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ON THE COEFFICIENT RING OF A TORUS EXTENSION

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Introduction. S. Abhyankar, W. Heinzer and P. Eakin treated the following problem in [1]; if A[X]=B[Y], when is A isomorphic or identical to B? Replacing the polynomial ring by the torus extension we shall take up the following problem; if $A[X, X^{-1}]=B[Y, Y^{-1}]$, when is A isomorphic or identical to B? We say that A is torus invariant (resp. strongly torus invariant) whenever $A[X, X^{-1}]=B[Y, Y^{-1}]$ implies $A\cong B$ (resp. A=B). The rolles played by polynomial rings in [1] are played by the graded rings in our theory. A graded ring $A=\sum A_i, i\in \mathbb{Z}$, with the property that $A_i \neq 0$ for each $i\in \mathbb{Z}$, will be called a \mathbb{Z} -graded ring. Main results are the followings.

An affine domain A of dimension one over a field k is always torus invariant. Moreover A is not strongly torus invariant if and only if A has a graded ring structure. An affine domain of dimension two is not always torus invariant. We shall construct an affine domain of dimension two which is not torus invariant. Let A be an affine domain over k of dimension two. Assume that the field k contains all roots of "unity" and is of characteristic zero. If A is not torus invariant, then A is a Z-graded ring such that there exist invertible elements of non-zero degree.

In Section 1 we study elementary properties of graded rings. Especially we are interested in \mathbb{Z} -graded rings with invertible elements of non-zero degree. In Section 2 we discuss some conditions for A to be torus invariant. In Section 3 we give several sufficient conditions for an integral domain to be strongly torus invariant. Some relevant results will be found in S. Iitaka and T. Fujita [2]. Section 4 is devoted to the proof of the main results mentioned above. In Section 5 we fix an integral domain D and we treat only D-algebras and D-isomorphisms there. We shall prove the following two results. When A is a D-algebra of $tr. deg_D A=1$ and A is not D-torus invariant, A is a \mathbb{Z} -graded ring such that D is contained in A_0 . If A is a \mathbb{Z} -graded ring such as $D=A_0$, then the number of elements of the set of $\{D$ -isomorphic classes of D-algebras B such that $A[X, X^{-1}]=B[Y, Y^{-1}]\}$ is $\Phi(d)$, where d is the smallest positive integer among the degrees of units in A and Φ is the Euler function.

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1. Some properties of graded rings

Let R be commutative ring with indentity. The ring R is said to be a graded ring if R is a graded module, $R = \sum R_i$, and $R_n R_m \subseteq R_{n+m}$.

Lemma 1.1. Let R be a graded domain. Then we have the following.

(1) The unity element of R is homogeneous.

(2) If a is homogeneous and a=bc, then b and c are both homogeneous. In particular every invertible element is homogeneous.

(3) If R contains a field k, then k is a subring of R_0 .

Proof. (1) follows immediately from the relation $1^2=1$. The proof of (2) is easy and will be omitted. To prove (3) we can assume k is different from F_2 by (1). Let a be an element of k different from 1. Then 1-a is homogeneous from (2). The unity 1 is homogeneous of degree 0 by (1). Hence a should be homogeneous of degree 0.

We call a graded ring $R = \sum R_i$ to be a Z-graded ring if $R_i \neq 0$, for some $i \in \mathbb{Z}^+$ and \mathbb{Z}^- .

Proposition 1.2. Let R be a Z-graded domain. Let $S = \{i \in Z; R_i \neq 0\}$. Then S = nZ for a certain integer n.

Proof. Since R is a domain, S is a semi-group. Hence (1.2) is immediately seen by the following lemma.

Lemma 1.3. Let $S \subseteq \mathbb{Z}$ be a semi-group. If $S \cap \mathbb{Z}^+ \neq 0$ and $S \cap \mathbb{Z}^- \neq 0$, then S is a subgroup of \mathbb{Z} .

If R is a Z-graded domain, then we may assume $R_i \neq 0$ for any $i \in \mathbb{Z}$.

Proposition 1.4. Let R be a graded ring. If there is an invertible element x in R_1 , then $R = R_0[x, x^{-1}]$.

Proof. For any $r \in R_n$, $r=r(x^{-1}x)^n=rx^{-n}x^n$ and rx^{-n} is in R_0 , therefore $r \in R_0x^n$. Hence $R=R_0[x, x^{-1}]$.

Corollary. Let R be a Z-graded domain. If R_0 is a field, so $R=R_0[x, x^{-1}]$ for every $x \in R_1, x \neq 0$.

Proof. Choose non-zero elements $x \in R_1$, and $y \in R_1$. Since R_0 is a field, $0 \neq xy$ is invertible, therefore x and y are units in R, hence $R = R_0[x, x^{-1}]$.

2. Torus invariant rings

A ring A is said to be torus invariant provided that A has the following property:

If there exist a ring B, a variable Y over B, and a variable X over A such that $A[X, X^{-1}]$ is isomorphic to $B[Y, Y^{-1}]$,

$$\Phi\colon A[X, X^{-1}]\to B[Y, Y^{-1}],$$

then A is always isomorphic to B.

Especially if we have always $\Phi(A)=B$ in such case, we say that the ring A is strongly torus invariant.

To show A is torus invariant (resp. strongly torus invariant) it suffices to prove that A is isomorphic to B (resp. A=B) under the assumption: $A[X, X^{-1}]=B[Y, Y^{-1}]$.

(2.0) We begin with some elementary observations. Assume that

(1)
$$R = A[X, X^{-1}] = B[Y, Y^{-1}].$$

Then X and Y are units of R. It follows from (1.1) that we have

(2)
$$X = vY^f$$
 and $Y = uX^{f'}$, $v \in B$ and $u \in A$,

or equivalently

(3)
$$v = u^{-f} X^{1-ff'}$$
 and $u = v^{-f'} Y^{1-ff'}$.

In the rest of our paper we shall use the letters u and v to denote the elements of A and B respectively satisfying the relations (2) and (3) whenever we encounter the situation (1).

(2.1) The element u is in B if and only if ff'=1. In this case we have $A[X, X^{-1}]=B[X, X^{-1}]$, thus we have $A \simeq B$.

