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# ASYMPTOTIC PROPERTIES FOR CONTINUOUS AND JUMP TYPE'S FEYNMAN-KAC FUNCTIONALS

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#### 1. Introduction

During the past years, theory of measure perturbations of Dirichlet forms and the associated perturbed semigroups - so called generalized Feynman-Kac semigroups have been studied by many authors ([1],[2],[3],[7],[24],[25],[31],[32]). Furthermore, Dirichlet forms perturbed by jumps were recently investigated ([3],[24]), and unlike the case of the perturbation by smooth measures, we needed some new concepts and tools to study them. Indeed, when the underlying Markov process  $X_t$  is discontinuous, the following important discontinuous additive functionals can be considered ;

(1.1) 
$$A_t^F = \sum_{s \le t} F(X_{s-}, X_s),$$

where F is a Borel function on  $X \times X$  vanishing on the diagonal. In this case, there are some difficulties in studying the Feynman-Kac semigroup for (1.1) and the corresponding bilinear form, because the basic tools used in the continuous case are not available in this discontinuous case. Song [24] studied the additive functionals of the forms

$$A_t^{\mu,F} = A_t^{\mu} + A_t^F$$

for the symmetric  $\alpha$ -stable processes, where  $A^{\mu}$  is the continuous additive functional (or abbreviated as CAF) with  $\mu$  as its Revuz measure. For  $f \in \mathcal{B}(\mathbb{R}^d)$ , he proved that if  $\mu$  is a measure in the Kato class and F is a bounded "admissible" function with respect to the base process, then the Feynman-Kac semigroup

$$p_t^{\mu,F} f(x) = E_x \left( \exp\left( -A_t^{\mu,F} \right) f(X_t) \right)$$

is strongly continuous on  $L^p(\mathbb{R}^d)$ . As a generalized approach, Ying [31] introduced the concepts of additive functionals of *extended Kato class* and showed that, if an additive functional  $A_t$  of symmetric Markov process  $X_t$  belongs to this class, the Feynman-Kac semigroup  $p_t^{-A}f(x) = E_x((\exp(A))_t f(X_t))$  may be extended to a strongly continuous

semigroup of bounded operator, where  $\exp(A) \left(=e^{A_t^c} \prod_{s \leq t} (1 + \Delta A_s)\right)$  is the Stieltjes exponential of A. He also characterized the perturbed bilinear form associated with  $(p_t^{-A})$ .

In 1951, Kac [18] obtained a remarkable relation between the principal eigenvalue of Schrödinger operators and an asymptotics of the so called Feynman-Kac functional. More precisely, let  $(B_t, P_x^W)$  be the d-dimensional Brownian motion and V a function on  $\mathbb{R}^d$  such that  $\lim_{x\to\infty} V(x) = \infty$ . Then, it holds

(1.3)  
$$\lim_{t \to \infty} \frac{1}{t} \log P_x^W \left( \exp\left(-\int_0^t V(B_s) ds\right) \right)$$
$$= -\inf_{\substack{u \in H^1(R^d) \\ ||u||_2 = 1}} \left( \frac{1}{2} \mathbf{D}(u, u) + \int_{R^d} u^2(x) V(x) dx \right).$$

Here **D** is the classical Dirichlet integral and  $(1/2)\mathbf{D}$  plays a role as the rate function of the large deviation principle for the Brownian motion. Afterward, this formula became a motivation of Donsker-Varadhan large deviation theory which tells us an asymptotics of occupation time distribution  $L_t(\omega, \cdot) = (1/t) \int_0^t \chi \cdot ((B_s(\omega))ds)$ , and nowadays, the formula (1.3) also can be derived as a corollary of Donsker-Varadhan large deviation theory.

On the other hand, many people recently studied the so called generalized Schrödinger operator with a signed smooth measure  $\mu$  as a potential,  $-\frac{1}{2}\Delta + \mu$ , and the associated semigroup  $E_x^W (\exp(-A_t^{\mu}) f(B_t))$  ([1],[2],[3],[6]). Moreover, Carmona-Masters-Simon [7] considered the relativistic Hamiltonian operator  $\sqrt{-\Delta + m^2} - m$ , m > 0instead of Laplacian. Note that if we want to extend the formula (1.3) to the relativistic Hamiltonian operator and also to the generalized Schrödinger operator, Donsker-Varadhan large deviation principle is no longer available because the additive functional  $A_t^{\mu,F}$  is not a function of the occupation time distribution  $L_t(\omega, \cdot)$  while  $\int_0^t V(X_s) ds$ can be written as  $t \int_X V(x) L_t(\omega, dx)$ .

An objective of this paper is extend the formula (1.3) with continuous and jump type's Feynman-Kac functionals  $\exp\left(-A_t^{\mu,F}\right)$  in the framework of regular Dirichlet space. To do this, we shall concentrate our attention on the modification of "Donsker-Varadhan large deviation principle for symmetric Markov processes with multiplicative functional  $\exp\left(-A_t^{\mu,F}\right)$ ". The two methods may be considered. One can immediately expect on account of [26] that, if  $\mu$  and F are positive and a symmetric Markov process as a base process explodes so fast that  $R_1^{\mu,F}$ 1 the 1-resolvent of the identity function 1 belongs to  $C_{\infty}(X)$  the space of continuous functions on the state space X vanishing at infinity, then we can use the full large deviation principle<sup>1</sup>(Theorem 1.1 and 4.4 in [26]). The other is the choice of special Lévy process satisfying the exponentially

<sup>&</sup>lt;sup>1</sup>Throughout this paper, "large deviation principle" means a "Donsker-Varadhan's type", otherwise, we shall give a comment.

localized condition (cf. [7]), under which, the  $L^{p}$ -spectral radius of the Feynman-Kac semigroup of kernels be ones *p*-independent. We combine this independent property with the large deviation principle.

More precisely, let  $(\mathcal{E}, \mathcal{F})$  be a regular Dirichlet form on  $L^2(X; m)$  and  $\mathbf{M} = (\Omega, X_t, P_x, \zeta, \theta_t)$  the associated Hunt process, where X is a locally compact separable metric space and m is a positive Radon measure on X with full support. Let  $\mu$  be a signed smooth measure associated with  $(\mathcal{E}, \mathcal{F})$  and F a Borel function of bounded below on  $X \times X$  vanishing on the diagonal. First we show that if  $\overline{F} = (1 - e^{-F}) \in L^2(X \times X \setminus d; Nm)$  and  $A^{\mu, \overline{F}}$  is the additive functional of extended Kato class, then the Hunt process  $\mathbf{M}$  can be transformed into the  $\phi^2 m$ -symmetric ergodic process whose Dirichlet form on  $L^2(X; \phi^2 m)$  is written as

$$\mathcal{E}^{\phi}(u,u) = \frac{1}{2} \int_{X} \Gamma^{c}(u,u) \phi^{2} dm + \int \int_{X \times X \setminus d} (\tilde{u}(x) - \tilde{u}(y))^{2} \phi(x) \phi(y) e^{-F} N(x,dy) m(dx)$$

by means of a supermartingale multiplicative functional

(1.4) 
$$N_t^{\phi} = \exp\left(-A_t^{\mu,F}\right) \frac{\phi(X_t)}{\phi(X_0)} \exp\left(-\int_0^t \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_s)ds\right).$$

Using this fact, we prove the following formula for the symmetric Lévy process on  $R^d$  with its exponent being the so-called  $\alpha$ -relativistic Hamiltonian

(1.5)  

$$(p^{2} + m^{2})^{\frac{\alpha}{2}} - m^{\alpha}, \qquad m > 0, \ 0 < \alpha \le 2;$$

$$\lim_{t \to \infty} \frac{1}{t} \log E_{x} \left( \exp\left(-A_{t}^{\mu,F}\right) \right)$$

$$= -\inf_{\substack{u \in \mathcal{F}^{\mu,F} \\ ||u||_{2} = 1}} \left( \mathcal{E}(u, u) + \int_{R^{d}} u(x)^{2} d\mu(x) + \int_{R^{d} \times R^{d} \setminus d} u(x)u(y)(1 - e^{-F})J(dx \, dy) \right)$$

holding for all  $x \in \mathbb{R}^d$ , where  $\mathcal{F}^{\mu,F} = \{u \in \mathcal{F}; u \in L^2(\mathbb{R}^d; |\mu| + |\rho|)\}, \ \rho(A) = \int_A \cdot \int_{\mathbb{R}^d} (1 - e^{-F(x,y)}) J(dxdy), \ A \in \mathcal{B}(\mathbb{R}^d).$ 

We shall also discuss some of its applications. When the base process is a symmetric stable process of index  $\alpha$ ,  $(1 < \alpha \le 2)$ , the surface measure of the sphere on  $R^3$  is in the (extended) Kato class ([6]). Since the pseudo-differential operator  $|-\Delta|^{\alpha/2}(1 < \alpha \le 2)$  and the  $\alpha$ -relativistic Hamiltonian operator  $(-\Delta + m^2)^{\alpha/2} - m^{\alpha} (m > 0, 1 < \alpha \le 2)$  define the same Kato class, the formula (1.5) gives us the asymptotic behaviour of the local time for the surface measure of the sphere on  $R^3$  (see Example 5.1 and 5.2). As mentioned above, when  $\mu$  and F are positive and  $R_1^{\mu,F}$  1 belongs to  $C_{\infty}(X)$ ,

we can also consider the full large deviation formula. We shall describe the asymptotics of the number of jumps for symmetric Markov chains on a symmetric region of the state space and give some sufficient conditions to ensure  $R_1^F 1 \in C_{\infty}(X)$  in establishing full large deviation principle (see Example 5.3 and Theorem 5.2).

