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ON MULTIPLY TRANSITIVE GROUPS XIV

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1. Introduction

Let G be a 4-fold transitive group on Ω . If the stabilizer of four points i, j, k, l in G has an orbit of length one in $\Omega - \{i, j, k, l\}$, then G is S_5 , A_6 or M_{11} by a theorem of H. Nagao [2]. We now consider the case in which the stabilizer of four points in G has an orbit of length two and have the following results.

Theorem 1. *Let G be a 4-fold transitive group on $\Omega = \{1, 2, \dots, n\}$. If the stabilizer of four points in G has an orbit of length two, then G is S_6 .*

Corollary. *Let D be a $4-(v, k, \lambda)$ design, where $k=5$ or 6 and $\lambda=1$ or 2 . If an automorphism group G of D is 4-fold transitive on the set of points of D , then D is a $4-(11, 5, 1)$ design or a trivial design: a $4-(5, 5, 1)$ design, a $4-(6, 5, 2)$ design or a $4-(6, 6, 1)$ design.*

The case $k=5$ and $\lambda=1$ has proved by H. Nagao [2]. Hence in this paper we shall prove the remaining cases.

We shall use the same notations as in [3].

2. Proof of Theorem 1

Let G be a group satisfying the assumption of Theorem 1. If the order of the stabilizer of four points in G is not divisible by three, then $G = S_6$ by Theorem of [7]. Hence from now on we assume that the order of the stabilizer of four points in G is divisible by three and proceed by way of contradiction.

Let $\{i_1, i_2\}$ be a G_{1234} -orbit of length two and P be a Sylow 3-subgroup of G_{1234} . Then $I(P) \supseteq \{1, 2, 3, 4, i_1, i_2\}$. By a theorem of E. Witt [9], $N_G(P)^{I(P)}$ is 4-fold transitive on $I(P)$. By assumption, $3 \nmid |(N_G(P)^{I(P)})_{1234}|$ and $(N_G(P)^{I(P)})_{1234}$ has an orbit $\{i_1, i_2\}$. Hence by Theorem of [7], $N_G(P)^{I(P)} = S_6$ and $|I(P)| = 6$.

Hence G_{1234} has exactly one orbit $\{i_1, i_2\}$ of length two and lengths of orbits in $\Omega - \{1, 2, 3, 4, i_1, i_2\}$ are all divisible by three. Furthermore any Sylow 3-subgroup of G_{1234} fixes exactly six points $1, 2, 3, 4, i_1, i_2$.

Let Q be a 3-subgroup of G such that the order of Q is maximal among all 3-subgroups of G fixing more than six points. Set $N=N_c(Q)^{I(Q)}$. Then for any four points i, j, k, l of $I(Q)$ any Sylow 3-subgroup of $N_{i j k l}$ fixes exactly six points i, j, k, l, i', j' , where $\{i', j'\}$ is the unique $G_{i j k l}$ -orbit of length two, and is semiregular on $I(Q) - \{i, j, k, l, i', j'\}$. Hence to complete the proof of Theorem 1, it will suffice to show the following

Lemma. *Let G be a permutation group on $\Omega = \{1, 2, \dots, n\}$, where $n > 6$. Then it is impossible that G satisfies the following condition:*

Let i, j, k, l be any four points of Ω . Then there exist two points i_1, i_2 in $\Omega - \{i, j, k, l\}$ such that any Sylow 3-subgroup of $G_{i j k l}$ fixes six points i, j, k, l, i_1, i_2 and is semiregular on $\Omega - \{i, j, k, l, i_1, i_2\}$.

Proof. Suppose by way of contradiction that G is a counter-example to Lemma. For four points i, j, k, l let i_1, i_2 be two points in $\Omega - \{i, j, k, l\}$ uniquely determined by Sylow 3-subgroups of $G_{i j k l}$. Set $\Delta(i, j, k, l) = \{i, j, k, l, i_1, i_2\}$. Then $\{\Delta(i, j, k, l) \mid i, j, k, l \in \Omega\}$ forms a $4-(v, 6, 1)$ design on Ω and G is an automorphism group of this design.

