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Author(s)	Ohkuma, Tadashi
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Ohkuma, Tadashi
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Comparability between Ramified Sets

By Tadashi OHKUMA

CONTENTS

Introduction

By "a ramified set", we mean a partially ordered set X in which for any element a, the set of all elements less than a makes a well-ordered subset of X. Such a set is called "un tableau ramifié" in [4] and [5] or "a tree" in [6]. (In [4], by "un ensemble ramifié" is meant a rather general set which is called "a tree" in [2]).

In connection with *Souslin's Problem*, investigation of ramified sets has been proceeded by many authors including especially Prof. George Kurepa whose contribution in this branch is distinguished ([4], [5], [6]). But it seems that most works concerning those sets are concentrated to the problem of finding conditions in order that a ramified set becomes countable, or of finding propositions about ramified sets, which turn out to be equivalent to Souslin's Problem, and that few results are obtained about internal structures of ramified sets themselves or about reciprocal relations which take place among them.

In this paper we are interested in the structures of ramified sets and a relation between them, especially comparison between them and processes by which larger ramified sets are constructed of smaller ones, and we obtained some results which seem fundamental in the theory of structures of ramified sets themselves.

It is well-known that any topological space is uniquely decomposed into the union of its perfect part and scattered part. Especially all scattered sets, i.e., sets with void perfect parts, in the real line are well-ordered by the order of homeomorphic imbedding. Similar situations occur about ramified sets. A ramified set is uniquely decomposed into the union of its resoluble part and perfectly irresoluble part (see Def. 7 and Th. 4).

As the means of comparison between ramified sets, in place of homeomorphic imbedding for topological spaces, it seems suitable to apply the relation $\propto$ such that $X \propto Y$ implies the existence of an increasing mapping of X into Y, which appears in the so-called *comparison theorem* in the usual proof of *Lusin's 2nd Principle* (see [7], pp. 208–221) in the theory of analytic sets. Since $\propto$ is a quasi-ordering between ramified sets, X and Y such that $X \propto Y$ and $Y \propto X$ are regarded as equivalent. Then we shall see in Theorem A all resoluble ramified sets with potencies less than a given regular cardinal number $\aleph_\beta > \aleph_0$ are well-ordered by $\propto$ under the identification of equivalent sets, and that any resoluble ramified set can be compared with any other one (resoluble or not).

Speaking of general ramified sets with potencies less than $\aleph_\beta$, including irresoluble ones, they do not seem to be well-ordered by $\propto$. In fact, under the assumption of continuum hypothesis, they make neither a ramified family nor a totally ordered family, and we shall see in Theorem B continuum hypothesis implies the existence of a countable descending sequence of ramified sets, and a pair of ramified sets which are not comparable with each other.

It is the main purpose of this paper to prove Theorem A and Theorem B.

Contents and composition of this paper are as follows.

We provide some preliminaries concerning ordinal numbers in Chapter I. In § 1, functions $\alpha_\nu(\lambda)$ assigned to each ν such that $1 \leqq \nu < \omega_\beta$ are defined. They are defined so that an ordinal number λ with $\lambda = \omega_\beta^\lambda$ is characterized by means of the function α_2, and when $2 < \nu < \omega_\beta$, a number λ with $\lambda = \alpha_\eta(\omega_\beta^\lambda)$ for any $\eta < \nu$ is characterized by means of α_ν. For a limit number λ, the least ordinal number $\nu = \mathrm{gn}(\lambda)$ such that $\lambda < \alpha_\nu(\omega_\beta^\lambda)$ is especially interesting, and according to the number $\mathrm{gn}(\lambda)$ and some properties of λ relating to functions α_ν, we shall set up a classification of ordinal numbers less than a certain number $\beta^{\#}$ in § 2.

In Chapter II, central notion and Main Theorems are mentioned. In § 1 definitions of ramified sets, relation $\propto$ and several notations are given. In § 2 our Main Theorems A and B are stated, and then the proof of Theorem A is oriented. Briefly speaking, Theorem A is attributed to the existence of a sequence $\mathfrak{N}$ of resoluble ramified sets N_λ, $\lambda<\beta^\sharp$, each of which satisfies certain conditions D. 1), D. 2) and D. 3) mentioned there. Principle 1 to construct N_λ is stated at the end of § 2, where by several operations on ramified sets similar to cardinal or ordinal arithmetic operations on general partially ordered sets (see [2] or [3]) are studied in advance.

Chapter III is devoted to show that every N_λ constructed according to Principle 1 satisfies D. 1), D. 2) and D. 3). In § 1 some lemmas concerning N_λ are prepared. § 2, § 3 and § 4 correspond to the three cases respectively to which every number less than $\beta^\sharp$ is allotted by Definition 3, and consequently to the different form of N_λ given by Principle 1 in each case.

Chapter IV consists of two parts. In § 1 we are interested in a ramified set S_ζ^ν defined for each $\lambda=\alpha_\nu(\omega_\beta^{\zeta+1})$. S_ζ^ν is irresoluble, but it is situated by $\propto$ at the least upper bound to the assending sequence N_ξ, $\xi<\lambda$, within the family of all ramified sets with potencies less than $\aleph_\beta$, and accordingly it is comparable by $\propto$ with any other ramified set (resoluble or not). § 2 is devoted to prove Theorem B, and examples to confirm Theorem B are introduced by modifications of S_1^1.

Finally we add an appendix where we specify case $\beta=1$. In this case not only the family of all resoluble ramified sets but also the family of all ramified sets including irresoluble ones with potencies less than $\aleph_1$ is well-ordered by $\propto$ (Theorem C). The proof is obtained similarly to Theorem A. By inserting S_ζ^ν defined in II, § 1 among sequence $\mathfrak{N}$, we get a sequence $\mathfrak{M}$ where any term M_λ satisfies a certain condition D. 2′) besides D. 1). Theorem C follows from the existence of such a sequence.

Chapter I. Preliminaries on ordinal numbers

In this chapter, we shall provide some preliminaries concerning ordinal numbers. In § 1 a function $\alpha_\nu(\lambda)$ of ordinal numbers λ is defined, which is assigned to each ν such that $1\leqq\nu<\omega_\beta$ where ω_β is a fixed regular initial number greater than ω_0. In § 2 we shall set up a classification of ordinal numbers less than a certain number $\beta^\sharp$, referring to these functions α_ν. This classification, as well as functions α_ν, will become a basis in constructing a certain sequence $\mathfrak{N}$ of ramified sets in a latter chapter.

1. Certain fuctions α_ν on ordinal numbers

Throughout this paper small Greek letters are used to denote ordinal numbers or ordinal-number-valued functions without stating special notice in each occurrence. Letters ω and ω_β are used with usual meaning. i, k, m and n stand for finite numbers. In terminologies: "limit numbers", "isolated numbers", "regular numbers", "singular numbers", "segments", "rests", "cofinal to" etc. and operations $\mathrm{cf}(\lambda)$, $\xi+\zeta$, $\xi\zeta$, ξ^ζ etc., we follow usual definitions (for example see [1]). To avoid confusion, we distinguish between terms "a power" and "the potency" of an ordinal number λ: the former means λ^ξ, while the latter means $\bar{\lambda}$. Capital Greek letters are used to denote sets or sequences (not necessarily countable) of ordinal numbers. Besides usual notations we shall define the following.

DEFINITION 1. $\mathrm{ls}(\lambda)$ denotes the greatest limit segment of λ, i.e., $\mathrm{ls}(\lambda)$ is the greatest limit number which does not exceed λ.

$\mathrm{fr}(\lambda)$ denotes the greatest finite rest of λ, i.e., $\mathrm{fr}(\lambda)$ is the finite number n with $\lambda=\mathrm{ls}(\lambda)+n$.

ρ_λ denotes the characteristic function of limit numbers, i.e., $\rho_\lambda=1$ if λ is a limit number, and $\rho_\lambda=0$ if λ is an isolated number.

Let ω_β be an arbitrarily given regular initial number greater than ω_0. β will be fixed throughout this paper (however in proving Theorem A (ii), it is utilized that β is arbitrarily given). Now we shall define functions $\alpha_\nu(\lambda)$ of λ assigned to each ν such that $1\leqq\nu<\omega_\beta$. These functions will play a main rôle in this paper.

DEFINITION 2. (i). Put $\alpha_1(\lambda)=\lambda$, and $\alpha_\nu(0)=0$ for any ν such that $1\leqq\nu<\omega_\beta$.

Assume that $\nu=\eta+1$ and $\alpha_\eta(\lambda)$ is already defined for any λ, put

$$\alpha_\nu(\mu+1) = \alpha_\eta(\omega_\beta{}^{\alpha_\nu(\mu)+\rho_\mu}), \qquad \text{and}$$
$$\alpha_\nu(\mu) = \sup_{\xi<\mu} \alpha_\nu(\xi) \quad \text{for a limit number } \mu.$$

Assume that ν is a limit number such that $1\leqq\nu<\omega_\beta$, and $\alpha_\eta(\lambda)$ is already defined for any λ and η such that $1\leqq\eta<\nu$, put $\mu=\nu\delta+\zeta$ where $0\leqq\zeta<\nu$ and

$$\alpha_\nu(\mu) = \alpha_\zeta(\omega_\beta{}^{\alpha_\nu(\nu\delta)+1}) \qquad \text{if} \quad \zeta>0,$$
$$\alpha_\nu(\mu) = \sup_{\xi<\mu} \alpha_\nu(\xi) \qquad \text{if} \quad \zeta=0.$$

Finally $$\beta^\# = \sup_{\nu<\omega_\beta} \alpha_\nu(1).$$

(ii). If ν is an isolated number less than ω_β, then Φ_ν^ζ denotes the class of all limit numbers and Φ_ν^i denotes the class of all isolated numbers.

If ν is a limit number such that $0<\nu<\omega_\beta$, then Φ_ν^l denotes the class of all numbers with forms $\nu\delta$ and Φ_ν^i denotes the class of all numbers with forms $\nu\delta+\zeta$ where $0<\zeta<\nu$.

For $\lambda<\beta^\#$, $\varphi_\nu^s(\lambda)$ denotes the greatest number in Φ_ν^l which does not exceed λ and $\varphi_\nu^r(\lambda)$ denotes the number ζ such that $\lambda=\varphi_\nu^s(\lambda)+\zeta$. The representation $\lambda=\xi+\zeta$ where $\xi=\varphi_\nu^s(\lambda)$ and $\zeta=\varphi_\nu^r(\lambda)$ is called the φ_ν-decomposition of λ.

REMARK. If ν is an isolated number, then $\varphi_\nu^s(\lambda)=\mathrm{ls}(\lambda)$ and $\varphi_\nu^r(\lambda)=\mathrm{fr}(\lambda)$. In general if $\mu\in\Phi_\nu^l$, then $\varphi_\nu^r(\mu+\xi)=\varphi_\nu^r(\xi)$ for any $\xi<\beta^\#$.

It follows from the definition that, if $\lambda\in\Phi_\nu^l$, then the function $\alpha_\nu(\mu)$ of μ with a constant ν is continuous at $\mu=\lambda$.

EXAMPLES. $\alpha_2(1)=\omega_\beta$, $\alpha_2(2)=\omega_\beta{}^{\omega_\beta}$, $\alpha_2(3)=\omega_\beta{}^{\omega_\beta{}^{\omega_\beta}}$ etc.. In general $\alpha_{\nu+1}(1)=\alpha_\nu(\omega_\beta)$. If ν is a limit number and $0<\zeta<\nu<\omega_\beta$, then $\alpha_\nu(\zeta)=\alpha_\zeta(\omega_\beta)=\alpha_{\zeta+1}(1)$.

Hereafter we assume that a number denoted by ν or η, which is mainly used as the suffix of a function α_ν or as the index of a power ω^ν of ω, is greater than 0 and less than ω_β, without mentioning special notices in each occurance.

Lemma 1.1. *The function $\alpha_\nu(\omega_\beta^\mu)$ of μ with a constant ν is continuous at any limit number $\mu>0$.*

Proof. Since $\nu<\omega_\beta$, $\omega_\beta^\mu\in\Phi_\nu^l$. Hence the function $\alpha_\nu(\lambda)$ of λ is continuous at $\lambda=\omega_\beta^\mu$. Since the function ω_β^μ of μ is continuous at any limit number $\mu>0$, $\alpha_\nu(\omega_\beta^\mu)$ is continuous at any limit number $\mu>0$.

Lemma 1.2. $\mu\leqq\alpha_\nu(\mu)$.

Proof. In the case where either $\nu=1$ or $\mu=0$, our lemma is trivial.

Assume $\theta\leqq\alpha_\eta(\theta)$ for any θ and η with $1\leqq\eta<\nu$, and $\xi\leqq\alpha_\nu(\xi)$ for any $\xi<\mu$.

Case 1. $\nu=\eta+1$ and $\mu=\xi+1$.

In this case $\alpha_\nu(\xi+1)=\alpha_\eta(\omega_\beta{}^{\alpha_\nu(\xi)+\rho_\xi})\geqq\omega_\beta{}^{\alpha_\nu(\xi)+\rho_\xi}\geqq\omega_\beta^{\xi+\rho_\xi}$. If ξ is an isolated number, then $\omega_\beta^\xi>\xi$, since ω_β^ξ is a limit number. If ξ is a limit number, then $\rho_\xi=1$ and $\omega_\beta^{\xi+1}>\xi+1$. Hence in either case $\alpha_\nu(\xi+1)\geqq\xi+1=\mu$.

Case 2. ν is a limit number and $\mu=\nu\delta+\zeta$ where $0<\zeta<\nu$.

In this case $\alpha_\nu(\mu)=\alpha_\zeta(\omega_\beta{}^{\alpha_\nu(\nu\delta)+1})\geqq\omega_\beta^{\nu\delta+1}>\omega_\beta^{\nu\delta}\cdot 2>\nu\delta+\zeta$.

Case 3. $\mu\in\Phi_\nu^l$.

In this case $\alpha_\nu(\mu)=\sup\limits_{\xi<\mu}\alpha_\nu(\xi)\geqq\sup\limits_{\xi<\mu}\xi=\mu$.

Hence our lemma is proved by double induction on μ and ν.

Lemma 1.3. *If* $\eta<\nu$ *and* $\mu>0$, *then* $\alpha_\eta(\omega_\beta^\mu)\leqq\alpha_\nu(\omega_\beta^\mu)$.

Proof. Since $\omega_\beta^\mu\in\Phi_\nu^\zeta$ for any μ, $\xi<\omega_\beta^\mu$ implies $\alpha_\nu(\xi)\leqq\alpha_\nu(\omega_\beta^\mu)$.

If $\nu=\eta+1$ and $\mu=\xi+1$, then $\omega_\beta^\mu>\mu+1$ and $\alpha_\nu(\omega_\beta^\mu)\geqq\alpha_\nu(\mu+1)$ $=\alpha_\eta(\omega_\beta{}^{\alpha_\nu(\mu)+\rho_\mu})\geqq\alpha_\eta(\omega_\beta^\mu)$ by Lemma 1.2.

If ν is a limit number and $\mu=\xi+1$, then $\alpha_\nu(\omega_\beta^\mu)\geqq\alpha_\nu(\omega_\beta^\xi+\eta)$ $=\alpha_\eta(\omega_\beta{}^{\alpha_\nu(\omega_\beta^\xi)+1})\geqq\alpha_\eta(\omega_\beta^{\xi+1})=\alpha_\eta(\omega_\beta^\mu)$.

If μ is a limit number and eithr $\nu=\eta+1$ or ν is a limit number, then $\alpha_\nu(\omega_\beta^\mu)=\sup_{\xi<\mu}\alpha_\nu(\omega_\beta^{\xi+1})\geqq\sup_{\xi<\mu}\alpha_\eta(\omega_\beta^{\xi+1})=\alpha_\eta(\omega_\beta^\mu)$ (refer to the cases above).

Therefore our lemma is proved by induction on ν.

REMARK. In general $\eta<\nu$ does not imply $\alpha_\eta(\mu)\leqq\alpha_\nu(\mu)$. For example, $\alpha_3(1)=\alpha_2(\omega_\beta)\geqq\alpha_2(2)=\omega_\beta{}^{\omega_\beta}>\omega_\beta$ while $\alpha_\omega(1)=\alpha_1(\omega_\beta)=\omega_\beta$.

Lemma 1.4. *If* $\xi<\mu$, *then* $\alpha_\nu(\xi)\leqq\alpha_\nu(\mu)$.

Proof. It is trivial for $\nu=1$. For $\mu\in\Phi_\nu^\zeta$ this lemma follows from definition. If $\nu=\eta+1$ and $\mu=\xi+1$, then $\alpha_\nu(\mu)=\alpha_\eta(\omega_\beta{}^{\alpha_\nu(\xi)+\rho_\xi})\geqq\alpha_\nu(\xi)$. Hence in the case $\nu=\eta+1$ our lemma is proved by induction on μ.

Assume that ν is a limit number and $\mu=\nu\delta+\zeta$ where $0<\zeta<\nu$, then $\alpha_\nu(\mu)=\alpha_\zeta(\omega_\beta{}^{\alpha_\nu(\nu\delta)+1})\geqq\alpha_\nu(\nu\delta)=\sup_{\xi<\nu\delta}\alpha_\nu(\xi)$.

Hence $\xi\leqq\nu\delta$ implies $\alpha_\nu(\xi)\leqq\alpha_\nu(\mu)$.

If $\nu\delta<\xi<\mu$ and $\xi=\nu\delta+\zeta'$ where $0<\zeta'<\zeta$, then $\alpha_\nu(\mu)=\alpha_\zeta(\omega_\beta{}^{\alpha_\nu(\nu\delta)+1})$ $\geqq\alpha_{\zeta'}(\omega_\beta{}^{\alpha_\nu(\nu\delta)+1})=\alpha_\nu(\xi)$ by Lemma 1.3. Hence our lemma is proved also in the case where ν is a limit number.

Lemma 1.5. *We have* $\omega_\beta^\lambda=\lambda$ *if and only if* $\lambda=\alpha_2(\mu)$ *for a limit number* $\mu>0$.

Proof. Assume that μ is a limit number, then $\alpha_2(\mu)=\sup_{\xi<\mu}\alpha_2(\xi+2)$ $=\sup_{\xi<\mu}\omega_\beta{}^{\alpha_2(\xi+1)}=\omega_\beta{}^{\sup_{\xi<\mu}\alpha_2(\xi+1)}=\omega_\beta{}^{\alpha_2(\mu)}$ by Lemma 1.1.

Conversely assume $\omega_\beta^\lambda=\lambda$. Since $\alpha_2(\mu)$ is a continuous function of μ and unbounded by $\mu\leqq\alpha_2(\mu)$, there exists the greatest number μ such that $\alpha_2(\mu)\leqq\lambda$. If μ is an isolated number, then $\rho_\mu=0$ and $\lambda<\alpha_2(\mu+1)$ $=\omega_\beta{}^{\alpha_2(\mu)}\leqq\omega_\beta^\lambda$ contradicting $\lambda=\omega_\beta^\lambda$. Hence μ is a limit number. If $\alpha_2(\mu)<\lambda$, then putting $\lambda=\alpha_2(\mu)+\zeta$ where $0<\zeta$, we have $\omega_\beta^\lambda=\omega_\beta{}^{\alpha_2(\mu)+\zeta}\geqq$ $\omega_\beta{}^{\alpha_2(\mu)+1}=\alpha_2(\mu+1)>\lambda$ contradicting $\omega_\beta^\lambda=\lambda$. Hence $\omega_\beta^\lambda=\lambda$ implies $\lambda=\alpha_2(\mu)$ for a limit number μ.

Theorem 1. *If* $\varepsilon+1\leqq\nu$, *then for any* $\mu>0$, *except the case where* ν *is a limit number*, $\mu=\nu\delta+\zeta$ *and* $0<\zeta<\varepsilon$, *there exists an ordinal number* $\lambda_{\mu,\nu,\varepsilon}>0$ *such that* $\alpha_\nu(\mu)=\alpha_\varepsilon(\omega_\beta{}^{\lambda_{\mu,\nu,\varepsilon}})$.

Proof. If $\nu=\varepsilon+1$ and $\mu=\xi+1$, then $\alpha_\nu(\mu)=\alpha_\varepsilon(\omega_\beta{}^{\alpha_\nu(\xi)+\rho_\xi})$ and $\alpha_\nu(\xi)+\rho_\xi>0$. Hence $\lambda_{\mu,\nu,\varepsilon}=\alpha_\nu(\xi)+\rho_\xi$. If $\nu=\varepsilon+1$ and μ is a limit number greater than 0, then $\alpha_\nu(\mu)=\sup_{\xi<\mu}\alpha_\nu(\xi+2)=\sup_{\xi<\mu}\alpha_\varepsilon(\omega_\beta^{\alpha_\nu(\xi+1)})=\alpha_\varepsilon(\omega_\beta^{\alpha_\nu(\mu)})$ by Lemma 1.1. Hence $\lambda_{\mu,\nu,\varepsilon}=\alpha_\nu(\mu)$.

Assume that our assertion is true for any ν such that $\varepsilon<\nu<\eta$, and we shall inductively show that there exists a $\lambda_{\mu,\eta,\varepsilon}>0$ such that $\alpha_\eta(\mu)=\alpha_\varepsilon(\omega_\beta^{\lambda_{\mu,\eta,\varepsilon}})$.

Case 1. $\eta=\nu+1$ and $\mu=\xi+1$.

In this case put $\zeta=\omega_\beta^{\alpha_\eta(\xi)+\rho_\xi}$, then since ζ is a power of ω_β, the arguments ζ and ν of $\alpha_\nu(\zeta)$ would not fall into the excepted case, and there exists a $\lambda_{\zeta,\nu,\varepsilon}$ such that $\alpha_\nu(\zeta)=\alpha_\varepsilon(\omega_\beta^{\lambda_{\zeta,\nu,\varepsilon}})$. Since $\alpha_\eta(\mu)=\alpha_\nu(\zeta)$, putting $\lambda_{\mu,\eta,\varepsilon}=\lambda_{\zeta,\nu,\varepsilon}$, $\lambda_{\mu,\eta,\varepsilon}$ satisfies the condition.

Case 2. $\eta=\nu+1$ and μ is a limit number.

As we saw in Case 1, for any $\xi<\mu$ there exists a $\lambda_{\xi+1,\eta,\varepsilon}$ such that $\alpha_\eta(\xi+1)=\alpha_\varepsilon(\omega_\beta^{\lambda_{\xi+1,\eta,\varepsilon}})$. Put $\lambda_{\mu,\eta,\varepsilon}=\sup_{\xi<\mu}\lambda_{\xi+1,\eta,\varepsilon}$ then $\alpha_\eta(\mu)=\sup_{\xi<\mu}\alpha_\eta(\xi+1)=\sup_{\xi<\mu}\alpha_\varepsilon(\omega_\beta^{\lambda_{\xi+1,\eta,\varepsilon}})=\alpha_\varepsilon(\omega_\beta^{\lambda_{\mu,\eta,\varepsilon}})$ by Lemma 1.1. Hence $\lambda_{\mu,\eta,\varepsilon}$ is the required ordinal number.

Case 3. η is a limit number and $\mu=\eta\delta+\varepsilon$.

In this case $\alpha_\eta(\mu)=\alpha_\varepsilon(\omega_\beta^{\alpha_\eta(\eta\delta)+1})$, and putting $\lambda_{\mu,\eta,\varepsilon}=\alpha_\eta(\eta\delta)+1$, $\lambda_{\mu,\eta,\varepsilon}$ satisfies the condition.

Case 4. η is a limit number and $\mu=\eta\delta+\xi$ where $\varepsilon<\xi<\eta$.

In this case put $\zeta=\omega_\beta^{\alpha_\eta(\eta\delta)+1}$, then since ζ is a power of ω_β, the arguments ζ and ξ of $\alpha_\xi(\zeta)$ would not fall into the excepted case, and there exists a $\lambda_{\zeta,\xi,\varepsilon}>0$ such that $\alpha_\xi(\zeta)=\alpha_\varepsilon(\omega_\beta^{\lambda_{\zeta,\xi,\varepsilon}})$. Since $\alpha_\eta(\mu)=\alpha_\xi(\zeta)$, putting $\lambda_{\mu,\eta,\varepsilon}=\lambda_{\zeta,\xi,\varepsilon}$, $\lambda_{\mu,\eta,\varepsilon}$ satisfies the condition.

Case 5. η is a limit number and $\mu=\eta\delta$.

Put $\Xi=\{\xi\,|\,\xi=\eta\delta'+\xi',\ \delta'<\delta,\ \varepsilon\leqq\xi'<\eta\}$, then Ξ is cofinal to $\eta\delta$. As we saw in Case 3 and 4 for any $\xi\in\Xi$ there exists a $\lambda_{\xi,\eta,\varepsilon}>0$ such that $\alpha_\eta(\xi)=\alpha_\varepsilon(\omega_\beta^{\lambda_{\xi,\eta,\varepsilon}})$. Put $\lambda_{\mu,\eta,\varepsilon}=\sup_{\xi\in\Xi}\lambda_{\xi,\eta,\varepsilon}$, then $\alpha_\eta(\mu)=\sup_{\xi<\mu}\alpha_\eta(\xi)=\sup_{\xi\in\Xi}\alpha_\eta(\xi)$ $=\sup_{\xi\in\Xi}\alpha_\varepsilon(\omega_\beta^{\lambda_{\xi,\eta,\varepsilon}})=\alpha_\varepsilon(\omega_\beta^{\lambda_{\mu,\eta,\varepsilon}})$. Hence $\lambda_{\mu,\eta,\varepsilon}$ is the number required.

Therefore in any case we can find a $\lambda_{\mu,\eta,\varepsilon}$ required in our assertion, and the proof is completed.

Especially putting $\varepsilon=2$, we can assert that, if $\nu\geqq3$, then for any $\mu>0$, except where ν is a limit number and $\mu=\nu\delta+1$, there exists a $\lambda_{\mu,\nu,2}$ such that $\alpha_\nu(\mu)=\alpha_2(\omega_\beta^{\lambda_{\mu,\nu,2}})$. Since $\lambda=\omega_\beta^{\lambda_{\mu,\nu,2}}$ is a limit number, following Corollaries 1–3 immediately follow from Lemma 1.5.

Corollary 1. *If $\nu\geqq3$ and $\mu>0$, except the case where ν is a limit number and $\mu=\nu\delta+1$, we have $\omega_\beta^{\alpha_\nu(\mu)}=\alpha_\nu(\mu)$.*

Corollary 2. *If $\nu\geqq2$ and $\mu>0$, then $\alpha_\nu(\mu)$ is a power of ω_β.*

Corollary 3. *If* $\nu \geqq 2$ *and* $\mu > 0$, *then* $\alpha_{\nu+1}(\mu+1) = \alpha_\nu(\alpha_{\nu+1}(\mu) \cdot \omega_\beta^{\rho_\mu})$. *If* ν *is a limit number greater than* 0, $\delta \geqq 1$ *and* $2 \leqq \xi < \nu$, *then* $\alpha_\nu(\nu\delta + \xi) = \alpha_\xi(\alpha_\nu(\nu\delta) \cdot \omega_\beta)$.

Corollary 4. *If* μ *is a limit number and* $2 \leqq \nu < \omega_\beta$, *then* $\alpha_\nu(\alpha_{\nu+1}(\mu)+1) \geqq \alpha_{\nu+1}(\mu) \cdot \omega_\beta$.

Proof. If ν is a limit number, then $\alpha_{\nu+1}(\mu) = \omega_\beta^{\alpha_{\nu+1}(\mu)}$, and hence $\nu\alpha_{\nu+1}(\mu) = \alpha_{\nu+1}(\mu)$. Hence by Corollary 3 above,
$\alpha_\nu(\alpha_{\nu+1}(\mu)+1) = \alpha_1(\alpha_\nu(\alpha_{\nu+1}(\mu)) \cdot \omega_\beta) \geqq \alpha_{\nu+1}(\mu) \cdot \omega_\beta$.

If ν is an isolated number $\eta+1$, then $\alpha_\nu(\alpha_{\nu+1}(\mu)+1) = \alpha_\eta(\alpha_\nu(\alpha_{\nu+1}(\mu)) \cdot \omega_\beta) \geqq \alpha_{\nu+1}(\mu) \cdot \omega_\beta$.

Lemma 1.6. *Let* ν *be an ordinal number greater than* 1. *Then* $\alpha_\nu(\lambda) = \lambda$ *if and only if* $\lambda = \alpha_{\nu+1}(\mu)$ *for a limit number* μ.

