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CONJUGACY CLASSES AND REPRESENTATION GROUPS

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Let D be a conjugacy class of a finite group G, and H a (finite) central extension of G. In the first section of the paper, we investigate how D is extended in H. Let ϕ be the homomorphism of H to G. Then, there exist a group H_0 and epimorphisms $\phi_1: H \to H_0$ and $\phi_2: H_0 \to G$ such that $\phi = \phi_1 \phi_2$, that $\phi_2^{-1}(D) = E_1 \cup \cdots \cup E_n$ with conjugacy classes E_i of H_0 with $|E_i| = |D|$ (i.e., D splits completely in H_0), and that $\phi_1^{-1}(E_i) = C_i$ with a conjugacy class C_i of H with $|C_i| = e|D|$ for every i. e is called the (covering) multiplicity of D in H. Especially, when H is a representation group of G, we can show that e is equal to $|M|/|M_0|$ where M is the Schur multiplier of G and M_0 is a subgroup of M consisting of all cohomology classes that split over D. In the second section of the paper, we investigate the structure of a group or of a conjugacy class of a group with respect to inner automorphisms. An algebraic system which is the abstraction of a group with inner automorphisms as operations is called a p.s. set (or, a pseudosymmetric set). We show that all representation groups of G are isomorphic with respect to inner automorphisms, i.e., as p.s. sets, that every conjugacy class of a central extension of G is a homomorphic image of a conjugacy class of a representation group of G, and that the multiplicity e given in the above divides the order of the Schur multiplier of G. As an application, we obtain a criterion for a p.s. set to be a conjugacy class of a group, using which we can find a class of exceptional transitive p.s. sets of orders $n(n-1)(n-2)/2$, $(n \geq 5)$.

1. Central extensions of conjugacy classes

Proposition 1. *Let H be a group, Z a subgroup of H contained in the center of H, and C a conjugacy class of H. Let $Z_0 = \{z \in Z \mid zC = C\}$. Then, $Z_0 = [a, H] \cap Z$ for any element a in ZC. If $\{z_i\}$ is a representative system of Z/Z_0, then ZC is a union of conjugacy classes C_i where $C_i = z_i C$ and $C_i \neq C_j$ if $i \neq j$. Thus, ZC is a union of conjugacy classes of the same order.*

Proof. In the following, we denote $y^{-1}xy$ by $x \circ y$. So, $C = c \circ H$ with c in C. An element z of Z belongs to Z_0 if and only if $zc = c \circ x$ for some element x in H, which implies that $z = [c, x]$. Hence, $Z_0 = [c, H] \cap Z$. On the other hand, $Z_0=\{z\in Z\,|\,zC=C\}=\{z\in Z\,|\,z(z_iC)=z_iC\}$ for every i. Therefore, by the first argument where we use z_iC in place of C, we have $Z_0=[a, H]\cap Z$ for any element a in z_iC, i.e., in ZC. The remaining part of Proposition 1 is almost clear.

Let G be a finite group, and H a central extension of G with the homomorphism ϕ. Let Z be the kernel of ϕ. Z is contained in the center of H. For a conjugacy class D of G, let C be a conjugacy class of H such that $\phi(C)=D$. Then, $\phi^{-1}(D)=ZC$. By Proposition 1, $\phi^{-1}(D)=\cup C_i$ where $|C_i|=|C|$ for all i. Moreover, we can show that $|D|$ divides $|C|$. For, let d and d' be elements of D. Then, $\{x\in C\,|\,\phi(x)=d\}$ and $\{y\in C\,|\,\phi(y)=d'\}$ have the same order, since $\{y\in C\,|\,\phi(y)=d'\}=\{x\in C\,|\,\phi(x)=d\}\circ t$ where t is an element of H such that $d\circ\phi(t)=d'$.

Theorem 1. *Let H be a central extension of a finite group G with the homomorphism ϕ. If D is a conjugacy class of G, then $\phi^{-1}(D)=C_1\cup\cdots\cup C_n$ with conjugacy classes C_i of H where $|C_1|=\cdots=|C_n|$. Furthermore, there exist a group H_0 and epimorphisms ϕ_1: $H\to H_0$ and ϕ_2: $H_0\to G$ such that $\phi=\phi_1\phi_2$, that $\phi_2^{-1}(D)=E_1\cup\cdots\cup E_n$ with conjugacy classes E_i of H_0 where $|E_i|=|D|$ for every i, and that $\phi_1^{-1}(E_i)=C_i$.*

Proof. The first part was explained in the above. For the second part, let Z be the kernel of ϕ and $Z_0=\{z\in Z\,|\,zC=C\}$ where $C=C_1$. If we let $H_0=H/Z_0$ and ϕ_1 and ϕ_2 the natural homomorphisms of H to H_0 and of H_0 to G, respectively, then the second part of Theorem 1 follows easily.