## 2. Preliminaries and Assumptions

Let X be a locally compact separable metric space and m a positive Radon measure on X with full support. Let  $(\mathcal{E}, \mathcal{F})$  be a regular Dirichlet form on  $L^2(X; m)$  and  $\mathbf{M} = (\Omega, X_t, P_x, \zeta)$  a corresponding Hunt process.<sup>2</sup> Here  $\Omega$  is the space of all right continuous functions from  $[0, \infty)$  to  $X_\Delta$  with the left hand limit on  $(0, \infty)$  such that  $\omega(t) = \Delta$  for all  $t \ge \zeta(\omega) = \inf\{s \ge 0; \omega(s) = \Delta\}$  and  $X_t$  is the coordinate maps,  $X_t(\omega) = \omega(t)$ . The stopping time  $\zeta$  is called the *life time* of  $\mathbf{M}$  and  $\Delta$  is called a *cemetery point* adjoined to X. Let  $(p_t)_{t>0}$  be the transition semigroup of  $\mathbf{M}$ ,  $p_t f(x) = E_x(f(X_t))$ , and  $(R_\alpha)_{\alpha>0}$  the resolvent of  $\mathbf{M}$ ,  $R_\alpha f(x) = E_x(\int_0^\infty e^{-\alpha t} f(X_s) ds)$ . The *capacity* associated with the Dirichlet form  $(\mathcal{E}, \mathcal{F})$  is defined by

$$\operatorname{Cap}(O) = \inf \left\{ \mathcal{E}(u, u) + \int_X u^2 dm \; ; \; u \in \mathcal{F}, u \ge 1 \; m \text{-a.e. on } O \right\}$$

for an open set O, and is extended to any set as an outer capacity. We say that an increasing sequence  $\{F_n\}$  of closed sets is a generalized nest (simply say, a nest) if  $\lim_{n\to\infty} \operatorname{Cap}(K \setminus F_n) = 0$  for any compact set K. A positive Borel measure  $\mu$  on X is said to be smooth if  $\mu$  charges no set of zero capacity and there exists a nest  $\{F_n\}$  such that  $\mu(F_n) < \infty$ , for all n (§2.2 in [16]). Let us denote by S the totality of smooth measures. Given a signed Borel measure  $\mu = \mu^+ - \mu^-$ , we say  $\mu$  the signed smooth measure and write  $\mu \in S - S$  if  $\mu^+ \in S$  and  $\mu^- \in S$ . For a Borel function f on X, we define the quasi-norm by

$$\|f\|_q = \inf_{\operatorname{Cap}(N)=0} \sup_{x \in X \setminus N} |f(x)|.$$

Note that, if f is quasi-continuous,  $||f||_q$  is the same as  $||f||_{\infty}$  the usual  $L^{\infty}$ -norm.

A signed Radon measure  $\mu$  is said to be in the Kato class  $S_K$  if

(2.1) 
$$\lim_{t \downarrow 0} \|E_x (A_t^{\mu})\|_q = 0,$$

which is the extension of the classical definition of the Kato class for Brownian motion. Now, we introduce more general concepts of the Kato class (2.1)

<sup>&</sup>lt;sup>2</sup>The Dirichlet form is defined as a Markovian closed "symmetric" form and hence, the associated Hunt process is always assumed to be "*m*-symmetric" (cf. [16]).

DEFINITION 2.1. A positive additive functional A of M is said to be in the *extended Kato class* if it belongs to

$$\mathcal{A}_{K}^{ad} = \left\{ A \; ; \; \inf_{t>0} \|E_{x}(A_{t})\|_{q} < \frac{1}{2} \right\}.$$

A signed smooth measure  $\mu = \mu^+ - \mu^-$  (resp. A Borel function  $F = F^+ - F^-$  on  $X \times X \setminus d$ ) is said to belong to the *extended Kato class* if  $A^{\mu} = A^{\mu^+} - A^{\mu^-} \in \mathcal{A}_K^{ad} - \mathcal{A}_K^{ad}$  (resp.  $A^F = A^{F^+} - A^{F^-} \in \mathcal{A}_K^{ad} - \mathcal{A}_K^{ad}$ ).

From now on, we assume that F is a symmetric (i.e. F(x,y) = F(y,x)) Borel function on  $X \times X \setminus d$  and for convenience,  $F(x, \Delta) = 0$ ,  $x \in X$ . Put  $\overline{F} = (1 - e^{-F})$ . For the additive functional  $A^{\mu,\overline{F}}$  which belongs to the extended Kato class, we define the symmetric bilinear form  $(\mathcal{E}^{\mu,F}, \mathcal{F}^{\mu,F})$  on  $L^2(X;m)$  by

$$\begin{cases} \mathcal{E}^{\mu,F}(u,v) = \mathcal{E}(u,v) + \int_X \tilde{u}(x)\tilde{v}(x)\,\mu(dx) + \iint_{X\times X\setminus d} \tilde{u}(x)\tilde{v}(y)\bar{F}J(dx\,dy) \\ for \ u,v\in\mathcal{F}^{\mu,F} \\ \mathcal{F}^{\mu,F} = \{u\in\mathcal{F}\ ;\ \tilde{u}\in L^2(X;|\mu|+|\rho|)\}, \end{cases}$$

where  $\rho(B) = \int_B \int_X \bar{F}(x,y) J(dxdy), B \in \mathcal{B}(X)$ , i.e., the marginal measure of  $\bar{F}J$ and  $\tilde{u}$  means a quasi continuous *m*-version of *u* with respect to the capacity Cap. Then  $(\mathcal{E}^{\mu,F}, \mathcal{F}^{\mu,F})$  is a lower semibounded closed symmetric bilinear form on  $L^2(X;m)$ because its perturbed semigroup, the so-called *Feynman-Kac semigroup*,

(2.2) 
$$p_t^{\mu,F}f(x) = E_x\left(\exp\left(-A_t^{\mu,F}\right)f(X_t)\right) \quad x \in X.$$

is a strongly continuous symmetric semigroup of bounded operators on  $L^2(X; m)$  (Theorem 3.2 in [31]).

Now, in order to obtain our main results, we make the following assumptions on  $(\mathcal{E}, \mathcal{F})$  and M which are always available throughout this paper.

#### Assumptions

(I) The energy measure corresponding to  $(\mathcal{E}, \mathcal{F})$  is absolutely continuous with respect to the base measure m, that is,  $\mathcal{E}$  can be written as

$$\begin{split} \mathcal{E}(u,u) &= \frac{1}{2} \int_X \Gamma^c(u,u) dm + \frac{1}{2} \iint_{X \times X \setminus d} (u(x) - u(y))^2 N(x,dy) m(dx) \\ &+ \int_X u(x)^2 k(x) m(dx) \quad \text{ for } u \in \mathcal{F} \cap C_0(X), \end{split}$$

where  $C_0(X)$  is the set of continuous functions on X with compact support.

(II) (Irreducibility of M) If a Borel set  $B \in \mathcal{B}(X)$  satisfies

$$\chi_B p_t f = p_t(\chi_B f)$$
 for  $\forall f \in L^2(X; m) \cap \mathcal{B}(X)$  and  $t > 0$ ,

then m(B) = 0 or  $m(X \setminus B) = 0$ .

(III) The transition probability  $p_t(x, dy)$  of M is absolutely continuous with respect to m for each t > 0 and  $x \in X$ .

(IV)  $p_t$  is a bounded operator from  $L^1(X;m)$  to  $L^{\infty}(X;m)$  for any t > 0.

The next lemma asserts that under the extended Kato class conditions for  $A^{\mu,\bar{F}}$  determined by  $\mu$  and F, the Feynman-Kac semigroup (2.2) inherits the  $L^p$ -boundedness and  $L^p$ -smoothing properties of the original semigroup.

**Lemma 2.1.** Suppose  $A_t^{\mu,\bar{F}} \in \mathcal{A}_K^{\mathrm{ad}} - \mathcal{A}_K^{\mathrm{ad}}$ . Then

(i) There exist positive constants c and  $\beta(\mu, F)$  such that for all t > 0,

$$\left\|p_t^{\mu,F}\right\|_{p,p} \leq c e^{\beta(\mu,F)t}, \quad 1 \leq \forall p \leq \infty,$$

where  $\| \|_{p,p}$  means the operator norm from  $L^p$  to  $L^p$ . (ii) For any  $1 \le p, q \le \infty$  and t > 0,  $\left\| p_t^{\mu,F} \right\|_{p,q} < \infty$ .

Proof. (i) Since  $\exp(-A^{\mu,\bar{F}})_t = e^{-A_t^{\mu}}\prod_{s \le t}(1+e^{-F(X_{s-},X_s)}-1) = \exp(-A_t^{\mu,F})$ , we see by Lemma 2.1 in [31] that there exist positive constants c and  $\beta(\mu,F)$  such that for any t > 0,

$$\left\| E_x \left( \exp\left( -A_t^{\mu,F} \right) \right) \right\|_q \le c e^{\beta(\mu,F)t} \qquad x \in X.$$

Now, we have the result by using interpolations.

(ii) It is enough to show that  $p_t^{\mu,F}$  is bounded operator from  $L^1(X;m)$  to  $L^p(X;m)$ ,  $(1 \le p \le \infty)$  for any t > 0. The assumption (IV) say that there exists a positive constant  $C_t$  such that for any t > 0,

$$||p_t||_{1,\infty} = C_t < \infty.$$

Now, for any  $f \in L^q(X; m), (1 \le q \le \infty)$ , we have

$$\left| p_t^{\mu,F} f \right| \le \left( E_{\cdot} \left( e^{-pA_t^{\mu,F}} \right) \right)^{\frac{1}{p}} (p_t|f|^q)^{\frac{1}{q}} \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Using (i), we get  $\left\| p_t^{\mu,F} \right\|_{q,\infty} \leq \exists C'_t, t > 0$  and its dual property give the conclusion.

# 3. Transformation to Ergodic Processes by Multiplicative Functional

In this section, we transform the Hunt process M into the ergodic subprocess by certain multiplicative functional, which will play a crucial role in the coming section. First, we define the resolvent  $\{R^{\mu,F}_{\alpha}\}_{\alpha>\beta(\mu,F)}$  by

$$R^{\mu,F}_{\alpha}f(x) = E_x\left(\int_0^\infty \exp(-\alpha t - A^{\mu,F}_t)f(X_t)dt\right) \quad \text{for } f \in \mathcal{B}_b(X)$$

and the generator  $\mathcal{L}^{\mu,F}$  by

$$\mathcal{L}^{\mu,F}u = \alpha u - f$$
 for  $u = R^{\mu,F}_{\alpha}f$ ,  $f \in C_b(X)$ 

where  $\mathcal{B}_b(X)$  (resp.  $C_b(X)$ ) is the set of all bounded Borel (resp. continuous) functions and  $\beta(\mu, F)$  is the constant in Lemma 2.1. Set the domain of  $\mathcal{L}^{\mu,F}$ 

$$\mathcal{D}^{2}_{+}(\mathcal{L}^{\mu,F}) = \left\{ R^{\mu,F}_{\alpha}f \; ; \; \alpha > \beta(\mu,F), \; f \in L^{2}(X;m) \cap C_{b}(X), \; f \ge 0 \; \text{and} \; f \neq 0 \right\}.$$

Note that any function in  $\mathcal{D}^2_+(\mathcal{L}^{\mu,F})$  is strictly positive on X by assumption (II) and (III) (cf. Theorem 4.6.6 in [16]).