We may assume that $\Delta(1, 2, 3, 4) = \{1, 2, 3, 4, 5, 6\}$. Let a be an element of order three in G_{1234} . Then we may assume that

$$a = (1)(2) \cdots (6)(7\ 8\ 9) \cdots$$

Since $a \in N_G(G_{1789})$, there is an element b of order three in G_{1789} such that $ab = ba$. Then we may assume that

$$b = (1)(2)(3)(4\ 5\ 6)(7)(8)(9) \cdots$$

Similarly G_{4789} has an element c of order three such that $ac = ca$. Since $\Delta(1, 7, 8, 9) \neq \Delta(4, 7, 8, 9)$, $\Delta(1, 7, 8, 9) \cap \Delta(4, 7, 8, 9) = \{7, 8, 9\}$. Hence $\Delta(4, 7, 8, 9) = \{4, 5, 6, 7, 8, 9\}$ and we may assume that

$$c = (1\ 2\ 3)(4)(5)(6)(7)(8)(9) \cdots$$

Then a Sylow 3-subgroup of $\langle a, b, c \rangle$ has the same form as $\langle a, b, c \rangle$ on $\{1, 2, \dots, 9\}$. By assumption, $3 \nmid |\langle a, b, c \rangle_{1,2,\dots,9}|$ and so the order of a Sylow 3-subgroup of $\langle a, b, c \rangle$ is 3^3 . Hence we may assume that $\langle a, b, c \rangle$ is a 3-group. Then since $I(b^c b^{-1}) \supseteq \{1, 2, \dots, 9\}$, $b^c b^{-1} = 1$. Hence $\langle a, b, c \rangle$ is an elementary abelian 3-group.

Let r be the number of $\langle a, b \rangle$ -orbits of length three. For a point i of a $\langle a, b \rangle$ -orbit of length three, $\langle a, b \rangle_i$ is of order three and so exactly two elements of $\langle a, b \rangle - \{1\}$ fixes 1, 2, 3 and three points of the $\langle a, b \rangle$ -orbit containing i . Hence $2r \leq |\langle a, b \rangle - \{1\}| = 8$ and so $r \leq 4$. Since $\{4, 5, 6\}$ and $\{7, 8, 9\}$ are $\langle a, b \rangle$ -orbits of length three, $r = 2, 3$ or 4 .

Suppose that $\langle a, b, c \rangle$ has an orbit of length nine. Since $\langle a, b, c \rangle$ is an abelian group of order twenty-seven, $\langle a, b, c \rangle$ has an element of order three fixing nine points of this $\langle a, b, c \rangle$ -orbit of length nine, contrary to the assumption. Thus the lengths of $\langle a, b, c \rangle$ -orbits are three or twenty-seven and every $\langle a, b \rangle$ -orbit of length three is a $\langle a, b, c \rangle$ -orbit.

Case I. $r=4$.

Assume that $\langle a, b \rangle$ -orbits of length three are $\{4, 5, 6\}$, $\{7, 8, 9\}$, $\{10, 11, 12\}$ and $\{13, 14, 15\}$. Then we may assume that

$$\begin{aligned} a &= (1)(2) \cdots (6)(7\ 8\ 9)(10\ 11\ 12)(13\ 14\ 15) \cdots, \\ b &= (1)(2)(3)(4\ 5\ 6)(7)(8)(9)(10\ 11\ 12) \cdots, \\ c &= (1\ 2\ 3)(4)(5)(6)(7)(8)(9)(10\ 11\ 12) \cdots. \end{aligned}$$

Since $|I(ab^{-1})| \leq 6$ and $|I(ac^{-1})| \leq 6$, $b=c=(13\ 15\ 14)$ on $\{13, 14, 15\}$. Then $I(bc^{-1}) \supseteq \{7, 8, \dots, 15\}$ and so $|I(bc^{-1})| \geq 9$, contrary to the assumption.

Case II. $r=2$ or 3 .

When $r=2$, $\langle a, b \rangle$ -orbits of length three are $\{4, 5, 6\}$ and $\{7, 8, 9\}$. If $|\Omega|=9$, then we have a 4-(9, 6, 1) design. Then the number of blocks is $\frac{\binom{9}{4}}{\binom{6}{4}} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{6 \cdot 5 \cdot 4 \cdot 3}$, which is not an integer. Thus $|\Omega| > 9$. Hence we may assume that

$$\begin{aligned} a &= (1)(2) \cdots (6)(7\ 8\ 9)(10\ 11\ 12)(13\ 14\ 15)(16\ 17\ 18) \cdots, \\ b &= (1)(2)(3)(4\ 5\ 6)(7)(8)(9)(10\ 13\ 16)(11\ 14\ 17)(12\ 15\ 18) \cdots. \end{aligned}$$

Then

$$ab = (1)(2)(3)(4\ 5\ 6)(7\ 8\ 9)(10\ 14\ 18)(11\ 15\ 16)(12\ 13\ 17) \cdots,$$

and $I(ab) = \{1, 2, 3\}$. Since $\langle a, b, c \rangle$ has no orbit of length nine, the lengths of $\langle a, b, c \rangle$ -orbits in $\{10, 11, \dots, n\}$ are twenty-seven. Thus $n \equiv 9 \pmod{27}$ and $n > 9$.