Proof is easy and is omitted.

Proposition 2.2. Let k be a field and A be a k-algebra. If A^* (the set of all invertible elements in A)= k^* , then the ring A is torus invariant.

Proof. Let $R = A[X, X^{-1}] = B[Y, Y^{-1}]$. By (1.1) the field k is contained in B. Since $A^* = k^*$, the unit element u of A is in k, hence in B, It follows from (2.1) that A is torus invariant.

Proposition 2.3. Let $A = A_0[t_1, t_2, \dots, t_n, (t_1t_2\cdots t_n)^{-1}]$ where t_i 's are independent variables over k-algebra A_0 and $A_0^* = k^*$, then A is torus invariant.

Proof. Let $R=A[X, X^{-1}]=B[Y, Y^{-1}]$. Then by the lemma (1.1) $Y=uX^{f'}$ and $X=vY^{f}$. Since u is invertible in $A=A_0[t_1, t_2, \cdots, t_n, (t_1\cdots t_n)^{-1}]$, $Y=rt_1^{e_1}\cdots t_n^{e_n}, r\in A_0^*=\mathbf{k}^*$. We may assume that r=1, so $Y=t_1^{e_1}\cdots t_n^{e_n}X^{f'}$.

On the other hand as t_i is invertible in $R=B[Y, Y^{-1}]$, $t_i=b_iY^{f_i}$, $b_i\in B^*$. Then we have that

$$ff' + \sum e_i f_i = 1$$
.

Therefore the following natural homomorphism is surjective.

$$j: \mathbf{Z}^{(n+1)} = \mathbf{Z} \oplus \cdots \oplus \mathbf{Z} \to \mathbf{Z}$$
$$j(i_0, i_1, \cdots, i_n) = i_0 f' + \sum i_j e_j.$$

Since Z is P.I.D., we can construct a basis of $Z^{(n+1)}$ containing this vector (f', e_1, \dots, e_n) . Put this basis

$$e_0 = (f', e_1, \dots, e_n)$$

$$\vdots$$

$$e_i = (f_i, f_{i1}, \dots, f_{in})$$

and put $u_i = t_1^{f_{i1}} \cdots t_n^{f_{in}} X^{f_i}$.

$$R = A_0[u_1, \cdots, u_n, (u_1 \cdots u_n)^{-1}] [Y, Y^{-1}] = A[X, X^{-1}] = B[Y, Y^{-1}].$$

Therefore A is isomorphic to B. Hence A is torus invariant.

(2.4) Let $R = A[X, X^{-1}] = B[Y, Y^{-1}]$. An ideal *I* of *R* is said to be vertical relative to *A* if there exists an ideal *J* of *A* such that JR = I. If *J* is an ideal of *A* such that *JR* is vertical relative to *B*, then we will simply say that *J* is vertical relative to *B*. If *A* is a *k*-affine domain, the prime ideals defined by the singular locus of Spec *A* are vertical relative to *B*.

Proposition 2.5. Let $R = A[X, X^{-1}] = B[Y, Y^{-1}]$. If there exists a maximal ideal of A which is vertical relative to B, then $A[X, X^{-1}] = B[X, X^{-1}]$. In particular A and B are isomorphic.

Proof. Let *m* be a maximal ideal of *A* which is vertical elative to *B*. Then there exists an ideal *n* of *B* such that mR = nR. Therefore $R/mR = A/m[X, X^{-1}] = R/nR = B/n[Y, Y^{-1}]$, where $X = vY^f$ and $Y = uX^{f'}$. Since *m* is a maximal ideal, A/m is a field. Hence *u* is in B/n by (1.1). Therefore we obtain $f = \pm 1$ by (2.1). Thus *A* is isomorphic to *B*.

Corollary 2.6. Let A be a k-affine domain with isolated singular points, then A is torus invariant.

3. Strongly torus invariant rings

In this section we investigate strongly torus invariant rings.

Proposition 3.1. Let $A[X, X^{-1}] = B[Y, Y^{-1}]$. If $Q(A) \subseteq Q(B)$, then A = B, where Q(R) is the total quotient field of R.

Proof. Let x be an element of A, then there exist two elements b and b' of B such as x=b/b'. Hence b=b'x. In the graded ring $B[Y, Y^{-1}]$ the elements b and b' are homogeneous of degree zero, thus x is also degree zero. Hence we have $A \subseteq B$. Let b be an element of B. Then $b=\sum a_j X^j$, $a_j \in A$. By (2) of (2.0) we have that $b=\sum a_j v^j Y^{jf}$. If f=0, then $X \in B$. Thus $A[X, X^{-1}] \subseteq B$,

it's a contradiction, hence $f \neq 0$. Since $a_j v^j \in B$ and Y is a variable over B, $b=a_0 \in A$. Thus A=B.

Corollary 3.2. Let \overline{A} denote the integral closure of A. If \overline{A} is strongly torus invariant, then A is also so.

Proof. It is easily seen that if $A[X, X^{-1}] = B[Y, Y^{-1}]$ then $\overline{A}[X, X^{-1}] = \overline{B}[Y, Y^{-1}]$. Since \overline{A} is strongly torus invariant, $\overline{A} = \overline{B}$. Hence Q(A) = Q(B), and we have that A = B.

Proposition 3.3. Let A be a domain with $J(A) \neq 0$, where J(D) is the Jacobson radical of a ring D. Then A is strongly torus invariant.

Proof. Let *a* be a non-zero element of J(A). Then 1+a is unit, so in the graded ring $B[Y, Y^{-1}]$, 1+a is homogeneous. Since the "unity 1" is a homogeneous element of degree 0, the element *a* is also so. Thus the element *a* is contained in *B*.

Let x be any element of A. Since xa is contained in J(A), xa is in B. Hence A is contained in Q(B). By (3.1), we have that A=B.

Corollary 3.4. If A is a local domain, then A is strongly torus invariant.

Proposition 3.5. Let A be an affine ring over a field k and let $A[X, X^{-1}] = B[Y, Y^{-1}]$. Then A=B if and only if every maximal ideals of A is vertical relative to B.