Theorem 1 implies that D splits completely in H_0 and each component E_i does not split at all in H. We call $e=|C_i|/|D|$ (which is common to all i) the (covering) multiplicity of D in H. Note also that $e=|H|/|H_0|=|Z_0|$.

Next, we determine the condition of the splitting of a conjugacy class D of G in H in terms of cohomologies. Let $C=C_1$ as above. By Proposition 1, $Z_0=\{z\in Z\,|\,zC=C\}=[c, H]\cap Z$ for an element c of C. In the following, we fix c. For $x\in H$, $[c, x]\in Z$ if and only if $cx\equiv xc$ mod Z. The latter condition is equivalent with $\phi(c)\,\phi(x)=\phi(x)\,\phi(c)$. In the following, we denote $\phi(c)$ by d so that $D=d\circ G$. Thus, $Z_0=\{[c, x]\,|\,d\,\phi(x)=\phi(x)\,d,\ x\in H\}$. Denote elements of G by u, v, etc, and let $\{t_u\,|\,u\in G\}$ be a representative system of H/Z. It is clear that if $\phi(x)=u$ then $[c, x]=[c, t_u]$. So, $Z_0=\{[c, t_u]\,|\,du=ud,\ u\in G\}$. Now, let $z(u, v)$ be a cocycle corresponding to the extension H/Z, i.e., $t_u\,t_v=z(u, v)\,t_{uv}$ where $z(u, v)\in Z$. Since $[c, t_u]=z(u, d)^{-1}\,z(d, u)$ as we can easily verify, we obtain that $Z_0=\{z(u, d)^{-1}\,z(d, u)\,|\,du=ud,\ u\in G\}$. We can conclude that D splits completely in H if and only if $Z_0=1$ or $z(u, d)=z(d, u)$ for all u such that $du=ud$.

Now, we consider, as H, the standard representation group of G. The standard representation group is defined to be $Q=\sum\hat{M}\,t_u$, $u\in G$, where $\hat{M}=$

Hom$(M, K^{\times})$, M being the Schur multiplier of G and K the complex number field. (For this part, see [1].) Here, $t_u t_v = z(u, v) t_{uv}$ with an element $z(u, v)$ in $\hat{M}$ such that $z(u, v)(\alpha) = \alpha(u, v)$ where α is an element of M. Let d be a fixed element of D. We say that α splits over D if $\alpha(u, d)^{-1} \alpha(d, u) = 1$ for every u such that $ud = du$. It follows that α splits over D if and only if α is mapped to 1 by every element of Z_0. We obtained

Theorem 2. *Let Q be the standard representation group of G, and let D be a conjugacy class of G. If M_0 denotes the subgroup of M consisting of cohomology classes that split over D, then $|M|/|M_0| = e =$ the multiplicity of D in Q*

In the following section, we show that all representation groups of G are isomorphic with each other as p.s. sets. Therefore, Theorem 2 holds for any representation groups of G.

2. Unions of conjugacy classes

Let U be a union of conjugacy classes of a group. It is closed under the operation $\circ$ where $a \circ b = b^{-1} ab$. The binary system $(U, \circ)$ satisfies

(1) The right multiplication of an element a of U is a permutation on U.
(2) $a \circ a = a$ for every element a of U.
(3) $(a \circ b) \circ c = (a \circ c) \circ (b \circ c)$ for $a, b, c \in U$.

Generally, a binary system which satisfies (1), (2) and (3) is called a pseudosymmetric set, or briefly, a p.s. set. (When especially it satisfies (4) $(x \circ a) \circ a = x$ for any x and a of U, we say it is a symmetric set.) Any group is a p.s. set in the above sense. A p.s. subset of a group is called a special p.s. set. Thus, a union of conjugacy classes of a group is a special p.s. set. A p.s. set which is not special is said to be exceptional.

Proposition 2. *Let ϕ be an epimorphism of a group H to a group G. If ϕ induces an isomorphism of the commutator subgroup H' of H onto the commutator subgroup G' of G, then every conjugacy class of H is mapped to a conjugacy class of G bijectively. More generally, if $U = C_1 \cup \cdots \cup C_n$ is a union of conjugacy classes C_i of H and if $\phi(C_i) \neq \phi(C_j)$ whenever $i \neq j$, then U is mapped isomorphically (as a p.s. set) onto a union of conjugacy classes of G by ϕ.*

Proof. It is sufficient to show that ϕ is injective on a conjugacy class C of H. Since $C = c \circ H$, it is sufficient to show that $\phi(c \circ a) = \phi(c \circ b)$ implies $c \circ a = c \circ b$. Assume $\phi(c \circ a) = \phi(c \circ b)$. Then, $\phi([c, a]) = \phi([c, b])$ as $[c, a] = c^{-1}(c \circ a)$ and ϕ is a (group) homomorphism. Now, the assumption in Proposition 2 implies that $[c, a] = [c, b]$, from which we can easily conclude that $c \circ a = c \circ b$.