For  $\phi = R^{\mu,F}_{\alpha}g \in \mathcal{D}^2_+(\mathcal{L}^{\mu,F})$ , put

$$M_t^{\mu,F,\phi} = e^{-A_t^{\mu,F}} \phi(X_t) - \phi(X_0) - \int_0^t e^{-A_s^{\mu,F}} \mathcal{L}^{\mu,F} \phi(X_s) \, ds.$$

Then,  $M_t^{\mu,F,\phi}$  is a martingale with respect to  $P_x$ ,  $x \in X$  because

$$E_x(M_t^{\mu,F,\phi}) = 0$$
 and  $M_{s+t}^{\mu,F,\phi} = M_s^{\mu,F,\phi} + e^{-A_s^{\mu,F}} M_t^{\mu,F,\phi}(\theta_s).$ 

**Lemma 3.1.**  $M_t^{\mu,F,\phi}$  can be also written as

$$M_t^{\mu,F,\phi} = \int_0^t e^{-A_{s-}^{\mu,F}} dM_s^{[\phi]} - \int_0^t e^{-A_{s-}^{\mu,F}} dL_s^{\phi,\bar{F}}, \quad P_x\text{-a.e.} \quad x \in X,$$

where

$$L_t^{\phi,\bar{F}} = \sum_{s \le t} \phi(X_s) \bar{F}(X_{s-}, X_s) - \int_0^t \int_X N(X_s, dy) \phi(y) \bar{F}(X_s, y) \, ds$$

and  $M^{[\phi]}$  is the martingale part for Fukushima decomposition of  $\phi(X_t) - \phi(X_0)$ .

Proof. We apply Itô formula to G(x, y) = xy. Since  $de^{-A_t^{\mu, F}} = -(e^{-A_t^{\mu, F}} dA_t^{\mu} + e^{-A_{t-}^{\mu, F}} dA_t^{\bar{F}})$ , we get

$$(3.1) \qquad e^{-A_t^{\mu,F}}\phi(X_t) - \phi(X_0) = G(e^{-A_t^{\mu,F}}, \phi(X_t)) - G(e^{-A_0^{\mu,F}}, \phi(X_0)) \\ = \int_0^t e^{-A_{s-}^{\mu,F}} dM_s^{[\phi]} + \int_0^t e^{-A_{s-}^{\mu,F}} dN_s^{[\phi]} \\ - \int_0^t \phi(X_s) e^{-A_s^{\mu,F}} dA_s^{\mu} - \int_0^t \phi(X_s) e^{-A_{s-}^{\mu,F}} dA_s^{\bar{F}},$$

where  $A_t^{\bar{F}} = \sum_{s \leq t} \bar{F}(X_{s-}, X_s)$ . Put  $A_t^{\phi, \bar{F}} = \sum_{s \leq t} \phi(X_s) \bar{F}(X_{s-}, X_s)$  and consider its dual predictable projection  $\left(A_t^{\phi, \bar{F}}\right)^p = \int_0^t \int_X N(X_s, dy) \phi(y) \bar{F}(X_s, y) ds$ . Then

$$L_t^{\phi,\bar{F}} = \sum_{s \le t} \phi(X_s)\bar{F}(X_{s-}, X_s) - \int_0^t \int_X N(X_s, dy)\phi(y)\bar{F}(X_s, y)ds$$

is a martingale with respect to  $P_x, x \in X$ , and hence, the last term of the right hand side of (3.1) (discontinuous part) equals

$$\int_{0}^{t} e^{-A_{s-}^{\mu,F}} dA_{s}^{\phi,\bar{F}}$$
  
= 
$$\int_{0}^{t} e^{-A_{s-}^{\mu,F}} dL_{s}^{\phi,\bar{F}} + \int_{0}^{t} e^{-A_{s-}^{\mu,F}} \int_{X} N(X_{s},dy)\phi(y)\bar{F}(X_{s},y)ds.$$

We have the lemma.

Let us define the multiplicative functional  $N^{\phi}$  by

(3.2) 
$$N_t^{\phi} = \exp\left(-A_t^{\mu,F}\right) \frac{\phi(X_t)}{\phi(X_0)} \exp\left(-\int_0^t \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_s)ds\right).$$

Then  $N^{\phi}$  is a supermartingale multiplicative functional. Indeed, put  $K_n = \{x \in X ; \phi(x) \ge \frac{1}{n}\}$  and denote by  $K_n^o$  the fine interior of  $K_n$ . By noting

$$d\left(e^{-A_t^{\mu,F}}\phi(X_s)\exp\left(-\int_0^t\frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u)du\right)\right)$$
  
=  $\exp\left(-\int_0^t\frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u)du\right)\left(d(e^{-A_t^{\mu,F}}\phi(X_s))-e^{-A_t^{\mu,F}}\mathcal{L}^{\mu}\phi(X_s)ds\right),$ 

we can immediately check that for each n

(3.3) 
$$N_{t\wedge\tau_n}^{\phi} - 1$$
$$= \int_0^{t\wedge\tau_n} \frac{1}{\phi(X_0)} \exp\left(-\int_0^s \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u) du\right) dM_s^{\mu,F,\phi} \quad P_x\text{-a.e.} \quad x \in X,$$

where  $\tau_n$  is the first leaving time from  $K_n^o$ ,  $\tau_n = \inf\{t > 0 : X_t \notin K_n^o\}$ . Therefore, we see from (3.3) that

$$E_x(N_t^{\phi}) = E_x(N_{t\wedge\zeta}^{\phi}) \le \liminf_{n\to\infty} E_x(N_{t\wedge\tau_n}^{\phi}) = 1, \quad x \in X.$$

Let us denote by  $\mathbf{M}^{\phi} = (\Omega, X_t, P_x^{\phi}, \zeta)$  the transformed process of  $\mathbf{M}$  by  $N^{\phi}$ .

**Lemma 3.2.**  $\mathbf{M}^{\phi}$  is a  $\phi^2 m$ -symmetric right process on X.

Proof. For a path  $\omega \in \Omega$  with  $t < \zeta(\omega)$ , let  $r_t$  be a reversal operator on  $\Omega$  which is defined by

$$r_t(\omega)(s) = \omega((t-s)-)$$
 if  $(0 \le s \le t)$ ,  $r_t(\omega)(s) = \omega(0)$  if  $(s > t)$ .

Note that  $\mathbf{M}^{\phi}$  is a right process (see [21]) and is reversible under  $P_m^{\phi}$ -a.e., because for any  $\mathcal{F}_t$ -measurable function f,

$$E_m(f(r_t \cdot) ; t < \zeta) = E_m(f(\cdot) ; t < \zeta).$$

Since F is symmetric on  $X \times X \setminus d$ ,  $A_t^{\mu,F}(r_t\omega) = A_t^{\mu,F}(\omega)$ ,  $P_m$ -a.e. on account of Theorem 5.4.1 in [16]. Now for  $f, g \in \mathcal{B}(X)$ ,

$$\begin{aligned} &(p_t^{\phi}f,g)_{\phi^2 m} \\ &= \left( E. \left( e^{-A_t^{\mu,F}} \frac{\phi(X_t)}{\phi(X_0)} \exp\left( -\int_0^t \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_s) \, ds \right) f(X_t) \right), g \right)_{\phi^2 m} \\ &= E_m \left( e^{-A_t^{\mu,F}} \phi(X_t) \phi(X_0) f(X_0) g(X_t) \exp\left( -\int_0^t \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_s) \, ds \right) \right) \\ &= (p_t^{\phi}g, f)_{\phi^2 m}, \end{aligned}$$

which completes the proof.

Denote by  $(\mathcal{E}^{\phi}, \mathcal{F}^{\phi})$  the Dirichlet form on  $L^2(X; \phi^2 m)$  associated with  $\mathbf{M}^{\phi}$ . Note that  $K_n^o$  is also the fine interior of  $K_n$  with respect to  $\mathbf{M}^{\phi}$  because  $P_x = P_x^{\phi}$  on  $\mathcal{F}_{0+}^* (= \bigcap_{t>0} \mathcal{F}_t^*, \mathcal{F}_t^*$  is the universal completion of  $\mathcal{F}_t^0 = \sigma\{X_s; 0 \le s \le t\}$ ). Let  $\mathbf{M}^n$  (resp.  $\mathbf{M}^{\phi,n}$ ) be the part process of  $\mathbf{M}$  (resp.  $\mathbf{M}^{\phi}$ ) on  $K_n^o$  and  $(\mathcal{E}^n, \mathcal{F}^n)$  (resp.  $(\mathcal{E}^{\phi,n}, \mathcal{F}^{\phi,n}))$ ) the Dirichlet form generated by  $\mathbf{M}^n$  (resp.  $\mathbf{M}^{\phi,n}$ ).

D. Kim

**Lemma 3.3** ([19]). Under the identification of  $L^2(K_n^o; \phi^2 m)$  with  $L^2_{K^2}(X; \phi^2 m) = \{u \in L^2(X; \phi^2 m) ; u = 0 \text{ m-a.e. on } X \setminus K_n^o\},$ 

$$\mathcal{F}^{\phi,n} = \{ u \in \mathcal{F}^{\phi} ; u = 0 \ \phi^2 m \text{-a.e. on } X \setminus K_n \}$$

and  $\mathcal{E}^{\phi,n} = \mathcal{E}^{\phi}$ . In particular,  $\mathcal{F}^{\phi,n}$  is included in  $\mathcal{F}^{\phi}$ .

In order to reach our final goal of this section, we make full use of the following expression of transformed Dirichlet form. We would like to emphasize that  $\mathcal{F}^{\phi}$  the domain of the transformed Dirichlet form in Proposition 3.1 includes  $\mathcal{F}$ , which makes it possible to show that the identity function 1 belongs to  $\mathcal{F}^{\phi}$  even if the expression (3.4) itself follows from Itô formula.

