y^i \), therefore we obtain \( A_0[x, x^{-1}] X, X^{-1}] = A_0[y', y'^{-1}] Y, Y^{-1}] \). Since \( \sigma(X) = X \) and \( \sigma(Y) = Y \), we have \( A[X, X^{-1}] = B_1[Y, Y^{-1}] \), hence \( B[Y, Y^{-1}] = B_1[Y, Y^{-1}] \). Thus we have \( B \cong B_1 \).

References


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