In the following, U denotes a special or exceptional p.s. set. A subset N

of U is called a normal p.s. subset of U if $N\circ U\subseteq N$. A union of conjugacy classes of a group is a normal p.s. subset of the group. Let N be a normal p.s. subset of U, and N' a copy of N. Denote elements of N' by a' which are the copies of a of N. Consider the set-theoretic union $V=U\cup N'$. We define a binary operation on V which is an extension of $\circ$ on U as follows. Let u denotes an element of U, and a and b elements of N. We define: $u\circ a'=u\circ a$, $a'\circ u=(a\circ u)'$ and $a'\circ b'=(a\circ b)'$. It can be verified that V is a p.s. set with this operation. Naturally, U is a normal p.s. subset of V. We call V an augmentation of U (by N). For example, let U be a union of conjugacy classes of a group, and let C be a conjugacy class contained in U. Suppose that z is an element of the center of the group and that the conjugacy class zC is not contained in U. Then, $U\cup zC$ is (isomorphic with) an augmentation of U by C. A p.s. set which is obtained from U by several augmentations is called an expansion of U by augmentations. Let Z be a subgroup of the center of a finite group H, and let ϕ be the natural homomorphism of H to $H/Z=G$. Let $G=D_1\cup\cdots\cup D_n$ be the conjugacy class decomposition of G. For each i, we take a conjugacy class C_i of H such that $\phi(C_i)=D_i$. Let $K=C_1\cup\cdots\cup C_n$. Then, we can see that H is (isomorphic with) an expansion of K by augmentations. Here, K is uniquely (up to within isomorphisms) determined by H and Z. We call K a G-core of H. C_i is called a component of K, and the number of augmentations we need to obtain H for each C_i is called the (augmentation) multiplicity of C_i. It is the number of different conjugacy classes zC_i for all $z\in Z$. Thus, we can conclude that the multiplicity of $C_i=|Z|/|Z_0|$, where Z_0 is as given in 1, taking $C=C_i$. Therefore, the multiplicity of C_i is equal to $|Z|\,|D_i|/|C_i|$.

Theorem 3. *All representation groups of a finite group are isomorphic with each other as p.s. sets.*

Proof. Let R be a representation group of a finite group G. First, we determine the G-core of R. Let F be a free central extension of G with the homomorphism ψ of F to G. There exist homomorphisms ψ_1: $F\rightarrow R$ and ψ_2: $R\rightarrow G$ such that $\psi=\psi_1\psi_2$ and that ψ_1 induces an isomorphism of F' to R'. (See [3].) Let $G=D_1\cup\cdots\cup D_n$ be the conjugacy class decomposition of G, and let $W=X_1\cup\cdots\cup X_n$ be a G-core of F, where X_i is a conjugacy class of F and $\psi(X_i)=D_i$. By Proposition 2, ψ_1 maps W isomorphically to $\psi_1(W)=K$, where $K=C_1\cup\cdots\cup C_n$ with conjugacy classes C_i of R such that $\psi_1(X_i)=C_i$ and $\psi_2(C_i)=D_i$. Thus, K is a G-core of R. This shows that G-cores of all representation groups are isomorphic with W and hence are isomorphic with each other. Now, let K be as above, and $C=C_i$. As we noted before, the augmentation multiplicity of C is equal to $|Z|\,|D|/|C|$ where $D=D_i$. It depends only on the orders of Z, of C and of D. It is well known that the order of Z is equal to the order of

M=the Schur multiplier of G. Thus, it does not depend on the choice of a representation group. Now, we can see that Theorem 3 holds.

Theorem 4. *Let R be a representation group of a finite group G, and H a central extension of G. Then, a G-core of H is a homomorphic image of a G-core of R. Especially, a conjugacy class of H is a homomorphic image of a conjugacy class of R (as a p.s. set).*

Proof. We use the same notation as in the proof of Theorem 3. Let ϕ be the homomorphism of H to G. Since F is a free central extension of G, there exists a homomorphism θ of F to H such that $\psi=\theta\phi$. Let $J=\theta(W)$, where W is a G-core of F as given before. It is easy to see that J is a G-core of H and is a homomorphic image of W, the latter being isomorphic with a G-core of R. This proves the first part. The second part is almost clear.