**Proposition 3.1.** Fix a Borel function F of bounded below on  $X \times X \setminus d$ . Suppose that  $\overline{F} \in L^2(X \times X \setminus d; Nm)$ , where Nm(dxdy) = N(x, dy)m(dx) and  $A_t^{\mu, \overline{F}} \in \mathcal{A}_K^{\mathrm{ad}} - \mathcal{A}_K^{\mathrm{ad}}$ . Then the Dirichlet space  $\mathcal{F}^{\phi}$  includes  $\mathcal{F}$  and for  $u \in \mathcal{F}$ ,

(3.4) 
$$\begin{aligned} \mathcal{E}^{\phi}(u,u) &= \frac{1}{2} \int_{X} \Gamma^{c}(u,u) \phi^{2} dm \\ &+ \int \int_{X \times X \setminus d} (\tilde{u}(x) - \tilde{u}(y))^{2} \phi(x) \phi(y) e^{-F} N(x,dy) m(dx). \end{aligned}$$

Proof. First, for  $\phi = R^{\mu,F}_{\alpha}g \in \mathcal{D}^2_+(\mathcal{L}^{\mu,F})$  and  $u \in \mathcal{F}^n \cap L^{\infty}(X;m)$ , we put

$$(u - E.(N_t^{\phi}u(X_t) ; t < \tau_n), u)_{\phi^2 m}$$
  
=  $(u - E.(u(X_t) ; t < \tau_n), u)_{\phi^2 m} - (E.((N_t^{\phi} - 1)u(X_t) ; t < \tau_n), u)_{\phi^2 m}$   
=  $(I)_t - (II)_t.$ 

From (3.3),  $(II)_t$  equals

$$E_{u\phi m}\left(u(X_t)\int_0^t \exp\left(-\int_0^s \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u)du\right)dM_s^{\mu,F,\phi} \; ; \; t < \tau_n\right).$$

Put

$$(III)_t = E_{u\phi m} \left( u(X_t) \int_0^{t \wedge \tau_n} \exp\left(-\int_0^s \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u) du\right) dM_s^{\mu,F,\phi} \; ; \; t \ge \tau_n \right).$$

Then, by Lemma 3.1,

(3.5) 
$$\frac{1}{t^2} (III)_t^2 \leq \frac{1}{t} E_{|u|\phi m} \left( \int_0^{t \wedge \tau_n} \exp\left(-2A_s^{\mu,F} -2\int_0^s \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u) du \right) d \langle M^{[\phi]} - L^{\phi,\bar{F}} \rangle_s \right) \\ \times \frac{1}{t} E_{|u|\phi m} (u(X_t)^2 ; t \geq \tau_n)$$

For  $\phi, \varphi \in \mathcal{F}$ , we let

$$\Gamma(\phi, arphi)(x) = \Gamma^c(\phi, arphi)(x) + \int_X ( ilde{\phi}(x) - ilde{\phi}(y)) ( ilde{arphi}(x) - ilde{arphi}(y)) N(x, dy) + ilde{\phi}(x) ilde{arphi}(x) k(x).$$

Then the joint quadratic variation  $\langle M^{[\phi]}, M^{[\varphi]} \rangle_t$  is equal to

$$\langle M^{[\phi]}, M^{[\varphi]} \rangle_t = \int_0^t \Gamma(\phi, \varphi)(X_s) ds, \qquad \phi, \ \varphi \in \mathcal{F}$$

(Theorem 5.3.1 in [16]). Now we claim that the first factor of the right hand side of (3.5) is bounded. By Lemma 2.1 (i) and the fact that  $\left|\frac{\mathcal{L}^{\mu,F}\phi}{\phi}\right| \leq \exists c < \infty \text{ on } K_n^o$ ,

$$\begin{split} &\frac{1}{t}E_{|u|\phi m}\left(\int_{0}^{t\wedge\tau_{n}}\exp\left(-2A_{s}^{\mu,F}-2\int_{0}^{s}\frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_{u})du\right)d\langle M^{[\phi]}\rangle_{s}\right)\\ &\leq &\frac{1}{t}\exp(2ct)E_{|u|\phi m}\left(\int_{0}^{t}\exp(-2A_{s}^{\mu,F})\Gamma(\phi,\phi)(X_{s})ds\right)\\ &\leq &\frac{1}{t}\exp(2ct)\|u\phi\|_{\infty}\int_{0}^{t}\|p_{s}^{2\mu,2F}(\Gamma(\phi,\phi))\|_{1}ds\\ &\leq &\frac{1}{t}\exp(2ct)\|u\phi\|_{\infty}\int_{0}^{t}\|p_{s}^{2\mu,2F}\|_{1,1}\|\Gamma(\phi,\phi)\|_{1}ds\\ &\leq &c'\frac{1}{t}\exp(2ct)\|u\phi\|_{\infty}\|\Gamma(\phi,\phi)\|_{1}\int_{0}^{t}e^{\beta(2\mu,2F)s}ds<\infty. \end{split}$$

Also, since  $\langle L^{\phi,\bar{F}} \rangle_t = \int_0^t \int_X \phi^2(y) \bar{F}^2(X_s,y) N(X_s,dy) ds$ ,

$$\begin{split} &\frac{1}{t}E_{|u|\phi m}\left(\int_{0}^{t\wedge\tau_{n}}\exp\left(-2A_{s}^{\mu,F}-2\int_{0}^{s}\frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_{u})du\right)d\langle L^{\phi,\bar{F}}\rangle_{s}\right)\\ &\leq \quad \frac{1}{t}\exp(2ct)E_{|u|\phi m}\left(\int_{0}^{t}\exp\left(-2A_{s}^{\mu,F}\right)\int_{X}\phi^{2}(y)\bar{F}^{2}(X_{s},y)N(X_{s},dy)ds\right)\\ &\leq \quad \frac{1}{t}\exp(2ct)\|u\phi^{3}\|_{\infty}\int_{0}^{t}\|p_{t}^{2\mu,2F}(\int_{X}\bar{F}^{2}(\cdot,y)N(\cdot,dy))\|_{1}ds \end{split}$$

$$\leq \frac{1}{t} \exp(2ct) \| u\phi^3 \|_{\infty} \int_0^t \| p_t^{2\mu,2F} \|_{1,1} \| \int_X \bar{F}^2(\cdot,y) N(\cdot,dy) \|_1 ds$$
  
 
$$\leq c' \frac{1}{t} \exp(2ct) \| u\phi^3 \|_{\infty} \int_0^t e^{\beta(2\mu,2F)s} ds \iint_{X \times X \setminus d} \bar{F}^2(x,y) N(x,dy) m(dx) < \infty .$$

Now, Kunita-Watanabe inequality gives the boundedness of the first factor of (3.5). The second factor of (3.5) is equal to  $\frac{1}{t}(|u|\phi, p_tu^2 - u^2)_m - \frac{1}{t}(|u|\phi, p_t^n u^2 - u^2)_m$ , which tends to  $\mathcal{E}(|u|\phi, u^2) - \mathcal{E}^n(|u|\phi, u^2) = 0$  as  $t \to 0$  because  $|u|\phi$  and  $u^2$  are elements of  $\mathcal{F}^n$  and we can appeal to Theorem 4.4.2 in [16]. Hence we get

$$\lim_{t \to 0} \frac{1}{t} (II)_t$$

$$= \lim_{t \to 0} \frac{1}{t} E_{u\phi m} \left( u(X_t) \int_0^{t \wedge \tau_n} \exp\left(-\int_0^s \frac{\mathcal{L}^{\mu,F} \phi}{\phi}(X_u) du\right) dM_s^{\mu,F,\phi}\right)$$

$$(3.6)$$

$$= \lim_{t \to 0} \frac{1}{t} E_{u\phi m} \left( (u(X_t) - u(X_0)) \times \int_0^{t \wedge \tau_n} \exp\left(-\int_0^s \frac{\mathcal{L}^{\mu,F} \phi}{\phi}(X_u) du\right) dM_s^{\mu,F,\phi}\right).$$

Now, let  $u(X_t) - u(X_0) = M_t^{[u]} + N_t^{[u]}$  be the Fukushima decomposition (cf. Theorem 5.2.2 in [16]). Then,

$$\lim_{t \to 0} \frac{1}{t} E_{u\phi m} \left( N_t^{[u]} \int_0^{t \wedge \tau_n} \exp\left( -\int_0^s \frac{\mathcal{L}^{\mu,F} \phi}{\phi}(X_u) du \right) dM_s^{\mu,F,\phi} \right) = 0$$

because  $N^{[u]}$  is of zero-energy (p.201 in [16]), and thus the right hand side of (3.6) is equal to

(3.7) 
$$\lim_{t \to 0} \frac{1}{t} E_{u\phi m} \left( \int_0^{t \wedge \tau_n} \exp\left( -A_s^{\mu, F} - \int_0^s \frac{\mathcal{L}^{\mu, F} \phi}{\phi}(X_u) du \right) d \langle M^{[u]}, M^{[\phi]} - L^{\phi, \bar{F}} \rangle_s \right).$$

Moreover, since

$$\left|\frac{\exp\left(-\int_0^s \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u)du\right)-1}{s}\right| \leq \exists M < \infty, \quad s \leq t \wedge \tau_n.$$

**GENERALIZED FEYNMAN-KAC FUNCTIONALS** 

$$\begin{split} &\frac{1}{t^2} E_{u\phi m} \left( \int_0^{t\wedge\tau_n} e^{-A_s^{\mu,F}} \left( e^{-\int_0^s \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u)du} - 1 \right) d\left\langle M^{[u]}, L^{\phi,\bar{F}} \right\rangle_s \right)^2 \\ &\leq \frac{1}{t^2} \|u\phi\|_\infty^2 E_m \left( \int_0^t e^{-A_s^{\mu,F}} d\left\langle L^{\phi,\bar{F}} \right\rangle_s \right) E_m \left( \int_0^{t\wedge\tau_n} \left( e^{-\int_0^s \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u)du} - 1 \right) d\left\langle M^{[u]} \right\rangle_s \right) \\ &\leq M^2 \|u\phi\|_\infty^2 E_m \left( \int_0^t e^{-A_s^{\mu,F}} d\left\langle L^{\phi,\bar{F}} \right\rangle_s \right) E_m \left( \int_0^t \Gamma(u,u)(X_s)ds \right) \\ &\leq M^2 \|u\phi^3\|_\infty^2 \int_0^t \|p_s^{\mu,F} \left( \int_X \bar{F}^2(\cdot,y)N(\cdot,dy) \right) \|_1 ds \int_0^t \|p_s\left(\Gamma(u,u)\right)\|_1 ds \\ &\leq M' \|u\phi^3\|_\infty^2 \int_0^t e^{\beta s} ds \int \int_{X\times X\setminus d} \bar{F}^2(x,y)N(x,dy)m(dx) \int_0^t \|p_s\left(\Gamma(u,u)\right)\|_1 ds \\ &\to 0 \quad \text{as} \quad t \to 0. \end{split}$$

