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CONJUGACY CLASSES AND REPRESENTATION GROUPS

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Let D be a conjugacy class of a finite group G , and H a (finite) central extension of G . In the first section of the paper, we investigate how D is extended in H . Let ϕ be the homomorphism of H to G . Then, there exist a group H_0 and epimorphisms $\phi_1: H \rightarrow H_0$ and $\phi_2: H_0 \rightarrow G$ such that $\phi = \phi_1 \phi_2$, that $\phi_2^{-1}(D) = E_1 \cup \dots \cup E_n$ with conjugacy classes E_i of H_0 with $|E_i| = |D|$ (i.e., D splits completely in H_0), and that $\phi_1^{-1}(E_i) = C_i$ with a conjugacy class C_i of H with $|C_i| = e|D|$ for every i . e is called the (covering) multiplicity of D in H . Especially, when H is a representation group of G , we can show that e is equal to $|M|/|M_0|$ where M is the Schur multiplier of G and M_0 is a subgroup of M consisting of all cohomology classes that split over D . In the second section of the paper, we investigate the structure of a group or of a conjugacy class of a group with respect to inner automorphisms. An algebraic system which is the abstraction of a group with inner automorphisms as operations is called a p.s. set (or, a pseudosymmetric set). We show that all representation groups of G are isomorphic with respect to inner automorphisms, i.e., as p.s. sets, that every conjugacy class of a central extension of G is a homomorphic image of a conjugacy class of a representation group of G , and that the multiplicity e given in the above divides the order of the Schur multiplier of G . As an application, we obtain a criterion for a p.s. set to be a conjugacy class of a group, using which we can find a class of exceptional transitive p.s. sets of orders $n(n-1)(n-2)/2$, ($n \geq 5$).

1. Central extensions of conjugacy classes

Proposition 1. *Let H be a group, Z a subgroup of H contained in the center of H , and C a conjugacy class of H . Let $Z_0 = \{z \in Z \mid zC = C\}$. Then, $Z_0 = [a, H] \cap Z$ for any element a in ZC . If $\{z_i\}$ is a representative system of Z/Z_0 , then ZC is a union of conjugacy classes C_i where $C_i = z_i C$ and $C_i \neq C_j$ if $i \neq j$. Thus, ZC is a union of conjugacy classes of the same order.*

Proof. In the following, we denote $y^{-1}xy$ by $x \circ y$. So, $C = c \circ H$ with c in C . An element z of Z belongs to Z_0 if and only if $zc = c \circ x$ for some element x in H , which implies that $z = [c, x]$. Hence, $Z_0 = [c, H] \cap Z$. On the other

hand, $Z_0 = \{z \in Z \mid zC = C\} = \{z \in Z \mid z(z_i C) = z_i C\}$ for every i . Therefore, by the first argument where we use $z_i C$ in place of C , we have $Z_0 = [a, H] \cap Z$ for any element a in $z_i C$, i.e., in ZC . The remaining part of Proposition 1 is almost clear.

Let G be a finite group, and H a central extension of G with the homomorphism ϕ . Let Z be the kernel of ϕ . Z is contained in the center of H . For a conjugacy class D of G , let C be a conjugacy class of H such that $\phi(C) = D$. Then, $\phi^{-1}(D) = ZC$. By Proposition 1, $\phi^{-1}(D) = \cup C_i$ where $|C_i| = |C|$ for all i . Moreover, we can show that $|D|$ divides $|C|$. For, let d and d' be elements of D . Then, $\{x \in C \mid \phi(x) = d\}$ and $\{y \in C \mid \phi(y) = d'\}$ have the same order, since $\{y \in C \mid \phi(y) = d'\} = \{x \in C \mid \phi(x) = d\} \circ t$ where t is an element of H such that $d \circ \phi(t) = d'$.

Theorem 1. *Let H be a central extension of a finite group G with the homomorphism ϕ . If D is a conjugacy class of G , then $\phi^{-1}(D) = C_1 \cup \dots \cup C_n$ with conjugacy classes C_i of H where $|C_1| = \dots = |C_n|$. Furthermore, there exist a group H_0 and epimorphisms $\phi_1: H \rightarrow H_0$ and $\phi_2: H_0 \rightarrow G$ such that $\phi = \phi_1 \phi_2$, that $\phi_2^{-1}(D) = E_1 \cup \dots \cup E_n$ with conjugacy classes E_i of H_0 where $|E_i| = |D|$ for every i , and that $\phi_1^{-1}(E_i) = C_i$.*

Proof. The first part was explained in the above. For the second part, let Z be the kernel of ϕ and $Z_0 = \{z \in Z \mid zC = C\}$ where $C = C_1$. If we let $H_0 = H/Z_0$ and ϕ_1 and ϕ_2 the natural homomorphisms of H to H_0 and of H_0 to G , respectively, then the second part of Theorem 1 follows easily.