Theorem 5. *Let H be a central extension of G with the homomorphism ϕ. If C is a conjugacy class of H, then $|C|$ divides $|\phi(C)|\,|M|$, where M is the Schur multiplier of G. Thus, the covering multiplicity e of the conjugacy class $\phi(C)$ divides $|M|$.*

Proof. Let ψ_1, ψ_2, θ, D_i and X_i be as before. Suppose $\phi(C)=D_i$. We may assume that $C=\theta(X_i)$. Let $\psi(X_i)=E_i$=a conjugacy class of R. Then, $|E_i|$ divides $|\psi_2^{-1}(D_i)|=|D_i|\,|M|=|\phi(C)|\,|M|$. On the other hand, $|C|$ divides $|X_i|$ which is equal to $|E_i|$.

Let a be an element of a p.s. set U, and denote by a_R the right multiplication of a. Let $U_R=\{a_R | a\in U\}$. U_R is a subset of the permutation group of U. In fact, it is a p.s. subset of the permutation group of U. Let $G(U)$ be the subgroup of the permutation group of U generated by U_R. When $G(U)$ is a transitive permutation group of U, we say that U is transitive. Next, let U be transitive and special. U is a p.s. subset of a group H. Without losing generalities, we may assume that U generates H (as a group). In this case, an inner automorphism of H is uniquely determined by its effect on elements of U, and we can conclude that $G(U)$ is isomorphic with the inner automorphism group of H, the isomorphism being induced by the mapping: $a_R\to\tilde{a}$ (=the inner automorphism by a). Let G be the inner automorphism group of H, and we identify G with $G(U)$ through the above isomorphism. We can also conclude that in the above case, the transitivity of U implies U is a conjugacy class of H. Now, let ϕ be the natural homomorphism of H to G, i.e., ϕ: $a\to\tilde{a}(=a_R$ if $a\in U)$. Since $\phi(U)=U_R$, we have the following by Theorem 5.

Corollary. *If U is a special transitive p.s. set, then $|U|$ divides $|U_R|\,|M|$, where M is the Schur multiplier of $G(U)$.*

Lastly, we apply Corollary to obtain a class of exceptional transitive p.s. sets of orders $n(n-1)(n-2)/2$. Let $N=\{1, 2, \cdots, n\}$, and S=the symmetric group of degree n operating on N. Let $N\times S=\{(i, \sigma) \mid i\in N, \sigma\in S\}$. We define a binary operation on $N\times S$ by

$$(i, \sigma)\circ(j, \tau)=(i^{\sigma^{-1}\tau}, \sigma\circ\tau).$$

We can verify that $(N\times S, \circ)$ is a p.s. set. (For a more theoretical approach of the above p.s. set, see [2].) In the following, we assume that $n\geq 5$. Let C be the conjugacy class of S consisting of all transpositions (i, j), and consider $U=\{(k, (i, j))\in N\times C \mid k$ is different from i and $j\}$. U is verified to be a normal p.s. subset of $N\times S$. We want to show that U is an exceptional transitive p.s. set. Clearly, the order of U is $n(n-1)(n-2)/2$.

When (i, σ) and (j, τ) are elements of U, it follows that $(i, \sigma)\circ(j, \tau)=(i^{\tau}, \sigma\circ\tau)$. Hence, the mapping: $a_R\to\tau$ where $a=(j, \tau)$ gives an isomorphism of $G(U)$ to S. Therefore, the Schur multiplier of $G(U)$ has the order 2. (See [3].) Next, we show that U is transitive. Let (i, σ) and (j, τ) be any elements of U. From the above definition, it is easy to see that $(i, \sigma)^{G(U)}$ contains (k, τ) for some k in N. If $k=j$, we are done. So, assume $k\neq j$. Let $\rho=(j, k)\in C$. For any element (h, ρ) of U, we have $(k, \tau)\circ(h, \rho)=(j, \tau)$ since, if $\tau=(i', j')$, i' and j' are different from j and k. We have shown that U is transitive. Now, we can conclude that U is exceptional. For, if U is special, then by Corollary the order of U must divide $|U_R||M|$, i.e., $n(n-1)(n-2)/2$ must divide $(n(n-1)/2)2=n(n-1)$, which is impossible as $n\geq 5$.

In [2], we obtained an exceptional transitive p.s. set of order 90. In this paper, we obtained two exceptional transitive p.s. sets of smaller orders, i.e., of orders 30 and 60 corresponding to $n=5$ and 6 in the above. So far, the above p.s. set of order 30 seems to be of the smallest order of exceptional transitive p.s. sets.

References

[1] H. Nagao and Y. Tsushima: Representations of Finite Groups, Academic Press, 1989.

[2] N. Nobusawa: *On a construction of exceptional pseudosymmetric sets*, Osaka J. Math. **21** (1984), 701–706.

[3] M. Suzuki: Group Theory, Springer-Verlag, 1982.

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