Next when $r=3$, we may assume that $\langle a, b \rangle$ -orbits of length three are $\{4, 5, 6\}$, $\{7, 8, 9\}$ and $\{10, 11, 12\}$. If $|\Omega|=12$, then we have a 4-(12, 6, 1) design. Then the number of blocks containing a given point is $\frac{\binom{11}{3}}{\binom{5}{3}} = \frac{11 \cdot 10 \cdot 9}{5 \cdot 4 \cdot 3}$,

which is not an integer. Thus $|\Omega| > 12$. Hence we may assume that

$$\begin{aligned} a &= (1)(2) \cdots (6)(7\ 8\ 9)(10\ 11\ 12)(13\ 14\ 15)(16\ 17\ 18)(19\ 20\ 21) \cdots, \\ b &= (1)(2)(3)(4\ 5\ 6)(7)(8)(9)(10\ 11\ 12)(13\ 16\ 19)(14\ 17\ 20) \end{aligned}$$

(15 18 21)

Then

$$ab = (1)(2)(3)(4\ 5\ 6)(7\ 8\ 9)(10\ 12\ 11)(13\ 17\ 21)(14\ 18\ 19) \\ (15\ 16\ 20) \dots,$$

and $I(ab) = \{1, 2, 3\}$. Then by the same reason as above $n \equiv 12 \pmod{27}$ and $n > 12$.

For any two points i, j , of $\{2, 3, \dots, 6\}$ set $\Gamma(i, j) = \{(i_1\ i_2\ i_3) \mid a = (i_1\ i_2\ i_3) \dots \text{ and } i, j \in \Delta(1, i_1, i_2, i_3)\}$. For any 3-cycle $(i_1\ i_2\ i_3)$ of a $\Delta(1, i_1, i_2, i_3)^a = \Delta(1, i_1, i_2, i_3)$. Hence there are two points i, j in $\{2, 3, \dots, 6\}$ such that $\Delta(1, i_1, i_2, i_3) = \{1, i, j, i_1, i_2, i_3\}$. Thus $(i_1\ i_2\ i_3) \in \Gamma(i, j)$.

For any 3-cycle $(i'\ j'\ k')$ of ab $\Delta(i', j', k', 1)^{ab} = \Delta(i', j', k', 1)$. Hence $\Delta(i', j', k', 1) = \{i', j', k', 1, 2, 3\}$. Thus for any point i' of $\{4, 5, \dots, n\}$ $\Delta(1, 2, 3, i') = \{1, 2, 3, i', j', k'\}$, where $(i'\ j'\ k')$ is a 3-cycle of ab . Hence $\Gamma(2, 3) = \{(7\ 8\ 9)\}$ or $\{(7\ 8\ 9), (10\ 11\ 12)\}$ for $r=2$ or 3 respectively.

Conversely for any two points i, j of $\{2, 3, \dots, 6\}$ if there is a 3-cycle $(i_1\ i_2\ i_3)$ of a such that $(i_1\ i_2\ i_3) \in \Gamma(i, j)$, then $G_{i_1\ i_2\ i_3}$ has an element b' of order three such that $ab' = b'a$. Since $\Delta(i, i_1, i_2, i_3) \cap \Delta(1, i_1, i_2, i_3) \supseteq \{i, i_1, i_2, i_3\}$, $\Delta(i, i_1, i_2, i_3) = \Delta(1, i_1, i_2, i_3) = \{1, i, j, i_1, i_2, i_3\}$. Then ab' is of order three and $I(ab') = \{1, i, j\}$. Hence by the same argument as is used for $\Gamma(2, 3)$, $|\Gamma(i, j)| = 1$ or 2 for $r=2$ or 3 respectively.