Proof. If μ is a limit number, then $\alpha_{\nu+1}(\mu) = \sup_{\xi<\mu} \alpha_{\nu+1}(\xi+2) = \sup_{\xi<\mu} \alpha_\nu(\alpha_{\nu+1}(\xi+1)) = \alpha_\nu(\alpha_{\nu+1}(\mu))$ by Corollary above.

Conversely assume $\lambda = \alpha_\nu(\lambda)$. Since the function $\alpha_{\nu+1}(\mu)$ of μ is continuous and unbounded, there exists the greatest number μ such that $\alpha_{\nu+1}(\mu) \leqq \lambda$. If μ is an isolated number, then $\rho_\mu = 0$ and $\lambda < \alpha_{\nu+1}(\mu+1) = \alpha_\nu(\alpha_{\nu+1}(\mu)) \leqq \alpha_\nu(\lambda)$, contradicting $\lambda = \alpha_\nu(\lambda)$. Hence μ is a limit number. If $\lambda \geqq \alpha_{\nu+1}(\mu)+1$, then by Corollary 4 above $\lambda = \alpha_\nu(\lambda) \geqq \alpha_\nu(\alpha_{\nu+1}(\mu)+1) \geqq \alpha_{\nu+1}(\mu) \cdot \omega_\beta$. But then, since μ is a limit number, $\alpha_\nu(\lambda) \geqq \alpha_\nu(\alpha_{\nu+1}(\mu)\omega_\beta) = \alpha_{\nu+1}(\mu+1) > \lambda$ contradicting $\lambda = \alpha_\nu(\lambda)$. Hence $\lambda = \alpha_{\nu+1}(\mu)$ where μ is a limit number.

Corollary 1. *If* $\eta+2 \leqq \nu$, *then* $\alpha_\eta(\alpha_\nu(\mu)) = \alpha_\nu(\mu)$ *for any* $\mu > 0$, *except where* ν *is a limit number,* $\mu = \nu\delta + \xi$ *and* $0 < \xi \leqq \eta$.

Corollary 2. *If* ν *is a limit number greater than* 0 *and* $0 < \xi < \nu$, *then* $\alpha_\nu(\nu\delta + \xi) = \alpha_{\xi+1}(\alpha_\nu(\nu\delta)+1)$.

Proof. By Corollary 1 above $\alpha_\nu(\nu\delta) = \alpha_{\xi+1}(\alpha_\nu(\nu\delta))$, and hence $\alpha_\nu(\nu\delta + \xi) = \alpha_\xi(\omega_\beta^{\alpha_\nu(\nu\delta)+1}) = \alpha_\xi(\omega_\beta^{\alpha_{\xi+1}(\alpha_\nu(\nu\delta))+1}) = \alpha_{\xi+1}(\alpha_\nu(\nu\delta)+1)$.

Lemma 1.7. *Let* ν *be a limit number greater than* 0. *In order that* $\lambda = \omega_\beta^\lambda = \alpha_\eta(\lambda)$ *for any* $\eta < \nu$, *it is necessary and sufficient that* $\lambda = \alpha_\nu(\nu\delta)$ *with a* $\delta > 0$.

Proof. If $\lambda = \alpha_\nu(\nu\delta)$ and $\delta > 0$, then $\lambda = \omega_\beta^\lambda$ by Corollary 1 of Theorem 1. Assume $\eta < \nu$ and put $\Xi = \{\xi \mid \xi = \nu\delta' + \xi',\ \delta' < \delta,\ \eta < \xi' < \nu\}$, then Ξ is cofinal to $\nu\delta$. By Corollary 1 of Lemma 1.6, $\alpha_\eta(\alpha_\nu(\xi)) = \alpha_\nu(\xi)$ for any $\xi \in \Xi$. Hence $\alpha_\eta(\lambda) = \sup_{\xi \in \Xi} \alpha_\eta(\alpha_\nu(\xi)) = \sup_{\xi \in \Xi} \alpha_\nu(\xi) = \alpha_\nu(\nu\delta) = \lambda$.

Conversely assume $\lambda = \omega_\beta^\lambda = \alpha_\eta(\lambda)$ for any $\eta < \nu$. By definition of

$\alpha_\nu(\nu\delta)$ for $\delta>0$, the function $\alpha_\nu(\nu\delta)$ of δ, with a fixed ν, is continuous. Hence there exists the greatest number δ such that $\alpha_\nu(\nu\delta)\leqq\lambda$. If $\delta=0$, then there exists a ξ such that $\xi<\nu$ and $\lambda<\alpha_\nu(\xi)$. But then, since $\alpha_\nu(\xi)=\alpha_\xi(\omega_\beta)=\alpha_{\xi+1}(1)$ and $1<\lambda$, we have $\lambda<\alpha_{\xi+1}(1)\leqq\alpha_{\xi+1}(\lambda)$ contradicting $\lambda=\alpha_{\xi+1}(\lambda)$. Hence $\delta>0$. Since $\alpha_\nu(\nu(\delta+1))>\lambda$, there exists the least number ξ such that $1\leqq\xi<\nu$ and $\lambda<\alpha_\nu(\nu\delta+\xi)$. If $\xi>1$, then $\alpha_\nu(\nu\delta+1)\leqq\lambda$ and $\lambda<\alpha_\nu(\nu\delta+\xi)=\alpha_\xi(\omega_\beta{}^{\alpha_\nu(\nu\delta)+1})=\alpha_\xi(\alpha_\nu(\nu\delta+1))\leqq\alpha_\xi(\lambda)$ contradicting $\lambda=\alpha_\xi(\lambda)$. Hence $\xi=1$ and $\alpha_\nu(\nu\delta)\leqq\lambda<\alpha_\nu(\nu\delta+1)$. If $\alpha_\nu(\nu\delta)+1\leqq\lambda$, then $\lambda<\alpha_\nu(\nu\delta+1)=\omega_\beta{}^{\alpha_\nu(\nu\delta)+1}\leqq\omega_\beta^\lambda$ contradicting $\lambda=\omega_\beta^\lambda$. Hence we have $\lambda=\alpha_\nu(\nu\delta)$ where $\delta>0$, which is to be proved.

Summing up Lemma 1. 5, 1. 6 and 1. 7, we have

Theorem 2. *In order that* $\lambda=\alpha_\eta(\omega_\beta^\lambda)$ *for any* $\eta<\nu$, *it is necessary and sufficient that* $\lambda=\alpha_\nu(\mu)$ *with* $\mu\in\Phi_\nu^l$ *and* $\mu>0$.

Lemma 1. 8. $\alpha_\nu(\mu)<\alpha_\nu(\mu+1)$ *for any* μ *and* ν.

Proof. $\alpha_\nu(\mu)\leqq\alpha_\nu(\mu+1)$ is already proved in Lemma 1. 4. Our assertion is trivial for $\nu=1$.

If $\nu=\eta+1$ where $\eta\geqq1$, then $\alpha_\nu(\mu+1)=\alpha_\eta(\omega_\beta{}^{\alpha_\nu(\mu)+\rho_\mu})$. If further μ is a limit number, then $\rho_\mu=1$ and $\alpha_\eta(\omega_\beta{}^{\alpha_\nu(\mu)+1})\geqq\alpha_\nu(\mu)+1>\alpha_\nu(\mu)$. If μ is an isolated number, then $\alpha_\eta(\omega_\beta{}^{\alpha_\nu(\mu)})>\alpha_\nu(\mu)$ by Theorem 2. Hence in either case $\alpha_\nu(\mu+1)>\alpha_\nu(\mu)$.

Assume that ν is a limit number. If further $\mu\in\Phi_\nu^l$, then $\alpha_\nu(\mu+1)=\alpha_1(\omega_\beta{}^{\alpha_\nu(\mu)+1})>\alpha_\nu(\mu)$. If $\mu\in\Phi_\nu^i$, then letting $\mu=\xi+\zeta$ be the φ_ν-decomposition of μ, $\zeta>0$. $\alpha_\nu(\mu+1)=\alpha_\nu(\xi+\zeta+1)=\alpha_{\zeta+1}(\omega_\beta{}^{\alpha_\nu(\xi)+1})>\alpha_{\zeta+1}(\alpha_\nu(\xi)+1)$, since our lemma is already proved for an isolated number $\zeta+1$. Furthermore, by Corollary 2 of Lemma 1. 6, $\alpha_{\zeta+1}(\alpha_\nu(\xi)+1)=\alpha_\nu(\xi+\zeta)=\alpha_\nu(\mu)$. Hence our lemma is proved also for a limit number ν.

For $\nu\geqq2$, $\alpha_\nu(\mu)$ is a limit number by Corollary 2 of Theorem 1. Here we have

Theorem 3. *Assume* $\nu\geqq2$, *then* $\mathrm{cf}\,(\alpha_\nu(\mu))=\beta$ *for* $\mu=\Phi_\nu^i$, *and* $\mathrm{cf}\,(\alpha_\nu(\mu))=\mathrm{cf}\,(\mu)$ *for* $\mu\in\Phi_\nu^l$.

Proof. First we shall consider the case $\mu\in\Phi_\nu^i$.

Assume that ν is an isolated number $\eta+1$ where $\eta>0$. If μ is the next number to a limit number ξ, then $\alpha_\nu(\mu)=\alpha_\eta(\omega_\beta{}^{\alpha_\nu(\xi)+1})=\sup\limits_{\zeta<\omega_\beta}\alpha_\eta(\omega_\beta{}^{\alpha_\nu(\xi)}\cdot\zeta)$. Since the function $\alpha_\eta(\lambda)$ of λ is increasing by Lemma 1. 8, for a sequence Λ of ordinal number $\zeta<\omega_\beta$ cofinal to ω_β, the sequence $\{\alpha_\eta(\omega_\beta{}^{\alpha_\nu(\xi)}\cdot\zeta)\mid\zeta\in\Lambda\}$ is also cofinal to $\alpha_\nu(\mu)$. Hence $\mathrm{cf}\,(\alpha_\nu(\mu))=\beta$.

Assume $\mu=\xi+1$ where ξ is an isolated number and $\mathrm{cf}\,(\alpha_\nu(\xi))=\beta$, then $\alpha_\nu(\mu)=\alpha_\eta(\omega_\beta{}^{\alpha_\nu(\xi)})=\sup_{\zeta<\alpha_\nu(\xi)}\alpha_\eta(\omega_\beta^\zeta)$. Similarly as the above, for a sequence Λ of ordinal numbers $\zeta<\alpha_\nu(\xi)$ cofinal to $\alpha_\nu(\xi)$, the sequence $\{\alpha_\eta(\omega_\beta^\zeta)\mid\zeta\in\Lambda\}$ is also confinal to $\alpha_\nu(\mu)$. Hence $\mathrm{cf}(\alpha_\nu(\mu))=\mathrm{cf}(\alpha_\nu(\xi))=\beta$. Hence it is inductively proved that for any isolated number ν we have $\mathrm{cf}(\alpha_\nu(\mu))=\beta$.

Next let ν be a limit number such that $0<\nu<\omega_\beta$. If $0<\xi<\nu$, then by Corollary 2 of Theorem 1 $\alpha_\nu(\nu\delta+\xi)=\alpha_{\xi+1}(\alpha_\nu(\nu\delta)+1)$, and since both $\xi+1$ and $\alpha_\nu(\nu\delta)+1$ are isolated numbers, $\mathrm{cf}(\alpha_{\xi+1}(\alpha_\nu(\nu\delta)+1))=\beta$. Hence in either case that ν is an isolated number or a limit number, $\nu\geqq 2$ and $\mu\in\Phi_\nu^\xi$ imply $\mathrm{cf}(\alpha_\nu(\mu))=\beta$.

Next consider the case $\mu\in\Phi_\nu^\zeta$. For $\mu=0$, our statement is trivial. If $\mu>0$, then since $\alpha_\nu(\mu)=\sup_{\zeta<\mu}\alpha_\nu(\zeta)$ and the function $\alpha_\nu(\zeta)$ of ζ is increasing, we have immediately $\mathrm{cf}(\alpha_\nu(\mu))=\mathrm{cf}(\mu)$, and the whole proof is completed.

Lemma 1.9. *For any $\lambda<\beta^\#$ there exists a ν such that $0<\nu<\omega_\beta$ and $\lambda<\alpha_\nu(\omega_\beta^\lambda)$.*

Proof. If $\lambda=\alpha_\nu(\omega_\beta^\lambda)$ for any $\nu<\omega_\beta$, then $\lambda=\sup_{\nu<\omega_\beta}\alpha_\nu(\omega_\beta^\lambda)\geqq\sup_{\nu<\omega_\beta}\alpha_\nu(1)=\beta^\#$.

2. A classification of ordinal numbers

Hereafter any number denoted by a small Greek letter in this paper is assumed less than $\beta^\#$.

Lemma 1.10. *Let $\gamma(\mu)$ denote the least number ξ such that $\mu<\xi\omega$, then the following three conditions on μ are mutually equivalent.*

(*a*). $\gamma(\mu)=\mu$. (*b*). *Any non-zero rest of μ is equal to μ.*

(*c*). *There exists a $\xi\leqq\mu$ with $\mu=\omega^\xi$.*

For the proof refer to [1] pp. 67-68. Here we omit it.

DEFINITION 3. (i). $\gamma(\mu)$ denotes the number defined in Lemma 1.10. A number μ which satisfies conditions in Lemma 1.10 is called a γ-*number* (see [1], p. 67).

(ii). For $\lambda<\beta^\#$, the least number ν such that $\mathrm{ls}(\lambda)<\alpha_\nu(\omega_\beta^{\mathrm{ls}(\lambda)})$ (refer to Lemma 1.9) is called the *genus* of λ and denoted by $\mathrm{gn}(\lambda)$.

(iii). If $\lambda=\alpha_\nu(\mu)$, then we write $\mu=\alpha_\nu^{-1}(\lambda)$.

Put $\mathrm{de}(\lambda)=\alpha_{\mathrm{gn}(\lambda)}^{-1}(\mathrm{ls}(\lambda))$, (derivation of λ).

(iv). Let Γ denote the set of all ordinal numbers less than $\beta^\#$.

Γ_ν denotes the set of all $\lambda\in\Gamma$ with $\mathrm{gn}(\lambda)=\nu$.

Γ^0 denotes the set of all $\lambda\in\Gamma$ with $\mathrm{cf}(\mathrm{ls}(\lambda))<\beta$.

Γ^1 denotes the set of all $\lambda\in\Gamma$ such that $\mathrm{cf}(\mathrm{ls}(\lambda))=\beta$ and $\mathrm{de}(\lambda)$ is not a γ-number.

Γ^2 denotes the set of all $\lambda \in \Gamma$ such that $\mathrm{cf}(\mathrm{ls}(\lambda))=\beta$ and $\mathrm{de}(\lambda)$ is a γ-number.

Put $\Gamma_\nu^i=\Gamma^i \cap \Gamma_\nu$, $\tilde{\Gamma}_\nu^i=\bigcup_{\nu \leqq \eta<\omega_\beta} \Gamma_\eta^i$ for $i=0,1,2$ and $\tilde{\Gamma}_\nu=\bigcup_{\nu \leqq \eta<\omega_\beta} \Gamma_\eta$.

REMARK. (i). $\mathrm{gn}(\lambda)$ and $\mathrm{de}(\lambda)$ depend only on $\mathrm{ls}(\lambda)$, or in other wards, $\mathrm{gn}(\lambda+n)=\mathrm{gn}(\lambda)$ and $\mathrm{de}(\lambda+n)=\mathrm{de}(\lambda)$.

(ii). If $\mathrm{gn}(\lambda)=1$, then $\mathrm{de}(\lambda)=\mathrm{ls}(\lambda)$.

If $1<\nu=\mathrm{gn}(\lambda)$, then $\eta<\nu$ implies $\alpha_\eta(\omega_\beta^{\mathrm{ls}(\lambda)})=\mathrm{ls}(\lambda)$. Hence $\mathrm{ls}(\lambda)=\alpha_\nu(\mu)$ where $\mu \in \Phi_\nu^l$ by Theorem 2, and $\mu=\mathrm{de}(\lambda)$.

Therefore $\mathrm{de}(\lambda)$ is defined for any $\lambda<\beta^\#$, and $\mathrm{de}(\lambda) \in \Phi_{\mathrm{gn}(\lambda)}^l$ in general.

Especially if $\nu=\mathrm{gn}(\lambda)>1$, then $\omega_\beta^{\mathrm{ls}(\lambda)}=\mathrm{ls}(\lambda)$ and hence $\alpha_\nu(\mathrm{de}(\lambda))=\mathrm{ls}(\lambda)<\alpha_\nu(\omega_\beta^{\mathrm{ls}(\lambda)})=\alpha_\nu(\mathrm{ls}(\lambda))$. Since α_ν is increasing, $\mathrm{de}(\lambda)<\mathrm{ls}(\lambda)$.

(iii). If $\eta<\nu=\mathrm{gn}(\lambda)$, then especially $1<\nu$ and $\mathrm{ls}(\lambda)=\omega_\beta^{\mathrm{ls}(\lambda)}$. Hence $\mathrm{ls}(\lambda)=\alpha_\eta(\omega_\beta^{\mathrm{ls}(\lambda)})=\alpha_\eta(\mathrm{ls}(\lambda))$.

(iv). In general $\lambda=\alpha_{\mathrm{gn}(\lambda)}(\mathrm{de}(\lambda))+\mathrm{fr}(\lambda)$.

Lemma 1.11. (i). $\mathrm{gn}(\mathrm{de}(\lambda)) \leqq \mathrm{gn}(\lambda)$. (ii). $\mathrm{cf}(\mathrm{de}(\lambda))=\mathrm{cf}(\mathrm{ls}(\lambda))$.

Proof. (i). If $\mathrm{gn}(\lambda)=1$, then $\mathrm{de}(\lambda)=\mathrm{ls}(\lambda)$ and $\mathrm{gn}(\mathrm{de}(\lambda))=\mathrm{gn}(\mathrm{ls}(\lambda))=\mathrm{gn}(\lambda)$. If $\mathrm{gn}(\lambda)=\nu>1$, then by Remark (ii) above, $\mathrm{de}(\lambda)<\mathrm{ls}(\lambda)$. Assume $\mathrm{gn}(\mathrm{de}(\lambda))=\eta>\nu$, then by Remark (iii) above, $\mathrm{de}(\lambda)=\alpha_\nu(\mathrm{de}(\lambda))=\mathrm{ls}(\lambda)$ contradictrily. Hence $\mathrm{gn}(\mathrm{de}(\lambda)) \leqq \mathrm{gn}(\lambda)$ in general.

(ii) is a consequence of Theorem 3 and $\mathrm{de}(\lambda) \in \Phi_{\mathrm{gn}(\lambda)}^l$.

Lemma 1.12. *If* $\xi>0$, *then the least* γ-*number* μ *such that* $\mathrm{cf}(\mu)=\beta$ *and* $\mu>\xi$ *is* $\xi\omega_\beta$.

Proof. Let ζ be the least number with $\mu \leqq \xi\zeta$. $\xi<\mu$ implies $1<\zeta$. If $\mathrm{fr}(\zeta) \geqq 1$ (i.e., ζ is an isolated number), then $\xi(\zeta-1)<\mu$ and $\mu=\xi(\zeta-1)+\theta$ where $0<\theta \leqq \xi<\mu$ contradicting that μ is a γ-number (see (b) in Lemma 1.10). Hence ζ is a limit number, and necessarily $\mu=\xi\zeta$. If $\zeta<\omega_\beta$ then $\mathrm{cf}(\mu)=\mathrm{cf}(\zeta)<\beta$. Hence $\zeta \geqq \omega_\beta$. But $\xi\omega_\beta$ is obviously a γ-number with $\mathrm{cf}(\xi\omega_\beta)=\beta$ and $\xi\omega_\beta>\xi$.

Corollary. *In order that* μ *is a* γ-*number such that* $\mu>1$ *and* $\mathrm{cf}(\mu)=\beta$ *it is necessary and sufficient that* $\mu=\omega_\beta^\varepsilon$ *for an* ε *which is either an isolated number or a limit number with* $\mathrm{cf}(\varepsilon)=\beta$.

In general any $\lambda \in \Gamma$ is represented as

$$(1) \qquad \lambda=\alpha_\nu(\xi+\zeta)+n,$$

where $\nu=\mathrm{gn}(\lambda)$, $\xi=\gamma(\mathrm{de}(\lambda))$ and $n=\mathrm{fr}(\lambda)$. (Then ζ is uniquely determined).

Especially if $\lambda \in \Gamma^1$, then $\mathrm{de}(\lambda)$ is not a γ-number, and hence $0<\xi<\mathrm{de}(\lambda)$ and $0<\zeta<\mathrm{de}(\lambda)$.

If $\lambda \in \Gamma^2$, then $\mathrm{de}(\lambda)$ is a γ-number and $\xi = \mathrm{de}(\lambda)$ and $\zeta = 0$. Furthermore, since $\mathrm{cf}(\mathrm{de}(\lambda)) = \mathrm{cf}(\mathrm{ls}(\lambda)) = \beta$ by Lemma 1.11, (ii), it follows from Corollary of Lemma 1.12 that $\mathrm{de}(\lambda) = \omega_\beta^\mu$ and λ is represented as

$$(2) \qquad \lambda = \alpha_\nu(\omega_\beta^\mu) + n\,,$$

where $\nu = \mathrm{gn}(\lambda)$, $n = \mathrm{fr}(\lambda)$ and μ is either an isolated number or a limit number with $\mathrm{cf}(\mu) = \beta$.

DEFINITION 4. When $\lambda \in \Gamma^1$ or $\lambda \in \Gamma^2$, the right side of (1) or (2) respectively is called the *canonical decomposition* of λ.

REMARK. (i). When $\lambda \in \Gamma^1$, since $0 < \xi < \mathrm{de}(\lambda)$ and $0 < \zeta < \mathrm{de}(\lambda)$ in (1), we have $\alpha_\nu(\xi) < \alpha_\nu(\mathrm{de}(\lambda)) = \mathrm{ls}(\lambda)$ and $\alpha_\nu(\zeta) < \mathrm{ls}(\lambda)$. Furthermore since $\mathrm{cf}(\mathrm{ls}(\lambda)) = \beta$, we have $\alpha_\nu(\xi) + \omega < \mathrm{ls}(\lambda)$.

(ii). When $\lambda \in \Gamma^2$, since $\nu = \mathrm{gn}(\lambda)$ in (2), $\mu < \mathrm{ls}(\lambda)$.

If further μ is an isolated number, then $\mathrm{cf}(\omega\mu) = 0$ and $\mathrm{cf}(\omega_\beta^\mu) = \beta$, hence $\omega\mu \neq \omega_\beta^\mu$ and consequently $\omega\mu < \omega_\beta^\mu \leqq \mathrm{ls}(\lambda)$. If μ is a limit number with $\mathrm{cf}(\mu) = \beta$, then $\omega\mu = \mu$. (Indeed putting $\mu = \omega^\omega \cdot \delta + \theta$ where $0 \leqq \theta < \omega^\omega$, θ is necessarily 0. Hence $\omega\mu = \omega \cdot \omega^\omega \cdot \delta = \omega^\omega \cdot \delta = \mu$). Hence in either case $\omega\mu < \mathrm{ls}(\lambda)$.

This Remark (i) and (ii) will be recalled in Principle 1 later.

Lemma 1.13. *If $\lambda = \alpha_\nu(\mu) + \omega$, then the least number ξ in $\tilde{\Gamma}_\nu^1 \cup \tilde{\Gamma}_\nu^2$ greater than λ is $\alpha_\nu(\mu + \omega_\beta)$.*

Proof. $\xi \in \tilde{\Gamma}_\nu$ implies $\mathrm{ls}(\xi) = \alpha_\nu(\delta)$ for a $\delta \in \Phi_\nu^\zeta$. Since $\lambda < \xi$, $\mu < \delta$. Put $\delta = \mu + \zeta$. If $\zeta < \omega_\beta$, then $\mathrm{cf}(\mathrm{ls}(\xi)) = \mathrm{cf}(\delta) = \mathrm{cf}(\zeta) < \beta$ contradicting $\xi \notin \Gamma^0$. Hence $\zeta \geqq \omega_\beta$ and ξ is of the form $\alpha_\nu(\mu + \zeta) + n$. But among numbers of this form, $\xi = \alpha_\nu(\mu + \omega_\beta)$ is the least number and obviously contained in $\Gamma_\nu^1 \cup \Gamma_\nu^2$. (Especially if $\mu \geqq \omega_\beta$, then $\xi \in \Gamma_\nu^1$).

Lemma 1.14. *If $\lambda = \alpha_\nu(\omega_\beta^\mu) + \omega$, then the least number ξ in $\Gamma_\nu^2 \cup \tilde{\Gamma}_{\nu+1}$ greater than λ is $\alpha_\nu(\omega_\beta^{\mu+1})$.*

Proof. If $\zeta \in \tilde{\Gamma}_{\nu+1}$, then $\mathrm{ls}(\zeta) = \alpha_\nu(\omega_\beta^{\mathrm{ls}(\zeta)})$. Hence $\xi \in \Gamma_\nu^2 \cup \tilde{\Gamma}_{\nu+1}$ implies that $\mathrm{ls}(\xi)$ is of the form $\alpha_\nu(\omega_\beta^\delta)$ where $\lambda < \xi$ implies $\mu < \delta$. Hence $\xi \in \Gamma_\nu^2 \cup \tilde{\Gamma}_{\nu+1}$ and $\lambda < \xi$ imply that ξ is of the form $\alpha_\nu(\omega_\beta^{\mu+\varepsilon}) + n$ where $\varepsilon \geqq 1$. But among numbers of this form, $\xi = \alpha_\nu(\omega_\beta^{\mu+1})$ is the least number and contained in Γ_ν^2.

CHAPTER II. MAIN THEOREMS

In this chapter, we shall introduce central notions and state Main Theorems. Finally a brief orientation of the proof of Theorem A is mentioned.

1. Ramified sets and relation $\propto$

DEFINITION 5. A *ramified set* X (which is called "un tableau ramifié" in [4]) is a partially ordered set which satisfies the following condition: for any $x \in X$ the set $\{a \mid a \in X, a < x\}$ is a well-ordered subset of X.

By definition the following is obvious.

Lemma 2.1. (i). *A subset of a ramified set is also a ramified set with the original order-relation.* (ii). *A ramified set satisfies the descending chain condition* (see [2], p. 37). (iii). *A totally ordered ramified set is a well-ordered set. Conversely a well-ordered set is a ramified set.*

In this paper capital Latin letters, as well as these with suffixes, are used to denote ramified sets or their subsets. Especially W denotes a well-ordered set. W_λ is the set of all $\xi < \lambda$ with the natural order between them. Small Latin letters stand for elements of ramified sets or mappings into ramified sets (mostly from ramified sets), except i, k, m and n which are finite numbers. Especially f, g or h is used to denote a mapping of a ramified set into another (occasionally in itself). Capital German letters are used to denote families or sequences of ramified sets (or their subsets).

Concerning ramified sets we shall settle terminologies and notations followingly.

DEFINITION 6. (i). For $x \in X$, $\mathrm{Lb}(x; X)$, or simply $\mathrm{Lb}(x)$ (in the case where x is contained in a definite set or it is apparent what set X is referred to), denotes the set of all $a \in X$ with $a < x$.

$\mathrm{Lb}'(x; X)$, or simply $\mathrm{Lb}'(x)$, denotes $\mathrm{Lb}(x; X) \cup \{x\}$. (Lower bounds of x).

(ii). $\mathrm{Ub}(x; X)$, or simply $\mathrm{Ub}(x)$, denotes the set of all $a \in X$ with $a > x$.

$\mathrm{Ub}'(x; X)$ or $\mathrm{Ub}'(x)$ denotes $\mathrm{Ub}(x; X) \cup \{x\}$. (Upper bounds of x).

(iii). $\tau(x; X)$, or simply $\tau(x)$, denotes the order-type, which is an ordinal number, of $\mathrm{Lb}(x; X)$.

(iv). A subset Y of X is called a *cut* of X if $y \in Y$, $x \in X$ and $x < y$ imply $x \in Y$.

(v). $\mathrm{Seg}_\nu(X)$, $\mathrm{Lay}_\nu(X)$ and $\mathrm{Csg}_\nu(X)$ denote the sets $\{x \mid x \in X, \tau(x) < \omega^\nu\}$, $\{x \mid x \in X, \tau(x) = \omega^\nu\}$ and $\{x \mid x \in X, \tau(x) \geqq \omega^\nu\}$ respectively.