Proof. It suffices to prove the "if" part of the (3.5). By (3.3) we may assume that J(A)=0. Let x be an element of B and let $x=\sum_{j=s}^{t}a_{j}X^{j}$, where $s < t, a_{j} \in A$ and $a_{t} \neq 0$ and $a_{s} \neq 0$. For any maximal ideal m of A there exists a maximal ideal n of B such as mR=nR, where $R=A[X, X^{-1}]$. Let \bar{x} denote the residue class of x in B/n. Then \bar{x} is algebraic over the coefficient field k, hence there exist elements $\lambda_{0}, \lambda_{1}, \dots \lambda_{n-1}$ in k, such that $f(x)=x^{n}+\lambda_{n-1}x^{n-1}+\dots$ $+\lambda_{0}\in nR=mR$. If $t\neq 0$, then the highest degree term of f(x) with respect to X is $a_{i}^{n}x^{ni}\in mR$, thus a_{i} is contained in m for every maximal ideal in A. Since $J(A)=0, a_{t}=0$. It's a contradiction. Therefore t=0. By the same way, we have that s=0, hence x is in A. Thus A=B.

We denote the subring generated by all the units of A by A_{μ} .

Proposition 3.6. Let A be a k-affine domain with an isolated singular point. If A is algebraic over A_u , then A is strongly torus invariant.

Proof. Let $A[X, X^{-1}] = B[Y, Y^{-1}]$ and let *m* be the maximal ideal defined by the isolated singular point. Then there exists a maximal ideal *n* of *B* such as mR = nR. Let *a* be a unit element of *A*. In the graded ring $B[Y, Y^{-1}]$, the

element *a* is also invertible, so $a=bY^{j}$, for some invertible element *b* in *B* and a certain integer *j*. Since A/m is algebraic over *k*, there exist elements $\lambda_{0}, \lambda_{1}, \dots, \lambda_{n} \in \mathbf{k}$ such that $\lambda_{n}a^{n}+\dots+\lambda_{1}a+\lambda_{0}\in mR=nR$. If $j=0, \lambda_{n}b^{n}$ is in *n*, hence *b* is not invertible, it's a contradiction. Thus we have that $A_{\mu}\subseteq B$. By the following lemma our proof is over.

Lemma 3.7. Let $A[X, X^{-1}] = B[Y, Y^{-1}]$. If A is algebraic over $A \cap B$, then A = B.

Proof. Since A is algebraic over $A \cap B$, A is also algebraic over B, but B is algebraically closed in $B[Y, Y^{-1}]$, therefore A is contained in B. Thus we have that A=B.

Let A be an integral domain containing a field k. We denote the set of all automorphisms of A over k by $\operatorname{Aut}_k(A)$.

Proposition 3.8. Let A be an integral domain containing an infinite field k. If $Aut_k(A)$ is a finite set, then A is strongly torus invariant.

Proof. Let $R = A[X, X^{-1}] = B[Y, Y^{-1}]$. Let $\Phi_{\lambda}, \lambda \in k^*$, be an automorphism of R defined by $\Phi_{\lambda}(Y) = \lambda Y$ and $\Phi_{\lambda}(b) = b$ for $b \in B$. Following the notation of (2.0) we have $X = vY^f$, thus $\Phi_{\lambda}(X) = \lambda^f X$, therefore $R = \Phi_{\lambda}(A) [X, X^{-1}]$. Let p be the projection $A[X, X^{-1}] \to A$ defined by p(X) = 1 and i be the canonical injection $A \hookrightarrow A[X, X^{-1}]$. Define $\sigma_{\lambda} = q \circ \Phi_{\lambda} \circ i$. Then σ_{λ} is an endomorphism of A. We shall show that σ_{λ} is surjective. Let x be an element of A. Since $R = \Phi_{\lambda}(A) [X, X^{-1}]$, there exist elements a_j 's of A such as $x = \sum \Phi_{\lambda}(a_j)X^j$. Hence $x = p(x) = \sum p\Phi_{\lambda}(a_j)$. Let $x' = \sum a_j \in A$, then $\sigma_{\lambda}(x') = \sum p\Phi_{\lambda}(a_j) = x$. Thus σ_{λ} is surjective. Next we shall show that σ_{λ} is injective. Since $\Phi_{\lambda}^{-1}((X-1)R \cap \Phi_{\lambda}(A)) = \Phi_{\lambda}^{-1}(X-1)R \cap A = (\lambda^{-f}X-1)R \cap A = 0$, we have $(X-1)R \cap \Phi(A) = 0$, therefore σ_{λ} is injective. Hence σ_{λ} is an automorphism of A.

We shall prove that the set $\{\sigma_{\lambda} | \lambda \in \mathbf{k}^*\}$ is infinite when $A \neq B$. Since $u = v^{-f} Y^{1-ff'}$, $\sigma_{\lambda}(u) = \lambda^{1-ff'}u$. Therefore our assertion is proved when $1-ff' \neq 0$. Suppose ff'=1. Then we may assume that $R=A[X, X^{-1}]=B[X, X^{-1}]$. If $A \subseteq B$, then A=B, so there exists an element x of A not contained in B, say $x = \sum_{j=s}^{t} b_j X^j$, t > s. Since ker p = (X-1)R and $(X-1)R \cap B = 0$, $p(b_j) \neq 0$ for $b_j \neq 0$. Since $\sigma_{\lambda}(x) = \sum p(b_j)\lambda^j$ and $p(b_j) \neq 0$ for some $j \neq 0$, the set $\{\sigma_{\lambda}; \lambda \in \mathbf{k}^*\}$ is infinite.