Thus for any two points i, j of $\{2, 3, \dots, 6\}$ $|\Gamma(i, j)| \leq r-1$. On the other hand the number of unordered pairs (i, j) , where $i, j \in \{2, 3, \dots, 6\}$, is $\binom{5}{2} = 10$. Hence the number of 3-cycles of a is at most $10(r-1)$.

Suppose that $r=2$. Then $n \leq 6 + 3 \cdot 10 = 36$. Since $n \equiv 9 \pmod{27}$ and $n > 9$, $n=36$. Thus we have a $4-(36, 6, 1)$ design. Then the number of blocks

containing a given point is $\frac{\binom{35}{3}}{\binom{5}{3}} = \frac{35 \cdot 34 \cdot 33}{5 \cdot 4 \cdot 3}$, which is not an integer. Thus

$r \neq 2$.

Suppose that $r=3$. Then $n \leq 6 + 3 \cdot 20 = 66$. Since $n \equiv 9 \pmod{27}$ and $n > 12$, $n=39$ or 66 . If $n=39$, then we have a $4-(39, 6, 1)$ design. Then the number of

blocks is $\frac{\binom{39}{4}}{\binom{6}{4}} = \frac{39 \cdot 38 \cdot 37 \cdot 36}{6 \cdot 5 \cdot 4 \cdot 3}$, which is not an integer. Thus $n \neq 39$.

From now on we assume that $n=66$. Then for any two points i, j of $\{2, 3, \dots, 6\}$ there is a 3-subgroup fixing exactly three points $1, i, j$. Let i_1, i_2, i_3 be any three points of Ω . Then there is an element of order three which fixes exactly six points containing i_1, i_2, i_3 . Then by the same argument as is used for a , there is a 3-subgroup fixing exactly three points i_1, i_2, i_3 . Thus by a theorem of D.

Livingstone and A. Wagner [1], G is 3-fold transitive on Ω .

Let Γ_1, Γ_2 and Γ_3 are G_{123} -orbits such that $4 \in \Gamma_1, 7 \in \Gamma_2$ and $10 \in \Gamma_3$. For any point i of $\Omega - \{1, 2, 3\}$ $\Delta(1, 2, 3, i) = \{1, 2, 3, i, j, k\}$, where $ab = (i j k) \dots$. Hence any element of order three and fixing exactly three points 1, 2, 3 has a 3-cycle $(i j k)$ or $(i k j)$. Thus $\langle ab \rangle$ is a unique 3-subgroup of G_{123} which is of order three and semiregular on $\Omega - \{1, 2, 3\}$. Since G is 3-fold transitive on Ω , for any three points i_1, i_2, i_3 there is a unique 3-subgroup of order three which fixes exactly three points i_1, i_2, i_3 and is semiregular on $\Omega - \{i_1, i_2, i_3\}$.

Hence for a 3-cycle $(13 \ 17 \ 21)$ of ab , there is an element c' of order three such that $I(c') = \{13, 17, 21\}$ and $abc' = c'ab$. Then we may assume that

$$c' = (1 \ 2 \ 3)(13)(17)(21) \dots$$

Furthermore since $\{10, 11, 12\}$ is a $\langle a, b \rangle$ -orbit of length three, we may assume that

$$c = (1 \ 2 \ 3)(4)(5)(6)(7)(8)(9)(10 \ 11 \ 12) \dots,$$

and $\langle a, b, c \rangle$ is semiregular on $\Omega - \{1, 2, \dots, 12\}$.

Then since $c^{-1}c' \in G_{123}$ and c fixes Γ_1, Γ_2 and Γ_3 , c' fixes Γ_1, Γ_2 and Γ_3 .

Assume that c' fixes $\{4, 5, 6\}$. Then $\Delta(4, 5, 6, 13)^{c'} = \Delta(4, 5, 6, 13)$. Hence $\Delta(4, 5, 6, 13) = \{4, 5, 6, 13, 17, 21\}$. On the other hand ac is an element of order three fixing exactly three points 4, 5, 6. Hence ac fixes $\Delta(4, 5, 6, 13)$. This a contradiction since ac does not fix $\{13, 17, 21\}$. Thus c' does not fix $\{4, 5, 6\}$. Similarly c' fixes neither $\{7, 8, 9\}$ nor $\{10, 11, 12\}$. Since c' fixes Γ_1, Γ_2 and Γ_3 , $\{4, 5, 6\} \not\subseteq \Gamma_1$, $\{7, 8, 9\} \not\subseteq \Gamma_2$ and $\{10, 11, 12\} \not\subseteq \Gamma_3$.