(vi). Let Y be a cut of X. $\mathrm{Exp}_\nu(Y; X)$, or simply $\mathrm{Exp}_\nu(Y)$, denotes the subset of X which consists of all $y \in Y$ and all $x \in \mathrm{Csg}_\nu(X)$ such that $\mathrm{Lb}(x; X) \cap \mathrm{Seg}_\nu(X)$ is entirely included in Y.

By definition we have obviously

Lemma 2.2. (i). *If* $x \in Y \subset X$, *then* $\mathrm{Lb}(x\,;Y) = \mathrm{Lb}(x\,;X) \cap Y$. *Similar equalities hold for* Lb′, Ub *and* Ub′. (ii). *If* $x < y$, *then* $\tau(y\,;X) = \tau(x\,;X) + 1 + \tau(y\,;\mathrm{Ub}(x))$.

Concerning $\mathrm{Exp}_\nu(Y\,;X)$ we have

Lemma 2.3. *Let* Y *be a cut of* X. (i). $\mathrm{Exp}_\nu(Y)$ *is a cut of* X. (ii). $\mathrm{Exp}_\nu(\mathrm{Seg}_\nu(X)) = X$. (iii). $Y \subset \mathrm{Exp}_\nu(Y)$. (iv). *If* Z *is a cut of* Y, *then* $\mathrm{Exp}_\nu(Z\,;X) \subset \mathrm{Exp}_\nu(Y\,;X)$, $\mathrm{Exp}_\nu(Z\,;Y) = \mathrm{Exp}_\nu(Z\,;X) \cap Y$ *and* $\mathrm{Exp}_\nu(\mathrm{Exp}_\nu(Z\,;Y)\,;X) = \mathrm{Exp}_\nu(Z\,;X)$. (v). *If* Z *is a cut of* X, *then* $\mathrm{Exp}_\nu(Y \cup Z) = \mathrm{Exp}_\nu(Y) \cup \mathrm{Exp}_\nu(Z)$. (vi). $x \in \mathrm{Exp}_\nu(Y) \cap \mathrm{Csg}_\nu(X)$, *if and only if the type of* $\mathrm{Lb}(x) \cap Y$ *is not less than* ω^ν. (vii). *If* x *is a minimal element of* $X - \mathrm{Exp}_\nu(Y)$, *then* $x \in \mathrm{Seg}_\nu(X)$. (viii). $\mathrm{Exp}_\nu(\mathrm{Exp}_\nu(Y)) = \mathrm{Exp}_\nu(Y)$.

Proof. We shall show only (vii) and (viii), since the others are trivial.

(vii). Assume $\tau(x) \geqq \omega^\nu$. Since $x \in X - \mathrm{Exp}_\nu(Y)$, the type of $\mathrm{Lb}(x) \cap Y$ is less than ω^ν. Hence there exists an $a \in \mathrm{Lb}(x) - Y$ such that $\tau(a) < \omega^\nu$. Since x is minimal within $X - \mathrm{Exp}_\nu(Y)$, $a \in \mathrm{Exp}_\nu(Y)$. But by definition, $a \in \mathrm{Exp}_\nu(Y) - Y$ implies $a \in \mathrm{Csg}_\nu(X)$ contradicting $\tau(a) < \omega^\nu$. Hence $x \in \mathrm{Seg}_\nu(X)$.

(viii). $\mathrm{Exp}_\nu(\mathrm{Exp}_\nu(Y)) \supset \mathrm{Exp}_\nu(Y)$ follows from (iii) and (iv). Assume that $D = \mathrm{Exp}_\nu(\mathrm{Exp}_\nu(Y)) - \mathrm{Exp}_\nu(Y)$ is not void, and let x be a minimal element in D. Since x is minimal in $X - \mathrm{Exp}_\nu(Y)$, $\tau(x) < \omega^\nu$, while putting $Z = \mathrm{Exp}_\nu(Y)$, $x \in \mathrm{Exp}_\nu(Z) - Z$ implies $x \in \mathrm{Csg}_\nu(X)$ contradictrily. Hence D is void and $\mathrm{Exp}_\nu(\mathrm{Exp}_\nu(Y)) = \mathrm{Exp}_\nu(Y)$.

REMARK 1. By (iii), (iv), (v) and (viii) of Lemma 2.3 the operation Exp_ν on cuts of a ramified set satisfies the conditions of the so-called "finite-additive closure operation".

REMARK 2. In most cases where we are concerned with several ramified sets, each of them is a subset of one of them, for instance X. Then, when we write simply $\mathrm{Ub}(x)$, $\tau(x)$, $\mathrm{Exp}_\nu(Z)$ etc., it means $\mathrm{Ub}(x\,;X)$, $\tau(x\,;X)$, $\mathrm{Exp}_\nu(Z\,;X)$ etc. respectively, and according to Lemma 2.1 (i) and Lemma 2.2 (iv), if Y is a subset or a cut of X, then $\mathrm{Ub}(x\,;Y)$, $\mathrm{Exp}_\nu(Z\,;Y)$ etc. are mostly expressed by $\mathrm{Ub}(x) \cap Y$, $\mathrm{Exp}_\nu(Z) \cap Y$ etc. respectively.

DEFINITION 7. Let X be a non-void ramified set. X is called *resoluble* if for any non-void subset Y of X, there exists a $y \in Y$ such that $\mathrm{Ub}(y) \cap Y$ is totally ordered. (Of course a void set is regarded as a totally ordered set). If X is not resoluble, then it is called *irresoluble*. If for any $x \in X$, $\mathrm{Ub}(x)$ is not totally ordered, then X is called *perfectly irresoluble*.

For convenience a void set is regarded as resoluble and (perfectly) irresoluble in the same time.

For a subset Y of X, let $J(Y)$ denote the set of all $y \in Y$ such that $\mathrm{Ub}(y) \cap Y$ is not totally ordered. Put $J_0(X) = X$, $J_{\xi+1}(X) = J(J_\xi(X))$ for any ξ and $J_\xi(X) = \bigcap_{\zeta<\xi} J_\zeta(X)$ for a limit number ξ. If $J_{\xi+1}(X) \subsetneqq J_\xi(X)$ for any $\xi < \lambda$, then $\bar{\lambda}$ can not exceed the potency of $X - J_\lambda(X)$. Hence there exists a λ, whose potency does not exceed $\overline{\overline{X}}$, such that $J_{\lambda+1}(X) = J_\lambda(X)$. Put $K(X) = J_\lambda(X)$ for such a λ, then since $J(K(X)) = K(X)$, $K(X)$ is perfectly irresoluble. If a subset Y of X is not included in $K(X)$, then there exists the least number ξ such that $J_\xi(X)$ does not include Y. ξ is not a limit number or otherwise $Y \subset J_\zeta(X)$ for any $\zeta < \xi$ and $Y \subset \bigcap_{\zeta<\xi} J_\zeta(X)$ $= J_\xi(X)$ contradictorily. Put $\xi = \zeta + 1$, then $Y \subset J_\zeta(X)$. Let y be any element in $Y - J_\xi(X)$, then since $y \in J_\zeta(X) - J_\xi(X)$, $\mathrm{Ub}(y) \cap Y (\subset \mathrm{Ub}(y) \cap J_\zeta(X))$ is totally ordered, i.e., Y is not perfectly irresoluble. Hence $K(X)$ is the largest (in the sense of inclusion) perfectly irresoluble subset of X, which we shall call the *perfectly irresoluble part* of X. Similarly we can see that X is resoluble if and only if $K(X)$ is void. In general $X - K(X)$ is always resoluble, which we shall call the *resoluble part* of X.

Summing up we have

Theorem 4. *Any ramified set X is uniquely decomposed into the union of its perfectly irresoluble part and resoluble part. X is resoluble if and only if its perfectly irresoluble part is void.*

Corollary. *A subset of resoluble ramified set is also resoluble.*

DEFINITION 8. (i). A mapping f (many-to-one in general) which maps X into Y is called *increasing* if $a < b$ implies $f(a) < f(b)$ for any a and b in X.

(ii). If there exists an increasing mapping of X into Y, then X is called *smaller* than Y and Y is called *larger* than X; in symbol $X \propto Y$. If $X \propto Y$ and $Y \propto X$, then X is called *equivalent* to Y; in symbol $X \sim Y$. $X \not\propto Y$ and $X \not\sim Y$ denote the negations of $X \propto Y$ and $X \sim Y$ respectively. If $X \propto Y$ and $Y \not\propto X$, then we write $X \lnsim Y$.

(iii). A one-to-one increasing mapping of X onto whole Y with an increasing inverse is called an *isomorphism* of X to Y. If there exists an isomorphism of X to Y, then X is called *isomorphic* to Y; in symbol $X \equiv Y$. $X \not\equiv Y$ denotes the negation of $X \equiv Y$.

(iv). For an increasing mapping f of X into Y and a subset Z of X, $f(Z)$ denotes the set $\{f(x) \mid x \in Z\}$. f is called *reduced* if $f(Z)$ is a cut of Y for any cut Z of X.

(v). $\kappa(X)$ denotes the least ordinal number ξ such that $W_\xi \not\propto X$.

Lemma 2.4. *Let f be an increasing mapping of X into Y. Then,* (i) $\tau(f(x))\geqq\tau(x)$ *for any* $x\in X$, (ii) *f is reduced if and only if* $\tau(f(x))=\tau(x)$ *for any* $x\in X$.

Proof. (i). Since $\mathrm{Lb}(x)$ is a well-ordered set and f is increasing, $\mathrm{Lb}(x)$ is isomorphic to $f(\mathrm{Lb}(x))$ which is a subset of $\mathrm{Lb}(f(x))$. Hence the order-type $\tau(x)$ of $\mathrm{Lb}(x)$ does not exceed the type $\tau(f(x))$ of $\mathrm{Lb}(f(x))$ by a well-known theorem on well-ordered sets. Hence $\tau(x)\leqq\tau(f(x))$.

(ii). Assume that there exists an $x\in X$ such that $\tau(f(x))>\tau(x)$. Let x be such a minimal element (see Lemma 2.1 (ii)). Since $\tau(f(x))>\tau(x)$, there exists a unique element b such that $\tau(b)=\tau(x)$ and $b<f(x)$. If $a<x$, then $\tau(f(a))=\tau(a)<\tau(x)=\tau(b)$ and hence $f(a)\neq b$. Hence there is no element $a\in\mathrm{Lb}'(x)$ such that $f(a)=b$. $\mathrm{Lb}'(x)$ is a cut of X, while since $f(\mathrm{Lb}'(x))$ contains $f(x)$ and does not contain $b<f(x)$, $f(\mathrm{Lb}'(x))$ is not a cut of Y. Hence f is not reduced.

Conversely assume $\tau(f(x))=\tau(x)$ for any $x\in X$, and let Z be a cut of X. If $b<f(z)$ where $z\in Z$, then, since $\tau(b)<\tau(f(z))=\tau(z)$, there exists an $a<z$ such that $\tau(a)=\tau(b)$. Since $\tau(f(a))=\tau(a)=\tau(b)$ and both $f(a)$ and b are contained in $\mathrm{Lb}(f(z))$, $f(a)$ coincides with b. Since Z is a cut of X, $a\in Z$ and $b\in f(Z)$. Hence $f(Z)$ is a cut of Y and f is reduced.

Lemma 2.5. *If $X\propto Y$, then there exists a reduced increasing mapping of X into Y.*

Proof. $X\propto Y$ implies the existence of an increasing mapping g of X into Y. For any $x\in X$, since $\tau(x)\leqq\tau(g(x))$ by Lemma 2.4 (i), an element $f(x)$ of Y is uniquely determined by x in such a way that $f(x)\leqq g(x)$ and $\tau(f(x))=\tau(x)$. If $x<x'$ where $x, x'\in X$, then $f(x)\leqq g(x)<g(x')$, and hence $f(x)$ and $f(x')$ are contained in $\mathrm{Lb}'(g(x'))$. Since $\mathrm{Lb}'(g(x'))$ is well-ordered and $\tau(f(x))=\tau(x)<\tau(x')=\tau(f(x'))$, we have $f(x)<f(x')$ and f is increasing. Further f is reduced by Lemma 2.4 (ii), and our lemma is proved.

Corollary. *If $X\propto Y$, then* $\mathrm{Seg}_\nu(X)\propto\mathrm{Seg}_\nu(Y)$ *and* $\mathrm{Csg}_\nu(X)\propto\mathrm{Csg}_\nu(Y)$.

It is easily seen that the relation $\propto$ is a quasi-ordering between ramified sets and the relation $\sim$ is an equivalence relation (see [2], p. 4).

Theorem 5. (i). *If $\kappa(X)=\mu+1$, then $X\sim W_\mu$.* (ii). *If $\kappa(X)<\kappa(Y)$, then $X\not\approx Y$.*

Proof. (i). By the definition of $\kappa(X)$, $W_\mu\propto X$.

For any element $x\in X$, $\tau(x)<\mu$, since $W_{\mu+1}\not\propto\mathrm{Lb}'(x)$. Put $f(x)=\tau(x)$ for any $x\in X$, then obviously f is an increasing mapping of X into W_μ. Hence $X\propto W_\mu$, which shows $X\sim W_\mu$.

(ii). Put $\mu=\kappa(X)$. Since $\mu<\kappa(Y)$, $W_\mu\propto Y$ and there exists an increasing mapping g of W_μ into Y. Similarly as the above, the mapping f of X such that $f(x)=\tau(x)$ for any $x \in X$ is an increasing mapping of X into W_μ. Since gf is an increasing mapping of X into Y, $X \propto Y$. If $Y \propto X$, then $W_\mu \propto X$ contradicting $\kappa(X)=\mu$. Hence $X \lessapprox Y$.

We shall say that X is *comparable* with Y if either $X \propto Y$ or $Y \propto X$.

Corollary. *If $\kappa(X)$ is an isolated number, then X is comparable with any Y.*

2. Main Theorems

DEFINITION 9. Let ω_β be a fixed regular initial number greater than ω_0 as we assumed in Chapter 1, § 1.

(i). $\mathfrak{R}_\beta$ denotes the family of all ramified sets with potencies less than $\aleph_\beta$.

(ii). $\mathfrak{S}_\beta$ denotes the family of all resoluble ramified sets in $\mathfrak{R}_\beta$.

(iii). $[\mathfrak{R}_\beta]$ denotes the family of all equivalence classes of sets $X \in \mathfrak{R}_\beta$.

(iv). A class $\mathfrak{X}$ in $[\mathfrak{R}_\beta]$ is called resoluble if it contains a resoluble set. The family of all resoluble classes in $[\mathfrak{R}_\beta]$ is denoted by $[\mathfrak{S}_\beta]$.

(v). For $\mathfrak{X}$ and $\mathfrak{Y}$ in $[\mathfrak{R}_\beta]$, $\mathfrak{X} \propto \mathfrak{Y}$ means that there exists an $X \in \mathfrak{X}$ and a $Y \in \mathfrak{Y}$ such that $X \propto Y$.

REMARK 1. $[\mathfrak{S}_\beta]$ is not defined as the family of all equivalence classes of sets in $\mathfrak{S}_\beta$. But it is not a matter of much difference, and $[\mathfrak{S}_\beta]$ may be taken for such a family without any modification in the succeeding mention.

REMARK 2. Since the relation $\propto$ between ramified sets is a quasi-ordering, it is an order-relation between classes in $[\mathfrak{R}_\beta]$. It is obvious that if $\mathfrak{X} \propto \mathfrak{Y}$ for $\mathfrak{X}$ and $\mathfrak{Y}$ in $[\mathfrak{R}_\beta]$, then $X \propto Y$ for any $X \in \mathfrak{X}$ and $Y \in \mathfrak{Y}$.

Lemma 2.6. *$X \in \mathfrak{R}_\beta$ implies $\kappa(X) < \omega_\beta$.*

Proof. Since ω_β is regular and $\overline{\overline{X}} < \aleph_\beta$, numbers $\tau(x)$ with $x \in X$, which are obviously less than ω_β, are not cofinal to ω_β. Hence putting $\xi = \sup_{x \in X} \tau(x)$, $\xi < \omega_\beta$ and $W_{\xi+1} \not\propto X$, i.e., $\kappa(X) \leqq \xi + 1 < \omega_\beta$.

Our main purpose of this paper is to prove the following:

Main Theorem A. (i). *The family $[\mathfrak{S}_\beta]$ of resoluble classes is well-ordered by $\propto$.* (ii). *A resoluble ramified set is comparable with any other ramified set (resoluble or not).*

Main Theorem B. *Continuum Hypothesis* (see [2], p. 45 or [8]) *implies that,* (i) *there exist ramified sets which are not comparable with each other,* *and*, (ii) *there exists a sequence of ramified sets* X_i, $i=1, 2, \cdots$ *such that* $X_{i+1} \precnsim X_i$ *for any* $i<\omega$.

Of course examples to confirm Theorem B can not be obtained within the confines of resoluble ramified sets as far as Theorem A is valid. Theorem B will be proved in Chapter IV, § 2, while hereafter up to the end of Chapter III, we shall exclusively discuss about Theorem A.

Speaking of only Theorem A, (ii), it seems to be proved rather easily by induction on the least number ξ with which $J_\xi(X)$ vanishes for a resoluble ramified set X (see the mention above Theorem 4). But by this induction the proof of Theorem A, (i) seems at least as laborious as the discussion we shall proceed henceforth.

In this paper, in order to prove Theorem A, we shall construct a certain sequence $\mathfrak{N}=\{N_\lambda | \lambda<\beta^\sharp\}$ of sets $N_\lambda \in \mathfrak{S}_\beta$ starting from $N_0=\emptyset$, and show that

C) *for any* $\delta<\omega_\beta$, *there exists a* $\lambda<\beta^\sharp$ *with* $\kappa(N_\lambda)>\delta$, *and*

D) *each* N_λ *satisfies the following* D. 1), D. 2) *and* D. 3):

D. 1) $N_\mu \precnsim N_\lambda$ *for any* $\mu<\lambda$.

D. 2) *If* $X \in \mathfrak{S}_\beta$ *and* $X \not\propto N_\mu$ *for any* $\mu<\lambda$, *then* $N_\lambda \propto X$.

D. 3) N_λ *is comparable with any* $X \in \mathfrak{R}_\beta$.

Note that

Lemma 2.7. *If a* N_λ *in* $\mathfrak{N}$ *satisfies* D. 1) *and* D. 2′) *mentioned below, then* N_λ *satisfies* D. 2) *and* D. 3).

D. 2′). *If* $X \in \mathfrak{R}_\beta$ *and* $X \not\propto N_\mu$ *for any* $\mu<\lambda$, *then* $N_\lambda \propto X$.

Proof. It is obvious that N_λ satisfies D. 2). Let X be any set in $\mathfrak{R}_\beta$. If there exists a $\mu<\lambda$ with $X \propto N_\mu$, then by D. 1) $X \propto N_\lambda$. If $X \not\propto N_\mu$ for any $\mu<\lambda$, then $N_\lambda \propto X$ by D. 2′). Hence N_λ is comparable with X, and N_λ satisfies D. 3).

REMARK. Hereafter, when we say that a set in $\mathfrak{N}$, for instance N_ξ, satisfies D. 1), D. 2), D. 3) or D. 2′), it means that N_ξ satisfies it in which λ is replaced by ξ.

Before we actually construct this sequence $\mathfrak{N}$, we shall assume that there exists a sequence $\mathfrak{N}$ which satisfies C) and D), and consider the consequence of its existence.

Lemma 2.8. *If a sequence* $\mathfrak{N}$ *of* $N_\lambda \in \mathfrak{S}_\beta$ *with* $N_0=\emptyset$ *satisfies* C) *and* D), *then for any* $X \in \mathfrak{S}_\beta$, *there exists a* $\lambda<\beta^\sharp$ *such that* $X \sim N_\lambda$.

Proof. Since $\kappa(X)<\omega_\beta$, there exists a $\lambda<\beta^\sharp$ such that $\kappa(X)<\kappa(N_\lambda)$, which implies $X \propto N_\lambda$ by Theorem 5, (ii). Let λ be the least number such that $X \propto N_\lambda$. If $\lambda=0$, then $N_0=\emptyset$ and hence $X=\emptyset$, i.e., $X \equiv N_0$. If $\lambda>0$, then $X \not\propto N_\mu$ for any $\mu<\lambda$, which implies $N_\lambda \propto X$ by D. 2). Hence $X \sim N_\lambda$.

Corollary. $[\mathfrak{S}_\beta]$ *is well-ordered by* $\propto$ (*Theorem A*, (*i*)).

Proof. The order-type of $[\mathfrak{S}_\beta]$ is same as the type of sequence $\mathfrak{N}$ by the lemma above, i.e., $[\mathfrak{S}_\beta]$ is well-ordered by $\propto$ in the type β^*.

Lemma 2.9. *Any* $X \in \mathfrak{S}_\beta$ *is comparable with any* $Y \in \mathfrak{R}_\beta$.

Proof. By Lemma 22, there exists a $\lambda < \beta^*$ with $X \sim N_\lambda$. By D. 3), N_λ is comparable with any $Y \in \mathfrak{R}_\beta$. Hence X is also comparable with Y.

Corollary. *The existence of* $\mathfrak{N}$ *for any regular number* $\omega_\beta > \omega_0$, *which satisfies* C) *and* D), *implies Theorem A*, (ii).

Proof. For any ramified sets X and Y, there exists sufficiently large regular number $\omega_\beta > \omega_0$ such that X and Y are contained in $\mathfrak{R}_\beta$. Especially if X is resoluble, then $X \in \mathfrak{S}_\beta$. Hence X is comparable with Y by the lemma above.

Thus Theorem A is proved if a certain sequence $\mathfrak{N}$ of $N_\lambda \in \mathfrak{S}_\beta$ with $N_0 = \emptyset$ satisfies C) and D) (associating to every regular number $\omega_\beta > \omega_0$). Now we shall consider the construction of $\mathfrak{N}$, and for this purpose we shall define several operations on ramified sets.

DEFINITION 10. (i). Let Λ be a set of indecies, or especially a set of ordinal numbers, and assume that a ramified set X_λ is assigned to each $\lambda \in \Lambda$. $\bigvee_{\lambda \in \Lambda} X_\lambda$, or simply $\bigvee_\Lambda X_\lambda$ (or occationally $\bigvee_{\nu < \lambda < \xi} X_\lambda$, etc.), denotes the set of all pairs (λ, x) with $\lambda \in \Lambda$ and $x \in X_\lambda$, where $(\lambda, x) < (\mu, y)$ holds if and only if $\lambda = \mu$ and $x < y$ within X_λ.

For a subset Y of X_λ, the set $\{(\lambda, x) \mid x \in Y\}$ is denoted by (λ, Y).

(ii). For an ordinal number λ and a ramified set X, $W_\lambda + X$ denotes the set which consists of all $\mu \in W_\lambda$ and all terms $\lambda + x$ with $x \in X$, where the order-relation preserves original meaning within W_λ, $\mu < \lambda + x$ for any $\mu \in W_\lambda$ and $x \in X$, and $\lambda + x < \lambda + y$ if and only if $x < y$ within X.

For a subset Y of X the set $\{\lambda + x \mid x \in Y\}$ is denoted by $\lambda + Y$.

(iii). For ramified sets X and Y and an ordinal number $\nu < \omega_\beta$, the *ramified product* $X \odot_\nu Y$ is a set which consists of all $x \in X$ and pairs (x, y) with $x \in \mathrm{Seg}_\nu(X)$ and $y \in Y$, where order within X preserves orginal meaning, $x < (x', y)$ if and only if $x \leqq x'$ within X, and $(x, y) < (x', y')$ if and only if $x = x'$ and $y < y'$ within Y.

For a subset Z of Y, the set $\{(x, z) \mid z \in Z\}$ is denoted by (x, Z).

REMARK 1. $\bigvee_\Lambda X_\lambda$ and $W_\lambda + X$ are called *a cardinal sum* and *an ordinal sum* respectively in [2] or [3]. But the first term of $W_\lambda + X$ is restricted to a well-ordered set in order that the resultant be a ramified set.

REMARK 2. In $W_\lambda + X$, the notation $\lambda + x$ is used only to denote a term, and the signeture $+$ has no special meaning. But when $X = W_\xi$ and

$x=\mu\in W_\xi$, $\lambda+\mu$ is taken for the usual sum of ordinal numbers. Thus we have

$$(3) \qquad W_\lambda + W_\xi = W_{\lambda+\xi},$$

not only the equivalency of both sides.

Lemma 2.10. *$\bigvee_\Lambda X_\lambda$, $W_\lambda+X$ and $X\odot_\nu Y$ are ramified sets, and if X_λ (for any $\lambda\in\Lambda$), X and Y are resoluble, then $\bigvee_\Lambda X$, $W_\lambda+X$ and $X\odot_\nu Y$ are resoluble.*

We omit the proof since it is obvious.

Lemma 2.11. *If $X_\lambda\propto Y$ for any $\lambda\in\Lambda$, then $\bigvee_\Lambda X_\lambda\propto Y$.*

Proof. For any $\lambda\in\Lambda$ there exists an increasing mapping f_λ of X_λ into Y. Put $f((\lambda, x))=f_\lambda(x)$ for any $\lambda\in\Lambda$ and $x\in X_\lambda$, then obviously f is an increasing mapping of $\bigvee_\Lambda X_\lambda$ into Y and $\bigvee_\Lambda X_\lambda\propto Y$.

Corollary. *Let X_λ and $Y_{\lambda'}$ be ramified sets assigned to each $\lambda\in\Lambda$ and $\lambda'\in\Lambda'$ respectively. If for any $\lambda\in\Lambda$ there exists a $\lambda'\in\Lambda'$ such that $X_\lambda\propto Y_{\lambda'}$, then $\bigvee_\Lambda X_\lambda\propto\bigvee_{\Lambda'} Y_{\lambda'}$.*

Further the following lemma is easily seen and we omit the proof.

Lemma 2.12. (i). *If $X\propto X'$ and $Y\propto Y'$, then $W_\lambda+X\propto W_\lambda+X'$ and $X\odot_\nu Y\propto X'\odot_\nu Y'$.*

(ii). *Putting* $A=\mathrm{Lay}_\nu(X)$, $\mathrm{Csg}_\nu(X)\equiv\bigvee_{a\in A}\mathrm{Ub}'(a)$.

(iii). $X\propto W_{\omega^\nu}+Y$, *if and only if* $\mathrm{Csg}_\nu(X)\propto Y$.

(iv). $(\bigvee_\Lambda X_\lambda)\odot_\nu Y\equiv\bigvee_{\lambda\in\Lambda}(X_\lambda\odot_\nu Y)$.

(v). $\kappa(\bigvee_\Lambda X_\lambda)=\sup_{\lambda\in\Lambda}\kappa(X_\lambda)$. *If* $\max(\kappa(X),\ \kappa(Y),\ \omega^\eta)\leqq\omega^\nu$, *then* $\kappa(X\odot_\eta Y)\leqq\omega^\nu$. *If* $\kappa(X)\leqq\omega^\nu$ *and* $\xi<\omega^\nu$ *then* $\kappa(W_\xi+X)\leqq\omega^\nu$.

(vi). $\mathrm{Csg}_\nu(\bigvee_\Lambda X_\lambda)\sim\bigvee_{\lambda\in\Lambda}\mathrm{Csg}_\nu(X_\lambda)$. *If $X\propto Y$ and $\eta\leqq\nu$, then* $\mathrm{Csg}_\nu(X\odot_\eta Y)\sim\mathrm{Csg}_\nu(Y\odot_\eta X)\sim\mathrm{Csg}_\nu(Y)$. *If $\xi<\omega^\nu$, then* $\mathrm{Csg}_\nu(W_\xi+X)\equiv\mathrm{Csg}_\nu(X)$.

(vii). *If Z is a cut of X, then* $\mathrm{Exp}_\nu(Z\,;X\odot_\nu Y)=\mathrm{Exp}_\nu(Z\,;X)$. *If Z is a cut of $\bigvee_\Lambda X_\lambda$, then* $\mathrm{Exp}_\nu(Z\,;\bigvee_\Lambda X_\lambda)\equiv\bigvee_{\lambda\in\Lambda}\mathrm{Exp}_\nu(Z\cap X_\lambda\,;X_\lambda)$.

Lemma 2.13. *If η is an isolated number less then ω^ν, then $W_\eta+X\sim W_\eta\odot_\nu X$.*

Proof. W_η has a greatest number ξ, and $\xi\in\mathrm{Seg}_\nu(W_\eta)$. The mapping f such that $f(\mu)=\mu$ for $\mu\in W_\eta$ and $f(\eta+x)=(\xi, x)$ for $x\in X$ is obviously an increasing mapping of $W_\eta+X$ into $W_\eta\odot_\nu X$. Conversely the mapping g such that $g(\mu)=\mu$ for $\mu\in W_\eta$, and $g((\mu, x))=\eta+x$ for any $\mu\in W_\eta$ and $x\in X$ is an increasing mapping of $W_\eta\odot_\nu X$ into $W_\eta+X$. Hence $W_\eta+X\sim W_\eta\odot_\nu X$.