Next we shall give two cases of rings which are not strongly torus invariant. If A has a non-trivial locally finite iterative higher derivation $\psi: A \rightarrow A[T]$, then A[T]=B[T], where $B=\psi(A)$ and $A \neq B$, as is proved in [4]. Hence we have that $A[T, T^{-1}]=B[T, T^{-1}]$ and $A \neq B$. If A is a graded ring, then A is not strongly torus invariant. Indeed, let X be a variable over A and let

 $B_i = \{a_i X^i; a_i \in A_i\}$. Then B_i is an A_0 -module contained in $A[X, X^{-1}]$. Let $B = \sum B_i$. Then B is a graded ring and we easily see that $A[X, X^{-1}] = B[X, X^{-1}]$. We shall show that X is a variable over B. Assume that there exist elements b_0, b_1, \dots, b_n in B such that $b_n \neq 0$ and $b_n X^n + \dots + b_1 X + b_0 = 0$. By the definition of B we denote $b_i = \sum a_{ij} X^j$, $a_{ij} \in A_j$. In the graded ring $A[X, X^{-1}]$ the homogeneous term of degree t of this equation is that

$$(a_{n,t-n} + a_{n-1,t-n+1} + \dots + a_{0,t})X^{t} = 0.$$

Since A is a graded ring and a_{ij} is a homogeneous element of degree j, we obtain $a_{ij}=0$ for all index i and j, hence X is a variable over B.

By [4] we have that a k-algebra A has a non-trivial locally finite iterative higher derivation if and only if $\operatorname{Aut}_{k}(A)$ has a subgroup isomorphic to G_{a} = Spec k[T]. We easily see that A is a non-trivial graded ring if and only if $\operatorname{Aut}_{k}(A)$ has a subgroup isomorphic to $G_{m}=Spec(k[T, T^{-1}])$.

Proposition 3.9. A k-algebra A is not strongly torus invariant, if $Aut_k(A)$ has a subgroup isomorphic to G_a or G_m .

Assume that $\operatorname{Aut}_k(A)$ is an infinite group. If $\operatorname{Aut}_k(A)$ has an algebraic group structure, then there exists the following exact sequence;

$$0 \to T \to \operatorname{Aut}_{k}(A)_{0} \to \theta \to 0$$

where $\operatorname{Aut}_k(A)_0$ is the connected component containing the identity I_A , and T is a maximal torus subgroup of $\operatorname{Aut}_k(A)_0$ and θ is an abelian variety. Let P be an arbitrary closed point of Spec(A). If T=0, then there exists a regular map

$$\Phi: \operatorname{Aut}_{k}(A)_{0} \to \operatorname{Spec}(A)$$
$$\sigma \to \sigma(P).$$

Since $Im(\Phi)$ is a projective variety contained in the affine variety Spec(A), the set $Im(\Phi)$ consists of one point, it contradicts dim $Aut_k(A)_0 > 0$. Hence we have that $T \neq 0$. Since $T \supseteq G_a$ or G_m , we have the following result:

Proposition 3.10. If $Aut_k(A)$ is not a finite set and has an algebraic group structure, then A is not strongly torus invariant.

4. Affine domains of dimension ≤ 2

Let k be a field of characteristic zero which contains all roots of "unity". In this section let A be an affine domain over k. We shall see that if dim A=1, then A is always torus invariant. Moreover A is not strongly torus invariant if and only if $\operatorname{Aut}_k(A) \supseteq G_m$. Let dim $A \ge 2$. Then A is not always torus invariant. But if an integrally closed domain A is not a Z-graded ring, then A is torus invariant.

For the proof we need a lemma.

Lemma 4.1. Let K be a finite separable algebraic field extension of a field k. If A is a one-dimensional affine normal ring such that $k \subset A \subseteq K[X, X^{-1}]$, then A is a polynomial ring or a torus ring over k' where k' is the algebraic closure of k in A.

Proof. We may assume that $\mathbf{k} = \mathbf{k}'$. Following the similar device to the proof of (2.9) in [1, p 322], we have $Q(A) = \mathbf{k}(\theta)$ for some element θ of A.

Since $\mathbf{k}[\theta] \subseteq A \subset \mathbf{k}(\theta)$, $A = \mathbf{k}[\theta]$ or $A = \mathbf{k}\left[\theta, \frac{1}{f(\theta)}\right]$ for some polynomial $f(\theta) \in \mathbf{k}[\theta]$. Let $A = \mathbf{k}\left[\theta, \frac{1}{f(\theta)}\right]$. Then we may assume that $f(\theta)$ has no multiple factors. The element $f(\theta)$ is invertible in A, so is also invertible in $K[X, X^{-1}]$. Thus we have $f(\theta) = \beta X^r$, $\beta \in K$, $\theta \in K[X, X^{-1}]$. We may assume that $r \ge 0$, if necessary, by replacing X with X^{-1} . Then we easily see that $\theta \in K[X]$. The uniqueness of the irreducible decomposition in a polynomial ring implies that $\deg_{\theta} f(\theta) = 1$, since the polynomial $f(\theta)$ has not multiple factors and $f(\theta) = \beta X^r$. Hence we may assume that $f(\theta) = \theta$ and we obtain $A = \mathbf{k} \left[\theta, \frac{1}{\theta}\right]$.

Let A be an integral domain. If A is contained in $K[X, X^{-1}]$, then \overline{A} is a polynomial ring or a torus ring over k'.

Proposition 4.2. Let A be a one-dimensional affine domain over a field k of characteristics zero. Then we obtain that

(1) A is torus invariant,

(2) A is not strongly torus invariant if and only if $Aut_k(A)$ has a subgroup isomorphic to G_m . If A is not strongly torus invariant and A is integrally closed, then A is a polynomial ring or a torus ring over the algebraic closure of k in A.

Proof. At first we shall prove (2). The sufficiency follows from (3.9). Let $R = A[X, X^{-1}] = B[Y, Y^{-1}]$ in which $A \neq B$. If $mR \cap A \neq 0$ for any maximal ideal *m* of *B*, then *m* is vertical relative to *A*, and we have A = B by (3.5). Hence there exists a maximal ideal *m* such as $mR \cap A = 0$. Since *ch* k = 0, B/m = K is a finite separable algebraic field over k. The residue mapping of *R* to R/mR yields (up to isomorphism) $k \subset A \subseteq K[Y, Y^{-1}]$ where *Y* is algebraicly independent over *K*. Therefore *A* is a polynomial ring or *a* torus ring by the lemma (4.1). Thus the automorphism group $\operatorname{Aut}_k A$ contains a subgroup isomorphic to G_m .

Assume that A is not integrally closed. Then prime divisors in A of the conductor $t(\overline{A}|A)$ are vertical relative to B. Hence we may assume X=Y by

(2.5). The above lemma (4.1) implies that $\bar{A} = \mathbf{k}'[t, t^{-1}]$ or $\bar{A} = \mathbf{k}'[t]$ where \mathbf{k}' is the algebraic closure of \mathbf{k} in \bar{A} .