Assume that G_{123} has an orbit of length six. Then we may assume that $\{4, 5, 6, 7, 8, 9\} = \Gamma_1 = \Gamma_2$. Since c' commutes with ab and fixes Γ_1 , c' fixes $\{4, 5, 6\}$, which is a contradiction. Thus G_{123} has no orbit of length six.

Assume that Γ_1, Γ_2 and Γ_3 are three distinct G_{123} -orbits. Since c fixes Γ_1, Γ_2 and Γ_3 , $|\Gamma_i| \geq 3 + 27 = 30$ ($i=1, 2, 3$). Hence $|\Gamma_1 \cup \Gamma_2 \cup \Gamma_3| \geq 90$, which is a contradiction.

Assume that $\Gamma_1 = \Gamma_2 \neq \Gamma_3$. Since c fixes Γ_1 and Γ_3 , $|\Gamma_1| = 6 + 27 = 33$ and $|\Gamma_3| = 3 + 27 = 30$. Let R be a Sylow 11-subgroup of G_{123} . Then since $|\Gamma_1| = 33$, $I(R) \cap \Gamma_1 = \phi$. Hence $|I(R)| = 33, 22$ or 11 . Since $\langle a, b \rangle$ is the unique 3-subgroup of G_{123} which is of order three and is semiregular on $\Omega - \{1, 2, 3\}$, $R \leq C_G(\langle a, b \rangle)$. Hence ab fixes $I(R)$. Thus $|I(R)| \equiv 0 \pmod{3}$ and so $|I(R)| = 33$. Since R is an 11-group ($\neq 1$), for any four points i, j, k, l of $I(R)$ $\Delta(i, j, k, l) \subset I(R)$. Thus we have a 4-(33, 6, 1) design on $I(R)$. Then

the number of blocks containing given two points is $\frac{\binom{31}{2}}{\binom{4}{2}} = \frac{31 \cdot 30}{4 \cdot 3}$, which is

not an integer. Thus we have a contradiction.

Hence $\Gamma_1 = \Gamma_2 = \Gamma_3 \cong \{4, 5, \dots, 12\}$. Then since $\langle a, b \rangle < G_{123}$, the lengths of G_{123} -orbits in $\{4, 5, \dots, 66\}$ are all divisible by nine. On the other hand $|\{4, 5, \dots, 66\}|$ is not divisible by twenty-seven. Hence any Sylow 3-subgroup of G_{123} has an orbit of length nine. Let Q be a Sylow 3-subgroup of G_{123} and i a point of a Q -orbit of length nine. If $Q_i \neq 1$, then $|I(Q_i)| = 6$ and Q_i is semiregular on $\Omega - I(Q)$ by the assumption. Then since $|\Omega - I(Q_i)| = 60$, $|Q_i| = 3$. Thus $|Q| = 9$ or $9 \cdot 3$.

Assume that $\Gamma_1 = \{4, 5, \dots, 12\}$. Since c' fixes Γ_1 but does not fix $\{4, 5, 6\}$, we may assume that

$$c' = (1 \ 2 \ 3)(4 \ 7 \ 10)(5 \ 8 \ 12)(6 \ 9 \ 11) \dots$$

Then

$$c'c^{-1} = (1)(2)(3)(4 \ 7 \ 12 \ 5 \ 8 \ 11 \ 6 \ 9 \ 10) \dots$$

Since a Sylow 3-subgroup of $\langle a, b, c'c^{-1} \rangle$ has the same form as $\langle a, b, c'c^{-1} \rangle$ on $\{1, 2, \dots, 12\}$, we may assume that $\langle a, b, c'c^{-1} \rangle$ is a 3-subgroup. Then since $|\langle a, b, c'c^{-1} \rangle| = 9 \cdot 3$, $\langle a, b, c'c^{-1} \rangle$ is a Sylow 3-subgroup of G_{123} .