Corollary. *$W_n+X\sim W_n\odot_\nu X$ for any ν with $0<\nu<\omega_\beta$.*

Lemma 2.14. *If Y is not void, then $X\odot_\nu(Y\odot_\nu Z)\sim(X\odot_\nu Y)\odot_\nu Z$.*

Proof. Here temporarily let x, y and z denote elements in X, Y and Z respectively and x' and y' denote elements in $\mathrm{Seg}_\nu(X)$ and $\mathrm{Seg}_\nu(Y)$ respectively. $X\odot_\nu(Y\odot_\nu Z)$ consists of all terms x, (x', y) and $(x', (y', z))$ while $(X\odot_\nu Y)\odot_\nu Z$ consists of all terms x, (x', y), $((x', y'), z)$ and (x', z). (remark that $(x', y)\in\mathrm{Seg}_\nu(X\odot_\nu Y)$ if and only if $y\in\mathrm{Seg}_\nu(Y)$). The mapping f such that $f(x)=x$, $f((x', y))=(x', y)$ and $f((x', (y', z)))=((x', y'), z)$ is an increasing mapping of $X\odot_\nu(Y\odot_\nu Z))$ into $(X\odot_\nu Y)\odot_\nu Z$. Let y'_0 be any element in $\mathrm{Seg}_\nu(Y)$. The mapping g which is the inverse of f in the range of f and $g((x', z))=(x', (y'_0, z))$, is an increasing mapping of $(X\odot_\nu Y)\odot_\nu Z$ into $X\odot_\nu(Y\odot_\nu Z)$ and hence we have our equivalency.

Lemma 2.15. *If X is not void and $\eta<\omega^\nu$, then $W_\eta+(X\odot_\nu Y)\sim(W_\eta+X)\odot_\nu Y$.*

Proof. We use notations x, y, x' and y' with same meaning in the proof of Lemma 2.14, and μ as an element in W_η. $W_\eta+(X\odot_\nu Y)$ consists of all terms μ, $\eta+x$ and $\eta+(x', y)$ while $(W_\eta+X)\odot_\nu Y$ consists of all terms μ, $\eta+x$, $(\eta+x', y)$ and (μ, y). $W_\eta+(X\odot_\nu Y)\propto(W_\eta+X)\odot_\nu Y$ is obvious. Let x'_0 be any element in $\mathrm{Seg}_\nu(X)$, then the mapping f such that $f(\mu)=\mu$, $f(\eta+x)=\eta+x$, $f((\eta+x', y))=\eta+(x', y)$ and $f((\mu, y))=\eta+(x'_0, y)$ is an increasing mapping of $(W_\eta+X)\odot_\nu Y$ into $W_\eta+(X\odot_\nu Y)$. Hence we have our equivalency.

We assume that for any limit number $\lambda<\beta^\#$, a set Λ_λ of ordinal numbers $\mu<\lambda$ is selected by axiom of choice, so that $\overline{\overline{\Lambda}}_\lambda=\aleph_{\mathrm{cf}(\lambda)}$ and Λ_λ is cofinal to λ. For an isolated number λ put $\Lambda_\lambda=\Lambda_{\mathrm{ls}(\lambda)}$. Besides, put $\sigma_\xi=\omega$ if $\mathrm{cf}(\mathrm{ls}(\xi))=\beta$ and $\sigma_\xi=0$ if $\mathrm{cf}(\mathrm{ls}(\xi))<\beta$. N_λ in $\mathfrak{N}$ is inductively defined along the following principle.

PRINCIPLE 1. *Put $N_0=\emptyset$ and $N_n=W_n$.*

Case $\lambda\in\Gamma^0$. Then put

$$N_\lambda = W_{\mathrm{fr}(\lambda)}+(\bigvee_{\Lambda_\lambda} N_\mu).$$

Case $\lambda\in\Gamma^1$. Let $\lambda=\alpha_\nu(\xi+\zeta)+n$ be the canonical decomposition of λ (see Definition 4), *and put*

$$N_\lambda = N_{\alpha_\nu(\zeta)+n}\odot_\nu N_{\alpha_\nu(\xi)+\sigma_\xi}.$$

Case $\lambda\in\Gamma^2$. Let $\lambda=\alpha_\nu(\omega_\beta^\xi)+n$ be the canonical decomposition of λ. If ξ is a limit number (and necessarily $\mathrm{cf}(\xi)=\beta$*), then put*

$$N_\lambda = W_{\omega^\nu}+N_{\xi+n}.$$

If ξ is an isolated number $\zeta+1$, then put

$$N_\lambda = W_{\omega^\nu}+N_{\omega\zeta+\sigma_\zeta+n}.$$

REMARK 1. According to Remark below Definition 4, the index attached to each N which appears in the right sides of four formulae in Principle 1 is surely less than λ, and the definition of N_λ is inductive.

REMARK 2. Let λ be a limit number in Γ^0. Of course the set N_λ itself, as well as $N_{\lambda+n}$, is determined depending on the choice of Λ_λ, but it is easily seen that, under the assumption that N_μ with any $\mu<\lambda$ satisfies D. 1), the equivalence class which contains N_λ is determined independently of the choice of the sequence Λ_λ cofinal to λ. (Refer to Corollary of Lemma 2. 11).

Referring to Lemma 2. 10, we can see that any N_λ is contained in $\mathfrak{S}_\beta$. For $\lambda=\alpha_\nu(\omega_\beta)$, $N_\lambda=W_{\omega^\nu}+N_0=W_{\omega^\nu}$, and $\kappa(N_\lambda)=\omega^\nu+1$, from which it follows that $\mathfrak{N}$ satisfies C). Therefore in proving Main Theorem A, it is remained only to prove that each N_λ in $\mathfrak{N}$ satisfies D. 1), D. 2) and D. 3). Next chapter is devoted to this proof.

CHAPTER III. PROOF OF MAIN THEOREM A

In this chapter we shall show that any $N_\lambda\in\mathfrak{N}$ satisfies D. 1), D. 2) and D. 3) to complete the proof of Main Theorem A. First we prepare some lemmas on N_λ (§ 1), and next discuss about N_λ for each case $\lambda\in\Gamma^0$, $\lambda\in\Gamma^1$ and $\lambda\in\Gamma^2$ in § 2, § 3 and § 4 respectively.

1. Preliminaries on N_λ

Lemma 3. 1. *If* $\lambda\in\tilde{\Gamma}^1_\nu\cup\tilde{\Gamma}^2_\nu$ *and* $\eta<\omega^\nu$, *then*

$$W_\eta+N_\lambda\sim N_\lambda. \tag{4}$$

Proof. If $\lambda\in\tilde{\Gamma}^2_\nu$, then N_λ is of the form $W_{\omega^{\nu'}}+X$ where $\eta<\omega^\nu\leqq\omega^{\nu'}$. Since $W_\eta+W_{\omega^{\nu'}}=W_{\eta+\omega^{\nu'}}=W_{\omega^{\nu'}}$, we have (4) trivially. Especially (4) is true for $\lambda=\alpha_\nu(\omega_\beta)$ which is the least number in $\tilde{\Gamma}^1_\nu\cup\tilde{\Gamma}^2_\nu$ (see Lemma 1. 13). Assume $\lambda'\in\tilde{\Gamma}^1_\nu$ and (4) is true for any $\lambda\in\tilde{\Gamma}^1_\nu\cup\tilde{\Gamma}^2_\nu$ with $\lambda<\lambda'$. Let $\lambda'=\alpha_{\nu'}(\xi+\zeta)+n$ be the canonical decomposition of λ', and then $N_{\lambda'}=N_{\alpha_{\nu'}(\zeta)+n}\odot_{\nu'}N_{\alpha_{\nu'}(\xi)+\sigma_\xi}$. Since ζ is a limit number with $\mathrm{cf}(\zeta)=\beta$, $\alpha_{\nu'}(\zeta)+n\in\tilde{\Gamma}^1_\nu\cup\tilde{\Gamma}^2_\nu$ and by assumption $W_\eta+N_{\alpha_{\nu'}(\zeta)+n}\sim N_{\alpha_{\nu'}(\zeta)+n}$ which is not void. Hence by Lemma 2. 15, $W_\eta+N_{\lambda'}=W_\eta+(N_{\alpha_{\nu'}(\zeta)+n}\odot_{\nu'}N_{\alpha_{\nu'}(\xi)+\sigma_\xi})$ $\sim(W_\eta+N_{\alpha_{\nu'}(\zeta)+n})\odot_{\nu'}N_{\alpha_{\nu'}(\xi)+\sigma_\xi}\sim N_{\alpha_{\nu'}(\zeta)+n}\odot_{\nu'}N_{\alpha_{\nu'}(\xi)+\sigma_\xi}=N_{\lambda'}$, and (4) is proved for $\lambda=\lambda'$.

Corollary. *If* $\lambda\in\Gamma^1\cup\Gamma^2$, *then* $W_1+N_\lambda\sim N_\lambda$.

Assume $2\leqq\nu<\omega_\beta$, and letting $\mu=\xi+\zeta$ be the φ_ν-decomposition of μ (see Definition 2, (ii)), $\zeta=\varphi^r_\nu(\mu)>0$. Now we define $\mathrm{ta}_\nu(\mu)$ (tail of μ) by

$$\mathrm{ta}_\nu(\mu) = \omega^\eta \cdot \zeta \qquad \textit{if} \quad \nu = \eta + 1 \text{ (and accordingly } \zeta < \omega)$$

and

$$\mathrm{ta}_\nu(\mu) = \omega^\zeta \qquad \textit{if } \nu \textit{ is a limit number.}$$

Lemma 3.2. *Under the same assumption as above, putting* $\mathrm{ta}_\nu(\mu) = \varepsilon$,

$$N_{\alpha_\nu(\mu)+n} = W_\varepsilon + N_{\alpha_\nu(\xi)+\sigma_\xi+n}\,. \tag{5}$$

Proof. We shall prove (5) only for the case $\nu = \eta + 1$, since the proof for the case that ν is a limit number is similarly obtained.

$\alpha_\nu(\xi+1) = \alpha_\eta(\omega_\beta^{\alpha_\nu(\xi)+1}) \in \Gamma_\eta^2$, and hence $N_{\alpha_\nu(\xi+1)+n} = W_{\omega^\eta} + N_{\alpha_\nu(\xi)+\sigma_\xi+n}$ (since $\omega\alpha_\nu(\xi) = \alpha_\nu(\xi)$; refer to Corollary 2 of Theorem 1), which shows (5) for $\zeta = 1$. Assume that (5) is true for $\zeta = k \geqq 1$. $\alpha_\nu(\xi+k+1) = \alpha_\eta(\omega_\beta^{\alpha_\nu(\xi+k)}) \in \Gamma_\eta^2$ and $N_{\alpha_\nu(\xi+k+1)+n} = W_{\omega^\eta} + N_{\alpha_\nu(\xi+k)+n} = W_{\omega^\eta} + W_{\omega^\eta \cdot k} + N_{\alpha_\nu(\xi)+\sigma_\xi+n}$ $= W_{\omega^\eta(k+1)} + N_{\alpha_\nu(\xi)+\sigma_\xi+n}$ which shows (5) for $\zeta = k+1$. Hence (5) is proved by induction on $\zeta < \omega$.

Corollary. *If* $2 \leqq \nu < \omega_\beta$, $\mu > 0$, $\varphi_\nu^\xi(\mu) = 0$ *and* $\mathrm{ta}_\nu(\mu) = \varepsilon$, *then*

$$N_{\alpha_\nu(\mu)+n} = W_{\varepsilon+n}\,. \tag{6}$$

In proving that N_λ satisfies D.1), D.2) and D.3), it is assumed that N_ξ with any $\xi < \lambda$ satisfies them. In the following three lemmas, which are used to prove that N_λ satisfies D.1), D.2) and D.3), especially D.1) is assumed for N_ξ with any $\xi < \lambda$, and under this assumption, for a limit number $\zeta \in \Gamma^0$ with $\zeta \leqq \lambda$ and for any sequence Δ'_ζ of ordinal numbers less than and cofinal to ζ, $\bigvee_{\Delta'_\zeta} N_\mu \sim \bigvee_{\Delta_\zeta} N_\mu \sim N_\zeta$, as we remarked below Principle 1.

Lemma 3.3. *If* ξ *is a* γ*-number greater than* 1 *and* $0 < \zeta < \xi\omega$, *then putting* $\lambda = \xi + \sigma_\xi + \zeta$,

$$N_\lambda \sim N_\zeta \odot_1 N_{\xi+\sigma_\xi}\,. \tag{7}$$

Proof. If ζ is a finite number greater than 0, then $N_{\xi+\sigma_\xi+\zeta} = W_\zeta + N_{\xi+\sigma_\xi}$ $\sim N_\zeta \odot_1 N_{\xi+\sigma_\xi}$ by Corollary of Lemma 2.13, and (7) is proved for finite numbers ζ. Now assume that $\omega \leqq \zeta < \xi\omega$ and $N_{\xi+\sigma_\xi+\mu} \sim N_\mu \odot_1 N_{\xi+\sigma_\xi}$ for any μ with $0 < \mu < \zeta$.

In the case $\zeta \in \Gamma^0$, $N_\lambda = W_n + \bigvee_{\Delta_\lambda} N_\mu$ where $n = \mathrm{fr}(\lambda)$, and referring to the remark above,

$$\begin{aligned} N_\lambda &\sim W_n + \bigvee\nolimits_{\mu\in\Delta_\zeta} N_{\xi+\sigma_\xi+\mu} \\ &\sim W_n + \bigvee\nolimits_{\mu\in\Delta_\zeta} (N_\mu \odot_1 N_{\xi+\sigma_\xi}) && \text{(by assumption),} \\ &\sim W_n + ((\bigvee\nolimits_{\mu\in\Delta_\zeta} N_\mu) \odot_1 N_{\xi+\sigma_\xi}) && \text{(by Lemma 2.12, (iv)),} \end{aligned}$$

$$\sim (W_n+(\bigvee_{\mu\in\Lambda_\zeta} N_\mu)) \odot_1 N_{\xi+\sigma_\xi} \quad \text{(by Lemma 2.15)},$$
$$\sim N_\zeta \odot_1 N_{\xi+\sigma_\xi},$$

and (7) is proved for $\zeta\in\Gamma^0$.

If $\zeta\in\Gamma^1\cup\Gamma^2$ and accordingly $\mathrm{cf}(\mathrm{ls}(\zeta))=\beta$, then $\lambda\in\Gamma_1^1$ and $\gamma(\lambda)=\xi$, and (7) immediately follows from the definition of N_λ.

Lemma 3.4. *Assume that ξ is a γ-number not less than ω_β, $0<\zeta<\xi\omega$ and $2\leqq\nu<\omega_\beta$, then putting $\lambda=\alpha_\nu(\xi+\zeta)+n$,*

$$N_\lambda\sim N_{\alpha_\nu(\zeta)+n}\odot_\nu N_{\alpha_\nu(\xi)+\sigma_\xi}. \tag{8}$$

except where $n=\varphi_\nu^s(\zeta)=0$.

Proof. Since $\xi\in\Phi_\nu^l$, $\varphi_\nu^r(\xi+\zeta)=\varphi_\nu^r(\zeta)$. Hence putting $\varepsilon=\mathrm{ta}_\nu(\zeta)$, $\varepsilon=\mathrm{ta}_\nu(\xi+\zeta)$.

If $\varphi_\nu^s(\zeta)=0$, then $\varphi_\nu^r(\xi+\zeta)=\zeta$. In this case, excepting where $n=0$,

$$\begin{aligned}
N_\lambda &\sim W_\varepsilon+N_{\alpha_\nu(\xi)+\sigma_\xi+n} && \text{(by (5))},\\
&\sim W_\varepsilon+W_n+N_{\alpha_\nu(\xi)+\sigma_\xi} && \text{(since } \alpha_\nu(\xi)+\sigma_\xi\in\Gamma^0\text{)},\\
&\sim W_{\varepsilon+n}\odot_\nu N_{\alpha_\nu(\xi)+\sigma_\xi} && \text{(by Lemma 2.13 and } n>0\text{)},\\
&\sim N_{\alpha_\nu(\zeta)+n}\odot_\nu N_{\alpha_\nu(\xi)+\sigma_\xi} && \text{(by (6))},
\end{aligned}$$

and we have (8).

Assume $\varphi_\nu^s(\zeta)>0$ and $N_{\alpha_\nu(\xi+\theta)+n}\sim N_{\alpha_\nu(\theta)+n}\odot_\nu N_{\alpha_\nu(\xi)+\sigma_\xi}$ for any θ such that $0<\theta<\zeta$ with similar exceptions. We distinguish three cases.

Case 1. Putting $\varphi_\nu^s(\zeta)=\delta$, $\mathrm{cf}(\delta)<\beta$.

In this case, since $\varphi_\nu^s(\xi+\zeta)=\xi+\delta$,

$$\begin{aligned}
N_\lambda &\sim W_\varepsilon+N_{\alpha_\nu(\xi+\delta)+n} && \text{(by (5) and } \mathrm{cf}(\delta)<\beta)\\
&\sim W_\varepsilon+W_n+(\bigvee_{\theta\in\Lambda_\delta} N_{\alpha_\nu(\xi+\theta)+1}) && \text{(by } \mathrm{cf}(\delta)<\beta \text{ and } \delta\neq 0)\\
&\sim W_{\varepsilon+n}+(\bigvee_{\theta\in\Lambda_\delta}(N_{\alpha_\nu(\theta)+1}\odot_\nu N_{\alpha_\nu(\xi)+\sigma_\xi})) && \text{(by assumption)},\\
&\sim (W_{\varepsilon+n}+(\bigvee_{\theta\in\Lambda_\delta} N_{\alpha_\nu(\theta)+1}))\odot_\nu N_{\alpha_\nu(\xi)+\sigma_\xi} && \text{(by Lemma 2.15)},\\
&\sim (W_\varepsilon+N_{\alpha_\nu(\delta)+n})\odot_\nu N_{\alpha_\nu(\xi)+\sigma_\xi}\\
&\sim N_{\alpha_\nu(\zeta)+n}\odot_\nu N_{\alpha_\nu(\zeta)+\sigma_\xi} && \text{(by (5))},
\end{aligned}$$

and we have (8).

Case 2. ζ is a limit number with $\mathrm{cf}(\zeta)=\beta$ (then necessarily $\zeta\in\Phi_\nu^l$).

In this case, $\lambda\in\Gamma_\nu^1$ and $\gamma(\xi+\zeta)=\xi$. Hence (8) follows from the definition of N_λ.

Case 3. Letting $\delta+\eta$ be the φ_ν-decomposition of ζ, $\mathrm{cf}(\delta)=\beta$ and $\eta>0$.

In this case,

$$\begin{aligned}
N_\lambda &\sim W_\varepsilon + N_{\alpha_\nu(\xi+\delta)+\omega+n} && \text{(by (5))},\\
&\sim W_\varepsilon + W_n + (\bigvee_{k<\omega} N_{\alpha_\nu(\xi+\delta)+k})\\
&\sim W_{\varepsilon+n} + (\bigvee_{k<\omega} (N_{\alpha_\nu(\delta)+k} \odot_\nu N_{\alpha_\nu(\xi)+\sigma_\xi})) && \text{(by assumption)},\\
&\sim W_{\varepsilon+n} + ((\bigvee_{k<\omega} N_{\alpha_\nu(\delta)+k}) \odot_\nu N_{\alpha_\nu(\xi)+\sigma_\xi}) && \text{(by Lemma 2. 12, (iv))},\\
&\sim (W_{\varepsilon+n} + (\bigvee_{k<\omega} N_{\alpha_\nu(\delta)+k})) \odot_\nu N_{\alpha_\nu(\xi)+\sigma_\xi} && \text{(by Lemma 2. 15)},\\
&\sim (W_\varepsilon + N_{\alpha_\nu(\delta)+\omega+n}) \odot_\nu N_{\alpha_\nu(\xi)+\sigma_\xi}\\
&\sim N_{\alpha_\nu(\zeta)+n} \odot_\nu N_{\alpha_\nu(\xi)+\sigma_\xi} && \text{(by (5))},
\end{aligned}$$

and we have (8).

Hence for any possible case concerning ζ we have (8) and the proof is completed.

Corollary. *Assume that ξ is a γ-number not less than ω_β, $0<\nu<\omega_\beta$, $\omega\leqq\zeta<\xi\omega$ and $\zeta\in\Phi^\xi_\nu$. Put $\lambda=\xi+\sigma_\xi+\zeta+\sigma_\zeta$ if $\nu=1$, and $\lambda=\alpha_\nu(\xi+\zeta)+\sigma_\zeta$ if $\nu>1$, then*

$$N_\lambda \sim N_{\alpha_\nu(\zeta)+\sigma_\zeta} \odot_\nu N_{\alpha_\nu(\xi)+\sigma_\xi}\,. \tag{9}$$

Proof. If either $\nu=1$ or $\nu>1$ and $\mathrm{cf}(\zeta)<\beta$ (and accordingly $\sigma_\zeta=0$), (9) is a special case of (7) or (8). For the case where $\nu>1$, $\mathrm{cf}(\zeta)=\beta$ and accordingly $\sigma_\zeta=\omega$, see the proof of Case 3 in the lemma above.

REMARK. Assume that λ is a limit number in Γ^1, and let $\lambda=\alpha_\nu(\xi+\zeta)$ be the canonical decomposition of λ. Put $\Psi_\lambda=\{\mu\,|\,\omega^2<\mu<\zeta\}$ if $\nu=1$, $\Psi_\lambda=\{\mu\,|\,\omega<\mu<\zeta\}$ if ν is an isolated number greater than 1 and $\Psi_\lambda=\{\mu\,|\,\nu<\mu<\zeta\}$ if ν is a limit number. Then the numbers $\alpha_\nu(\xi+\mu)$ with $\mu\in\Psi_\lambda$ are cofinal to λ. For $\nu=1$, Ψ_λ is defined so that $\mu\in\Psi_\lambda$ implies $\xi+\sigma_\xi+\mu=\xi+\mu$. Hence it follows from Lemmas 3. 3 and 3. 4 that in general $\mu\in\Psi_\lambda$ implies

$$N_{\alpha_\nu(\xi+\mu)+n} \sim N_{\alpha_\nu(\mu)+n} \odot_\nu N_{\alpha_\nu(\xi)+\sigma_\xi}\,. \tag{10}$$

This Remark is the foundation of the proof that N_λ with a limit number $\lambda\in\Gamma^1$ satisfies D. 1), D. 2) and D. 3).

We shall say that a ramified set X satisfies condition E$_\nu$) if *there exists no maximal totally ordered subset of X whose type is less than ω^ν.*

Lemma 3. 5. *If $\lambda=\alpha_\nu(\mu)+\theta$ where $\omega_\beta\leqq\mu<\beta^*$ and $0\leqq\theta<\omega_\beta$, then there exists a $N'_\lambda\in\mathfrak{S}_\beta$ such that $N'_\lambda\sim N_\lambda$ and N'_λ satisfies* E$_\nu$).

Proof. If $\mu=\omega_\beta$ and $\theta=n<\omega$, then $N_\lambda=W_{\omega^\nu+n}$ which itself is a totally ordered set with type $\omega^\nu+n\geqq\omega^\nu$ and satisfies E$_\nu$).

Assume that μ and θ where $\omega \leqq \theta < \omega_\beta$ are given and that for any θ' with $0 \leqq \theta' < \theta$, putting $\lambda' = \alpha_\nu(\mu) + \theta'$, there exists a $N'_{\lambda'} \in \mathfrak{S}_\beta$ such that $N'_{\lambda'} \sim N_{\lambda'}$ and $N'_{\lambda'}$ satisfies E_ν). Put $N'_\lambda = W_{\mathrm{fr}(\theta)} + \bigvee_{\theta' \in \Lambda_\theta} N'_{\alpha_\nu(\mu)+\theta'}$, obviously $N'_\lambda \sim N_\lambda$. Since every maximal totally ordered subset A of N'_λ has a from $W_{\mathrm{fr}(\theta)} + (\theta', A')$ where $\theta' \in \Lambda_\theta$ and A' is a maximal totally ordered subset of $N'_{\alpha_\nu(\mu)+\theta'}$, $\bar{A} = \mathrm{fr}(\theta) + \bar{A}' \geqq \omega^\nu$. Hence N'_λ satisfies E_ν).

Hence in order to prove our lemma, we need to consider only the case $\theta = n < \omega$.

Assume $\omega_\beta < \mu < \beta^\sharp$ and that for any μ' with $\omega_\beta \leqq \mu' < \mu$ and θ' with $0 \leqq \theta' < \omega_\beta$, putting $\lambda' = \alpha_\nu(\mu') + \theta'$, there exists a $N'_{\lambda'} \in \mathfrak{S}_\beta$ such that $N'_{\lambda'} \sim N_{\lambda'}$ and $N'_{\lambda'}$ satisfies E_ν). Put $\theta = n < \omega$. We distinguish five cases.

Case 1. $\mu \in \Phi^i_\nu$.

Put $\varphi^s_\nu(\mu) = \xi$ and $\mathrm{ta}_\nu(\mu) = \varepsilon$ where $\mu \in \Phi^i_\nu$ implies $\varepsilon > 0$. Put $N'_\lambda = W_\varepsilon + N'_{\alpha_\nu(\xi)+\sigma_\xi+n}$, then by (5), $N'_\lambda \sim N_\lambda$. Since any maximal totally ordered subset A of N'_λ has a form $W_\varepsilon + A'$ where A' is a maximal totally ordered subset of $N'_{\alpha_\nu(\xi)+\sigma_\xi+n}$, the type $\bar{A}$, as well as $\bar{A}'$, is not less than ω^ν, i.e., N'_λ satisfies E_ν).

Case 2. $\mu \in \Phi^l_\nu$ and $\mathrm{cf}(\mu) < \beta$.

In this case, the sequence $\Lambda'_\lambda = \{\alpha_\nu(\mu') \mid \omega_\beta \leqq \mu' < \mu,\ \mu' \in \Lambda_\mu\}$ is cofinal to $\mathrm{ls}(\lambda) = \alpha_\nu(\mu)$. Put $N'_\lambda = W_{\mathrm{fr}(\lambda)} + \bigvee_{\Lambda'_\lambda} N'_{\lambda'}$, then $N'_\lambda \sim N_\lambda$ and N'_λ satisfies E_ν) similarly as we saw in the case $\omega \leqq \theta < \omega_\beta$.

Case 3. $\mu \in \Phi^l_\nu$, $\mathrm{cf}(\mu) = \beta$ and $\mu = \alpha_\nu(\mu) = \mathrm{ls}(\lambda)$.

In this case $\mu \in \Gamma_{\nu'}$ where $\nu' > \nu$. Put $N'_\lambda = W_{\omega^\nu} + N_\lambda$, then by Lemma 3.1, $N'_\lambda \sim N_\lambda$. Since any maximal totally ordered subset of N'_λ includes W_{ω^ν}, N'_λ satisfies E_ν).

Case 4. $\mu \in \Phi^l_\nu$, $\mathrm{cf}(\mu) = \beta$, $\mu < \alpha_\nu(\mu)$ and μ is not a γ-number.

In this case $\lambda \in \Gamma^1_\nu$ and putting $\mu = \xi + \zeta$ where $\xi = \gamma(\mu)$ and $N'_\lambda = N'_{\alpha_\nu(\zeta)+n} \odot_\nu N'_{\alpha_\nu(\xi)+\sigma_\xi}$, we have $N'_\lambda \sim N_\lambda$. Since any maximal totally ordered subset of N'_λ is either included in $N'_{\alpha_\nu(\zeta)+n}$ or of the form $\mathrm{Lb}'(x\,; N'_{\alpha_\nu(\zeta)+n}) + A'$ where $x \in \mathrm{Seg}_\nu(N'_{\alpha_\nu(\zeta)+n})$ and A' is a maximal totally ordered subset of $(x, N'_{\alpha_\nu(\xi)+\sigma_\xi})$, N'_λ satisfies E_ν).

Case 5. $\mu \in \Phi^l_\nu$, $\mathrm{cf}(\mu) = \beta$, $\mu < \alpha_\nu(\mu) = \mathrm{ls}(\lambda)$ and μ is a γ-number.