Firstly let $\overline{A} = \mathbf{k}'[t]$. Since $\overline{A} \simeq \overline{B}$, there exists an element s in \overline{B} such as $\overline{B} = \mathbf{k}'[s]$. Since $\overline{R} = \overline{A}[X, X^{-1}] = \overline{B}[X, X^{-1}]$, we have $\mathbf{k}'[X, X^{-1}][t] = \mathbf{k}'[X, X^{-1}]$ [s], hence we easily see that $t = f_1(X)s + f(X)$ and $s = g_1(X)t + g(X)$ where $f_1(X)$ $g_1(X) = 1$ and f(X), $g(X) \in \mathbf{k}'[X, X^{-1}]$. We may assume that $t = X^n s + f(X)$ and $s = X^{-n}t + g(X)$. Let \overline{n} be a prime divisor in \overline{A} of the conductor $t(\overline{A}/A)$.

Then there exists a maximal ideal \overline{m} of \overline{B} such as $\overline{n}\overline{R}=\overline{m}\overline{R}$. Since $\overline{A}/\overline{n}$ is algebraic over k, there exist elements $\lambda_0, \lambda_1, \dots, \lambda_{d-1} \in k$ such that $t^{d-1}+\lambda_{d-1}$ $t^{d-1}+\dots+\lambda_0 \in \overline{m}\overline{R}=\overline{n}\overline{R}$. Hence we have that $(X^ns+f(X))^d+\lambda_{d-1}(X^ns+f(X))^{d-1}$ $+\dots+\lambda_0 \in \overline{n}\overline{R}$. The constant term of this polynomial with respect to s is the following;

$$f(X)^d + \lambda_{d-1}f(X)^{d-1} + \cdots + \lambda_0 \in \bar{n}k'[s] [X, X^{-1}].$$

Therefore $f(X)=f \in \mathbf{k}'$. Hence we may assume that $t=X^n s$. We shall show that A is a graded ring. Let a be an element of A. Since a is contained in $\overline{A}=\mathbf{k}'[t]$ and $t=X^n s$, we have that $a=\sum \lambda_j t^j=\sum \lambda_j s^j X^{jn}, \lambda_j s^j \in \overline{B}$. On the other hand, as the element a is contained in $B[X, X^{-1}], a=\sum b_i X^i, b_i \in B$. Comparing the coefficient of the each term in the following; $\sum \lambda_j s^j X^{jn} = \sum b_i X^i$, we have $b_i=\lambda_j s^j$ (i=jn) and $b_i=0$ $(i \in n\mathbb{Z})$. If $b_i \neq 0$, then $b_i X^i = \lambda_j s^j X^{jn} =$ $\lambda_j t^j \in B[X, X^{-1}] \cap \overline{A} = A[X, X^{-1}] \cap \overline{A} = A$. Therefore A has a graded ring structure.

Secondary let $\overline{A} = \mathbf{k}'[t, t^{-1}]$. Then $\overline{B} = \mathbf{k}'[s, s^{-1}]$. Since t and s are invertible in \overline{R} , we may assume that $t = s^i X^n$ and $s = t^j X^m$, then $t = (t^j X^m)^i X^n = t^{ij} X^{im+n}$, therefore ij=1. Hence we may assume $t = sX^n$. By the same method as in the case $\overline{A} = \mathbf{k}'[t]$ we have that A is a graded ring.

Proof of (1). If A is not integrally closed, then the prime divisors of the conductor $t(\overline{A}|A)$ are vertical relative to B. Since non-zero prime ideals of A are maximal, the ring A is isomorphic to B by (2.5). If A is integrally closed and A is neither a polynomial ring nor a torus ring, then A is strongly torus invariant, hence A is torus invariant. If A is either a polynomial ring or a torus ring, A is torus invariant by (2.2) and (2.3).

Next we shall consider the case; the coefficient field k has all roots of "unity" and its characteristic is zero. Then we prove the following:

Theorem 4.3. Let A be an integrally closed k-affine domain of dimension two, where the field k has all roots of "unity" and ch k=0. If A is not torus invariant, then A is a Z-graded ring which contains units of non-zero degree.

Proof. Assume that A is not torus invariant. Then there exist a k-algebra B and independent variables X, Y such that A is not isomorphic to B and $R = A[X, X^{-1}] = B[Y, Y^{-1}]$. By (2.0) and (2,1) we obtain $ff' \neq 1$. We shall show

that it follows from $ff' \neq 1$ that A is a Z-graded ring. We may only consider the case 1-ff'>0. Let x be a (1-ff')-th root of u and let $y=x^{-f}/X$. Then $y^{1-ff'}=v$. Since $(y^{-f'}y)^{1-ff'}=u$, $x=\lambda y^{-f'}Y$ for some (1-ff')-th root λ of "unity". From the relations; $y=x^{-f}X$ and $Y=uX^{f'}$, we have $\lambda=1$.

Since $y = x^{-f}X$ and $x = y^{-f'}Y$ are invertible, we have $A[x][X, X^{-1}] = B[y][Y, Y^{-1}] = A[x][y, y^{-1}] = B[y][x, x^{-1}]$. Define a surjective homomorphism $j: A[x][y, y^{-1}] \rightarrow A[x]$ by j(y)=1. Let $A_0=j(B[y])\subseteq A[x]$. We shall show that $A[x]=A_0[x, x^{-1}]$. Let a be an element of A. Then $a=\sum b_i x^i$, $b_i \in B$. Since j(a)=a and j(x)=x, we have that $a=\sum j(b_i)x^i$, $j(b_j)\in A_0$. Thus $A[x]=A_0[x, x^{-1}]$ and x is algebraically independent over A_0 . By the same way $B[y]=B_0[y, y^{-1}]$.

Since the every (1-ff')-th roots of "unity" is contained in k and $ch \ k=0$ and A is normal, the extension A[x]/A is a Galois extension with a cyclic group $G=\langle \sigma \rangle$ (cf. [3] p 214). Indeed when |G|=n, n|1-ff' and there exists a primitive n-th root λ of "unity" such that $\sigma(x)=\lambda x$ and the invariant subring $(A[x])^{\sigma}=A$ and $A[x]=A+Ax+\dots+Ax^{n-1}$ is a free A-module.