Theorem 1 implies that D splits completely in H_0 and each component E_i does not split at all in H . We call $e = |C_i|/|D|$ (which is common to all i) the (covering) multiplicity of D in H . Note also that $e = |H|/|H_0| = |Z_0|$.

Next, we determine the condition of the splitting of a conjugacy class D of G in H in terms of cohomologies. Let $C = C_1$ as above. By Proposition 1, $Z_0 = \{z \in Z \mid zC = C\} = [c, H] \cap Z$ for an element c of C . In the following, we fix c . For $x \in H$, $[c, x] \in Z$ if and only if $cx \equiv xc \pmod{Z}$. The latter condition is equivalent with $\phi(c) \phi(x) = \phi(x) \phi(c)$. In the following, we denote $\phi(c)$ by d so that $D = d \circ G$. Thus, $Z_0 = \{[c, x] \mid d \phi(x) = \phi(x) d, x \in H\}$. Denote elements of G by u, v , etc, and let $\{t_u \mid u \in G\}$ be a representative system of H/Z . It is clear that if $\phi(x) = u$ then $[c, x] = [c, t_u]$. So, $Z_0 = \{[c, t_u] \mid du = ud, u \in G\}$. Now, let $z(u, v)$ be a cocycle corresponding to the extension H/Z , i.e., $t_u t_v = z(u, v) t_{uv}$ where $z(u, v) \in Z$. Since $[c, t_u] = z(u, d)^{-1} z(d, u)$ as we can easily verify, we obtain that $Z_0 = \{z(u, d)^{-1} z(d, u) \mid du = ud, u \in G\}$. We can conclude that D splits completely in H if and only if $Z_0 = 1$ or $z(u, d) = z(d, u)$ for all u such that $du = ud$.

Now, we consider, as H , the standard representation group of G . The standard representation group is defined to be $Q = \sum \dot{M} t_u, u \in G$, where $\dot{M} =$

$\text{Hom}(M, K^\times)$, M being the Schur multiplier of G and K the complex number field. (For this part, see [1].) Here, $t_u t_v = z(u, v) t_{uv}$ with an element $z(u, v)$ in \hat{M} such that $z(u, v)(\alpha) = \alpha(u, v)$ where α is an element of M . Let d be a fixed element of D . We say that α splits over D if $\alpha(u, d)^{-1} \alpha(d, u) = 1$ for every u such that $ud = du$. It follows that α splits over D if and only if α is mapped to 1 by every element of Z_0 . We obtained

Theorem 2. *Let Q be the standard representation group of G , and let D be a conjugacy class of G . If M_0 denotes the subgroup of M consisting of cohomology classes that split over D , then $|M|/|M_0| = e =$ the multiplicity of D in Q*

In the following section, we show that all representation groups of G are isomorphic with each other as p.s. sets. Therefore, Theorem 2 holds for any representation groups of G .

2. Unions of conjugacy classes

Let U be a union of conjugacy classes of a group. It is closed under the operation \circ where $a \circ b = b^{-1}ab$. The binary system (U, \circ) satisfies

- (1) The right multiplication of an element a of U is a permutation on U .
- (2) $a \circ a = a$ for every element a of U .
- (3) $(a \circ b) \circ c = (a \circ c) \circ (b \circ c)$ for $a, b, c \in U$.

Generally, a binary system which satisfies (1), (2) and (3) is called a pseudosymmetric set, or briefly, a p.s. set. (When especially it satisfies (4) $(x \circ a) \circ a = x$ for any x and a of U , we say it is a symmetric set.) Any group is a p.s. set in the above sense. A p.s. subset of a group is called a special p.s. set. Thus, a union of conjugacy classes of a group is a special p.s. set. A p.s. set which is not special is said to be exceptional.

Proposition 2. *Let ϕ be an epimorphism of a group H to a group G . If ϕ induces an isomorphism of the commutator subgroup H' of H onto the commutator subgroup G' of G , then every conjugacy class of H is mapped to a conjugacy class of G bijectively. More generally, if $U = C_1 \cup \dots \cup C_n$ is a union of conjugacy classes C_i of H and if $\phi(C_i) \neq \phi(C_j)$ whenever $i \neq j$, then U is mapped isomorphically (as a p.s. set) onto a union of conjugacy classes of G by ϕ .*

Proof. It is sufficient to show that ϕ is injective on a conjugacy class C of H . Since $C = c \circ H$, it is sufficient to show that $\phi(c \circ a) = \phi(c \circ b)$ implies $c \circ a = c \circ b$. Assume $\phi(c \circ a) = \phi(c \circ b)$. Then, $\phi([c, a]) = \phi([c, b])$ as $[c, a] = c^{-1}(c \circ a)$ and ϕ is a (group) homomorphism. Now, the assumption in Proposition 2 implies that $[c, a] = [c, b]$, from which we can easily conclude that $c \circ a = c \circ b$.