We get

$$\begin{split} &\lim_{t\to 0} \frac{1}{t} E_{u\phi m} \left( \int_0^{t\wedge\tau_n} \exp\left( -A_s^{\mu,F} - \int_0^s \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u) du \right) d\left\langle M^{[u]}, L^{\phi,\bar{F}} \right\rangle_s \right) \\ &= \lim_{t\to 0} \frac{1}{t} E_{u\phi m} \left( \int_0^{t\wedge\tau_n} \exp\left( -A_s^{\mu,F} \right) d\left\langle M^{[u]}, L^{\phi,\bar{F}} \right\rangle_s \right), \end{split}$$

and also similarly,

$$\begin{split} &\lim_{t\to 0} \frac{1}{t} E_{u\phi m} \left( \int_0^{t\wedge\tau_n} \exp\left( -A_s^{\mu,F} - \int_0^s \frac{\mathcal{L}^{\mu,F}\phi}{\phi}(X_u) du \right) d\left\langle M^{[u]}, M^{[\phi]} \right\rangle_s \right) \\ &= \lim_{t\to 0} \frac{1}{t} E_{u\phi m} \left( \int_0^{t\wedge\tau_n} \exp\left( -A_s^{\mu,F} \right) d\left\langle M^{[u]}, M^{[\phi]} \right\rangle_s \right). \end{split}$$

Therefore, (3.7) is equal to

$$\lim_{t\to 0} \frac{1}{t} E_{u\phi m} \left( \int_0^{t\wedge\tau_n} \exp\left(-A_s^{\mu,F}\right) \left( d\left\langle M^{[u]}, M^{[\phi]} \right\rangle_s - d\left\langle M^{[u]}, L^{\phi,\bar{F}} \right\rangle_s \right) \right).$$

Now, since  $\lim_{t\to 0} P_{|u|\phi m}(t \ge \tau_n) = 0$ ,

$$\begin{split} &\frac{1}{t^2} E_{|u|\phi m} \left( \int_0^t \exp(-A_s^{\mu,F}) d\left\langle M^{[u]}, L^{\phi,\bar{F}} \right\rangle_s; t \ge \tau_n \right)^2 \\ &\le \quad \frac{1}{t^2} P_{|u|\phi m} (t \ge \tau_n) E_{|u|\phi m} \left( \left( \int_0^t \exp(-A_s^{\mu,F}) d\left\langle M^{[u]}, L^{\phi,\bar{F}} \right\rangle_s \right)^2 \right) \\ &\le \quad \frac{1}{t^2} P_{|u|\phi m} (t \ge \tau_n) \|u\phi\|_{\infty}^2 E_m \left( \int_0^t \Gamma(u,u)(X_s) ds \right) E_m \left( \int_0^t \exp(-2A_s^{\mu,F}) d\left\langle L^{\phi,\bar{F}} \right\rangle_s \right) \end{split}$$

$$\leq \frac{1}{t^2} P_{|u|\phi m}(t \ge \tau_n) \|u\phi^3\|_{\infty}^2 \int_0^t \|p_s^{2\mu,2F}(\int_X \bar{F}^2(\cdot, y) N(\cdot, dy))\|_1 ds \\ \times \int_0^t \|p_s(\Gamma(u, u))\|_1 ds \\ \longrightarrow 0 \quad \text{as} \quad t \to 0,$$

and also

$$\lim_{t\to 0} \frac{1}{t} E_{u\phi m} \left( \int_0^{t\wedge \tau_n} \exp\left(-A_s^{\mu,F}\right) d \langle M^{[u]}, M^{[\phi]} \rangle_s; t \ge \tau_n \right) = 0.$$

Therefore, we conclude that

$$\begin{split} \lim_{t \to 0} \frac{1}{t} (II)_t \\ &= \lim_{t \to 0} \frac{1}{t} E_{u\phi m} \left( \int_0^t \exp(-A_s^{\mu,F}) \Gamma(u,\phi)(X_s) ds \right) \\ &- \lim_{t \to 0} \frac{1}{t} E_{u\phi m} \left( \int_0^t \exp(-A_s^{\mu,F}) \int_X \phi(y) \bar{u} \bar{F}(X_s,y) N(X_s,dy) ds \right) \\ &= \lim_{t \to 0} \frac{1}{t} (u\phi, \int_0^t p_s^{\mu,F} (\Gamma(u,\phi)) ds)_m \\ &- \lim_{t \to 0} \frac{1}{t} (u\phi, \int_0^t p_s^{\mu,F} (\int_X \phi(y) \bar{u} \bar{F}(\cdot,y) N(\cdot,dy)) ds)_m \\ &= (u\phi, \Gamma(u,\phi))_m - (u\phi, \int_X \phi(y) \bar{u} \bar{F}(\cdot,y) N(\cdot,dy))_m \\ &= \int_X u\phi \Gamma(u,\phi) dm - \iint_{X \times X \setminus d} u(x) \phi(x) \phi(y) \bar{u} \bar{F}(x,y) N(x,dy) m(dx), \end{split}$$

where  $\bar{u}(x,y) = u(y) - u(x)$ . On the other hand,  $\lim_{t\to 0} \frac{1}{t}(I)_t = \mathcal{E}(u, u\phi^2)$ . Hence,

$$\begin{split} &\lim_{t\to 0} \frac{1}{t} (u - E_{\cdot}(N_t^{\phi}u(X_t); t < \tau_n), u)_{\phi^2 m} \\ &= \frac{1}{2} \int_X \Gamma(u, u\phi^2) dm + \frac{1}{2} \int_X u^2 \phi^2 k dm - \int_X u\phi \Gamma(u, \phi) dm \\ &\quad + \iint_{X \times X \setminus d} u(x) \phi(x) \phi(y) \bar{u} \bar{F}(x, y) N(x, dy) m(dx) \\ &= \frac{1}{2} \int_X \Gamma^c(u, u) \phi^2 dm \\ &\quad + \iint_{X \times X \setminus d} (\tilde{u}(x) - \tilde{u}(y))^2 \phi(x) \phi(y) e^{-F(x, y)} N(x, dy) m(dx), \end{split}$$

and consequently,  $u \in \mathcal{F}^{\phi,n}$ , which implies that  $\mathcal{F}^n \cap L^{\infty}(X;m) \subset \mathcal{F}^{\phi}$  by virtue of Lemma 3.3. Noting that

$$\mathcal{E}^{\phi}(u,u) \leq \|\phi\|_{\infty}^{2} \mathcal{E}(u,u) \qquad u \in \mathcal{F}^{n} \cap L^{\infty}(X;m).$$

Since  $\cup_n (\mathcal{F}^n \cap L^\infty(X; m))$  is dense in  $\mathcal{F}$ , we arrive at the lemma.

**Proposition 3.2.** For  $\phi \in \mathcal{D}^2_+(\mathcal{L}^{\mu,F})$ , the identity function 1 belongs to  $\mathcal{F}^{\phi}$  and  $\mathcal{E}^{\phi}(1,1) = 0$ .

Proof. This proposition follows from Proposition 3.1 with the same argument as in Lemma 6.3.3 in [16].  $\hfill \Box$ 

**Lemma 3.4** ([14],[19],[22]). Let  $(\mathcal{E}, \mathcal{F})$  be the Dirichlet form associated with an *m*-symmetric right process  $(\Omega, X_t, P_x)$  on X. Suppose that

$$1 \in \mathcal{F}$$
 and  $\mathcal{E}(1,1) = 0$ .

Then, the following statements are equivalent each other;

(i)  $(\mathcal{E}, \mathcal{F})$  is irreducible.

(ii) If  $\mathcal{E}(u, u) = 0$ , then u is constant m-a.e.

(iii) If  $T_t u = u$  m-a.e. for all t > 0, then u is constant m-a.e.

(iv)  $(\Omega, P_m, \mathcal{F}^0, \theta_t)$  is ergodic, (i.e., if  $\Lambda \in \mathcal{F}^0$  is  $\theta_t$ -invariant,  $(\theta_t)^{-1}(\Lambda) = \Lambda$ , then  $P_x(\Lambda) = 0$  for all  $x \in X$  or  $P_x(\Lambda) = 1$  for all  $x \in X$ . Here  $\mathcal{F}^0 = \sigma\{X_t : 0 \le t < \infty\}$  and  $\theta_t$  is the shift operator on  $\Omega$ ).

**Theorem 3.1.** The transformed process  $\mathbf{M}^{\phi}$  is ergodic in the sense of Lemma 3.4.

Proof. On account of the positivity of  $N^{\phi}$  up to the life time  $\zeta$ ,  $(\mathcal{E}^{\phi}, \mathcal{F}^{\phi})$  is irreducible. Hence, it follows from Lemma 3.4 and Proposition 3.2 that  $P^{\phi}_{\phi^2 m}(\Lambda) = 0$  or  $P^{\phi}_{\phi^2 m}(\Omega \setminus \Lambda) = 0$ . Moreover, by assumption (III),  $\mathbf{M}^{\phi}$  also admits a transition density. Hence  $P^{\phi}_{x}(\Lambda) = 0$  or 1.

Assume that  $\mu$  and F are positive. Then by combining Theorem 3.2 in [31] for the multiplicative functional  $\exp\left(-A_t^{\mu,F}\right)$  with Theorem 6.3.1 in [16], we can immediately check that Theorem 3.1 is also derived without the assumption (I).

#### 4. Lower and Upper Estimations

In this section, we now describe the asymptotic behaviour of  $E_x\left(\exp\left(-A_t^{\mu,F}\right)\right)$  by applying the results of previous sections. We first give the lower estimate of its large

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deviation principle governed by the perturbed Dirichlet form  $\mathcal{E}^{\mu,F}$ , and next consider the upper estimate for the symmetric Lévy process with its exponent (4.1) below.