Since the element u is a unit of A and ch(k)=0, the extension A[x]/A is étale. Since A is a normal domain, A[x], hence $A_0[x, x^{-1}]$, is also a normal domain. From this we see that A_0 is always normal.

We shall show that there exists a subring A'_0 in A[x] such that $A[x]=A'_0[x, x^{-1}]$ and $\sigma(A'_0)=A'_0$. If A_0 is strongly torus invariant, then $\sigma(A_0)=A_0$; for $\sigma(A_0)[x, x^{-1}]=A_0[x, x^{-1}]$, therefore A_0 satisfies the conditions. If A_0 is not strongly torus invariant, then $A_0=\mathbf{k}'[t]$ or $=\mathbf{k}'[t, t^{-1}]$ by (4.2). Firstly let $A_0=\mathbf{k}'[t]$. Since $\mathbf{k}'[x, x^{-1}][t]=\mathbf{k}'[x, x^{-1}][\sigma(t)]$, we easily see that $\sigma(t)=\mu x^i t+f(x), \mu \in \mathbf{k}^*$ and $f(x)\in \mathbf{k}'[x, x^{-1}]$. The order of σ is n, i.e. $\sigma^n=$ Identity, so $\sigma^n(t)=t$, the other hand $\sigma^n(t)=\mu^n\lambda^{(1+\dots n-1)i}x^{in}t+g(x), g(x)\in \mathbf{k}'[x, x^{-1}]$, therefore we have that i=0, thus $\sigma(t)=\mu t+f(x)$ and $\mu^n=1$. Let $f(x)=\sum f_i x^i$ and define the set $\Delta=\{j\in Z; \lambda^j \neq \mu\}$. Let $h(x)=\sum_{j\in \Delta}h_j x^j$, where $h_j=f_j(\mu-\lambda^j)^{-1}$, and put s=t+h(x). Then $\sigma(s)=\mu s+\sum_{i\notin \Delta}f_j x^i$, hence $\sigma^n(s)=\mu^n s+n\mu^{n-1}(\sum_{i\notin \Delta}f_i x^i)=s+n\mu^{n-1}$ ($\sum_{i\notin \Delta}f_i x^i$). Since $\sigma^n(s)=s$, we have $\sigma(s)=\mu s$. We set $A'_0=\mathbf{k}'[s]$, then A'_0 satisfies the conditions.

Secondary let $A_0 = \mathbf{k}'[t, t^{-1}]$. Since $\mathbf{k}'[x, x^{-1}][t, t^{-1}] = \mathbf{k}'[x, x^{-1}][\sigma(t), \sigma(t)^{-1}]$, we easily see that $\sigma(t) = \mu x^i t$ or $\sigma(t) = \mu x^i t^{-1}$, $\mu \in \mathbf{k}'^*$.

Case (i); $\sigma(t) = \mu x^{i}t$. Since $\sigma^{n}(t) = \mu^{n} \lambda^{(1+\dots+n-1)i} x^{n}t$ and $\sigma^{n}(t) = t$, we have that $\sigma(t) = \mu t$, so $\sigma(A_{0}) = A_{0}$.

Case (ii); $\sigma(t) = \mu x^{i}t^{-1}$. If *n* is odd, 'say n=2m+1, then $\sigma^{n}(t) = \mu \lambda^{im}x^{i}t^{-1}$, but this is imposible for $\sigma^{n}(t)=t$. Therefore *n* is even, say n=2m. Then $\sigma^{n}(t)=\lambda^{im}t$. Since λ is a primitive n-th root of "unity", the integer *i* is even, say i=2j. Let $s=x^{-j}t$ and $A'_{0}=k'[s, s^{-1}]$. Then A'_{0} satisfies the conditions.

Next we shall show that A has a Z-graded ring structure. Let a be an element of A. Since $a \in A'_0[x, x^{-1}]$, $a = \sum a_i x^i$. Then $a = \sigma(a) = \sum \sigma(a_i) \lambda^i x^i$ and $\sigma(a_i) \in A'_0$. Comparing the coefficient of each lterm in the equality; $\sum a_i x^i = \sum \sigma(a_i) \lambda^i x^i$, we have that $a_i = \sigma(a_i) \lambda^i$, then $\sigma(a_i x^i) = a_i x^i$. Thus $a_i x^i$ is an element of A. Therefore A is a graded ring. Since there exists units of non-zero degree, A has a Z-graded ring structure.

REMARK. The converse of this theorem is false. Indeed we find by (2.3) that the ring $k[T][X, X^{-1}]$ is a Z-graded ring with respect to X which is torus invariant.

EXAMPLE. We shall construct an example of an affine dimension A of dimension two which is not torus invariant.