In this case $\lambda \in \Gamma^2_\nu$ and N_λ is of a form $W_{\omega^\nu} + X$. Hence N_λ itself satisfies E_ν).

Thus in any case we can inductively find a N'_λ required, and we complete the proof.

2. Case $\lambda \in \Gamma^0$

By the aid of preliminary lemmas shown in §1, we shall prove that for any $\lambda < \beta^\sharp$,

Proposition 1. N_λ *satisfies* D. 1), D. 2) *and* D. 3).

As we have seen in Lemma 2. 8 and 2. 9 and their Corollaries, if we complete the proof of Proposition 1 for any $\lambda<\beta^\#$, then Main Theorem A is also proved. In proving Proposition 1 for a $\lambda<\beta^\#$, it is assumed that any N_ξ with $\xi<\lambda$ satisfies Proposition 1 in which λ is replaced by ξ, and hence Lemma 3. 3 and 3. 4 can be applied on any $N_{\lambda'}$ with $\lambda'\leqq\lambda$ without any restriction. Occasionally for some λ we can show

Proposition 2. N_λ *satisfies* D. 1) *and* D. 2′) (see Lemma 2. 7), and as we noticed in Lemma 2. 7, Proposition 2 implies Proposition 1.

Proposition 2 trivially holds for N_n with $n<\omega$. We shall show that any N_λ with $\lambda\in\Gamma^0$ satisfies Proposition 2.

Proof of Proposition 2 *for a limit number* $\lambda\in\Gamma^0$.

Assume $\mu<\lambda$. Let ξ be the least number in Λ_λ such that $\mu<\xi$, then by assumption D. 1) on N_ξ, $N_\mu \not\approx N_\xi$. By the definition of N_λ, it includes a subset isomorphic to N_ξ. Hence $N_\mu \not\approx N_\xi \propto N_\lambda$ and N_λ satisfies D. 1).

Let X be a set in $\mathfrak{R}_\beta$ such that $X\not\propto N_\mu$ for any $\mu<\lambda$. By assumption D. 3) on N_μ with $\mu<\lambda$, $N_\mu\propto X$. Hence $N_\lambda=\bigvee_{\Lambda_\lambda} N_\mu\propto X$ by Lemma 2. 11, and N_λ satisfies D. 2′).

Lemma 3. 6. *Let* λ *be a limit number in* Γ^0. *If* X *has the least element and* $X\propto N_\lambda$, *then there exists a* $\mu<\lambda$ *such that* $X\propto N_\mu$.

Proof. Let a be the least element of X and f be an increasing mapping of X into N_λ, then there exists a $\mu\in\Lambda_\lambda$ such that $f(a)\in(\mu, N_\mu)$, and $X=\mathrm{Ub}'(a)$ is entirely mapped into (μ, N_μ) which is isomorphic to N_μ, i.e., $X\propto N_\mu$.

Corollary. *If* λ *is a limit number in* Γ^0 *and* $a\in N_\lambda$, *then there exists a* $\mu<\lambda$ *such that* $\mathrm{Ub}'(a)\propto N_\mu$.

Lemma 3. 7. *Let* λ *be a limit number in* Γ^0. *If* $\mu<\lambda$, *then* $W_1+N_\mu\propto N_\lambda$.

Proof. If $\mu\in\Gamma^0$, then $W_1+N_\mu=N_{\mu+1}$ by definition. If $\mu\in\Gamma^1\cup\Gamma^2$, then $W_1+N_\mu\sim N_\mu$ by Lemma 3. 1. Hence our lemma follows from D. 1) on N_λ.

Corollary. *Let* λ *be a limit number in* Γ^0. *If* $N_\lambda\propto W_1+X$, *then* $N_\lambda\propto X$.

Proof. Since $W_1+N_\mu\propto N_\lambda\propto W_1+X$, $N_\mu\propto X$ for any $\mu<\lambda$. Hence $N_\lambda\propto X$.

Proof of Proposition 2 *for an isolated number* $\lambda=\lambda'+1\in\Gamma^0$.

Put $\xi=\mathrm{ls}(\lambda')=\mathrm{ls}(\lambda)$, $n=\mathrm{fr}(\lambda')$ then $N_{\lambda'}=W_n+N_\xi$ and $N_\lambda=W_n+\{n\}+N_\xi$.

First we shall show $N_{\lambda'}\lessapprox N_\lambda$. $N_{\lambda'}\propto N_\lambda$ is obvious. Assume $N_\lambda\propto N_{\lambda'}$, and let f be an increasing mapping of N_λ into $N_{\lambda'}$, then $\tau(f(n))\geqq\tau(n)=n$. Hence there exists an $a\in N_\xi$ such that $f(n)=n+a$, and f maps $\mathrm{Ub}(n; N_\lambda)$ into $\mathrm{Ub}(n+a; N_{\lambda'})$ which is isomorphic to $\mathrm{Ub}(a; N_\xi)$. Since $\mathrm{Ub}(n; N_\lambda)\equiv N_\xi$, we have $N_\xi\propto\mathrm{Ub}(a; N_\xi)$ contradicting Corollary of Lemma 3.6. Hence $N_{\lambda'}\not\propto N_\lambda$ and $\mu\leqq\lambda'$ implies $N_\mu\lessapprox N_\lambda$ by D. 1) on $N_{\lambda'}$. Hence N_λ satisfies D. 1).

Assume $X\in\mathfrak{R}_\beta$, and $X\not\propto N_{\lambda'}$ (and accordingly $X\not\propto N_\mu$ for any $\mu\leqq\lambda'$ by assumption D. 1) on $N_{\lambda'}$). Let Y denote the set of all $y\in X$ such that $\tau(y)=n$. If $\mathrm{Ub}'(y)\propto N_\xi$ for any $y\in Y$, then similarly to Lemma 2. 11, we have $X\propto W_n+N_\xi=N_{\lambda'}$ contradicting $X\not\propto N_{\lambda'}$. Hence there exists a $y\in Y$ such that $\mathrm{Ub}'(y)\not\propto N_\xi$. By assumption D. 3) on N_ξ, $N_\xi\propto\mathrm{Ub}'(y)=\{y\}+\mathrm{Ub}(y)$. Hence $N_\xi\propto\mathrm{Ub}(y)$ by Corollary of Lemma 3. 7, and $N_\lambda=W_{n+1}+N_\xi\propto\mathrm{Lb}'(y)+\mathrm{Ub}(y)\subset X$ which shows that N_λ satisfies D. 2′).

3. Case $\lambda\in\Gamma^1$

In this section we shall show that for any $\lambda\in\Gamma^1$ Proposition 1 holds and for some cases Proposition 2 also holds. In order to show it, we prepare a definition of a notation.

DEFINITION 11. We shall say that an $x\in X$ *supports* Y if $Y\propto\mathrm{Ub}(x; X)$. $\mathrm{Spt}(X; Y)$ denotes the set of all $x\in X$ which support Y. Let $\mathfrak{Y}=\{Y_\lambda; \lambda\in\Lambda\}$ be a family of ramified sets. $\mathrm{Spt}(X; \mathfrak{Y})$ or $\mathrm{Spt}(X; Y_\lambda, \lambda\in\Lambda)$ (or occasionally $\mathrm{Spt}(X; Y_\lambda, \lambda<\mu)$ etc.) denotes the set of all $x\in X$ which supports every $Y_\lambda\in\mathfrak{Y}$. $\mathrm{Exp}_\nu(\mathrm{Spt}(X; \mathfrak{Y}))$ is denoted by $\overline{\mathrm{Spt}}^\nu(X; \mathfrak{Y})$ and $\mathrm{Seg}_\nu(\mathrm{Spt}(X; \mathfrak{Y}))$ is denoted by $\underline{\mathrm{Spt}}_\nu(X; \mathfrak{Y})$.

REMARK. Let Z be a subset of X. In general $\mathrm{Spt}(Z; Y)$ does not agree with $\mathrm{Spt}(X; Y)\cap Z$. If $x\in\mathrm{Spt}(Z; Y)$, then $Y\propto\mathrm{Ub}(x; Z)$, while $x\in\mathrm{Spt}(X; Y)\cap Z$ implies $Y\propto\mathrm{Ub}(x; X)$. Hence we can assert only $\mathrm{Spt}(Z; Y)\subset\mathrm{Spt}(X; Y)\cap Z$.

The following is obvious.

Lemma 3. 8. (i). $\mathrm{Spt}(X; \mathfrak{Y})$ *is a cut of* X (*and hence* $\mathrm{Exp}_\nu(\mathrm{Spt}(X; \mathfrak{Y}))$ *can be defind*).

(ii). *If* $X\propto X'$ *then* $\mathrm{Spt}(X; \mathfrak{Y})\propto\mathrm{Spt}(X'; \mathfrak{Y})$ *and similar relations hold for* $\overline{\mathrm{Spt}}^\nu$ *and* $\underline{\mathrm{Spt}}_\nu$.

(iii). *If for any* $Y\in\mathfrak{Y}$ *there exists a* $Y'\in\mathfrak{Y}'$ *such that* $Y\propto Y'$, *then* $\mathrm{Spt}(X; \mathfrak{Y}')\subset\mathrm{Spt}(X; \mathfrak{Y})$, *and especially if* $Y\propto Y'$ *then* $\mathrm{Spt}(X; Y')\subset\mathrm{Spt}(X; Y)$. *Similar inclusions hold for* $\overline{\mathrm{Spt}}^\nu$ *and* $\underline{\mathrm{Spt}}_\nu$.

(iv). *For a family* $\{X_{\lambda'}; \lambda' \in \Lambda'\}$, $\mathrm{Spt}(\bigvee_{\Lambda'} X_{\lambda'}; \mathfrak{Y}) \sim \bigvee_{\lambda' \in \Lambda'} \mathrm{Spt}(X_{\lambda'}; \mathfrak{Y})$. *Similar equivalencies hold for* $\overline{\mathrm{Spt}}^{\nu}$ *and* $\underline{\mathrm{Spt}}_{\nu}$.

Lemma 3.9. *If* $y \in Y$ *implies* $Y \not\propto \mathrm{Ub}(y)$, *then* $\overline{\mathrm{Spt}}^{\nu}(X \odot_{\nu} Y; Y) = X$.

Proof. For any $x \in \mathrm{Seg}_{\nu}(X)$, $\mathrm{Ub}(x; X \odot_{\nu} Y)$ include the subset (x, Y) isomorphic to Y, and $x \in \mathrm{Spt}(X \odot_{\nu} Y; Y)$. Hence $\mathrm{Seg}_{\nu}(X) \subset \mathrm{Spt}(X \odot_{\nu} Y; Y)$ and $X \subset \overline{\mathrm{Spt}}^{\nu}(X \odot_{\nu} Y; Y)$. On the other hand, for any $x \in \mathrm{Seg}_{\nu}(X)$ and $y \in Y$, $\mathrm{Ub}((x, y); X \odot_{\nu} Y) \equiv \mathrm{Ub}(y; Y)$, and (x, y) does not support Y by assumption, i.e., $\mathrm{Spt}(X \odot_{\nu} Y; Y) \subset X$. Hence $\overline{\mathrm{Spt}}^{\nu}(X \odot_{\nu} Y; Y) \subset X$ by Lemma 2.1, (vi) and we have the lemma.

Lemma 3.10. $\overline{\mathrm{Spt}}^{\nu}(X; Y) \odot_{\nu} Y \propto X$.

Proof. If $\mathrm{Spt}(X; Y)$ is void, then $\overline{\mathrm{Spt}}^{\nu}(X; Y)$ and $\overline{\mathrm{Spt}}^{\nu}(X; Y) \odot_{\nu} Y$ are void and the inequality is trivial. Assume that $\mathrm{Spt}(X; Y)$ is not void. Since $x \in \mathrm{Spt}(X; Y)$ implies $Y \propto \mathrm{Ub}(x)$, there exists an increasing mapping f_x of Y into $\mathrm{Ub}(x)$ for any $x \in \mathrm{Spt}(X; Y)$. Put $f(x) = x$ for any $x \in \overline{\mathrm{Spt}}^{\nu}(X; Y)$ and $f((x, y)) = f_x(y)$ for any $x \in \underline{\mathrm{Spt}}_{\nu}(X; Y)$ and any $y \in Y$, then obviously f is an increasing mapping of $\overline{\mathrm{Spt}}^{\nu}(X; Y) \odot_{\nu} Y$ into X and the lemma is proved.

Lemma 3.11. *If* Y *is comparable with any* $X' \in \mathfrak{R}_3$, *and* Z *satisfies* E_{ν}) (see Lemma 3.5), *then for any* $X \in \mathfrak{R}_3$, $\overline{\mathrm{Spt}}^{\nu}(X; Y) \propto Z$ *implies* $X \propto Z \odot_{\nu} Y$.

Proof. Let g be a reduced increasing mapping of $\overline{\mathrm{Spt}}^{\nu}(X; Y)$ into Z and M be the set of all minimal elements of $X - \overline{\mathrm{Spt}}^{\nu}(X; Y)$. $a \in M$ implies $\tau(a) < \omega^{\nu}$ by Lemma 2.3 (vii), and $\mathrm{Lb}(a) \subset \underline{\mathrm{Spt}}_{\nu}(X; Y)$. Since g is reduced, the type of the set $g(\mathrm{Lb}(a))$ is equal to $\tau(a)$ which is less than ω^{ν}. Hence by E_{ν}) on Z, $g(\mathrm{Lb}(a))$ is not maximal totally ordered subset of Z, and there exists an $a' \in Z$ such that $\tau(a') = \tau(a)$ and $g(\mathrm{Lb}(a)) = \mathrm{Lb}(a')$. Let g' be the extension of g such that $g'(a) = a'$ for any $a \in M$. Since $a \in M$ implies $Y \not\propto \mathrm{Ub}(a)$, $\mathrm{Ub}(a) \propto Y$ by assumption on Y. Let f_a be an increasing mapping of $\mathrm{Ub}(a)$ into Y for any $a \in M$. X is decomposed into a union $\bigvee_{a \in M} \mathrm{Ub}(a) \cup (M \cup \overline{\mathrm{Spt}}^{\nu}(X; Y))$. Let f be a mapping of X into $Z \odot_{\nu} Y$ such that $f(a) = g'(a)$ for $M \cup \overline{\mathrm{Spt}}^{\nu}(X; Y)$ and $f(b) = (g'(a), f_a(b))$ for $b \in \mathrm{Ub}(a)$ where $a \in M$, then obviously f is increasing and $X \propto Z \odot_{\nu} Y$.

Lemma 3.12. *Assume* $\lambda \in \Gamma^1$ *and let* $\lambda = \alpha_{\nu}(\xi + \zeta) + n$ *be the canonical decomposition of* λ, *then*

(i). $N_{\lambda} \propto X$ *implies* $N_{\alpha_{\nu}(\zeta) + n} \propto \overline{\mathrm{Spt}}^{\nu}(X; N_{\alpha_{\nu}(\xi) + \sigma_{\xi}})$.

(ii). $X \not\propto N_{\lambda}$ *implies* $\overline{\mathrm{Spt}}^{\nu}(X; N_{\alpha_{\nu}(\xi) + \sigma_{\xi}}) \not\propto N_{\alpha_{\nu}(\zeta) + n}$.

(iii). $\overline{\mathrm{Spt}}^{\nu}(X; N_{\alpha_\nu(\xi)+\sigma_\xi}) \not\approx N_{\alpha_\nu(\zeta)+n}$ *implies* $X \not\approx N_\lambda$.

(iv). $N_{\alpha_\nu(\zeta)+n} \propto \overline{\mathrm{Spt}}^{\nu}(X; N_{\alpha_\nu(\xi)+\sigma_\xi})$ *implies* $N_\lambda \propto X$.

Proof. (i). Since $\alpha_\nu(\xi)+\sigma_\xi \in \Gamma^0$, $a \in N_{\alpha_\nu(\xi)+\sigma_\xi}$ implies $N_{\alpha_\nu(\xi)+\sigma_\xi} \not\propto \mathrm{Ub}(a)$ by Corollary of Lemma 3.6. Hence $\overline{\mathrm{Spt}}^{\nu}(N_\lambda; N_{\alpha_\nu(\xi)+\sigma_\xi}) = N_{\alpha_\nu(\zeta)+n}$ by Lemma 3.9. Hence (i) follows from 3.8 (ii).

(ii). Since $\omega_\beta \leqq \zeta < \beta^\#$, there exists a $N'_{\alpha_\nu(\zeta)+n} \in \mathfrak{S}_\beta$ which is equivalent to $N_{\alpha_\nu(\zeta)+n}$ and satisfies E$_\nu$) by Lemma 3.5. By Lemma 3.11, $\overline{\mathrm{Spt}}^{\nu}(X; N_{\alpha_\nu(\xi)+\sigma_\xi}) \propto N_{\alpha_\nu(\zeta)+n} \sim N'_{\alpha_\nu(\zeta)+n}$ implies $X \propto N'_{\alpha_\nu(\zeta)+n} \odot_\nu N_{\alpha_\nu(\xi)+\sigma_\xi} \sim N_\lambda$.

(iii) is a consequence of (i) and (ii).

(iv) follows from Lemma 3.10 and Lemma 2.12 (i).

Now referring to the remark below Lemma 3.4, we shall obtain the *Proof of Proposition* 1 *for* $\lambda \in \Gamma^1$. (Besides let $\lambda = \alpha_\nu(\xi+\zeta)+n$ be the canonical decomposition of λ. Proposition 2 holds for λ if Proposition 2 in which λ is replaced by θ holds for $\theta = \alpha_\nu(\zeta)+n$. In brackets of the following proof, we shall consider about this case.)

Let $\lambda = \alpha_\nu(\xi+\zeta)+n$ be the canonical decomposition of λ. First we shall show that N_λ satisfies D.1).

Case $n = \mathrm{fr}(\lambda) = 0$. If $\mu < \lambda = \alpha_\nu(\xi+\zeta)$, then there exists a $\psi \in \Psi_\lambda$ such that $\mu < \alpha_\nu(\xi+\psi)$ (see the remark bellow Lemma 3.4), and by assumption D.1) on $N_{\alpha_\nu(\xi+\psi)}$, $N_\mu \not\approx N_{\alpha_\nu(\xi+\psi)}$. Since $N_{\alpha_\nu(\psi)} \propto N_{\alpha_\nu(\zeta)}$, $N_{\alpha_\nu(\xi+\psi)} = N_{\alpha_\nu(\psi)} \odot_\nu N_{\alpha_\nu(\xi)+\sigma_\xi} \propto N_{\alpha_\nu(\zeta)} \odot_\nu N_{\alpha_\nu(\xi)+\sigma_\xi} = N_\lambda$ by Lemma 2.12, (i). Hence $N_\mu \not\approx N_\lambda$ and N_λ satisfies D.1).

Case $n = \mathrm{fr}(\lambda) > 0$. It suffices to show $N_{\alpha_\nu(\xi+\zeta)+n-1} \not\approx N_{\alpha_\nu(\xi+\zeta)+n}$. $\overline{\mathrm{Spt}}^{\nu}(N_{\alpha_\nu(\xi+\zeta)+n-1}; N_{\alpha_\nu(\xi)+\sigma_\xi}) = N_{\alpha_\nu(\zeta)+n-1} \not\approx N_{\alpha_\nu(\zeta)+n}$ by Lemma 3.9 and assumption D.1) on $N_{\alpha_\nu(\zeta)+n}$. Hence $N_{\alpha_\nu(\xi+\zeta)+n-1} \not\approx N_\lambda$ by Lemma 3.12, (iii) and N_λ satisfies D.1).

Next we shall show that N_λ satisfies D.2) (resp. D.2′) in the case where $N_{\alpha_\nu(\zeta)+n}$ satisfies D.2′)). Assume $X \in \mathfrak{S}_\beta$ (resp. $X \in \mathfrak{R}_\beta$) and that $X \not\propto N_\mu$ for any $\mu < \lambda$.

Case $n = \mathrm{fr}(\lambda) = 0$. By assumption D.3) on N_μ with $\mu < \lambda$, $N_\mu \propto X$ for any $\mu < \lambda$. Especially for any $\psi \in \Psi_\lambda$, $N_{\alpha_\nu(\xi+\psi)} \propto X$ and hence $N_{\alpha_\nu(\psi)} = \overline{\mathrm{Spt}}^{\nu}(N_{\alpha_\nu(\xi+\psi)}; N_{\alpha_\nu(\xi)+\sigma_\xi}) \propto \overline{\mathrm{Spt}}^{\nu}(X; N_{\alpha_\nu(\xi)+\sigma_\xi})$ by Lemma 3.9 and Lemma 3.8 (ii). Hence $N_{\alpha_\nu(\zeta)} \propto \overline{\mathrm{Spt}}^{\nu}(X; N_{\alpha_\nu(\xi)+\sigma_\xi})$ by assumption D.2) (resp. D.2′)) on $N_{\alpha_\nu(\zeta)}$. Hence $N_\lambda = N_{\alpha_\nu(\zeta)} \odot_\nu N_{\alpha_\nu(\xi)+\sigma_\xi} \propto X$ by Lemma 3.12 (iv) and N_λ satisfies D.2) (resp. D.2′)).

Case $n = \mathrm{fr}(\lambda) > 0$. It suffices to show that $X \not\propto N_{\alpha_\nu(\xi+\zeta)+n-1}$ implies $N_{\alpha_\nu(\xi+\zeta)+n} \propto X$. By Lemma 3.12 (ii), $X \not\propto N_{\alpha_\nu(\xi+\zeta)+n-1}$ implies $\overline{\mathrm{Spt}}^{\nu}(X; N_{\alpha_\nu(\xi)+\sigma_\xi}) \not\propto N_{\alpha_\nu(\zeta)+n-1}$ which implies $N_{\alpha_\nu(\zeta)+n} \propto \overline{\mathrm{Spt}}^{\nu}(X; N_{\alpha_\nu(\xi)+\sigma_\xi})$ by assumption D.2) (resp. D.2′)) on $N_{\alpha_\nu(\zeta)+n}$. Hence $N_\lambda \propto X$ by Lemma 3.12 (iv).

Finally we shall show that N_λ satisfies D. 3). Assume $X\in\mathfrak{R}_\beta$. By assumption D. 3) on $N_{\alpha_\nu(\zeta)+n}$, either $\overline{\mathrm{Spt}}^\nu(X;\,N_{\alpha_\nu(\xi)+\sigma_\xi})\approx N_{\alpha_\nu(\zeta)+n}$ or $N_{\alpha_\nu(\zeta)+n}\propto\overline{\mathrm{Spt}}^\nu(X;\,N_{\alpha_\nu(\xi)+\sigma_\xi})$ holds, hence either $X\approx N_\lambda$ or $N_\lambda\propto X$ holds accordingly by Lemma 3. 12, (iii) and (iv), and N_λ satisfies D. 3).

4. Case $\lambda\in\Gamma^2$

We distinguish three cases. First we shall consider the case $\lambda=\alpha_\nu(\omega_\beta)+n$, next the case $\lambda=\alpha_\nu(\omega_\beta^{\zeta+1})+n$ where $0<\zeta$ and finally the case $\lambda=\alpha_\nu(\omega_\beta^\zeta)+n$ where ζ is a limit number with $\mathrm{cf}(\zeta)=\beta$ and $\zeta<\mathrm{ls}(\lambda)$.

Lemma 3. 13. *Assume* $X\in\mathfrak{R}_\beta$, $\lambda\in\Gamma^2$, $\mathrm{fr}(\lambda)=0$ *and that any* N_μ *with* $\mu<\lambda$ *satisfies* D. 1). *Put* $Y=\mathrm{Spt}(X;\,N_\mu,\,\mu<\lambda)$, *then there exists a* $\xi<\lambda$ *such that for any* $x\in X$, $N_\xi\propto\mathrm{Ub}(x)$ *implies* $x\in Y$.

Proof. If $x\in X-Y$, then there exists a $\xi(x)<\lambda$ such that $N_{\xi(x)}\not\propto\mathrm{Ub}(x)$. Since $\overline{\overline{X}}<\aleph_\beta$ and $\mathrm{cf}(\lambda)=\beta$, numbers $\xi(x)$ with $x\in X-Y$ are not cofinal to λ. Hence there exists a $\xi<\lambda$ such that $\xi(x)<\xi$ and accordingly $N_{\xi(x)}\approx N_\xi$ for any $x\in X-Y$. Then $x\in X$ and $N_\xi\propto\mathrm{Ub}(x)$ imply $x\in Y$.

Proof of Proposition 1 *for* $\lambda=\alpha_\nu(\omega_\beta)$. $(\mathrm{fr}(\lambda)=0)$.

λ is the least number in $\tilde{\Gamma}_\nu^1\cup\tilde{\Gamma}_\nu^2$ (see Lemma 1. 13), i.e., $\mu<\lambda$ implies $\mu\in\Gamma^0\cup\bigcup_{\eta<\nu}\Gamma_\eta$. Hence applying Lemma 2. 12, (v), we can inductively see that $\mu<\lambda$ implies $\kappa(N_\mu)\leqq\omega^\nu$, while $\kappa(N_{\alpha_\nu(\omega_\beta)})=\kappa(W_{\omega^\nu})=\omega^\nu+1$. Hence by Theorem 5, $\mu<\lambda$ implies $N_\mu\approx N_\lambda$, i.e., N_λ satisfies D. 1).

Assume $X\in\mathfrak{S}_\beta$ and $X\not\propto N_\mu$ for any $\mu<\lambda$, then by assumption D. 3) on N_μ, $N_\mu\propto X$ for any $\mu<\lambda$. Put $Y=\mathrm{Spt}(X\,;\,N_\mu,\,\mu<\lambda)$, then by Lemma 3. 13, there exists a $\xi<\lambda$ such that $x\in X$ and $N_\xi\propto\mathrm{Ub}(x)$ imply $x\in Y$. Since ordinal numbers $\alpha_\nu(\theta)$ where $\theta\in\Phi_\nu^\zeta$ and $\theta<\omega_\beta$ are cofinal to $\lambda=\alpha_\nu(\omega_\beta)$, we may assume that $\xi=\alpha_\nu(\theta)$ where $\theta\in\Phi_\nu^\zeta$ and $\theta<\omega_\beta$ without loss of generality. Then $\mathrm{cf}(\xi)=\mathrm{cf}(\theta)<\beta$ by Theorem 3 and accordingly $\xi\in\Gamma^0$ and $\sigma_\xi=0$. Now we shall show that

(*a*). *for any* $\varepsilon<\omega^\nu$ *and* $y\in Y$, $W_\varepsilon+N_\xi\propto\mathrm{Ub}(y\,;\,X)$.

If $\nu=1$, then since ξ is a limit number in Γ^0, $W_n+N_\xi=N_{\xi+n}\propto\mathrm{Ub}(y)$ for any $n<\omega$.

Next assume $\nu\geqq2$. If $\nu=\eta+1$, then let ζ be the least finite number such that $\varepsilon<\omega^\eta\cdot\zeta$ and put $\varepsilon'=\omega^\eta\cdot\zeta$. If ν is a limit number, then let ζ be the least number such that $\varepsilon<\omega^\zeta$ and put $\varepsilon'=\omega^\zeta$. In either case put $\mu=\theta+\zeta$, then $\theta+\zeta$ is the φ_ν-decomposition of μ, $\mu<\omega_\beta$ and $\varepsilon<\varepsilon'=\mathrm{ta}_\nu(\mu)$ (see the mention above Lemma 3. 2). Hence by Lemma 3. 2, $W_\varepsilon+N_\xi\propto W_{\varepsilon'}+N_\xi=N_{\alpha_\nu(\mu)}\propto\mathrm{Ub}(y)$.

Hence Assertion (a) is proved for any ν with $0<\nu<\omega_\beta$. Next we shall show that (*b*). *for any* ε *with* $0<\varepsilon<\omega^{\nu}$ *and* $y\in Y$, *there exists a* $z\in \mathrm{Ub}(y)\cap Y$ *such that* $\tau(z\,;\mathrm{Ub}(y))=\varepsilon$.

Indeed, $W_{\varepsilon+1}+N_{\xi}\propto \mathrm{Ub}(y)$ by (a) above. Let f be a reduced increasing mapping of $W_{\varepsilon+1}+N_{\xi}$ into $\mathrm{Ub}(y)$, then f maps $\varepsilon\in W_{\varepsilon+1}$ onto a $z\in \mathrm{Ub}(y)$. Since $N_{\xi}\propto \mathrm{Ub}(\varepsilon\,; W_{\varepsilon+1}+N_{\xi})$, $N_{\xi}\propto \mathrm{Ub}(z)$ which implies $z\in Y$ by assumption on ξ. Since f is reduced, $\tau(z\,;\mathrm{Ub}(y))=\tau(\varepsilon\,; W_{\varepsilon+1})=\varepsilon$ and (b) is proved.