## The lower estimate

Let  $\mathcal{M}_1(X)$  be the set of probability measures on X equipped with the weak topology. Define the function  $I_{\mathcal{E}^{\mu,F}}$  on  $\mathcal{M}_1(X)$  by

$$I_{\mathcal{E}^{\mu,F}}(\nu) = \begin{cases} \mathcal{E}^{\mu,F}(\sqrt{f},\sqrt{f}) & \text{if } \nu = f \cdot m, \ \sqrt{f} \in \mathcal{F}^{\mu,F} \\ \infty & \text{otherwise.} \end{cases}$$

For  $\omega \in \Omega$  and  $0 < t < \zeta(\omega)$ , also define the occupation distribution

$$L_t(\omega)(A) = rac{1}{t} \int_0^t \chi_A(X_s(\omega)) ds, \quad A \in \mathcal{B}(X).$$

**Theorem 4.1.** Fix a Borel function F of bounded below on  $X \times X \setminus d$ . Suppose that  $\overline{F} \in L^2(X \times X; Nm)$  and  $A^{\mu, \overline{F}} \in \mathcal{A}_K^{\mathrm{ad}} - \mathcal{A}_K^{\mathrm{ad}}$ . Then for any open set  $G \subset \mathcal{M}_1(X)$ ,

$$\liminf_{t \to \infty} \frac{1}{t} \log E_x \left( \exp\left( -A_t^{\mu, F} \right) \; ; \; L_t \in G, t < \zeta \right) \ge -\inf_{\nu \in G} I_{\mathcal{E}^{\mu, F}}(\nu) \quad \text{for all } x \in X.$$

Proof. Using Theorem 3.1, we have the theorem with exactly the same argument as in [11],[27]. Take  $\phi = R^{\mu,F}_{\alpha}f \in \mathcal{D}^2_+(\mathcal{L}^{\mu,F})$  and  $\phi^2m \in G$ . For the multiplicative functional  $N^{\phi}$  defined by (3.2) and  $x \in X$ , we have

$$\begin{split} &E_x\left(\exp\left(-A_t^{\mu,F}\right)\;;\; L_t\in G,t<\zeta\right)\\ &= &E_x^{\phi}\left(\left(N_t^{\phi}\right)^{-1}\exp\left(-A_t^{\mu,F}\right)\;;\; L_t\in G\right)\\ &\geq &\exp\left(t\left(\int_X\phi\mathcal{L}^{\mu,F}\phi dm-\varepsilon\right)\right)E_x^{\phi}\left(\frac{\phi(X_0)}{\phi(X_t)}\;;\; S(t,\varepsilon)\right)\\ &\geq &\exp\left(t\left(\int_X\phi\mathcal{L}^{\mu,F}\phi dm-\varepsilon\right)\right)\frac{\phi(x)}{\|\phi\|_{\infty}}\left(1-P_x^{\phi}(\Omega\setminus S(t,\varepsilon))\right), \end{split}$$

where

$$S(t,\varepsilon) = \left\{ \omega \in \Omega \; ; \; \left| \int_X \frac{\mathcal{L}^{\mu,F} \phi}{\phi}(x) L_t(\omega, dx) - \int_X \phi \mathcal{L}^{\mu,F} \phi dm \right| < \varepsilon \right\} \cap \{ \omega \in \Omega \; ; \; L_t(\omega) \in G \}.$$

$$\begin{split} \Omega_1 &= \left\{ \omega \in \Omega \; ; \; \lim_{t \to \infty} \frac{1}{t} \int_0^t \frac{\mathcal{L}^{\mu,F} \phi}{\phi}(X_s(\omega)) ds = \int_X \phi \mathcal{L}^{\mu,F} \phi \, dm \right\}, \\ \Omega_2 &= \left\{ \omega \in \Omega \; ; \; L_t(\omega) \; \text{converges to} \; \phi^2 m \; \text{as} \; t \to \infty \right\}. \end{split}$$

Now, on account of the shift invariance of  $\Omega_i$  (i = 1, 2) due to Theorem 3.1, we know  $P_x^{\phi}(\Omega_i) = 1, \ \phi^2 m$ -a.e. Hence,

$$P_x^{\phi}(\Omega \setminus S(t,\varepsilon)) \longrightarrow 0 \quad \text{as} \quad t \to \infty \quad \text{for } \forall x \in X.$$

Consequently, we have

$$\liminf_{t\to\infty} \frac{1}{t} \log E_x \left( \exp\left(-A_t^{\mu,F}\right) \; ; \; L_t \in G, t < \zeta \right) \ge \int_X \phi \mathcal{L}^{\mu,F} \phi dm - \varepsilon.$$

Note that the set  $\{\phi \in \mathcal{D}^2_+(\mathcal{L}^{\mu,F}); \|\phi\|_2 = 1\}$  is dense in  $\{\phi \in \mathcal{F}^{\mu,F}; \phi \ge 0, \|\phi\|_2 = 1\}$  with respect to  $\mathcal{E}^{\mu,F}_{\alpha_0}$  ( $\alpha_0 > \beta(\mu,F)$ ) which completes the proof.

We can not treat the upper estimate of  $E_x \left(\exp\left(-A^{\mu,F}\right)\right)$  with large deviation theory without additional conditions. By the same argument of Proposition 4.2 in [27], we have the followings.

REMARK 4.1. Let us assume the hypotheses in Theorem 4.1 are satisfied. In addition, let assume that the symmetric Markov process  $\mathbf{M}$  as a base process is conservative and the transition probability  $p_t$  of  $\mathbf{M}$  satisfies the strong Feller property. Then for any compact subset K of  $\mathcal{M}_1(X)$ ,

$$\limsup_{t \to \infty} \frac{1}{t} \log E_x \left( \exp\left( -A_t^{\mu, F} \right) \; ; \; L_t \in K \right) \le -\inf_{\nu \in K} I_{\mathcal{E}^{\mu, F}}(\nu) \quad for \; all \; \; x \in X.$$

Moreover, if the base process explodes so fast in the sense that  $R_1^{\mu,F}$  belongs to  $C_{\infty}(X)$ , it can be extended for any closed subset of  $\mathcal{M}_1(X)$ .

We shall consider the upper estimate of large deviation principle without above two conditions. To do this, we apply the fact that the  $L^p$ -spectral radius of the Feynman-Kac semigroup of kernels is *p*-independent under the symmetric Lévy process whose Lévy exponent is the so-called  $\alpha$ -relativistic Hamiltonian (4.1) below.

# The upper estimate

Symmetric convolution semigroups  $\{\nu_t, t \ge 0\}$  of infinitely divisible probability measures on  $\mathbb{R}^d$  define a Markovian semigroup  $p_t$  by

$$p_t f(x) = \int_{R^d} f(x+y) \nu_t(dy) \qquad f \in \mathcal{B}_b(R^d),$$

which is symmetric with respect to the Lebesgue measure, and the Lévy-Khinchin formula under the above conditions for  $\nu_t$  leads as follows ;

$$\int_{R^d} e^{i(x,y)} \nu_t(dy) = \exp(-t\psi(x))$$
  
$$\psi(x) = \frac{1}{2}(Sx,x) + \int_{R^d} (1 - \cos(x,y)) J(dy),$$

where S is a non-negative definite  $d \times d$  symmetric matrix and J is a symmetric measure on  $\mathbb{R}^d \setminus \{0\}$  such that  $\int_{\mathbb{R}^d \setminus \{0\}} \min(1, |x|^2) J(dx) < \infty$ . Here, the function  $\psi$  is called the Lévy exponent and the measure J is called the Lévy measure. The Dirichlet form  $(\mathcal{E}, \mathcal{F})$  associated with the semigroup  $p_t$  is written as

$$\begin{aligned} \mathcal{E}(u,u) &= \frac{1}{2} \int_{R^d} (S\nabla u, \nabla u)(x) \, dx + \frac{1}{2} \iint_{R^d \times R^d \setminus d} (u(x) - u(y))^2 J(dy - x) \, dx, \\ \mathcal{D}(\mathcal{E}) &= \left\{ u \in L^2(R^d) \; ; \; \mathcal{E}(u,u) < \infty \right\}. \end{aligned}$$

The Hunt process associated with  $(\mathcal{E}, \mathcal{F})$  is called the Lévy process and denote it by  $\mathbf{M} = (\Omega, X_t, P_x)$ . In this situation, we add the following assumption ;

(V) 
$$\int_{R^d} e^{-t\psi(x)} dx < \infty, \qquad \forall t > 0.$$

Then, under this assumption, the function

$$p_t(x)=rac{1}{(2\pi)^d}\int_{R^d}e^{iy\cdot x}e^{-t\psi(y)}dy(\in C_\infty(R^d)),\qquad x\in R^d$$

is the density of the measure  $\nu_t$  for each t > 0 and is an analytic function of t on  $(0, \infty)$  ([7]). Furthermore, the transition density of  $\mathbf{M}$ ,  $p_t(x - y)$  satisfies all assumptions in section 2. Indeed, the irreducibility of  $\mathbf{M}$  is derived by noting the periodicity of  $\psi(x)$  on  $\mathbb{R}^d$ . Therefore, we can apply the lower estimate of large deviation principle for the Lévy process  $\mathbf{M}$ .

DEFINITION 4.1. A Lévy measure J is said to be exponentially localized if there exists a constant  $\delta > 0$  such that

$$\int_{|x|>1} e^{\delta|x|} J(dx) < \infty.$$

**Lemma 4.1** ([7]). Let the Lévy measure J of M be exponentially localized. Then there exist positive constants  $c_1, c_2$  such that

$$E_0\left(e^{\delta\sup_{0\leq s\leq t}|X_s|}\right)\leq c_1e^{c_2t}$$

We now concentrate on the symmetric Lévy process  $\mathbf{M} = (\Omega, X_t, P_x)$  on  $\mathbb{R}^d$  with the so-called  $\alpha$ -relativistic Hamiltonian as Lévy exponent

(4.1) 
$$(p^2 + m^2)^{\frac{\alpha}{2}} - m^{\alpha}, \quad m > 0, \ 0 < \alpha \le 2$$

which was recently investigated by Carmona, Masters, and Simon ([7]). Note that the Lévy measure corresponding to the Lévy exponent (4.1) is exponentially localized because (4.1) has an analytic continuation to some strip (see Proposition II.1 in [7]). With the Lévy process  $\mathbf{M}$  as the base process, we then have the following upper estimate.