Let D be an integrally closed domain of dimension one over an algebraically closed field k and $D^* = k^*$. Let a be a non-unit of D and $\alpha^5 = a, \alpha \in D$. Assume that D is noetherian and $D[\alpha]$ is strongly torus invariant. Since an affine domain of dimension one whose totally quotient field has a positive genus is strongly torus invariant by (4.2), this assumption can be satisfied for a suitable choice of D. Let T be a variable over D and $A=D[\alpha T, T^5, T^{-5}]$. Let X be a variable over A and $S=T^2X$ and $Y=T^5X^2$. Let $B=D[\alpha S^3, S^5, S^{-5}]$. Since T= $S^{-2}Y$ and $X=S^{5}Y^{-2}$, we have that $A[X, X^{-1}]=B[Y, Y^{-1}]$. By (1.1) invertible elements in the graded ring A are homogeneous. Since $D^* = k^*$, we obtain $A^* = \{\eta T^{5i}; \eta \in k^* \text{ and } i \in \mathbb{Z}\}$. Hence the quotient A^*/k^* is generated by T⁵. Similary B^*/k^* is generated by S⁵. We shall show that A is not isomorphic to B. We assume that there exists an isomorphism σ of A to B. Since σ is a group-isomorphism of A^* to B^* , we have $\sigma(T^5) = \mu S^5$ or $\sigma(T^5) = \mu S^{-5}$. $\mu \in k^*$. We shall only consider the case: $\sigma(T^5) = \mu S^5$, since the proof of the other case is the similar. Let $\bar{\sigma}$ be an isomorphism of A[T] to B[S] defined by $\bar{\sigma} = \sigma$ on A and $\bar{\sigma}(T) = \zeta S, \zeta^5 = \mu$. Then we have that $D[\alpha][S, S^{-1}] =$ $\bar{\sigma}(D[\alpha])$ [S, S⁻¹], therefore $\bar{\sigma}(D[\alpha]) = D[\alpha]$; for $D[\alpha]$ is strongly torus invariant. Since $\sigma(D) \subseteq D[\alpha] \cap B = D$, we have $\sigma(D) = D$, therefore we easily see that σ is an isomorphism as graded rings. Thus we have $\sigma((\alpha T)D) =$ $(\alpha^2 S)D$, hence $\sigma(a) \in a^2 D$. Since the element a is not a unit, $a^2 D \subseteq aD$, thus $\sigma(a)D \subseteq a^2D \subseteq aD$, so $aD \subseteq \sigma^{-1}(a)D$, hence we have a proper ascending chain $\{\sigma^{-n}(a)D\}$, but it contradicts the netherian assumption of D. Hence A is not torus invariant.

(4.4) Now let $A = \sum A_i$ be an integrally closed \mathbb{Z} -graded domain which contains invertible elements of non-zero degree. Let e be an invertible element of A with the smallest positive degree d. Let a be a unit of A, then a is a homogeneous elements with deg a=jd for some integer j, and there exists an element ξ of A_0^* such as $a=\xi e^j$. Let i be any positive integer and x be one of the ijd-th roots of a, say $x^{ijd}=a$. Since A[x] is a \mathbb{Z} -graded ring with the

invertible elements x of degree one, $A[x]=A'_0[x, x^{-1}]$ by (1.4) where A'_0 contains A_0 . Let f and f' be integers such as ff'+ijd=1 and let X be a variable over A. Put $y=x^{-f}X$ and $Y=aX^{f'}$. Then $x=y^{-f'}Y$ and $X=y^{ijd}Y^{f}$. Therefore $A'_0[x, x^{-1}] [X, X^{-1}]=A'_0[y, y^{-1}] [Y, Y^{-1}]$. Since the every *n*-th roots of "unity" is contained in k and A is integral closed, the extension A[x]/A is a Galois extension with a cyclic group $G=\langle \sigma \rangle$. Indeed |G|=di and there exists a primitive di-th root λ of "unity" such as $\sigma(x)=\lambda x$, and $(A[x])^{\sigma}=A$. Since A'_0 is algebraic over $A_0, \sigma(A'_0)$ is also so, hence $\sigma(A'_0)=A'_0$. Since $\sigma(y)=\lambda^{-f}y$, σ is an automorphism of $A'_0[x, x^{-1}]$, therefore $\sigma(A'_0)=X$ and $\bar{\sigma}$ be an automorphism of $A'_0[x, x^{-1}] [X, X^{-1}]$ defined by $\bar{\sigma}(X)=X$ and $\bar{\sigma}=\sigma$ over $A'_0[x, x^{-1}]$. Since $\bar{\sigma}(Y)=Y$ and $\bar{\sigma}(X)=X$, we obtain $B[Y, Y^{-1}]=A'_0[y, y^{-1}] [Y, Y^{-1}]^{\sigma}=A[X, X^{-1}]$.

Proposition 4.5. Let A be an integrally closed k-affine domain of dimension 2. If $A[X, X^{-1}] = B[Y, Y^{-1}]$ and $ff' \neq 1$, then A has a Z-graded ring structure and B is isomorphic to one of algebras constructed in (4.4).

Proof. The first statement is already mentioned in the proof of (4.3) and we obtained $A'_0[x, x^{-1}] [X, X^{-1}] = A'_0[y, y^{-1}] [Y, Y^{-1}]$ and $\sigma(A'_0) = A'_0$. Let $B' = A'_0[y, y^{-1}]^{\sigma}$. Then B' is one of algebras in (4.4). Since $B'[Y, Y^{-1}] = B[Y, Y^{-1}]$, B is isomorphic to B'.

5. D-torus invariant

Let D be an integral domain contining a field k of characteristic zero and A be a D-algebra. The ring A is called D-torus invariant; if $A[X, X^{-1}] = B[Y, Y^{-1}]$ for a certain D-algebra B and independent variables X and Y, then we have always $A \simeq_{D} B$. Then we have the following result:

Proposition 5.1. Let A be an integrally closed domain over D and tr. $deg_D A=1$. If A is not D-torus invariant, then A is a Z-graded ring containing units of non-zero degree.

Proof. Let $A[X, X^{-1}] = B[Y, Y^{-1}]$, where B is a D-algebra and not Disomorphic to A. By (2.0) and (2.1) we easily see that ff'=1. Then we may assume 1-ff'>0. Let x be a (1-ff')-th root of u and $y=x^{-f}X$. Then we have that $A[x]=A_0[x, x^{-1}]$ and $B[y]=B_0[y, y^{-1}]$ as the proof of (4.3), where A_0 and B_0 are respectively subalgebras of A[x] and B[y] containing D. Let σ be a generator of the cyclic Galois group of the extension A[x]/A. We shall show that $\sigma(A_0)=A_0$. Since tr. deg_D $A_0[x, X^{-1}]=1$, A_0 is algebraic over D, thus $\sigma(A_0)=A_0$. Following the similar devise to the proof of (4.3) we obtain that A is a Z-graded ring, and D is contained in A.

In the following we shall consider the case where A is a Z-graded ring and $A_0=D$. We consider only D-isomorphisms of D-algebras.

Theorem 5.2. Let A be an integrally closed Z-graded ring. Assume that the subring A_0 contains an algebraically closed field \mathbf{k} and that $A_0^* = \mathbf{k}^*$. Let d be the smallest positive integer among the set of degrees of units in A. Then the number of the isomorphic classes of A_0 -algebra as B such that $A[X, X^{-1}] = B[Y, Y^{-1}]$ equals to $\Phi(d)$, where Φ is the Euler function.