In the following, U denotes a special or exceptional p.s. set. A subset N

of U is called a normal p.s. subset of U if $N \circ U \subseteq N$. A union of conjugacy classes of a group is a normal p.s. subset of the group. Let N be a normal p.s. subset of U , and N' a copy of N . Denote elements of N' by a' which are the copies of a of N . Consider the set-theoretic union $V = U \cup N'$. We define a binary operation on V which is an extension of \circ on U as follows. Let u denotes an element of U , and a and b elements of N . We define: $u \circ a' = u \circ a$, $a' \circ u = (a \circ u)'$ and $a' \circ b' = (a \circ b)'$. It can be verified that V is a p.s. set with this operation. Naturally, U is a normal p.s. subset of V . We call V an augmentation of U (by N). For example, let U be a union of conjugacy classes of a group, and let C be a conjugacy class contained in U . Suppose that z is an element of the center of the group and that the conjugacy class zC is not contained in U . Then, $U \cup zC$ is (isomorphic with) an augmentation of U by C . A p.s. set which is obtained from U by several augmentations is called an expansion of U by augmentations. Let Z be a subgroup of the center of a finite group H , and let ϕ be the natural homomorphism of H to $H/Z = G$. Let $G = D_1 \cup \dots \cup D_n$ be the conjugacy class decomposition of G . For each i , we take a conjugacy class C_i of H such that $\phi(C_i) = D_i$. Let $K = C_1 \cup \dots \cup C_n$. Then, we can see that H is (isomorphic with) an expansion of K by augmentations. Here, K is uniquely (up to within isomorphisms) determined by H and Z . We call K a G -core of H . C_i is called a component of K , and the number of augmentations we need to obtain H for each C_i is called the (augmentation) multiplicity of C_i . It is the number of different conjugacy classes zC_i for all $z \in Z$. Thus, we can conclude that the multiplicity of $C_i = |Z|/|Z_0|$, where Z_0 is as given in 1, taking $C = C_i$. Therefore, the multiplicity of C_i is equal to $|Z|/|D_i|/|C_i|$.

Theorem 3. *All representation groups of a finite group are isomorphic with each other as p.s. sets.*

Proof. Let R be a representation group of a finite group G . First, we determine the G -core of R . Let F be a free central extension of G with the homomorphism ψ of F to G . There exist homomorphisms $\psi_1: F \rightarrow R$ and $\psi_2: R \rightarrow G$ such that $\psi = \psi_1 \psi_2$ and that ψ_1 induces an isomorphism of F' to R' . (See [3].) Let $G = D_1 \cup \dots \cup D_n$ be the conjugacy class decomposition of G , and let $W = X_1 \cup \dots \cup X_n$ be a G -core of F , where X_i is a conjugacy class of F and $\psi(X_i) = D_i$. By Proposition 2, ψ_1 maps W isomorphically to $\psi_1(W) = K$, where $K = C_1 \cup \dots \cup C_n$ with conjugacy classes C_i of R such that $\psi_1(X_i) = C_i$ and $\psi_2(C_i) = D_i$. Thus, K is a G -core of R . This shows that G -cores of all representation groups are isomorphic with W and hence are isomorphic with each other. Now, let K be as above, and $C = C_i$. As we noted before, the augmentation multiplicity of C is equal to $|Z|/|D|/|C|$ where $D = D_i$. It depends only on the orders of Z , of C and of D . It is well known that the order of Z is equal to the order of

M = the Schur multiplier of G . Thus, it does not depend on the choice of a representation group. Now, we can see that Theorem 3 holds.

Theorem 4. *Let R be a representation group of a finite group G , and H a central extension of G . Then, a G -core of H is a homomorphic image of a G -core of R . Especially, a conjugacy class of H is a homomorphic image of a conjugacy class of R (as a p.s. set).*

Proof. We use the same notation as in the proof of Theorem 3. Let ϕ be the homomorphism of H to G . Since F is a free central extension of G , there exists a homomorphism θ of F to H such that $\psi = \theta\phi$. Let $J = \theta(W)$, where W is a G -core of F as given before. It is easy to see that J is a G -core of H and is a homomorphic image of W , the latter being isomorphic with a G -core of R . This proves the first part. The second part is almost clear.