Similarly as the above, Y itself includes a subset isomorphic to W_{ε} for any $\varepsilon<\omega^{\nu}$, and especially Y is not void. Since $X\in\mathfrak{S}_{\beta}$, i.e., X is resoluble, there exists a $y\in Y$ such that $\mathrm{Ub}(y)\cap Y$ is a well-ordered subset of X. Since for any $\varepsilon<\omega^{\nu}$, there exits a $z\in\mathrm{Ub}(y)\cap Y$ such that $\tau(z\,;\mathrm{Ub}(y))=\varepsilon$ the type of $\mathrm{Ub}(y)\cap Y$ is at least ω^{ν} and hence $N_{\lambda}=W_{\omega^{\nu}}\propto\mathrm{Ub}(y)\cap Y\subset X$ and N_{λ} satisfies D. 2).

It follows from Corollary of Theorem 5 that N_{λ} satisfies D. 3).

By the way we shall show;

Lemma 3.14. *If either* ν *is an isolated number or a limit number with* $\mathrm{cf}(\nu)=0$, *then* N_{λ} *with* $\lambda=\alpha_{\nu}(\omega_{\beta})$ *satisfies* D. 2′). (*Hence Proposition* 2 *holds for* λ).

Proof. In either case $\mathrm{cf}(\omega^{\nu})=0$ and there exists a countable (strictly) ascending sequence $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m, \cdots$ cofinal to ω^{ν}.

Assume $X\in\mathfrak{R}_{\beta}$ and $X\not\propto N_{\mu}$ for any $\mu<\lambda$. It is all the same as (b) in the proof above that for any $y\in Y=\mathrm{Spt}(X\,; N_{\mu},\ \mu<\lambda)$ and $\varepsilon<\omega^{\nu}$, there exists a $z\in\mathrm{Ub}(y)\cap Y$ such that $\tau(z\,;\mathrm{Ub}(y))=\varepsilon$. Especially Y itself is not void. Let y_1 be an element in Y such that $\tau(y_1\,; X)=\varepsilon_1$, and after we have a $y_m\in Y$, let y_{m+1} be the element in $\mathrm{Ub}(y_m)\cap Y$ such that $\tau(y_{m+1}\,; X)=\varepsilon_{m+1}$, then we have a sequence $y_1, y_2, \cdots, y_m, \cdots$ of elements in Y such that $y_m<y_{m+1}$ for any $m<\omega$ and $\tau(y_m\,; X)=\varepsilon_m$. Let W be a maximal totally ordered subset of X, which contains every y_m, $m<\omega$, then since for any $\varepsilon<\omega^{\nu}$, W contains a w such that $\tau(w)=\varepsilon$, the type of W is at least ω^{ν}. Hence $N_{\lambda}=W_{\omega^{\nu}}\propto X$ and N_{λ} satisfies D. 2′), and the proof is completed.

Proof of Proposition 2 *for* $\lambda=\alpha_{\nu}(\omega_{\beta})+n$ *with* $n>0$.

$\kappa(N_{\lambda})=\kappa(W_{\omega^{\nu}+n})=\omega^{\nu}+n+1$ while $\kappa(N_{\alpha_{\nu}(\omega_{\beta})+n-1})=\kappa(W_{\omega^{\nu}+n-1})=\omega^{\nu}+n$. Hence $N_{\alpha_{\nu}(\omega_{\beta})+n-1}\not\approx N_{\alpha_{\nu}(\omega_{\beta})+n}=N_{\lambda}$ by Theorem 5 and N_{λ} satisfies D. 1).

Assume $X\in\mathfrak{R}_{\beta}$ and $X\not\propto N_{\alpha_{\nu}(\omega_{\beta})+n-1}$. Then since $\kappa(N_{\alpha_{\nu}(\omega_{\beta})+n-1})=\omega^{\nu}+n$ is an isolated number, $\omega^{\nu}+n<\kappa(X)$, i.e., $\omega^{\nu}+n+1\leqq\kappa(X)$. Hence by definition of $\kappa(X)$, $N_{\lambda}=W_{\omega^{\nu}+n}\propto X$, and N_{λ} satisfies D. 2′).

Next we shall consider the case $\lambda=\alpha_{\nu}(\omega_{\beta}^{\zeta+1})$ where $\zeta>0$.

Put $\zeta'=\omega_{\beta}^{\zeta}$ and $\sigma'_{\zeta}=\sigma_{\zeta'}$, i.e., $\sigma'_{\zeta}=0$ if ζ is a limit number with $\mathrm{cf}(\zeta)<\beta$ and $\sigma'_{\zeta}=\omega$ otherwise.

Lemma 3.15. *Put* $\varepsilon=\alpha_{\nu}(\omega_{\beta}^{\zeta})+\sigma'_{\zeta}$, *then* $\mathrm{Csg}_{\nu}(N_{\varepsilon})\sim N_{\omega\zeta+\sigma_{\zeta}}$.

Proof. Case 1. $\zeta=\theta+1$.

In this case $\sigma'_\zeta=\omega$, $\sigma_\zeta=\sigma_\theta$ and $N_{\alpha_\nu(\omega_\beta^\zeta)+n}=W_\omega{}^\nu+N_{\omega\theta+\sigma_\theta+n}$. Hence $\mathrm{Csg}_\nu(N_{\alpha_\nu(\omega_\beta^\zeta)+n})\equiv N_{\omega\theta+\sigma_\theta+n}$. Since $N_{\alpha_\nu(\omega_\beta^\zeta)+\omega}\sim\bigvee_{n<\omega}N_{\alpha_\nu(\omega_\beta^\zeta)+n}$ by Remark 2 below Principle 1, $\mathrm{Csg}_\nu(N_{\alpha_\nu(\omega_\beta^\zeta)+\sigma'_\zeta})\sim\bigvee_{n<\omega}\mathrm{Csg}_\nu(N_{\alpha_\nu(\omega_\beta^\zeta)+n})\sim\bigvee_{n<\omega}N_{\omega\theta+\sigma_\theta+n}\sim$ $N_{\omega\theta+\sigma_\theta+\omega}=N_{\omega(\theta+1)+\sigma_\theta}=N_{\omega\zeta+\sigma_\zeta}$ by Lemma 2.12, (vi).

Case 2. ζ is a limit number with $\mathrm{cf}(\zeta)=\beta$ and $\zeta=\alpha_\nu(\omega_\beta^\zeta)$.

In this case $\sigma'_\zeta=\sigma_\zeta=\omega$, $\omega\zeta=\zeta$ (see Remark (ii) below Definition 4) and $\zeta=\alpha_\nu(\omega_\beta^\zeta)\in\tilde\Gamma^1_{\nu+1}\cup\tilde\Gamma^2_{\nu+1}$. Hence $N_{\alpha_\nu(\omega_\beta^\zeta)+n}\sim W_\omega{}^\nu+N_{\zeta+n}$ by Lemma 3.1 and $\mathrm{Csg}_\nu(N_{\alpha_\nu(\omega_\beta^\zeta)+n})\sim N_{\zeta+n}$. Hence similarly as Case 1, $\mathrm{Csg}_\nu(N_{\alpha_\nu(\omega_\beta^\zeta)+\sigma'_\zeta})\sim$ $\bigvee_{n<\omega}N_{\zeta+n}\sim N_{\zeta+\omega}=N_{\omega\zeta+\sigma_\zeta}$.

Case 3. ζ is a limit number with $\mathrm{cf}(\zeta)=\beta$ and $\zeta<\alpha_\nu(\omega_\beta^\zeta)$.

In this case $\sigma'_\zeta=\sigma_\zeta=\omega$, $\omega\zeta=\zeta$ and $N_{\alpha_\nu(\omega_\beta^\zeta)+n}=W_\omega{}^\nu+N_{\zeta+n}$. Hence $\mathrm{Csg}_\nu(N_{\alpha_\nu(\omega_\beta^\zeta)+n})\equiv N_{\zeta+n}$ and similarly as Case 1, $\mathrm{Csg}_\nu(N_{\alpha_\nu(\omega_\beta^\zeta)+\sigma'_\zeta})\sim\bigvee_{n<\omega}N_{\zeta+n}$ $\sim N_{\zeta+\omega}=N_{\omega\zeta+\sigma_\zeta}$.

Case 4. ζ is a limit number with $\mathrm{cf}(\zeta)<\beta$.

In this case $\sigma'_\zeta=\sigma_\zeta=0$, $\alpha_\nu(\omega_\beta^\zeta)\in\Gamma^0$, and $N_\varepsilon\sim\bigvee_{\theta\in\Lambda_\zeta}N_{\alpha_\nu(\omega_\beta^{\theta+1})}=\bigvee_{\theta\in\Lambda_\zeta}(W_\omega{}^\nu$ $+N_{\omega\theta+\sigma_\theta})$. Hence $\mathrm{Csg}_\nu(N_{\alpha_\nu(\omega_\beta^\zeta)})\sim\bigvee_{\theta\in\Lambda_\zeta}\mathrm{Csg}_\nu(N_{\alpha_\nu(\omega_\beta^{\theta+1})})\sim\bigvee_{\theta\in\Lambda_\zeta}N_{\omega\theta+\sigma_\theta}\sim N_{\omega\zeta}$.

Lemma 3.16. *If $\alpha_\nu(\omega_\beta^\zeta)+\sigma'_\zeta\leqq\varepsilon<\alpha_\nu(\omega_\beta^{\zeta+1})$, then $\mathrm{Csg}_\nu(N_\varepsilon)\sim N_{\omega\zeta+\sigma_\zeta}$.*

Proof. $\alpha_\nu(\omega_\beta^{\zeta+1})$ is the least number in $\Gamma^2_\nu\cup\tilde\Gamma_{\nu+1}$ greater than $\alpha_\nu(\omega_\beta^\zeta)+\sigma'_\zeta$ by Lemma 1.14, i.e., $\varepsilon\in\Gamma^0\cup\bigcup_{\eta\leqq\nu}\Gamma^1_\eta\cup\bigcup_{\eta<\nu}\Gamma^2_\eta$. By Lemma 3.15 the equivalency is true for $\varepsilon=\alpha_\nu(\omega_\beta^\zeta)+\sigma'_\zeta$. Hence we can inductively prove our equivalency referring to Lemma 1.12 (vi).

Lemma 3.17. *Assume that $X\in\mathfrak{R}_\beta$ and $N_\mu\propto X$ for any $\mu<\lambda=\alpha_\nu(\omega_\beta^{\zeta+1})$. Put $Y'=\mathrm{Spt}(X;N_\mu,\ \mu<\lambda)$ and $Y=\mathrm{Exp}_\nu(Y';X)=\overline{\mathrm{Spt}}^\nu(X;N_\mu,\ \mu<\lambda)$, then for any $y\in Y\cap\mathrm{Seg}_\nu(X)$, $N_{\alpha_\nu(\omega_\beta^\zeta)+\sigma'_\zeta}\propto\mathrm{Ub}(y)\cap Y$.*

Proof. By Lemma 3.12, there exists a $\xi'<\lambda$ such that $x\in X$ and $N_{\xi'}\propto\mathrm{Ub}(x)$ imply $x\in Y'$. Without loss of generality we may assume $\xi'=\alpha_\nu(\xi)+\sigma_\xi$ where ξ is a γ-number such that $\omega_\beta^\zeta<\xi<\omega_\beta^{\zeta+1}$ (remark that if $\xi<\omega_\beta^{\zeta+1}$, then $\xi\omega<\omega_\beta^{\zeta+1}$ and $\xi\omega$ is a γ-number). Put $\theta=\xi+\sigma_\xi+\omega_\beta^\zeta+\sigma'_\zeta$ if $\nu=1$, and $\theta=\alpha_\nu(\xi+\omega_\beta^\zeta)+\sigma'_\zeta$ if $\nu\geqq2$, then by Corollary of Lemma 3.4, $N_\theta\sim N_{\alpha_\nu(\omega_\beta^\zeta)+\sigma'_\zeta}\odot_\nu N_{\alpha_\nu(\xi)+\sigma_\xi}$ of which we shall denote the right side by Z. If $y\in Y\cap\mathrm{Seg}_\nu(X)\subset Y'$, then since $\theta<\lambda$, $N_\theta\propto\mathrm{Ub}(y)$ and there exists a reduced increasing mapping f of Z into $\mathrm{Ub}(y)$. If $a\in\mathrm{Seg}_\nu(N_{\alpha_\nu(\omega_\beta^\zeta)+\sigma'_\zeta})$, then since $N_{\alpha_\nu(\xi)+\sigma_\xi}\propto\mathrm{Ub}(a;Z)\propto\mathrm{Ub}(f(a);\mathrm{Ub}(y))$, $f(a)\in Y'$ by the definition of ξ'. Hence f maps $\mathrm{Seg}_\nu(N_{\alpha_\nu(\omega_\beta^\zeta)+\sigma'_\zeta})$ entirely into $\mathrm{Ub}(y)\cap Y'$.

Hence f maps $N_{\alpha_\nu(\omega_\beta^\zeta)+\sigma'_\zeta}$ into $\mathrm{Ub}(y)\cap\mathrm{Exp}_\nu(Y')=\mathrm{Ub}(y)\cap Y$ (refer to Lemma 2.12, (vii)) and $N_{\alpha_\nu(\omega_\beta^\zeta)+\sigma'_\zeta}\propto\mathrm{Ub}(y)\cap Y$.

Lemm 3.18. *If X is comparable with any set in $\mathfrak{N}_\beta$, then $W_{\omega^\nu}+X$ is comparable with any $Y\in\mathfrak{N}_\beta$.*

Proof. If $\mathrm{Ub}'(a)\propto X$ for any $a\in\mathrm{Lay}_\nu(Y)$, then $Y\propto W_{\omega^\nu}+X$ by Lemmas 2.12, (iii). If there exists an $a\in\mathrm{Lay}_\nu(Y)$ such that $\mathrm{Ub}'(a)\not\propto X$, then $X\propto\mathrm{Ub}'(a)$ by assumption on X. Since $\mathrm{Lb}(a)\equiv W_{\omega^\nu}$, $W_{\omega^\nu}+X\propto\mathrm{Lb}(a)\cup\mathrm{Ub}'(a)\subset Y$. Hence in either case $W_{\omega^\nu}+X$ is comparable with Y.

Proof of Proposition 1 *for* $\lambda=\alpha_\nu(\omega_\beta^{\zeta+1})$. $(\mathrm{fr}(\lambda)=0)$.

If $\mu<\lambda$, then there exists an ε such that $\mu<\varepsilon$ and $\alpha_\nu(\omega_\beta^\zeta)+\sigma'_\zeta\leqq\varepsilon<\lambda$. Since $\mathrm{Csg}_\nu(N_\varepsilon)\sim N_{\omega\zeta+\sigma_\zeta}$ by Lemma 3.16, $N_\varepsilon\propto W_{\omega^\nu}+N_{\omega\zeta+\sigma_\zeta}=N_\lambda$ by Lemma 2.12 (ii). Hence $N_\mu\not\approx N_\varepsilon\propto N_\lambda$ by assumption D.1) on N_ε and N_λ satisfies D.1).

Assume $X\in\mathfrak{S}_\beta$ and $X\not\propto N_\mu$ for any $\mu<\lambda$. Put $Y=\overline{\mathrm{Spt}}^\nu(X;N_\mu,\mu<\lambda)$, then it follows from Lemma 3.16 that for any $y\in Y\cap\mathrm{Seg}_\nu(X)$, $N_{\alpha_\nu(\omega_\beta^\zeta)+\sigma'_\zeta}\propto\mathrm{Ub}(y)\cap Y$. Similarly $N_{\alpha_\nu(\omega_\beta^\zeta)+\sigma'_\zeta}\propto Y$ and Y is not void. Hence $Y\cap\mathrm{Seg}_\nu(X)$ is not void. Since $X\in\mathfrak{S}_\beta$ and X is resoluble, there exists a $y\in Y\cap\mathrm{Seg}_\nu(X)$ such that $\mathrm{Ub}(y)\cap Y\cap\mathrm{Seg}_\nu(X)$ is totally ordered. Put $Z=\mathrm{Ub}(y)\cap Y$, then since $N_{\alpha_\nu(\omega_\beta^\zeta)+\sigma'_\zeta}\propto Z$, $Z\cap\mathrm{Seg}_\nu(X)\equiv W_{\omega^\nu}$ and $N_{\omega\zeta+\sigma_\zeta}\sim\mathrm{Csg}_\nu(N_{\alpha_\nu(\omega_\beta^\zeta)+\sigma'_\zeta})\propto Z\cap\mathrm{Csg}_\nu(X)$. Hence $N_\lambda=W_{\omega^\nu}+N_{\omega\zeta+\sigma_\zeta}\propto W_{\omega^\nu}+(Z\cap\mathrm{Csg}_\nu(X))\equiv Z\subset X$ and N_λ satisfies D.2).

It follows from Lemma 3.18 that N_λ satisfies D.3).

Proof of Proposition 2 *for* $\lambda=\alpha_\nu(\omega_\beta^{\zeta+1})+n$ *where* $n>0$.

Obviously $N_{\lambda-1}=W_{\omega^\nu}+N_{\omega\zeta+\sigma_\zeta+n-1}\propto W_{\omega^\nu}+W_{\omega\zeta+\sigma_\zeta+n}=N_\lambda$. If $N_\lambda\propto N_{\lambda-1}$ then by Corollary of Lemma 2.5, $N_{\omega\zeta+\sigma_\zeta+n}\equiv\mathrm{Csg}_\nu(N_\lambda)\propto\mathrm{Csg}_\nu(N_{\lambda-1})\equiv N_{\omega\zeta+\sigma_\zeta+n-1}$ contradicting assumption D.1) on $N_{\omega\zeta+\sigma_\zeta+n}$. Hence $N_{\lambda-1}\not\approx N_\lambda$ and by assumption D.1) on $N_{\lambda-1}$, N_λ satisfies D.1).

Assume $X\in\mathfrak{N}_\beta$ and $X\not\propto N_{\lambda-1}$, then there exists an $a\in\mathrm{Lay}_\nu(X)$ such that $\mathrm{Ub}'(a)\not\propto N_{\omega\zeta+\sigma_\zeta+n-1}$. Since $\omega\zeta+\sigma_\zeta+n\in\Gamma^0$, it was already proved that $N_{\omega\zeta+\sigma_\zeta+n}$ satisfies D.2′). Hence $\mathrm{Ub}'(a)\not\propto N_{\omega\zeta+\sigma_\zeta+n-1}$ implies $N_{\omega\zeta+\sigma_\zeta+n}\propto\mathrm{Ub}'(a)$. Hence $N_\lambda=W_{\omega^\nu}+N_{\omega\zeta+\sigma_\zeta+n}\propto\mathrm{Ub}(a)\cup\mathrm{Lb}'(a)\subset X$ and N_λ satisfies D.2′).

Finally consider the case $\lambda=\alpha_\nu(\omega_\beta^\zeta)+n$ where ζ is a limit number such that $\zeta<\mathrm{ls}(\lambda)$ and $\mathrm{cf}(\zeta)=\beta$.

Proof of Proposition 1 *for* $\lambda=\alpha_\nu(\omega_\beta^\zeta)$ $(\mathrm{fr}(\lambda)=0)$. (Besides it can be proved that if Proposition 2 holds for $\lambda=\zeta$, then Proposition 2 also holds for $\lambda=\alpha_\nu(\omega_\beta^\zeta)$. In brackets of the following proof we shall consider this case).

If $\mu<\lambda$, then there exists a $\theta<\zeta$ such that $\mu<\alpha_\nu(\omega_\beta^{\theta+1})$. Since

$\omega^{\theta}+\sigma_{\theta}<\omega\zeta=\zeta$, $N_{\mu}\not\approx N_{\alpha_{\nu}(\omega_{\beta}^{\theta+1})}=W_{\omega^{\nu}}+N_{\omega\theta+\sigma_{\theta}}\propto W_{\omega^{\nu}}+N_{\zeta}=N_{\lambda}$ and N_{λ} satisfies D. 1).

Assume $X\in\mathfrak{S}_{\beta}$ (resp. $X\in\mathfrak{R}_{\beta}$ if N_{ζ} satisfies D. 2')) and $X\not\propto N_{\mu}$ for any $\mu<\lambda$. Put $A=\mathrm{Lay}_{\nu}(X)$. If for any $a\in A$ there exists a $\xi(a)<\zeta$ such that $\mathrm{Ub}'(a)\propto N_{\xi(a)}$, then the numbers $\xi(a)$, $a\in A$, are not cofinal to ζ since $\overline{\overline{X}}<\aleph_{3}$ and $\mathrm{cf}(\zeta)=\beta$. Hence there exists a $\xi'<\zeta$ with $\xi(a)<\xi'$ for any $a\in A$. Without loss of generality we may assume that ξ' has a form $\xi'=\omega\xi+\sigma_{\xi}$ where $\xi<\zeta$. Then since $\bigvee_{a\in A}\mathrm{Ub}'(a)\equiv\mathrm{Csg}_{\nu}(X)\propto N_{\omega\xi+\sigma_{\xi}}$, $X\propto W_{\omega^{\nu}}+N_{\omega\xi+\sigma_{\xi}}=N_{\alpha_{\nu}(\omega_{\beta}^{\xi+1})}$ while $\alpha_{\nu}(\omega_{\beta}^{\xi+1})<\lambda$ contradicting assumption. Hence there exists an $a\in A$ such that $\mathrm{Ub}'(a)\not\propto N_{\xi}$ for any $\xi<\zeta$. By assumption D. 2) (resp. D. 2')) on N_{ζ}, $N_{\zeta}\propto\mathrm{Ub}'(a)$. Hence $N_{\lambda}\propto W_{\omega^{\nu}}+\mathrm{Ub}'(a)$ $\equiv\mathrm{Lb}(a)\cup\mathrm{Ub}'(a)\subset X$ and N_{λ} satisfies D. 2) (resp. D. 2')).

It follows from Lemma 3. 18 that N_{λ} satisfies D. 3).

Proof of Proposition 1 *for* $\lambda=\alpha_{\nu}(\omega_{\beta}^{\zeta})+n$ *where* $\zeta<\mathrm{ls}(\lambda)$ *and* $n>0$. (Besides, under assumption that $N_{\zeta+n}$ satisfies D. 2'), Proposition 2 holds for λ).

Proof for D. 1) and D. 2) (resp. D. 2')) are obtained almost samely as the case $\lambda=\alpha_{\nu}(\omega_{\beta}^{\zeta+1})+n$ where $\zeta>0$ and $n>0$. For D. 3), refer to Lemma 3. 18.

Now we complete the proof that N_{λ} in $\mathfrak{N}$ satisfies Proposition 1 in any case, and as we noticed at the head of § 2 of this chapter, the proof of Theorem A is also completed.

REMARK. It will be worth to notice that for most $\lambda<\beta^{\sharp}$, Proposition 2 is satisfied, i.e., N_{λ} satisfies D. 1) and D. 2'). Indeed by a careful study of proofs in this chapter, we can see that $\lambda=\alpha_{\nu}(\omega_{\beta}^{\xi+1})$ is the only case where we can not assert Proposition 2 for $\lambda<\beta^{\sharp}$ even under the assumption that any N_{ζ} in $\mathfrak{N}$ which is refered to define N_{λ} satisfies D. 1) and D. 2'). Even for $\lambda=\alpha_{\nu}(\omega_{\beta}^{\xi+1})$, if $\zeta=0$ and ν is either an isolated number or a limit number with $\mathrm{cf}(\nu)=0$, then D. 2') also holds (Lemma 3. 14).

This remark will be recalled in the appendix at the end of this paper.

CHAPTER IV. PROOF OF MAIN THEOREM B

In § 1 we shall define an operation—the ramified power—of ramified sets. It is, in a sense, a limit operation of repeated ramified products. Applying it we shall find a ramified set S_{ζ}^{ν} for any ν and ζ with $0<\nu<\omega_{\beta}$ and $\zeta<\beta^{\sharp}$, which is situated by the order $\propto$ at the least upper bound of sets N_{μ} with $\mu<\alpha_{\nu}(\omega_{\beta}^{\zeta+1})$ within $\mathfrak{R}_{\beta}$. In § 2 we shall find sets which are examples to confirm Theorem B, (i) or (ii).

1. The sets S^ν_ζ

DEFINITION 12. Let $P^\nu_n(X)$ denote the set of all sequence $p=\{p(1), p(2), \cdots, p(n)\}$ of length n such that $p(k)\in \mathrm{Seg}_\nu(X)$ for $k=1, 2, \cdots, n-1$ and $p(n)\in X$. The *ramified power* $X^{\odot\nu}$ of X is the set $\bigvee_{n<\omega} P^\nu_n(X)$ within which we have $p<p'$ for $p\in P^\nu_n(X)$ and $p'\in P^\nu_{n'}(X)$, if and only if

$$\text{either } n<n',\ p(k)=p'(k) \text{ for any } k\leqq n-1 \text{ and } p(n)\leqq p'(n)$$
$$\text{or } n=n',\ p(k)=p'(k) \text{ for any } k\leqq n-1 \text{ and } p(n)<p'(n),$$

For $p\in P^\nu_n(X)$, $\mathrm{Dig}(p)$ (digitation of p) denotes the set of all $p'\in P^\nu_{n+1}(X)$ such that $p'(k)=p(k)$ for any $k\leqq n$. (If $p(n)\in \mathrm{Csg}_\nu(X)$, then $\mathrm{Dig}(p)$ is void.) $\mathrm{len}(p)$ denotes the number n such that $p\in P^\nu_n(X)$.

The set $P^\nu_n(X)$ and $\mathrm{Dig}(p)$ for $p\in X^{\odot\nu}$ are considered as a ramified subset of $X^{\odot\nu}$. If $\mathrm{Dig}(p)$ is not void, i.e., $p(\mathrm{len}(p))\in \mathrm{Seg}_\nu(X)$, then obviously $\mathrm{Dig}(p)$ is isomorphic to X. It is easily seen that a ramified power of a ramified set is always a ramified set and if $X\in\mathfrak{R}_\beta$ then $X^{\odot\nu}\in\mathfrak{R}_\beta$ for any ν with $0<\nu<\omega_\beta$. Further if $\kappa(X)\geqq\omega^\nu$, then $X^{\odot\nu}$ is irresoluble. The following is obvious, and we omit the proof.

Lemma 4.1. (i). *$\tau(p; X^{\odot\nu})<\omega^\nu$ if and only if $\tau(p(\mathrm{len}(p)); X)<\omega^\nu$.*
(ii). $\mathrm{Csg}_\nu(P^\nu_n(X))=P^\nu_n(X)\frown \mathrm{Csg}_\nu(X^{\odot\nu})$.
(iii). *$\bigvee_{k\leqq n} P^\nu_k(X)$ is a cut of $X^{\odot\nu}$, and $\mathrm{Exp}_\nu(\bigvee_{k\leqq n} P^\nu_k(X))=\bigvee_{k\leqq n} P^\nu_k(X)$.*

Corollary 1. $\mathrm{Csg}_\nu(X^{\odot\nu})\sim \mathrm{Csg}_\nu(X)$.

Corollary 2. *If $\kappa(X)>\omega^\nu$, then $\kappa(X^{\odot\nu})=\kappa(X)$.*

Corollary 3. $X^{\odot\nu}\propto W_{\omega^\nu}+\mathrm{Csg}_\nu(X)$.

Lemma 4.2. $P^\nu_n(X)\sim X$.

Proof. Obviously $P^\nu_1(X)\equiv X$. If $n=m+1$ where $m>0$, then $P^\nu_n(X)$ is the union of all $\mathrm{Dig}(q)$ where $q\in P^\nu_m(X)$. Since $p\in \mathrm{Dig}(q)$, $p'\in \mathrm{Dig}(q')$ and $q\neq q'$ for q and q' in $P^\nu_m(X)$ imply that p and p' are disordered, $P^\nu_n(X)=\bigvee_{q\in P^\nu_m(X)} \mathrm{Dig}(q)$. Since $\mathrm{Dig}(q)\sim X$ for $q\in P^\nu_m(X)\frown \mathrm{Seg}_\nu(X^{\odot\nu})$ and $\mathrm{Dig}(q)=\emptyset$ for $q\in P^\nu_m(X)\frown \mathrm{Csg}_\nu(X^{\odot\nu})$, $P^\nu_n(X)\sim X$.