Proof. Let *i* be an integer such as $1 \le i < d$ and (i, d) = 1. Since (i, d) = 1, ij+dh=1 for some integers *j* and *h*. Moreover we may assume $h \ge 0$. Fix a unit *e* of degree *d*. Let *x* be one of the *d*-th roots of *e*. Then we have that $A[x]=A_0[x, x^{-1}]$ for a subring A_0 containing A_0 by (1.4). Let σ be a generator of the cyclic Galois group of the extension A[x]/A. Then $\sigma(x)=\lambda x$, where λ is a primitive *d*-th root of "unity". Since A_0' is algebraic over A_0 and algebraically closed in $A_0'[x, x^{-1}]$, we obtain $\sigma(A_0')=A_0'$. Let *X* be a variable over *A* and let $y=x^{-i}X$ and $Y=e^hx^j$. Then we have that $A_0'[x, x^{-1}] [X, X^{-1}]=A_0'[y, y^{-1}]$ [*Y*, Y^{-1}]. Define $B_i=A_0'[y, y^{-1}]^{\sigma}$ and let σ be an isomorphism of $A_0'[x, x^{-1}] [X, X^{-1}] = d_0'[x, x^{-1}] = B_i[Y, Y^{-1}]$. Since $Y=e^hX^j$, $\sigma(Y)=Y$, therefore we obtain that $A[X, X^{-1}]=B_i[Y, Y^{-1}]$. We can easily see that B_i is a *X*-graded ring and $(B_i)_0=A_0$. Especially we have $B_1\cong A$.

Let i_1 and i_2 be integers such as $1 \le i_1 < i_2 < d$ and $(i_1, d) = (i_2, d) = 1$. Let $B' = A'_0[y, y^{-1}]^{\sigma}$ and $B'' = A'_0[z, z^{-1}]^{\sigma}$ where $\sigma(y) = \lambda^{-i_1}y$ and $\sigma(z) = \lambda^{-i_2}$ *i.e.*, $B' = B_{i_1}$ and $B'' = B_{i_2}$. We shall show that B' and B'' are not isomorphic. Assume that there exists an A_0 -isomorphism ψ of B' to B''. Let a be a unit in B' of non-zero degree, say degree a=n, $n \neq 0$. Let b be a homogeneous element of B' and degree b=t. Then we have $b^n = ra^t$ for an element r in the coefficient ring A_0 , hence $\psi(b^n) = \psi(b)^n = r\psi(a^t)$. Since r and $\psi(a^t)$ are homogeneous, $\psi(b)$ is also homogeneous by (1.1), therefore ψ is an isomorphism as graded rings.

Let c be a homogeneous element in B' of degree one. Then $c=s_1y$ for an element s_1 in A'_0 . Since $\sigma(c)=c$ and $\sigma(y)=\lambda^{-i_1}y$, we have $\sigma(s_1)=\lambda^{i_1}s_1$ hence s_1^d is in B'. Since $\psi(s_1y)$ is a homogeneous element of degree one, we obtain $\psi(s_1y)=s_2z$ for an element s_2 in A'_0 . Since $\sigma(s_2z)=s_2z$ and $\sigma(z)=\lambda^{-i_2}z$, we have $\sigma(s_2)=\lambda^{i_2}s_2$, hence s_2^d is in B''. By the relations; $s_1^d\psi(y^d)=\psi((s_1y)^d)=\psi((s_1y)^d)=\psi(s_1y)^d=s_2^dz^d$, we obtain $s_2^d=\psi(y^d)z^{-d}s_1^d$. Since $\psi(y^d)z^{-d}$ is an invertible element in B'' and degree zero, we have $\zeta=\psi(y^d)z^{-d}\in A_0^*=k^*$, therefore we have $s_2=\eta s_1$ for some $\eta \in k$, $\eta^d = \xi$. Hence $\sigma(s_2)=\lambda^{i_1}s_2$, but it contadicts the fact that $\sigma(s_2)=\lambda^{i_2}s_2$ and λ is a primitive d-th root of "unity". Therefore $B'\cong B''$.

Finally we shall show that if $A[X, X^{-1}] = B[Y, Y^{-1}]$ then B is isomorphic to B_i for some *i* satisfying 0 < i < d and (i, d) = 1. The invertible element u

in (2.0) is homogeneous. Let *n* be the degree of *u*. If n=0, then *A* is isomorphic to *B* by (2.1), hence $B \cong B_1$. Assume $n \neq 0$. Let *c* be a non-zero homogeneous element of degree 1 and put $\eta = c^n u^{-1}$. Then η is an element of A_0 . In the graded ring $B[Y, Y^{-1}]$ the elements *u* and η are homogeneous, hence *c* is also homogeneous, thus we denote $c=bY^j$ for some element *b* in *B* and some integer *j*. Then we obtain that $c^n = b^n Y^{nj}$. On the other hand we have $c^n = \eta u = \eta v^{-f'} Y^{1-ff'}$ by (2.0). Therefore we have 1-ff'=nj.

By the minimality of d we obtain n=ld for some integer l and $u=\xi e$, $\xi \in A_0^* = k^*$. Since the field k is algebraically closed, we may assume $\xi = 1$, then the d-th root x of e is an n-th root of u. Since the element λ is a primitive d-th root of "unity", there exists the unique integer i such that $\lambda^{-f} = \lambda^{-i}$, 0 < i < d, then (i, d) = 1 since (f, d) = 1. Let $y' = x^{-f}X^j$ and $B' = (A'_0[y', y'^{-1}])^{\overline{r}}$. Then $\sigma(y') = \lambda^{-f}y' = \lambda^{-i}y'$, hence $B' = B_i$. We can easily show that $x = y'^{-f'}Y^j$, therefore we obtain $A'_0[x, x^{-1}] [X, X^{-1}] = A'_0[y', y'^{-1}] [Y, Y^{-1}]$. Since $\sigma(X) = X$ and $\sigma(Y) = Y$, we have $A[X, X^{-1}] = B_i[Y, Y^{-1}]$, hence $B[Y, Y^{-1}] = B_i[Y, Y^{-1}]$. Thus we have $B \cong B_i$.

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