Suppose that $\langle a, b, c'c^{-1} \rangle$ has an orbit of length nine in $\{13, 14, \dots, 66\}$. Since $|\{13, 14, \dots, 66\}| = 54$, The number of $\langle a, b, c'c^{-1} \rangle$ -orbits of length nine in $\{3, 4, \dots, 66\}$ is at least four. Let $(i_1 \ i_2 \ i_3)$ be a 3-cycle of a contained in a $\langle a, b, c'c^{-1} \rangle$ -orbit of length nine. Then $|\langle a, b, c'c^{-1} \rangle_{123 i_1 i_2 i_3}| = 3$ and so two non-identity elements of $\langle a, b, c'c^{-1} \rangle_{123 i_1 i_2 i_3}$ fixes exactly six points $1, 2, 3, i_1, i_2, i_3$. Thus $\langle a, b, c'c^{-1} \rangle$ has at least $4 \cdot 3 \cdot 2 = 24$ elements which fix exactly six points. On the other hand since $(c'c^{-1})^3 = ab$, $\langle c'c^{-1} \rangle$ is semiregular on $\Omega - \{1, 2, 3\}$. Thus $|\langle a, b, c'c^{-1} \rangle| \geq 24 + 9 = 33$, which is a contradiction. Thus Γ_1 is the only $\langle a, b, c'c^{-1} \rangle$ -orbit of length nine.

Let i be any point in $\{4, 5, \dots, 66\}$. Then G_{123} has an element x of order three and fixing i . Then by the same argument as is used for $\langle a, b, c'c^{-1} \rangle$, there is a Sylow 3-subgroup of G_{123} which has exactly one orbit of length nine containing i . Since this Sylow 3-subgroup is conjugate to $\langle a, b, c'c^{-1} \rangle$ in G_{123} , G_{123} is transitive on $\{4, 5, \dots, 66\}$, which is a contradiction. Thus Γ_1 is not a G_{123} -orbit. This implies that G_{123} has no orbit of length nine.

Suppose that G_{123} is intransitive on $\{4, 5, \dots, 66\}$. Since c fixes Γ_1 , $|\Gamma_1| = 9 + 27 = 36$. Thus G_{123} -orbits on $\{4, 5, \dots, 66\}$ are Γ_1 and one orbit of length twenty-seven. Then G_{123} has a non-identity 3-element fixing a point of the G_{123} -orbit of length twenty-seven. Thus a Sylow 3-subgroup of G_{123} is of order more than twenty-seven, which is a contradiction.

Thus G_{123} is transitive on $\{4, 5, \dots, 66\}$. Hence G is 4-fold transitive on Ω . Let P be a Sylow 2-subgroup of G_{1234} . Then by Corollary of [4], $|I(P)| = 4$. Since $\Delta(1, 2, 3, 4) = \{1, 2, \dots, 6\}$, $\{5, 6\}$ is a G_{1234} -orbit. By Corollary of [5], $I(P_5) = \{1, 2, \dots, 6\} = I(a)$ and $|P:P_5| = 2$.

Suppose that P_5 is semiregular on $\{7, 8, \dots, 66\}$. Then since $|P:P_5| = 2$,

lengths of P -orbits on $\{7, 8, \dots, 66\}$ are $|P|$ or $|P|/2$. Hence for any point i of $\{7, 8, \dots, 66\}$, $|P_i| = 1$ or 2 . Then by Corollary 1 of [6], $|\Omega| \neq 66$, which is a contradiction. Thus P_5 is not semiregular on $\{7, 8, \dots, 66\}$.

For any three points i, j, k of $\{1, 2, \dots, 6\}$ there is an element x of order three fixing exactly three points i, j, k . Then since $\langle x \rangle$ is the unique 3-subgroup of $G_{i, j, k}$ which is of order three and semiregular on $\Omega - \{i, j, k\}$, $[a, x] = 1$ and so x fixes $I(a)$. Hence $x \in C_G(G_{I(a)})$. Let H be a subgroup generated by all elements of order three fixing exactly three points i, j, k , where i, j, k run over all three points of $\{1, 2, \dots, 6\}$. Then $H \leq C_G(G_{I(a)})$ and $H^{I(a)} = A_6$.