Lemma 4.3. *If $p\in \mathrm{Seg}_\nu(X^{\odot\nu})$, then $\mathrm{Ub}(p)\sim X^{\odot\nu}$.*

Proof. $\mathrm{Ub}(p)\propto X^{\odot\nu}$ is obvious. Put $n=\mathrm{len}(p)$. $p\in \mathrm{Seg}_\nu(X^{\odot\nu})$ implies $p(n)\in \mathrm{Seg}_\nu(X)$ (see Lemma 4.1 (i)). For $q\in X^{\odot\nu}$, put $f(q)=q'$ where $q'(k)=p(k)$ for $k\leqq n$ and $q'(n+k)=q(k)$ for $k\leqq \mathrm{len}(q)$, then it is easily seen that f is an increasing mapping of $X^{\odot\nu}$ into $\mathrm{Ub}(p)$ and $X^{\odot\nu}\propto \mathrm{Ub}(p)$.

Corollary 1. *If $\eta\leqq\nu$ then $X^{\odot\nu}\odot_\eta X^{\odot\nu}\sim X^{\odot\nu}$.*

Corollary 2. *If $\eta<\omega^\nu$ then $W_\eta+X^{\odot\nu}\sim X^{\odot\nu}$.*

Now we shall find a set $S_\zeta^\nu \in \mathfrak{R}_\beta$ which is situated by the order relation $\propto$ at the least upper bound of sets N_μ with $\mu < \alpha_\nu(\omega_\beta^{\zeta+1})$ within $\mathfrak{R}_\beta$. ($N_{\alpha_\nu(\omega_\beta^{\zeta+1})}$ is the least upper bound of them within $\mathfrak{S}_\beta$). It was already seen that if either ν is an isolated number or a limit number with $\mathrm{cf}(\nu)=0$, then $N_{\alpha_\nu(\omega_\beta)}$ is the least upper bound of sets N_μ with $\mu < \alpha_\nu(\omega_\beta)$ within $\mathfrak{R}_\beta$ (Lemma 3.14).

If ν is a limit number with $\mathrm{cf}(\nu)>0$ (where β is assumed greater than 1), then put $S_0^\nu = N^{\odot_\nu}_{\alpha_\nu(\nu)}$. In general (i.e., ν is any number with $0<\nu<\omega_\beta$), if $\zeta>0$, then put $S_\zeta^\nu = N^{\odot_\nu}_{\alpha_\nu(\omega_\beta^\zeta)+\sigma'_\zeta}$. And we shall show that a proposition similar to Proposition 2 for N_λ with $\lambda=\alpha_\nu(\omega_\beta^{\zeta+1})$ holds for S_ζ^ν.

Remark. If ν is a limit number with $\mathrm{cf}(\nu)>0$, then $\alpha_\nu(\nu)=\sup_{\mu<\nu}\alpha_\nu(\mu)$ $=\sup_{\mu<\nu}\alpha_\mu(\omega_\beta)$ and hence $N_{\alpha_\nu(\nu)} \sim \bigvee_{\mu\in\Lambda_\nu} N_{\alpha_\mu(\omega_\beta)} = \bigvee_{\mu\in\Lambda_\nu} W_{\omega^\mu}$. Hence $\alpha_\nu(\nu)$ is the least number among numbers ξ with $\kappa(N_\xi)=\omega^\nu$. Similarly when $\zeta>0$ (and ν is any number), $\alpha_\nu(\omega_\beta^\zeta)+\sigma'_\zeta$ is the least number among numbers ξ with $\mathrm{Csg}_\nu(N_\xi) \sim N_{\omega\zeta+\sigma'_\zeta} \equiv \mathrm{Csg}_\nu(N_{\alpha_\nu(\omega_\beta^{\zeta+1})})$.

Besides, Proposition 2 for above λ does not hold.

Hereafter until the end of this section, when we say $\zeta=0$, we automatically assume that ν is a limit number with $\mathrm{cf}(\nu)>0$. (When $\zeta>0$, this restriction is omitted).

Lemma 4.4. *Put* $\lambda=\alpha_\nu(\omega_\beta^{\zeta+1})$. (i). *If* $\mu<\lambda$ *then* $N_\mu \lneqq S_\zeta^\nu$.

(ii). *If* $X\in\mathfrak{R}_\beta$ *and* $X \not\propto N_\mu$ *for any* $\mu<\lambda$, *then* $S_\zeta^\nu \propto X$.

Proof. Put $\delta=\alpha_\nu(\nu)$ if $\zeta=0$ and $\delta=\alpha_\nu(\omega_\beta^\zeta)+\sigma'_\zeta$ if $\zeta>0$. λ is the least number greater than δ within $\Gamma_\nu^2 \cup \tilde{\Gamma}_{\nu+1}$ (see Lemma 1.14). Hence $\delta<\mu<\lambda$ implies $\mu\in\Gamma^0 \cup \bigcup_{\eta\leqq\nu}\Gamma_\eta^1 \cup \bigcup_{\eta<\nu}\Gamma_\eta^2$.

(i). Of course $\mu\leqq\delta$ implies $N_\mu \propto N_\delta^{\odot_\nu} = S_\zeta^\nu$. Assume $\delta<\mu<\lambda$ and $N_\theta \propto S_\zeta^\nu$ for any $\theta<\mu$, and we shall show $N_\mu \lneqq S_\zeta^\nu$.

Case $\mu\in\Gamma^0$. In this case $N_\mu \propto S_\zeta^\nu$ follows from D. 2′) on N_μ immediately.

Case $\mu\in\bigcup_{\eta\leqq\nu}\Gamma_\eta^1$. Let $\mu=\alpha_\eta(\xi+\theta)+n$ be the canonical decomposition of μ. By assumption $N_{\alpha_\eta(\theta)+n} \propto S_\zeta^\nu$ and $N_{\alpha_\eta(\xi)+\sigma_\xi} \propto S_\zeta^\nu$. Hence by Corollary 1 of Lemma 4.3 and $\eta\leqq\nu$, $N_\mu = N_{\alpha_\eta(\theta)+n} \odot_\eta N_{\alpha_\eta(\xi)+\sigma_\xi} \propto S_\zeta^\nu \odot_\eta S_\zeta^\nu \sim S_\zeta^\nu$.

Case $\mu\in\bigcup_{\eta<\nu}\Gamma_\eta^2$. Let $\mu=\alpha_\eta(\omega_\beta^\theta)+n$ be the canonical decomposition of μ. Put $\theta'=\theta$ if θ is a limit number with $\mathrm{cf}(\theta)=\beta$ and $\theta'=\omega(\theta-1)+\sigma_\theta$ if θ is an isolated number. By assumption $N_{\theta'+n} \propto S_\zeta^\nu$. Since $\eta<\nu$, $N_\mu = W_{\omega^\eta} + N_{\theta'+n} \propto W_{\omega^\eta} + S_\zeta^\nu \propto S_\zeta^\nu$ by Corollary 2 of Lemma 4.3.

Hence $N_\mu \propto S_\zeta^\nu$ for any $\mu<\lambda$, and hence $N_\mu \lneqq N_{\mu+1} \propto S_\zeta^\nu$ which proves (i).

(ii). Assume $X \in \mathfrak{R}_\beta$ and $X \not\propto N_\mu$ for any $\mu < \lambda$. By D. 3) on N_μ, $N_\mu \propto X$ for any $\mu < \lambda$. Put $Y = \overline{\mathrm{Spt}}^\nu(X; N_\mu, \mu < \lambda)$ then for any y in $Y \frown \mathrm{Seg}_\nu(X)$, $N_\delta \propto \mathrm{Ub}(y) \frown Y$ (refer to the proof of Proposition 1 for $\lambda = \alpha_\nu(\omega_\beta)$ in the case $\zeta = 0$ or to Lemma 3. 17 in the case $\zeta > 0$). Especially $N_\delta \propto Y$.

Let f_1 be a reduced increasing mapping of $P_1^\nu(N_\delta)$, which is equivalent to N_δ by Lemma 4. 2, into Y, and assume that f_1 is extended to a reduced increasing mapping f_n of $\bigcup_{k \leqq n} P_k^\nu(N_\delta)$ into Y. Put $A_n = P_n^\nu(N_\delta) \frown \mathrm{Seg}_\nu(S_\zeta^\nu)$, then $P_{n+1}^\nu(N_\delta) = \bigcup_{a \in A_n} \mathrm{Dig}(a)$. Put $y = f_n(a)$ for $a \in A_n$, then $y \in \mathrm{Seg}_\nu(X) \frown Y$. Since $\mathrm{Dig}(a) \equiv N_\delta \propto \mathrm{Ub}(y) \frown Y$, there exists a reduced increasing mapping g_a of $\mathrm{Dig}(a)$ into $\mathrm{Ub}(y) \frown Y$. Put $f_{n+1}(x) = g_a(x)$ for $x \in \mathrm{Dig}(a)$ and $a \in A_n$ and $f_{n+1}(x) = f_n(x)$ for $x \in \bigcup_{k \leqq n} P_k^\nu(N_\delta)$, then f_{n+1} is an extension of f_1 and a reduced increasing mapping of $\bigcup_{k \leqq n+1} P_k^\nu(N_\delta)$ into Y. Finally put $f(x) = f_n(x)$ for $x \in P_n^\nu(N_\delta)$, then f is an reduced increasing mapping of S_ζ^ν into $Y \subset X$. Hence $S_\zeta^\nu \propto X$ and (ii) is proved.

Corollary. *S_ζ^ν is comparable with any $X \in \mathfrak{R}_\beta$.*

Hence S_ζ^ν is the least upper bound of sets N_μ with $\mu < \alpha_\nu(\omega_\beta^{\zeta+1})$ within $\mathfrak{R}_\beta$ as we noticed. Moreover,

Lemma 4. 5. *$S_\zeta^\nu \not\approx N_\lambda$ where $\lambda = \alpha_\nu(\omega_\beta^{\zeta+1})$.*

Proof. Let δ be the same number defined in the proof of previous lemma.

Case $\zeta = 0$. It suffices to show $\kappa(S_0^\nu) = \omega^\nu$, since $N_\lambda = W_{\omega^\nu}$ and $\kappa(N_\lambda) = \omega^\nu + 1$ (see Theorem 5). Of course $\kappa(S_0^\nu) \geqq \kappa(N_\delta) = \omega^\nu$.

Let W be a maximal totally ordered subset of S_0^ν. If there exist a $p \in W$ and a $n < \omega$ such that $\mathrm{Ub}'(p) \frown W \subset P_n^\nu(N_\delta)$, then let p be such a least element. If $n = 1$ then $W \subset P_1^\nu(N_\delta) \sim N_\delta$ and $\overline{W} < \kappa(N_\delta) = \omega^\nu$. If $n > 1$, then p is contained in a $\mathrm{Dig}(q)$ where $q \in P_{n-1}^\nu(N_\delta)$, and obviously $\tau(p) = \tau(q) + 1$. Since $\mathrm{Dig}(q)$ is not void, $\tau(q) < \omega^\nu$ while the type of $\mathrm{Ub}'(p) \frown W$ is less than $\kappa(P_n^\nu(N_\delta)) = \kappa(N_\delta) = \omega^\nu$. Hence $\overline{W} = \tau(q) + 1 + \overline{\mathrm{Ub}'(p) \frown W} < \omega^\nu$.

If for any $n < \omega$, there exists a p_n in $P_n^\nu(N_\delta) \frown W$, then let p_n be such a least element. Similarly as the above $\tau(p_n; N_\delta^{\odot\nu}) < \omega^\nu$ for any $n < \omega$. But there is no $w \in W$ such that $p_n < w$ for any $n < \omega$, and numbers $\tau(p_n; N_\delta^{\odot\nu})$ with $n < \omega$ are cofinal to $\overline{W}$. By assumption ν is a limit number with $\mathrm{cf}(\nu) > 0$, and $\mathrm{cf}(\omega^\nu) = \mathrm{cf}(\nu) > 0$. Hence the number $\overline{W}$, to which a countable sequence of numbers less than ω^ν is cofinal, is less than ω^ν.

Hence $W \subset N_\delta^{\odot\nu}$ implies $\bar{W} < \omega^\nu$ and $\kappa(N_\delta^{\odot\nu}) = \omega^\nu$ which is to be proved.

Case $\zeta > 0$. Since $\mathrm{Csg}_\nu(S_\zeta^\nu) = \mathrm{Csg}_\nu(N_\delta) = N_{\omega\zeta+\sigma'_\zeta} = \mathrm{Csg}_\nu(N_\lambda)$ (see Corollary 1 of Lemma 4.1), $S_\zeta^\nu \propto W_{\omega^\nu} + N_{\omega\zeta+\sigma'_\zeta} = N_\lambda$ (Lemma 2.12, (iii)).

Assume $N_\lambda \propto S_\zeta^\nu$ and let f be a reduced increasing mapping of N_λ into S_ζ^ν. Let a be any element in $\mathrm{Lay}_\nu(N_\lambda)$, then $f(a)$ falls onto a $p \in P_n^\nu(N_\delta)$, and f maps the subset W_{ω^ν} of $N_\lambda = W_{\omega^\nu} + N_{\omega\zeta+\sigma'_\zeta}$ into $\bigcup_{k \leqq n} P_k^\nu(N_\delta)$. Hence $f(N_\lambda) \subset \mathrm{Exp}_\nu(f(W_{\omega^\nu}); S_\zeta^\nu) \subset \mathrm{Exp}_\nu(\bigcup_{k \leqq n} P_k^\nu(N_\delta); N_\delta^{\odot\nu}) = \bigcup_{k \leqq n} P_k^\nu(N_\delta)$ (see Lemma 4.1 (iii)). Besides there exists a $w \in W_{\omega^\nu}$ such that $f(w) \in P_n^\nu(N_\delta)$, or otherwise f maps any element w in W_{ω^ν} into $\bigcup_{k<n} P_k^\nu(N_\delta)$ and $f(N_\lambda) \subset \bigcup_{k<n} P_k^\nu(N_\delta)$ similarly as the above, contradicting $p = f(a) \in P_n^\nu(N_\delta)$. Then f maps $\mathrm{Ub}'(w; N_\lambda)$ entirely into $P_n^\nu(N_\delta)$ while $\mathrm{Ub}'(w; N_\lambda) = W_{\omega^\nu} + N_{\omega\zeta+\sigma'_\zeta} \sim N_\lambda$ contradicting $P_n^\nu(N_\delta) \sim N_\delta \not\approx N_\lambda$. Hence $S_\zeta^\nu \not\approx N_\lambda$ which is to be proved.

In Chapter 3 we did not assert that Proposition 2 holds for $\lambda = \alpha_\nu(\omega_\beta^{\zeta+1})$ where either $\zeta > 0$ or ν is a limit number with $\mathrm{cf}(\nu) > 0$, but Lemmas 4.4 and 4.5 shows that surely it does not hold for such λ.

Finally we add

Lemma 4.6. *Assume $\zeta > 0$ and an $X \in \mathfrak{R}_\beta$ satisfies the following conditions:* (*a*) *the potency of* $\mathrm{Lay}_\nu(X)$ *is at most* $\aleph_0$, *and* (*b*) *for any* $a \in \mathrm{Lay}_\nu(X)$, $\mathrm{Exp}_\nu(\mathrm{Lb}(a)) \propto N_{\alpha_\nu(\omega_\beta^\zeta)+\sigma'_\zeta}$; *then* $X \propto S_\zeta^\nu$.

Proof. Put $\delta = \alpha_\nu(\omega_\beta^\zeta) + \sigma'_\zeta$ similarly as previous lemmas. Let $\mathfrak{A}$ be the family of all maximal totally ordered subsets A of $\mathrm{Seg}_\nu(X)$ such that $\mathrm{Exp}_\nu(A) \cap \mathrm{Lay}_\nu(X)$ is not void. Then the type of $A \in \mathfrak{A}$ is ω^ν. By (*a*), the potency of $\mathfrak{A}$ is at most $\aleph_0$. Let $A_1, A_2, \cdots, A_n, \cdots$ $(n < \omega)$ be a sequence which consists of all $A \in \mathfrak{A}$. Here we shall consider only the case $\overline{\overline{\mathfrak{A}}} = \aleph_0$ and assume $A_k \neq A_n$ if $k \neq n$. For the case $\overline{\overline{\mathfrak{A}}} < \aleph_0$, the sequence ceases at a $n < \omega$ and we can proceed the following proof with slight modifications.

Since $\mathrm{Exp}_\nu(A_1) \propto N_\delta$, there exists a reduced increasing mapping g_1 of $\mathrm{Exp}_\nu(A_1)$ into $P_1^\nu(N_\delta)$. Assume that g_1 is extended to a reduced increasing mapping g_{n-1} of $\bigcup_{k<n} \mathrm{Exp}_\nu(A_k)$ into $\bigcup_{k<n} P_k^\nu(N_\delta)$ where $n > 1$. Since $A_n - \bigcup_{k<n} A_k$ is not void (or otherwise A_n coincides with an A_k where $k < n$, for n is finite, contradictrily), there exists a minimal element a in it.

If a is minimal within X, then let b be any minimal element in $\bigcup_{k<n} A_k$. If a is not minimal in X, then $\mathrm{Lb}(a)$ is included in an A_k where $k < n$. Since the type of A_k is ω^ν and $\tau(a) < \omega^\nu$, there exists uniquely a b in A_k such that $\tau(b)=\tau(a)$. In either case that a is minimal within X or not, we have a $b \in \bigcup_{k<n} A_k$ such that $\mathrm{Lb}(b)=\mathrm{Lb}(a)$. Since g_{n-1} is is reduced, $\tau(g_{n-1}(b))=\tau(b)=\tau(a)<\omega^{\nu}$. Since $\mathrm{Exp}_{\nu}(A_n)\cap \mathrm{Ub}(a)\propto N_{\delta}$, there exists a reduced increasing mapping g'_a of $\mathrm{Exp}_{\nu}(A_n)\cap \mathrm{Ub}(a)$ into $\mathrm{Dig}(g_{n-1}(b))$ which is isomorphic to N_{δ}. Put $g_n(x)=g_{n-1}(x)$ for $x\in \bigcup_{k<n} \mathrm{Exp}_{\nu}(A_k)$, $g_n(a)=g_{n-1}(b)$ and $g_n(x)=g'_a(x)$ for $x\in \mathrm{Exp}_{\nu}(A_n)\cap \mathrm{Ub}(a)$, then g_n is an extension of g_{n-1} and a reduced increasing mapping of $\bigcup_{k\leqq n}\mathrm{Exp}_{\nu}(A_k)$ into $\bigcup_{k\leqq n} P_k^{\nu}(N_{\delta})$.

Let M be the set of all minimal element of $X-\bigcup_{m<\omega}\mathrm{Exp}_{\nu}(A_m)$, then similarly as the above, for any $a\in M$ there exists a $n<\omega$ and a $b\in A_n$ such that $\mathrm{Lb}(a)=\mathrm{Lb}(b)$. $a\in M$ implies $\kappa(\mathrm{Ub}'(a))\leqq\omega^{\nu}+1$ while $\kappa(\mathrm{Exp}_{\nu}(A_n)$ $\cap\, \mathrm{Ub}'(b))\geqq\omega^{\nu}+2$. Hence there exists a reduced increasing mapping h_a of $\mathrm{Ub}'(a)$ into $\mathrm{Exp}_{\nu}(A_n)\cap \mathrm{Ub}'(b)$.

Put $f(x)=g_n(x)$ for $x\in \mathrm{Exp}_{\nu}(A_n)-\bigcup_{k<n} A_k$ and $f(x)=fh_a(x)$ for $x\in \mathrm{Ub}'(a)$ and $a\in M$, then f is a reduced increasing mapping of X into S_{ξ}^{ν}, and $X\propto S_{\xi}^{\nu}$.

2. Proof of Main Theorem B

Now we shall intend to find examples to confirm Theorem B i) or ii).

DEFINITION 13. i). Put $L=\bigvee_{3\leqq k<\omega} W_{\omega+k}$ and $S=L^{\odot_1}$.

ii). $\mathfrak{A}'$ denotes the family of all maximal totally ordered subsets of $\mathrm{Seg}_1(S)$ and put $\mathfrak{A}=\mathfrak{A}'-\{\mathrm{Lb}(a)\,|\,a\in \mathrm{Lay}_1(S)\}$.

iii). Let t_A denote a term assigned to each $A\in\mathfrak{A}$ and put $B=\{t_A\,|\,A\in\mathfrak{A}\}$.

Put $T=S\cup B$ where order within S preserves orginal relation and $a<t_A$ holds if and only if $a\in A$. ($t_A<x$ does not occur for any $x\in T$).

For any subset C of B, $S\cup C$ is regarded as a ramified subset of T.

iv). A subset C of B is called *barren* if $S\cup C\propto S$.

The followings are obvious and we omit the proofs.

Lemma 4.7. (i). *$L\sim N_{\omega_\beta+\omega}$ and hence $S\sim S_1^1$. Especially $L\not\approx S$.*

(ii). *If A is a maximal totally ordered subset of* S, *then either $\bar{A}=\omega$ or $\omega+3\leqq\bar{A}<\omega\cdot 2$.* (Remark that the type of any maximal totally ordered subset of L is at least $\omega+3$).

iii). $\overline{\overline{S}}=\aleph_0$ *and* $\overline{\overline{B}}=\overline{\overline{T}}=2^{\aleph_0}$.

(iv). *For any $A\in\mathfrak{A}'$,* $\mathrm{Exp}_1(A\,;\,T)\cap \mathrm{Lay}_1(T)$ *consists of one and only one element.*

(v). *If $a_n\in P_n^1(L)$ and $a_n<a_{n+1}$ for any $n<\omega$, then the sequence*

$\{a_n \mid n<\omega\}$ *uniquely determines a maximal totally ordered subset* W *of* $\mathrm{Seg}_1(T)$ *such that* $a_n \in W$ *for any* $n<\omega$, *and* W *is contained in* $\mathfrak{A}$.

(vi). *If* $a \in \mathrm{Seg}_1(T)$, *then* $\mathrm{Ub}(a) \sim T$. (See Lemma 4.3).

(vii). *If* $\overline{\overline{C}} \subset B$ *and* $C \leqq \aleph_0$, *then* C *is barren.* (See Lemma 4.6).

Lemma 4.8. $S \approx T$.

Proof. $S \propto T$ is obvious. Assume $T \propto S$ and that there exists a reduced increasing mapping f of T into S. Let a be any minimal element of T then $f(a_1) \in P_1^1(L)$. Assume that an $a_n \in T$ is already determined in such a way that $\tau(a_n)<\omega$ and $f(a_n) \in P_n^1(L)$. Since $\mathrm{Ub}(a_n) \sim T$ and $P_n^1(L) \sim L \approx S$, $\mathrm{Ub}(a_n) \not\propto P_n^1(L)$ and $f(\mathrm{Ub}(a_n))$ is not included in $P_n^1(L)$. Let a_{n+1} be a minimal element in $\mathrm{Ub}(a_n)$ which is mapped by f in $\mathrm{Ub}(f(a_n))-P_n(L)$, then obviously $f(a_{n+1}) \in P_{n+1}^1(L)$. Furthermore $\tau(a_{n+1})<\omega$, or otherwise, since $\mathrm{Lb}(a_{n+1}) \cap \mathrm{Seg}_1(T)$ is mapped by f into $\bigcup_{k \leqq n} P_k^1(L)$, a_{n+1} is mapped in $\mathrm{Exp}_1(\bigcup_{k \leqq n} P_k^1(L)\,;\,S)=\bigcup_{k \leqq n} P_k^1(L)$ contradictorily. Thus we have a sequence $a_1, a_2, \cdots, a_n, \cdots$ $(n<\omega)$ such that $a_n<a_{n+1}$, $a_n \in \mathrm{Seg}_1(T)$ and $f(a_n) \in P_n^1(L)$ for any $n<\omega$. By Lemma 4.7, (iv) and (v), there exists a $t \in \mathrm{Lay}_1(T)$ such that $a_n<t$ for any $n<\omega$. $f(t) \in S$ and hence $f(t)$ is contained in a $P_n^1(L)$, but then since $f(a_{n+1}) \in P_{n+1}^1(L)$, $f(a_{n+1})$ is not less than $f(t)$, contradicting $a_{n+1}<t$ and that f is increasing. Hence $T \not\propto S$ and accordingly $S \approx T$.

Lemma 4.9. *If* f *is a reduced increasing mapping of* T *into itself, then the restriction* f' *of* f *on* S *is a reduced increasing mapping of* S *into itself. Conversely if* f' *is a reduced increasing mapping of* S *into itself, then there exists a unique reduced increasing mapping of* T *into itself which is an extension of* f'.

Proof. Assume that f is a reduced increasing mapping of T into itself, but there exists an $x \in S$ such that $f(x) \in B=T-S$, then $f(x)=t_A$ for an $A \in \mathfrak{A}$. Since f is reduced, $\tau(x)=\tau(t_A)=\omega$. But by the definition of L and S, for any $x \in \mathrm{Lay}_1(S)$, $\mathrm{Ub}(x)$ is not void, while $\mathrm{Ub}(t_A)$ is void, contradicting $f(\mathrm{Ub}(x)) \subset \mathrm{Ub}(t_A)$. Hence $x \in S$ implies $f(x) \in S$ and the restriction f' of f on S is a reduced increasing mapping of S into itself.

Conversely let f' be a reduced increasing mapping of S into itself. For any $t_A \in B=T-S$, $A=\mathrm{Lb}(t_A) \in \mathfrak{A}$ and since f' is reduced, $f'(A) \in \mathfrak{A}'$. By Lemma 4.7 (iv) there exists a unique element $b_A \in \mathrm{Exp}_1(f'(A)\,;\,T) \cap \mathrm{Lay}_1(T)$. Put $f(t_A)=b_A$ and $f(x)=f'(x)$ for $x \in S$, then it is obvious that f is the unique reduced increasing mappping of T into itself, which is an extension of f'.

Corollary. *For any subsets* C *and* D *of* B *such that* $S \cup C \propto S \cup D$, *a reduced increasing mapping of* $S\cup C$ *into* $S\cup D$ *can be uniquely extended to a reduced increasing mapping of* T *into itself.*

Now we shall introduce topology in $\mathrm{Lay}_1(T)$ and B. For $x\in \mathrm{Seg}_1(T)$ put $V_x=\mathrm{Lay}_1(T)\cap \mathrm{Ub}(x)$. If V_x intersect with $V_{x'}$, then one is entirely included in the other, and if $V_x\subset V_{x'}$, then $x'\leqq x$. Hence if the intersection of a family $\{V_{x_k}|k\leqq n\}$ is not void, then the set $\{x_k|k\leqq n\}$ is totally ordered. Let x be the greatest element of this set, then $\bigcap_{k\leqq n} V_{x_k}=V_x$. Hence the family $\{V_x|x\in \mathrm{Seg}_1(T)\}$, which will be denoted by $\mathfrak{B}$, makes a basis of open sets, by which a topology is defined on $\mathrm{Lay}_1(T)$. Since $\mathrm{Seg}_1(T)$ is countable, $\mathfrak{B}$ is countable. It is easy to see (or rather well-known) that the topological space $\mathrm{Lay}_1(T)$ thus defined is a totally disconnected Hausdorff space.

B is a topological subspace of $\mathrm{Lay}_1(T)$ with relative topology. Put $V'_x=B\cap V_x$ for $x\in \mathrm{Seg}_1(T)$, and $\mathfrak{B}'=\{V'_x|x\in \mathrm{Seg}_1(T)\}$, then $\mathfrak{B}'$ is a basis of open sets of B.

For a reduced increasing mapping f of T, let f^{∇} denote the restriction of f on $\mathrm{Lay}_1(T)$. Since f is reduced, $f^{\nabla}(\mathrm{Lay}_1(T))\subset \mathrm{Lay}_1(T)$.

Lemma 4.10. *If f is a reduced increasing mapping of T into itself, then f^{∇} is a continuous mapping of* $\mathrm{Lay}_1(T)$ *into itself.*

Proof. Assume $b\in \mathrm{Lay}_1(T)$, $b'=f^{\nabla}(b)$ and that V is an open set which contains b', then there exists a $V_{x'}\in\mathfrak{B}$ such that $b'\in V_{x'}\subset V$. Let x be the unique element in $\mathrm{Lb}(b)$ such that $\tau(x)=\tau(x')$. Since f is reduced, $f(x)=x'$. Now $b\in V_x$ and $f^{\nabla}(V_x)\subset V_{x'}\subset V$ which shows that f^{∇} is a continuous mapping of $\mathrm{Lay}_1(T)$ into itself.

For a subset C of $\mathrm{Lay}_1(T)$, put $\tilde{C}=\bigcup_{b\in C}\mathrm{Exp}_1(\mathrm{Lb}(b))$.