**Theorem 4.2.** For any  $A^{\mu,\bar{F}} \in \mathcal{A}_K^{\mathrm{ad}} - \mathcal{A}_K^{\mathrm{ad}}$  and  $x \in \mathbb{R}^d$ ,

$$\limsup_{t \to \infty} \frac{1}{t} \log E_x \left( \exp\left( -A_t^{\mu,F} \right) \right) \le - \inf_{\substack{u \in \mathcal{F}^{\mu,F} \\ \|u\|_2 = 1}} \mathcal{E}^{\mu,F}(u,u).$$

Proof. Let

$$\lambda = -\log \left\|p_1^{\mu,F}
ight\|_{2,2} = \inf_{\substack{u\in\mathcal{F}^{\mu,F}\ \|u\|_2=1}}\mathcal{E}^{\mu,F}(u,u).$$

We also put

$$E_x \left( \exp\left(-A_t^{\mu,F}\right) \right)$$
  
=  $E_x \left( \exp\left(-A_t^{\mu,F}\right) \chi_{B_r(x)}(X_t) \right) + E_x \left( \exp\left(-A_t^{\mu,F}\right) \left(1 - \chi_{B_r(x)}\right) (X_t) \right)$   
=  $(I)_t + (II)_t,$ 

where  $B_r(x)$  is the ball centered at x with radius r. Then, from Lemma 4.1,

$$(II)_t \leq E_x \left( \exp\left(-2A_t^{\mu,F}\right) \left(1-\chi_{B_r(x)}\right) (X_t) \right)^{1/2} E_x \left( \left(1-\chi_{B_r(x)}\right) (X_t) \right)^{1/2}$$

$$\leq \left\| p_t^{2\mu,2F} \right\|_{\infty,\infty}^{1/2} P_0 \left( \sup_{0 \le s \le t} |X_s| \ge r \right)^{1/2}$$
  
 
$$\leq C \exp\left( \beta(2\mu,2F)t - \frac{\delta}{2}r + \frac{c_2}{2}t \right).$$

On the other hand,

$$(I)_{t} = p_{t}^{\mu,F} \chi_{B_{r}(x)}(x)$$
  

$$= p_{1}^{\mu,F} \cdot p_{t-1}^{\mu,F} \chi_{B_{r}(x)}(x)$$
  

$$\leq \left\| p_{1}^{\mu,F} \right\|_{2,\infty} \left\| p_{t-1}^{\mu,F} \chi_{B_{r}(x)} \right\|_{2}$$
  

$$\leq \left\| p_{1}^{\mu,F} \right\|_{2,\infty} e^{-\lambda(t-1)} \left\| \chi_{B_{r}(x)} \right\|_{2}$$
  

$$\leq C' r^{d/2} e^{-\lambda t},$$

where  $C' = e^{\lambda} \left\| p_1^{\mu,F} \right\|_{2,\infty}$ . Now, take k > 0 so that  $-\beta(2\mu, 2F) + \frac{\delta}{2}k - \frac{c_2}{2} > \lambda$  and put r = kt. We then have

$$E_x\left(\exp\left(-A_t^{\mu,F}\right)\right) \le K(1+t^{d/2})e^{-\lambda t},$$

for some positive constant K. The proof is complete.

Consequently, when the underlying process is the symmetric Lévy process with its Lévy exponent (4.1), Theorem 4.1 and Theorem 4.2 lead us to the following theorem.

**Theorem 4.3.** Fix a Borel function F of bounded below on  $\mathbb{R}^d \times \mathbb{R}^d \setminus d$ . Suppose that  $\overline{F} \in L^2(\mathbb{R}^d \times \mathbb{R}^d; Nm)$  and  $A^{\mu, \overline{F}} \in \mathcal{A}_K^{\mathrm{ad}} - \mathcal{A}_K^{\mathrm{ad}}$ . Then

(4.2) 
$$\lim_{t \to \infty} \frac{1}{t} \log E_x \left( \exp\left(-A_t^{\mu,F}\right) \right) = -\inf_{\substack{u \in \mathcal{F}^{\mu,F} \\ \|u\|_2 = 1}} \mathcal{E}^{\mu,F}(u,u) \quad for \ \forall x \in \mathbb{R}^d.$$

# 5. Applications

Let  $\mathbf{M}' = (\Omega, X_t, P_x)$  be a symmetric stable process of index  $\alpha, (1 < \alpha \leq 2)$  on  $\mathbb{R}^d$  with its exponent  $|z|^{\alpha}$ . It is known in [24] that the definition of Kato class measure  $\mu$ , (2.1) is equivalent to the classical definition of the Kato class

$$\lim_{\delta \to 0} \sup_{x \in R^d} \int_{|x-y| < \delta} \frac{1}{|x-y|^{d-\alpha}} |\mu|(dy) = 0, \qquad |\mu| = \mu^+ + \mu^-$$

166

under M'. So, we see from [6] that the surface measure  $\sigma$  of the sphere on  $\mathbb{R}^3$  is in the (extended) Kato class. We shall first apply the formula (4.2) to the surface measure  $\sigma$  and give the asymptotics of local time for  $\sigma$ . The problem is arised for the fact that at least, we cannot say anything about the formula (4.2) under M' because its Lévy measure is not exponentially localized. To show that the surface measure  $\sigma$  belongs to the Kato class corresponding to the  $\alpha$ -relativistic Hamiltonian operator, we need to apply the following fact essentially due to [7],[32]; the Kato class measure is same whether we deal with the pseudo-differential operator  $|-\Delta|^{\alpha/2}(1 < \alpha \leq 2)$  or the  $\alpha$ -relativistic Hamiltonian operator  $(-\Delta + m^2)^{\alpha/2} - m^{\alpha}(m > 0, 1 < \alpha \leq 2)$ .

Let us denote  $g_{\alpha}$  by the Green's density of M'. A measurable function f on  $\mathbb{R}^d$  is said to be in  $l^1(L^{\infty})$  if  $||f||_{l^1(L^{\infty})} = \sum_{k \in \mathbb{Z}^d} \sup_{x \in C(k)} |f(x)| < \infty$ , where C(k) denotes the cube centered at  $k \in \mathbb{Z}^d$  with sides of length 1. Note that the transition density  $p_t(\cdot)$  of M' belongs to  $l^1(L^{\infty})$  and for each fixed  $\delta > 0$ ,

(5.1) 
$$\sup_{t} \left\| \chi_{\{|y| > \delta\}} p_t u \right\|_{l^1(L^\infty)} < \infty$$

([29]). Note that most of the transition densities of the Lévy processes we are interested indeed, satisfy the above properties ([7]).

**Theorem 5.1.** Let  $\mu$  be a smooth measure such that  $\sup_{x \in \mathbb{R}^d} \mu(x + C(0)) < \infty$ . Suppose that the assumption (V) and the condition (5.1) are satisfied by the Lévy process. If

$$\lim_{\delta \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| < \delta} g_\alpha(x-y) \mu(dy) = 0,$$

then  $\mu$  belongs to the (extended) Kato class.

Proof. For fixed  $\delta > 0$ , arbitrary  $\varepsilon > 0$  and sufficiently large enough  $\beta > 0$ , we have from (5.1)

$$\begin{split} \sup_{x \in R^d} \int_{|x-y| \ge \delta} g_{\beta}(x-y)\mu(dy) \\ &= \sup_{x \in R^d} \int_{R^d} \chi_{\{|u| \ge \delta\}} p_s u \, \mu(x-du) \int_0^\infty e^{-\beta s} ds \\ &= \sup_{x \in R^d} \sum_{k \in Z^d} \int_{C(k)} \chi_{\{|u| \ge \delta\}} p_s u \, \mu(x-du) \int_0^\infty e^{-\beta s} ds \\ &\le \sup_{x \in R^d} \mu(x+C(0)) \sup_{s>0} \|\chi_{\{|u| \ge \delta\}} p_s u\|_{l^1(L^\infty)} \int_0^\infty e^{-\beta s} ds \\ &< \frac{\varepsilon}{4}. \end{split}$$

On the other hand, for such  $\beta > 0$ ,

$$\sup_{x\in R^d}\int_{|x-y|<\delta}g_{\beta}(x-y)\mu(dy)\leq \sup_{x\in R^d}\int_{|x-y|<\delta}g_{\alpha}(x-y)\mu(dy)\ (<\frac{\varepsilon}{4}).$$

Hence, we can choose  $\beta > 0$  large enough so that

$$\sup_{x \in R^{d}} E_{x} \left(A_{t}^{\mu}\right) \leq \sup_{x \in R^{d}} E_{x} \left(e^{\beta t} \int_{0}^{\infty} e^{-\beta s} dA_{s}^{\mu}\right)$$
$$= \sup_{x \in R^{d}} e^{\beta t} R_{\beta} \mu(x)$$
$$= \sup_{x \in R^{d}} e^{\beta t} \int_{R^{d}} g_{\beta}(x-y) \mu(dy) < e^{\beta t} \frac{\varepsilon}{2}$$

Now, to reach the conclusion, we only choose sufficiently small t so that  $e^{\beta t} < 2$ . The proof is complete.

In fact, the above theorem is held equivalently because for sufficiently large enough  $\beta > 0$  and some constant c > 0,

$$\int_{R^d} g_\beta(x-y)\mu(dy) \le \|p_s\|_{l^1(L^\infty)} \sup_{x \in R^d} \mu(x+C(0)) \left\{ \int_0^\infty e^{-\beta s} ds \right\} < \frac{\varepsilon}{c}$$

and there exists constant  $\delta > 0$  such that if  $|x - y| \le \delta$  then  $g_{\alpha}(x - y) \le cg_{\beta}(x - y)$ (Lemma III.3 in [7]).

EXAMPLE 5.1. Note that the Theorem 5.1 implies that the definition of the Kato class measure depends only on the behaviour of the exponent function  $\psi(u)$  when  $u \to \infty$  because the Green's density  $g_{\alpha}(x)$  is equal to

$$\frac{1}{(2\pi)^d} \int_{R^d} \frac{1}{\alpha + \psi(u)} e^{-iux} du = K_R + \frac{1}{(2\pi)^d} \int_{|u| \ge R} \frac{1}{\alpha + \psi(u)} e^{-iux} du.$$

So the Kato class is the same whether we deal with  $\psi(u) = |u|^{\alpha}$ ,  $(1 < \alpha \le 2)$  or  $\psi(u) = (u^2 + m^2)^{\alpha/2} - m^{\alpha}$ ,  $(m > 0, 1 < \alpha \le 2)$ . This means that the surface measure  $\mu = \sigma(|x| = R)$  of the sphere on  $R^3$  also belongs to the (extended) Kato class when the base process is the relativistic Hamiltonian of index  $\alpha$ ,  $(1 < \alpha \le 2)$ . Let us denote by  $\ell_R(t)$  the local time corresponding to  $\mu$ . We have

$$\lim_{t \to \infty} \frac{1}{t} \log E_x \left( \exp(-\ell_R(t)) \right) = - \inf_{\substack{u \in \mathcal{F}^{\mu} \\ \|u\|_2 = 1}} \left( \int_{R^3} \psi(x) |\hat{u}(x)|^2 dx + \int_{|x| = R} \tilde{u}(x)^2 d\mu(x) \right)$$

where  $\mathcal{F}^{\mu} = \{ u \in \mathcal{F} : \tilde{u} \in L^2(\mathbb{R}^3; |\mu|) \}, \ \psi(x) = (x^2 + m^2)^{\alpha/2} - m^{\alpha}(m > 0, \ 1 < \alpha \leq 2), \text{ and } \hat{\cdot} \text{ means its Fourier transformation.}$ 