Since $H \leq C_G(a)$, H induces a permutation group on the set of 3-cycles of a . Let K be a subgroup of H fixing $\{7, 8, 9\}$. Since $\Delta(7, 8, 9, 1) = \{7, 8, 9, 1, 2, 3\}$ and $\Delta(7, 8, 9, 4) = \{7, 8, 9, 4, 5, 6\}$, any element of K fixes or interchanges $\{1, 2, 3\}$ and $\{4, 5, 6\}$. Let \bar{K} be a subgroup of H which fixes or interchanges $\{1, 2, 3\}$ and $\{4, 5, 6\}$. Then $K \leq \bar{K}$ and $|\bar{K}^{I(a)}| = 6 \cdot 3 \cdot 2$. If $\{1, 2, 3, i, j, k\}$ is a block and $(i j k)$ is a 3-cycle of a , then $(i j k) = (7 \ 8 \ 9)$ or $(10 \ 11 \ 12)$. Similarly if $\{4, 5, 6, i, j, k\}$ is a block and $(i j k)$ is a 3-cycle of a , then $(i j k) = (7 \ 8 \ 9)$ or $(10 \ 11 \ 12)$. Hence any element of \bar{K} fixes or interchanges $\{7, 8, 9\}$ and $\{10, 11, 12\}$. Hence $|\bar{K}:K| = 1$ or 2 .

Since $|H:\bar{K}| = |A_6|/6 \cdot 3 \cdot 2 = 10$, $|H:K| = 10$ or 20 . Since a has twenty 3-cycles, H is transitive or has two orbits of length ten on the set consisting of 3-cycles of a . Furthermore for any 3-cycle $(i j k)$ of a H has a subgroup which is transitive on $\{i, j, k\}$. Hence H is transitive or has two orbits of length thirty on $\{7, 8, \dots, 66\}$.

Since P_5 is not semiregular on $\{7, 8, \dots, 66\}$, there is a non-identity element y in P_5 which has a fixed point i in $\{7, 8, \dots, 66\}$. Then since $H \leq C_G(y)$, $I(y) \cong i^H$. Hence H has two orbits on $\{7, 8, \dots, 66\}$ and so $|I(y)| = 6 + 30 = 36$. Then since G is a 4-fold transitive group of degree sixty-six, we have a contradiction by a theorem of W. A. Manning (See [8], Theorem 15.1). Thus we complete the proof of Theorem 1.

3. Proof of Corollary

(i) Let D be a $4-(v, 5, 2)$ design. Let $\{1, 2, 3, 4, i_1\}$ and $\{1, 2, 3, 4, i_2\}$ be two blocks containing $\{1, 2, 3, 4\}$. Then $G_{1, 2, 3, 4}$ fixes $\{i_1, i_2\}$. If $\{i_1\}$ and $\{i_2\}$ are $G_{1, 2, 3, 4}$ -orbits, then $G = A_6$ by a theorem of H. Nagao [2]. Hence D is a $4-(6, 5, 2)$ design. If $\{i_1, i_2\}$ is a $G_{1, 2, 3, 4}$ -orbit, then $G = S_6$ by Theorem 1. Hence D is also a $4-(6, 5, 2)$ design.

(ii) Let D be a $4-(v, 6, 1)$ design. Then by the same reason as (i), D is a $4-(6, 6, 1)$ design.

(iii) Let D be a $4-(v, 6, 2)$ design. Let $\{1, 2, 3, 4, i_1, j_1\}$ and $\{1, 2, 3, 4, i_2, j_2\}$ be two blocks containing $\{1, 2, 3, 4\}$. Then $G_{1, 2, 3, 4}$ fixes $\{i_1, j_1, i_2, j_2\}$ and $|\{i_1, j_1, i_2, j_2\}| = 3$ or 4 . If $G_{1, 2, 3, 4}$ has an element which has a 3-cycle on $\{i_1, j_1, i_2, j_2\}$,

then there are at least three blocks containing $\{1, 2, 3, 4\}$, which is a contradiction.

Let P be a Sylow 3-subgroup of G_{1234} . Then $I(P) \cong \{i, 2, 3, 4, i_1, j_1, i_2, j_2\}$. Hence by Theorem of [7], $|I(P)| = 11$ or 12 and $N_G(P)^{I(P)} = M_{11}$ or M_{12} . If $N_G(P)^{I(P)} = M_{12}$, then $(N_G(P)^{I(P)})_{1234}$ is transitive on $I(P) - \{1, 2, 3, 4\}$. This is a contradiction since the number of blocks containing $\{1, 2, 3, 4\}$ is two. If $N_G(P)^{I(P)} = M_{11}$, then a subgroup of $N_G(P)^{I(P)}$ fixing $\{1, 2, 3, 4\}$ as a set has two orbits of length one and six. Hence by the same reason as above, we have a contradiction. Thus we complete the proof of Corollary.

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