Theorem 5. *Let H be a central extension of G with the homomorphism ϕ . If C is a conjugacy class of H , then $|C|$ divides $|\phi(C)||M|$, where M is the Schur multiplier of G . Thus, the covering multiplicity e of the conjugacy class $\phi(C)$ divides $|M|$.*

Proof. Let $\psi_1, \psi_2, \theta, D_i$ and X_i be as before. Suppose $\phi(C) = D_i$. We may assume that $C = \theta(X_i)$. Let $\psi(X_i) = E_i =$ a conjugacy class of R . Then, $|E_i|$ divides $|\psi_2^{-1}(D_i)| = |D_i||M| = |\phi(C)||M|$. On the other hand, $|C|$ divides $|X_i|$ which is equal to $|E_i|$.

Let a be an element of a p.s. set U , and denote by a_R the right multiplication of a . Let $U_R = \{a_R | a \in U\}$. U_R is a subset of the permutation group of U . In fact, it is a p.s. subset of the permutation group of U . Let $G(U)$ be the subgroup of the permutation group of U generated by U_R . When $G(U)$ is a transitive permutation group of U , we say that U is transitive. Next, let U be transitive and special. U is a p.s. subset of a group H . Without losing generalities, we may assume that U generates H (as a group). In this case, an inner automorphism of H is uniquely determined by its effect on elements of U , and we can conclude that $G(U)$ is isomorphic with the inner automorphism group of H , the isomorphism being induced by the mapping: $a_R \rightarrow \tilde{a}$ (=the inner automorphism by a). Let G be the inner automorphism group of H , and we identify G with $G(U)$ through the above isomorphism. We can also conclude that in the above case, the transitivity of U implies U is a conjugacy class of H . Now, let ϕ be the natural homomorphism of H to G , i.e., $\phi: a \rightarrow \tilde{a}$ ($=a_R$ if $a \in U$). Since $\phi(U) = U_R$, we have the following by Theorem 5.

Corollary. *If U is a special transitive p.s. set, then $|U|$ divides $|U_R||M|$, where M is the Schur multiplier of $G(U)$.*

Lastly, we apply Corollary to obtain a class of exceptional transitive p.s. sets of orders $n(n-1)(n-2)/2$. Let $N = \{1, 2, \dots, n\}$, and $S =$ the symmetric group of degree n operating on N . Let $N \times S = \{(i, \sigma) \mid i \in N, \sigma \in S\}$. We define a binary operation on $N \times S$ by

$$(i, \sigma) \circ (j, \tau) = (i^{\sigma^{-1}\tau}, \sigma \circ \tau).$$

We can verify that $(N \times S, \circ)$ is a p.s. set. (For a more theoretical approach of the above p.s. set, see [2].) In the following, we assume that $n \geq 5$. Let C be the conjugacy class of S consisting of all transpositions (i, j) , and consider $U = \{(k, (i, j)) \in N \times C \mid k \text{ is different from } i \text{ and } j\}$. U is verified to be a normal p.s. subset of $N \times S$. We want to show that U is an exceptional transitive p.s. set. Clearly, the order of U is $n(n-1)(n-2)/2$.

When (i, σ) and (j, τ) are elements of U , it follows that $(i, \sigma) \circ (j, \tau) = (i^\tau, \sigma \circ \tau)$. Hence, the mapping: $a_R \rightarrow \tau$ where $a = (j, \tau)$ gives an isomorphism of $G(U)$ to S . Therefore, the Schur multiplier of $G(U)$ has the order 2. (See [3].) Next, we show that U is transitive. Let (i, σ) and (j, τ) be any elements of U . From the above definition, it is easy to see that $(i, \sigma)^{G(U)}$ contains (k, τ) for some k in N . If $k = j$, we are done. So, assume $k \neq j$. Let $\rho = (j, k) \in C$. For any element (h, ρ) of U , we have $(k, \tau) \circ (h, \rho) = (j, \tau)$ since, if $\tau = (i', j')$, i' and j' are different from j and k . We have shown that U is transitive. Now, we can conclude that U is exceptional. For, if U is special, then by Corollary the order of U must divide $|U_R| |M|$, i.e., $n(n-1)(n-2)/2$ must divide $(n(n-1)/2) 2 = n(n-1)$, which is impossible as $n \geq 5$.

In [2], we obtained an exceptional transitive p.s. set of order 90. In this paper, we obtained two exceptional transitive p.s. sets of smaller orders, i.e., of orders 30 and 60 corresponding to $n=5$ and 6 in the above. So far, the above p.s. set of order 30 seems to be of the smallest order of exceptional transitive p.s. sets.

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