EXAMPLE 5.2. Let  $B_R (= \{x; |x| < R\})$  be the open ball in  $R^3$  and  $\sigma$  the surface measure of the sphere  $\partial B_r, (r < R)$ . Let  $X_t$  be an absorbing symmetric stable process on  $B_R$  with index  $\alpha, (1 < \alpha \leq 2)$ . The surface measure  $\sigma$  is in the (extended) Kato class under the process  $X_t$ , and is the killing measure on  $B_R$ . Let us denote by  $\ell_r(t)$  the positive continuous additive functional corresponding to  $\sigma$ . Since the ball  $B_R$  is regular,  $\lim_{x\to\partial B_R} E_x (e^{-\tau_R}) = 1$ . That is,  $R_1^R 1 \in C_\infty(B_R)$  and which implies  $R_1^{R,r} 1 \in C_\infty(B_R)$ . In this case, we can derive the formula (4.2) by using the full large deviation principle. Therefore,

$$\lim_{t \to \infty} \frac{1}{t} \log E_x^R \left( \exp(-\ell_r(t)) \; ; \; t < \zeta \right) \\ = - \inf_{\substack{u \in \mathcal{F}^\sigma \\ ||u||_2 = 1}} \left( \int_{R^3} \psi(x) |\hat{u}(x)|^2 dx + \int_{|x|=r} \tilde{u}(x)^2 d\sigma(x) \right),$$

where  $\mathcal{F}^{\sigma} = \{ u \in \mathcal{F} ; \ \tilde{u} \in L^2(\mathbb{R}^3; |\sigma|) \}, \ \psi(x) = |x|^{\alpha}, (1 < \alpha \leq 2).$ 

Now, in the rest of this section, we shall consider the formula (4.2) with symmetric Markov chains. Let I be a countable set equipped with the discrete topology. Let  $Q = (q_{ij})$  be an  $I \times I$  matrix such that

$$q_{ij} \ge 0 \ (i \ne j), \qquad \sum_{k \ne i} q_{ik} \le -q_{ii} < \infty, \quad \forall i \in I$$

and  $m_i q_{ij} = m_j q_{ji}$  for some strictly positive function  $m_i$  on I. Let  $\mathcal{E}$  be the Dirichlet form on  $L^2(I;m)$  defined by

(5.2)  
$$\mathcal{E}(u,v) = \frac{1}{2} \sum_{i \neq j} q_{ij} m_i (u(j) - u(i)) (v(j) - v(i)) + \sum_i \left( -q_{ii} - \sum_{j \neq i} q_{ij} \right) m_i u(i) v(i).$$

Denote by  $\mathcal{F}^r$  the collection of functions u on I such that  $\mathcal{E}(u, u) < \infty$ . Let  $\mathcal{F}_e$  be the set of functions u in  $\mathcal{F}^r$  for which there exists  $u_n (n = 1, 2, \cdots)$  with finite support such that

$$u_n \to u$$
 and  $\sup_n \mathcal{E}(u_n, u_n) < \infty$ .

Then  $(\mathcal{E}, \mathcal{F})$  becomes a symmetric regular Dirichlet form on  $L^2(I; m)$  and the Dirichlet space  $\mathcal{F}$  is identified with the space  $\mathcal{F}_e \cap L^2(I; m)$  (cf. Theorem 17.2 in [22]). The space  $\mathcal{F}_e$  is said to be the *extended Dirichlet space of*  $\mathcal{F}$  (p.36 in [16]). Denote by  $\mathbf{M} = (P_i, X_t)$  the Hunt process associated with  $(\mathcal{E}, \mathcal{F})$ . Then  $\mathbf{M}$  is nothing but the minimal  $\mathcal{Q}$ -process constructed by W. Feller.

EXAMPLE 5.3. Let us consider a symmetric Markov chain  $\mathbf{M} = (P_i, X_t)$  on the finite state space  $I = \{1, 2, \dots, n\}$ . In this case, since the state space I is compact and the Dirichlet form on  $L^2(I;m)$  associated with  $\mathbf{M}$  is

$$\mathcal{E}(u,v) = rac{1}{2}\sum_{i
eq j}q_{ij}m_i(u(j)-u(i))(v(j)-v(i)),$$

we can establish the following full large deviation principle by Remark 4.1;

(5.3)  

$$\lim_{t \to \infty} \frac{1}{t} \log E_{i} \left( \exp \left( -\sum_{s \leq t} F(X_{s-}, X_{s}) \right) \right) \\
= - \inf_{\substack{u \in \mathcal{F}^{F} \\ ||u||_{2} = 1}} \left( \frac{1}{2} \sum_{i \neq j} q_{ij} m_{i} (u(j) - u(i))^{2} \\
+ \sum_{i \neq j} q_{ij} m_{i} u(i) u(j) (1 - e^{-F(i,j)}) \right) \\
= - \inf_{\substack{u \in \mathcal{F}^{F} \\ ||u||_{2} = 1}} \left( \frac{1}{2} \sum_{i \neq j} q_{ij} m_{i} e^{-F(i,j)} (u(j) - u(i))^{2} \\
+ \sum_{i} u^{2}(i) \sum_{j} q_{ij} m_{i} \left( 1 - e^{-F(i,j)} \right) \right),$$

where  $\mathcal{F}^F = \{u \in \mathcal{F}; u \in L^2(I; |\rho|)\}, \rho(A) = \sum_{i \in A} \cdot \sum_{i \in I} q_{ij} m_i \left(1 - e^{-F(i,j)}\right) \text{ for all } A \in \mathcal{B}(I).$  Now, let us take  $F(i, j) = \chi_B(i, j)$ , where  $B = \{(1, 2), (2, 3), \dots, (k, k+1), (k+1,k), (k, k-1), \dots, (2, 1)\}$  is the symmetric region on  $I \times I \setminus d$ . Then, the formula (5.3) gives to us the asymptotics of the number of jumps for the symmetric Markov chain M on B.

To establish the full large deviation principle in the case of infinite state space, we may need the following conditions on the Lévy kernel N.

**Theorem 5.2.** For a positive Borel function F on  $X \times X \setminus d$ , we put

(5.4) 
$$V(x) = \int_X \left( 1 - e^{-2F(x,y)} \right) N(x,dy).$$

Suppose that  $V(x) \to \infty$  as  $|x| \to \infty$  and

(5.5) 
$$\lim_{x \to \infty} P_x(\sigma_K \le t) = 0$$

for any compact subset  $K \subset X$ ,  $\sigma_K$  is the first hitting time of K. Then the full large deviation principle is established.

Proof. Let  $\tilde{F} = (e^{-2F} - 1)$ . For an increasing sequence of compact sets  $\{K_n\}$  on  $X \times X \setminus d$  such that  $\bigcup_{n=1}^{\infty} K_n = X \times X \setminus d$ , we put  $F_n = F\chi_{K_n}$ ,  $\tilde{F}_n = (e^{-2F_n} - 1)$  and  $B_t^{(n)} = \sum_{s \leq t} \tilde{F}_n(X_{s-}, X_s)$ . Then there exists the dual predictable projection of  $B_t^{(n)}$ ,

$$\left(B_t^{(n)}\right)^p = \int_0^t \int_X N(X_s, dy) \tilde{F}_n(X_s, y) \, ds.$$

Since  $B_t^{(n)} - (B_t^{(n)})^p$  is a  $P_x$ -martingale for every  $x \in X$ , Doléans-Dade exponential formula ([10]) implies that

$$Z_t = e^{B_t^{(n)} - (B_t^{(n)})^p} \prod_{s \le t} (1 + \tilde{F}_n(X_{s-}, X_s)) e^{-\tilde{F}_n(X_{s-}, X_s)}$$
$$= e^{-(B_t^{(n)})^p} \prod_{s \le t} (1 + \tilde{F}_n(X_{s-}, X_s))$$
$$= e^{-2\sum_{s \le t} F_n(X_{s-}, X_s) - (B_t^{(n)})^p}$$

is at least supermartingale multiplicative functional. From Schwarz inequality, we get

$$E_{x}\left(e^{-\sum_{s\leq t}F_{n}(X_{s-},X_{s})}\right)$$

$$= E_{x}\left(e^{-\sum_{s\leq t}F_{n}(X_{s-},X_{s})-(1/2)(B_{t}^{(n)})^{p}}e^{(1/2)(B_{t}^{(n)})^{p}}\right)$$

$$\leq \left(E_{x}\left(e^{-2\sum_{s\leq t}F_{n}(X_{s-},X_{s})-(B_{t}^{(n)})^{p}}\right)\right)^{1/2}\left(E_{x}\left(e^{(B_{t}^{(n)})^{p}}\right)\right)^{1/2}$$

$$\leq E_{x}\left(\exp\left(-\int_{0}^{t}\int_{X}N(X_{s},dy)\left(1-e^{-2F_{n}(X_{s},y)}\right)ds\right)\right)^{1/2}.$$

On account of positivity of F, if n tends to  $\infty$ , we have

$$E_x\left(e^{-\sum_{s\leq t}F(X_{s-},X_s)}\right)\leq E_x\left(e^{-\int_0^tV(X_s)\,ds}\right).$$

By assumption for V(x) and (5.5),

$$R_1^F 1(x) = \int_0^\infty e^{-\alpha t} E_x \left( e^{-\sum_{s \le t} F(X_{s-}, X_s)} \right) dt \longrightarrow 0$$

which implies that the full large deviation principle is derived by Remark 4.1.  $\Box$ 

EXAMPLE 5.4. The typical model of Markov chain which satisfies the hypotheses in Theorem 5.2 is a population model with birth rate  $\lambda_n = n\lambda$ ,  $(\lambda_0 = \lambda)$  and death rate  $\mu_n = n\mu$ , where  $\lambda, \mu$  are certain rates which satisfy  $\lambda > \mu$ . Indeed, since  $\liminf_{n\to\infty} m_n > 0, m_n = n^{-1}(\lambda/\mu)^n$  and  $E_n(\exp(-\sigma_K)) \in L^2(I;m)$ , we get

$$\sum_{n=1}^{\infty} m_n E_n (\exp(-\sigma_K))^2 < \infty$$

and which implies (5.5). Moreover, since N(i, j) of (5.4) is equals to  $n\lambda$  in this case, the same problem in Example 5.3 can be also considered on an infinite state space with the finite life time.

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