Lemma 4.11. *If a subset C of* $\mathrm{Lay}_1(T)$ *is closed, then* $\mathrm{Exp}_1(\tilde{C})=\tilde{C}$.

Proof. Assume $x\notin\tilde{C}$. If $\tau(x)<\omega$, then $x\notin \mathrm{Exp}_1(\tilde{C})$ by the definition of Exp_1. If $\tau(x)\geqq\omega$, then let b be the unique element in $\mathrm{Lay}_1(T)\cap \mathrm{Lb}'(x)$. Of course $b\notin\tilde{C}$. Since C is closed, there exists a $V_y\in\mathfrak{B}$ such that $b\in V_y$ and $V_y\cap C=\emptyset$. Since $x\in \mathrm{Ub}(y)$ and $\mathrm{Ub}(y)\cap\tilde{C}=\emptyset$, $x\notin \mathrm{Exp}_1(\tilde{C})$ and hence $\mathrm{Exp}_1(\tilde{C})\subset\tilde{C}$.

Lemma 4.12. *In order that a subset C of B is barren, it is necessary and sufficient that C is a subset of a union of at most countable barren subsets of B which are closed within B.*

Proof. If C is barren, then there exists a reduced increasing mapping f' of $S\cup C$ into S. f' is uniquely extended to a reduced increasing mapping f of T into itself (see Corollary of Lemma 4.9). f maps every $t\in C$ into $\mathrm{Lay}_1(S)$ which consists of countable elements $b_1, b_2, \cdots, b_n, \cdots$.

Let C_n be the set of all $t \in B$ such that $f^r(t)=b_n$, then since f^r is continuous, each C_n is closed in B. Since $f(S \cup C_n) \subset S$, each C_n is barren, and hence C is a subset of the union of countable barren subsets C_n of B each of which is closed in B.

Conversely assume that a subset C of B is included in a union C' of at most countable barren subsets C_n of B, where each C_n is closed in B. Let $\bar{C}_n$ be the closure of C_n in $\mathrm{Lay}_1(T)$, then since C_n is closed in B, $\bar{C}_n \cap B = C_n$. Put $C'' = \mathrm{Lay}_1(S) - \bigcup_{n<\omega} \bar{C}_n$, then C'' is at most countable. Let $\mathfrak{D}$ be the family of all sets $\bar{C}_n$ and all $\{d\}$ where $\{d\}$ is the set which consists of a single element in C''. Notice that any set D in $\mathfrak{D}$ is closed in $\mathrm{Lay}_1(T)$ and, since $D \cap B$ is either void or coincidental with a C_n which is barren, $D \cup S \propto S$.

$\mathfrak{D}$ contains at most countable sets (it can be proved that the potency of $\mathfrak{D}$ is not finite, but we need not use the fact here). Let $D_1, D_2, \cdots, D_n, \cdots$ $(n<\omega)$ be an arrangement of all sets in $\mathfrak{D}$ into a sequence, where repeats of same sets are admitted. $\mathrm{Lay}_1(S \cup C') = \bigcup_{n<\omega} D_n$, and
$S \cup C' = C' \cup \bigcup_{a \in \mathrm{Lay}_1(S)} \mathrm{Exp}_1(\mathrm{Lb}(a)) = \bigcup_{a \in \mathrm{Lay}_1(S \cup C')} \mathrm{Exp}_1(\mathrm{Lb}(a)) = \bigcup_{n<\omega} \bigcup_{a \in D_n} \mathrm{Exp}_1(\mathrm{Lb}(a))$
$= \bigcup_{n<\omega} \tilde{D}_n$.

Since $\tilde{D}_1 \subset D_1 \cup S \propto S$, there exists a reduced increasing mapping f_1 of $\tilde{D}_1$ into S. Assume that f_1 is extended to a reduced increasing mapping f_n of $\bigcup_{k \leq n} \tilde{D}_k$ into S. Put $E_n = \bigcup_{k \leq n} D_k$, then $\tilde{E}_n = \bigcup_{k \leq n} \tilde{D}_k$. Since E_n is closed, $\mathrm{Exp}_1(\tilde{E}_n) = \tilde{E}_n$ by Lemma 4.11. Hence for any minimal element a of $(S \cup C') - \tilde{E}_n$, $\tau(a) < \omega$ by Lemma 2.3 (vii). Let M'_n be the set of all minimal elements of $(S \cup C') - \tilde{E}_n$, then $(S \cup C') - \tilde{E}_n \subset \bigcup_{a \in M'_n} \mathrm{Ub}'(a\,;T)$. Put $D_{n+1,a} = D_{n+1} \cap \mathrm{Ub}'(a)$ for $a \in M'_n$ and let M_n be the set of all $a \in M'_n$ where $D_{n+1,a}$ is not void. (M_n itself may be void). Then $D_{n+1} = \bigcup_{a \in M_n} D_{n+1,a}$ and $\tilde{D}_{n+1,a} \subset S \cup D_{n+1} \propto S$. If $a \in M_n$, then since $\tau(a) < \omega$, either $\tau(a)=0$ or a has an immediate ascendant a' in $\tilde{E}_n$. If $\tau(a)=0$, then let g_a be any reduced increasing mapping of $\tilde{D}_{n+1,a}$ into S. If a has an immediate ascendant a' in $\hat{E}_n$, then since $\tau(f_n(a')\,;S) = \tau(a') < \omega$ and $\mathrm{Ub}(f_n(a')\,;S) \sim S$ by Lemma 4.3, we have a reduced increasing mapping g_a of $\tilde{D}_{n+1,a} \cap \mathrm{Ub}'(a)$ into $\mathrm{Ub}(f_n(a')\,;S)$. Put $f_{n+1}(x) = f_n(x)$ for $x \in \hat{E}_n$ and $f_{n+1}(x) = g_a(x)$ for $x \in \tilde{D}_{n+1,a} \cap \mathrm{Ub}'(a)$ and $a \in M_n$, then as easily seen, f_{n+1} is a reduced increasing mapping of $\bigcup_{k \leq n+1} \tilde{D}_k$ into S. Finally put $f(x) = f_n(x)$ for $x \in \tilde{D}_n$, then f is a reduced increasing mapping of $S \cup C' = \bigcup_{k<\omega} \tilde{D}_k$ into S and hence $S \cup C' \propto S$, i.e., C' is barren. Hence the subset C of C' is also barren.

Hereafter, until the end of this section, we assume continuum hypothesis. The following Lemma is essentially a consequence of Proposition P_8 of [8].

Lemma 4.13. *If a subset C of B is not barren, then there exists a subset D of C such that $S \not\approx S \cup D \not\approx S \cup C$.*

Proof. Let $\mathfrak{C}$ be the family of all barren subsets of B which are closed in B. Since the basis $\mathfrak{B}'$ of open sets of B is countable, the potency of $\mathfrak{C}$ is at most $\aleph_1$ by continuum hypothesis. Let $\mathfrak{F}$ be the family of all reduced increasing mappings of $S \cup C$ into itself, and $\mathfrak{F}'$ be the family of all reduced increasing mappings of S into itself. As we have seen in Lemma 4.9 and its Corollary, there exists a one-to-one correspondence between all mappings in $\mathfrak{F}$ and (not necessarily all) mappings in $\mathfrak{F}'$ such that, if $f \in \mathfrak{F}$ is corresponding to $f' \in \mathfrak{F}'$, then f' is the restriction of f on S. Since $\overline{\overline{S}} = \aleph_0$, the potency of $\mathfrak{F}'$ is at most $\aleph_1$ by continuum hypothesis. Hence the potency of $\mathfrak{F}$ is also at most $\aleph_1$. Since C is not barren, it is not countable (see Lemma 4.7 (vii)), while the potency of B is $2^{\aleph_0} = \aleph_1$ (see Lemma 4.7 (iii)). Hence $\overline{\overline{C}} = \aleph_1$

Let $\{C_\nu \mid \nu < \omega_1\}$ be a transfinite sequence formed of all sets in $\mathfrak{C}$, $\{f_\nu \mid \nu < \omega_1\}$ be a transfinite sequence formed of all mappings in $\mathfrak{F}$, and $\{b_\nu \mid \nu < \omega_1\}$ be a transfinite sequence formed of all elements in C, where in the former two sequences, repeats of same terms (sets or mappings) are admitted if they are necessary, but since $\overline{\overline{C}} = \aleph_1$, we may assume $b_\nu \neq b_{\nu'}$ if $\nu \neq \nu'$. Put $D_\xi = f_\xi(S \cup C) \cap B$. Since f_ξ is a reduced increasing mapping of $S \cup C$ into itself, $D_\xi \subset C$. D_ξ is not barren, or otherwise, there exists a reduced increasing mapping g of $S \cup D_\xi$ into S, and $g f_\xi$ is a reduced increasing mapping of $S \cup C$ into S, contradicting that C is not barren.

Now we shall define two increasing sequence $\{\lambda_\xi \mid \xi < \omega_1\}$ and $\{\mu_\xi \mid \xi < \omega_1\}$ of ordinal numbers less than ω_1, which satisfy following conditions: (i) if $\eta < \xi$, then $\mu_\eta < \lambda_\xi < \mu_\xi$, (ii) $b_{\lambda_\xi} \in D_\xi$ and (iii) $b_{\mu_\xi} \in C - \bigcup_{\nu<\xi} C_\nu$.

Since D_0 is not barren, it is not void. Let λ_0 be the least ordinal number such that $b_{\lambda_0} \in D_0$, and put $\mu_0 = \lambda_0 + 1$. Assume that for an ordinal number $\xi < \omega_1$, sequences $\{\lambda_\eta \mid \eta < \xi\}$ and $\{\mu_\eta \mid \eta < \xi\}$ are already obtained. Let ζ_ξ be the least ordinal number such that $\mu_\eta < \zeta_\xi$ for any $\eta < \xi$. Since the sequence $\{\mu_\eta \mid \eta < \xi\}$ is not cofinal to ω_1, $\zeta_\xi < \omega_1$ and $\overline{\zeta_\xi} = \aleph_0$. Let E_ξ be the union of the set $\{b_\nu \mid \nu < \zeta_\xi\}$ and all sets C_ν with $\nu < \xi$. Since E_ξ is a union of at most countable barren sets which are closed in B, E_ξ is also barren by Lemma 4.12. Since D_ξ is not barren, $D_\xi - E_\xi$ contains more than countable elements. Let λ_ξ be the least ordinal number such that $b_{\lambda_\xi} \in D_\xi - E_\xi$, and μ_ξ be the least ordinal number such that $b_{\mu_\xi} \in D_\xi - E_\xi - \{b_{\lambda_\xi}\}$, (remark $D_\xi \subset C$). Thus we can inductively define sequences $\{\lambda_\xi | \xi < \omega_1\}$ and $\{\mu_\xi | \xi < \omega_1\}$ which obviously satisfy (i), (ii) and (iii).

Let D be the set which consists of all b_{μ_ξ} with $\xi < \omega_1$. Since no repeats of same elements are admitted in the sequence $\{b_\nu | \nu < \omega_1\}$, and there is no number common to both sequences $\{\lambda_\xi | \xi < \omega_1\}$ and $\{\mu_\xi | \xi < \omega_1\}$ by (i), D does not contains b_{λ_ξ} for any $\xi < \omega_1$.

We shall show $S \lessapprox S \cup D \lessapprox S \cup C$. $S \propto S \cup D \propto S \cup C$ is obvious. First D is not barren, or otherwise D is included in a union of countable sets $C_{\nu_1}, C_{\nu_2}, \cdots$ in $\mathfrak{C}$, but since the sequence $\{\nu_n | n < \omega\}$ can not be cofinal to ω_1, there exists a $\xi < \omega_1$ such that $\nu_n < \xi$ for any $n < \omega$. D contains b_{μ_ξ} which is not contained in $\bigcup_{\nu<\xi} C_\nu$ by (iii), contradicting $D \subset \bigcup_{n<\omega} C_{\nu_n}$. Hence D is not barren and $S \lessapprox S \cup D$. Next we shall show $S \cup C \not\propto S \cup D$. Assume on the contrary, that there exists a reduced increasing mapping f of $S \cup C$ into $S \cup D$. Since $S \cup D$ is a subset of $S \cup C$, f itself may be regarded as a reduced increasing mapping of $S \cup C$ into itself. Hence f coincides with a f_ξ in $\mathfrak{F}$. But $b_{\lambda_\xi} \in f_\xi(S \cup C)$ by (ii), while b_{λ_ξ} is not contained in $D \cup \mathrm{Lay}_1(S)$, contradicting $f_\xi(\mathrm{Lay}_1(S \cup C)) \subset D \cup \mathrm{Lay}_1(S)$. Hence $S \cup C \not\propto S \cup D$ and accordingly $S \cup D \lessapprox S \cup C$.

Proof of Main Theorem B (ii).

The set B itself is not barren by Lemma 4.8. Hence putting $D_0 = B$, we can inductively define a sequence $D_0, D_1, \cdots$ of subsets of B such that $S \lessapprox S \cup D_{n+1} \lessapprox S \cup D_n$ for any $n < \omega$ by Lemma 4.13. Hence, putting $X_n = S \cup D_n$, the sets X_n, $n < \omega$, satisfy the condition of Theorem B (ii).

Proof of Main Theorem B (i).

By Lemma 4.13, there exists a subset C of B such that $S \lessapprox S \cup C \lessapprox T$. Let s_t be an element assigned to each $t \in C$, and put $E = \{s_t | t \in C\}$ and $U = S \cup C \cup E$ where each $s_t \in E$ is maximal in U and we have $x < s_t$ for $x \in S \cup C$ and $t \in C$ if and only if $x \leq t$ within $S \cup C$.

First assume $U \propto T$, and let f be a reduced increasing mapping of U into T. Since $\tau(s) = \omega + 1$ for any $s \in E$ and $\tau(t) = \omega$ for any $t \in B$, any $s \in E$ is not mapped by f (since f is reduced) on any $t \in B$, i.e., $f(s) \in S$. Since S is a cut of T and $f(t) < f(s_t)$ for any $t \in C$, C is mapped by f into S. Finally $f(S) \subset S$ by Lemma 4.9. Hence the restriction of f on $S \cup C$ is a reduced increasing mapping of $S \cup C$ into S contradicting $S \lessapprox S \cup C$. Hence $U \not\propto T$.

Next assume $T \propto U$, and let g be a reduced increasing mapping of T into U. Since g is reduced, any $t \in B$ is not mapped on any $s \in E$. Further by the definition of L and S, any $x \in S$ with $\tau(x) = \omega + 1$ is not maximal within T (see Lemma 4.7 (ii)), while $s \in E$ is maximal within U. Hence we have never $f(x) \in E$ for $x \in S$, and there is no element in T which is mapped by g into E. Hence g maps T entirely into the subset $S \cup C$ of U, contradicting $S \cup C \lessapprox T$. Hence $T \not\propto U$.

Thus we have ramified sets T and U which are not comparable with each other, completing the proof of Main Theorem B (i).

APPENDIX. CASE $\beta = 1$.

Here we shall consider the special case $\beta = 1$. Of course every statements hitherto mentioned holds unaffectedly, except these about S_0^ν (see Chapter 4, § 1) where we assumed $\beta > 1$. Besides, in the case $\beta = 1$, Main Theorem A can be fairly sharpened, and it will be proved that not only $[\mathfrak{S}_1]$ is well-ordered by $\propto$, but also

Theorem C. *$[\mathfrak{R}_1]$ is well-ordered by $\propto$.*

In order to prove this, we shall define a sequence $\mathfrak{M} = \{M_\lambda \mid \lambda < 1^\#\}$ such that each M_λ satisfies

D. 1) *$\eta < \lambda$ implies $M_\eta \lessapprox M_\lambda$, and*

D. 2′) *$X \in \mathfrak{R}_1$ and $X \not\propto M_\eta$ for any $\eta < \lambda$ imply $M_\lambda \propto X$.*

It is all the same as Lemma 2.8 and its Corollary that, under the assumption that $\mathfrak{M}$ is already constructed, for any $X \in \mathfrak{R}_1$ there exists a $\lambda < 1^\#$ such that $X \sim M_\lambda$ and hence $[\mathfrak{R}_1]$ is well-ordered by $\propto$ in the type $1^\#$.

DEFINITION 14. Δ_0 and Δ_1 denote sets of ordinal numbers less than $1^\#$ such that:

In the case $\lambda \in \Gamma^0$, always $\lambda \in \Delta_0$.

Assume that for any $\eta < \lambda$, it is decided whether $\eta \in \Delta_0$ or $\eta \in \Delta_1$.

In the case $\lambda \in \Gamma^1$, letting $\lambda = \alpha_\nu(\xi + \zeta) + n$ be the canonical decomposition of λ, $\lambda \in \Delta_0$ or $\lambda \in \Delta_1$ according as $\alpha_\nu(\zeta) \in \Delta_0$ or $\alpha_\nu(\zeta) \in \Delta_1$ respectively.

In the case $\lambda \in \Gamma^2$, letting $\lambda = \alpha_\nu(\omega_1^\mu) + n$ be the canonical decomposition of λ, we distinguish three cases;

if $\mu = 1$, then $\lambda \in \Delta_0$,

if μ is an isolated number greater than 1, then $\lambda \in \Delta_1$ and

if μ is a limit number with $\mathrm{cf}(\mu) = 1$, then $\lambda \in \Delta_0$ or $\lambda \in \Delta_1$ according as $\mu \in \Delta_0$ or $\mu \in \Delta_1$ respectively.

Then any number less than $1^\#$ is allotted to Δ_0 or Δ_1.

The sequence $\mathfrak{M}$ is defined according to the following Principle.

PRINCIPLE II.

If $\lambda \in \Delta_0$, then $M_\lambda = N_\lambda$.

For $\lambda \in \Delta_1$, *we distinguish four cases*:

In the case that λ *is an isolated number* $\lambda'+1$, *then* $M_\lambda = N_{\lambda'}$.

In the case $\lambda = \alpha_\nu(\omega_1^{\zeta+1})$ *where* $\zeta > 0$, *then* $M_\lambda = S_\zeta^\nu = N^{\odot_\nu}_{\alpha_\nu(\omega_1^\zeta)+\sigma'_\zeta}$.

Assume that λ *is a limit number in* Δ_1 *and for any* $\eta < \lambda$, M_η *is already defined.*

In the case $\lambda \in \Gamma^1$, *letting* $\lambda = \alpha_\nu(\xi+\zeta)$ *be the canonical decomposition of* λ, $M_\lambda = M_{\alpha_\nu(\zeta)} \odot_\nu N_{\alpha_\nu(\xi)+\sigma_\xi}$.

In the case $\lambda \in \Gamma^2$ *and, letting* $\lambda = \alpha_\nu(\omega_1^\mu)$ *be the canonical decomposition of* λ, μ *is a limit number with* $\mathrm{cf}(\mu)=1$, *then* $M_\lambda = W_{\omega^\nu} + M_\mu$.

REMARK. In $\mathfrak{M}$, all sets N_λ in $\mathfrak{N}$ are disposed retaining the original order within $\mathfrak{N}$, and for any limit number λ in Δ_1 a set M_λ, which is utterly new and not contained in $\mathfrak{N}$, is inserted. Remark that for any limit number λ, numbers $\eta < \lambda$ such that $M_\eta \in \mathfrak{N}$ are cofinal to λ, and hence if $M_\lambda = N_{\lambda'}$ (either $\lambda = \lambda'$ or $\lambda = \lambda'+1$) and $N_{\lambda'}$ satisfies D. 1) and D. 2′) within $\mathfrak{N}$, then M_λ satisfies them within $\mathfrak{M}$, reserving the case where λ is next to a limit number in Δ_1 for the proof mentioned later.

If $\lambda \in \Gamma^1$ and $\lambda = \alpha_\nu(\xi+\zeta)+n$ is the canonical decomposition of λ, then put $\theta(\lambda) = \alpha_\nu(\zeta)+n$. If $\lambda \in \Gamma^2$ and $\alpha_\nu(\omega_1^\mu)+n$ is the canonical decomposition of λ, where μ is a limit number with $\mathrm{cf}(\mu)=1$, then put $\theta(\lambda) = \mu+n$. $\lambda \in \Gamma^1 \cup \Gamma^2$ implies $\theta(\lambda) \in \Gamma^1 \cup \Gamma^2$. If further, either it is the case $\theta(\lambda) \in \Gamma^1$ or the case $\theta(\lambda) \in \Gamma^2$ and, letting $\alpha_{\nu_1}(\omega_1^{\mu_1})+n$ be the canonical decomposition of $\theta(\lambda)$, μ_1 is a limit number, then we can define $\theta^2(\lambda)$ $\theta = (\theta(\lambda))$. In general, putting $\theta^{k+1}(\lambda) = \theta(\theta^k(\lambda))$, since $\theta^{k+1}(\lambda) < \theta^k(\lambda)$, we shall finally arrive at a number $m < \omega$ such that $\theta^m(\lambda) \in \Gamma^2$, and letting $\alpha_{\nu_m}(\omega_1^{\mu_m})+n$ be the canonical decomposition of $\theta^m(\lambda)$, μ_m is an isolated number. Then whether $\theta^m(\lambda) \in \Delta_0$ or $\theta^m(\lambda) \in \Delta_1$ is decided by $\theta^m(\lambda)$ itself; the former in the case $\mu_m = 1$ and the latter in the case $\mu_m > 1$. Put $\bar{\theta}(\lambda) = \theta^m(\lambda)$ for such a final m, where $m=0$ and $\bar{\theta}(\lambda) = \lambda$, if $\lambda \in \Gamma^2$ and, latting $\lambda = \alpha_\nu(\omega_1^\mu)+n$ be the canonical decomposition of λ, μ itself is an isolated number.

For $\lambda \in \Gamma^1 \cup \Gamma^2$ we have $\lambda \in \Delta_0$ or $\lambda \in \Delta_1$ according as $\bar{\theta}(\lambda) \in \Delta_0$ or $\bar{\theta}(\lambda) \in \Delta_1$ respectively.

Now we shall briefly show that any M_λ in $\mathfrak{M}$ satisfies D. 1) and D. 2′).

First assume $\lambda \in \Gamma^1 \cup \Gamma^2$. It was proved in each case for $\lambda < 1^\#$, that if $N_{\theta(\lambda)}$ satisfies D. 1) and D. 2′), then N_λ also satisfies D. 1) and D. 2′). Hence if $N_{\bar{\theta}(\lambda)}$ satisfies D. 1) and D. 2′), then N_λ satisfies them. Especially if $\bar{\theta}(\lambda) \in \Delta_0$ then $\bar{\theta}(\lambda) \in \Gamma^2$ and letting $\alpha_{\nu'}(\omega_1^{\mu'})+n$ be the canonical decomposition of $\bar{\theta}(\lambda)$, it falls to $\mu'=1$. If $n=0$, then since ν' is either isolated number or a limit number with $\nu' < \omega_1$, and accordingly $\mathrm{cf}(\nu')=0$, $N_{\bar{\theta}(\lambda)}$

satisfies D. 1) and D. 2′) by Lemma 3. 14.

It is already seen that if $\lambda=\alpha_\nu(\omega_\beta^\mu)+n$ where $n>0$ and μ is an isolated number (then $\lambda\in\Gamma^2$), then N_λ satisfies D. 1) and D. 2′) in any case.

If $\lambda\in\Delta_0\cap(\Gamma^1\cup\Gamma^2)$, then $\bar\theta(\lambda)\in\Delta_0\cap\Gamma^2$. Hence $N_{\bar\theta(\lambda)}$, as well as N_λ, satisfies D. 1) and D. 2′).

Since $\lambda\in\Gamma^0$ (and accordingly $\lambda\in\Delta_0$) implies that N_λ satisfies D. 1) and D. 2′), for any $\lambda\in\Delta_0$, M_λ satisfies D. 1) and D. 2′). Hence for any $\lambda\in\Delta_0$, $M_\lambda(=N_\lambda)$ is seen to satisfy D. 1) and D. 2′) under assumption of induction (remark that $\lambda<\omega_1^2$ implies $\lambda\in\Delta_0$).

Similarly for any isolated number λ, N_λ is seen to satisfy D. 1) and D. 2′). For $\lambda\in\Delta_1$, if $\mathrm{fr}(\lambda)\geqq 2$, then, putting $\lambda=\lambda'+1$, $M_\lambda=N_{\lambda'}$ and $\mathrm{fr}(\lambda')>0$. Hence under assumption of induction, it follows from results on $N_{\lambda'}$ that M_λ satisfies D. 1) and D. 2′) (About the mention above, see the remark at the end of Chapter III).

Now it remains the case where $\lambda\in\Delta_1$ and $\mathrm{fr}(\lambda)=0$ or 1.

First consider the case $\bar\theta(\lambda)=\lambda$, i.e., $\lambda\in\Gamma^2$ and letting $\lambda=\alpha_\nu(\omega_\beta^\mu)+n$ be the canonical decomposition of λ, $\mu=\zeta+1$ where $\zeta>0$.

Case $n=0$. $M_\lambda=S_\zeta^\nu=N^{\odot_\nu}_{\alpha_\nu(\omega_1^\zeta)+\sigma'_\zeta}$. Then by Lemma 4. 4 (i), $\xi<\lambda$ implies $N_\xi\approx M_\lambda$. Since the numbers $\xi<\lambda$ with $M_\xi=N_\xi$ are cofinal to λ, it follows from assumption D. 1) on M_ξ with $\xi<\lambda$ that M_λ satisfies D. 1). Similarly it follows from Lemma 4. 4 (ii) that M_λ satisfies D. 2′).

Case $n=1$. Since $M_{\lambda-1}=S_\zeta^\nu$ and $M_\lambda=N_{\alpha_\nu(\omega_1^{\zeta+1})}$, it follows from Lemma 4. 5 that $M_{\lambda-1}\approx M_\lambda$, i.e., M_λ satisfies D. 1).

Assume $X\in\mathfrak{N}_1$ and $X\not\propto M_{\lambda-1}$. Since $\overline{\overline{X}}\leqq\aleph_0$, X satisfies (a) of Lemma 4. 6. Hence X does not satisfies (b) of Lemma 4. 6, or otherwise, $X\propto S_\zeta^\nu=M_{\lambda-1}$. Hence there exists an $a\in\mathrm{Lay}_\nu(X)$ such that $\mathrm{Exp}_\nu(\mathrm{Lb}(a))\not\propto N_{\alpha_\nu(\omega_1^\zeta)+\sigma'_\zeta}$. By D. 3) on $N_{\alpha_\nu(\omega_1^\zeta)+\sigma'_\zeta}$, $N_{\alpha_\nu(\omega_1^\zeta)+\sigma'_\zeta}\propto\mathrm{Exp}_\nu(\mathrm{Lb}(a))$. Since $N_{\omega\zeta+\sigma'_\zeta}\sim\mathrm{Csg}_\nu(N_{\alpha_\nu(\omega_1^\zeta)+\sigma'_\zeta})\propto\mathrm{Csg}_\nu(\mathrm{Exp}_\nu(\mathrm{Lb}(a)))$, $M_\lambda=W_{\omega^\nu}+N_{\omega\zeta+\sigma'_\zeta}\propto\mathrm{Exp}_\nu(\mathrm{Lb}(a))\subset X$, and M_λ satisfies D. 2′).

Therefore $\lambda\in\Delta_1$ with $\theta(\lambda)=\lambda$, we have proved that M_λ satisfies D. 1) and D. 2′).

For $\lambda\in\Delta_1$ with $\theta(\lambda)<\lambda$, it is inductively proved that M_λ satisfies D. 1) and D. 2′) all the same as we considered for $N_\lambda\in\mathfrak{N}$.

Hence any M_λ with $\lambda<1^\#$ satisfies D. 1 and D. 2′) and this fact implies Theorem C as we noticed below it.

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Bibliography

[1] H. Bachman: Transfinite Zahlen, Berlin-Göttingen-Heidelberg (1955).
[2] G. Birkhoff: Lattice Theory, New-York (1948).
[3] M. M. Day: Arithmetic of ordered systems, Trans. Am. Math. Soc. **58** (1945) 1-43.
[4] G. Kurepa: Ensemble ordonnés et ramifiés, Belgrade (1935).
[5] ————: Transfarmations monotones des ensembles partiellement ordonnés, C. R. Ac. Paris **205** (1937), 1033-35.
[6] ————: General continuum hypothèsis and ramifications, Fund. Math. **47** (1959) 29-33.
[7] N. Lusin: Leçons sur les ensembles analytiques et leur application, Gauthier-Villars Paris (1930).
[8] W. Sierpiński: Hypothèse du continu, Warzawa-Lwów (1934).