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Osaka University

**On Unions of Knots**

By Shin'ichi KINOSHITA and Hidetaka TERASAKA

**Introduction**

If two knots  $\kappa$  and  $\kappa'$  with a common arc  $\alpha$ , of which  $\kappa$  lies inside a cube  $Q$  and  $\kappa'$  outside of it,  $\alpha$  lying naturally on the boundary of  $Q$ , are joined together along  $\alpha$ , that is, if  $\alpha$  is deleted to obtain a single knot out of  $\kappa$  and  $\kappa'$ , then we have by definition the product of  $\kappa$  and  $\kappa'$ . If a knot  $\kappa$  cannot be the product of any two non trivial knots, then  $\kappa$  is said to be prime. It is H. Schubert [9] who showed that the genus of the product of two knots is equal to the sum of their genera and that every non trivial knot is decomposable in a unique way into prime knots.

Now a close inspection through the table of knots by Alexander and Briggs [1] as reproduced in the book of Reidemeister [8], where only prime knots are given, or rather a simple experiment by a thread, will show that there are a number of knots composed of prime knots in a more complicated way. Thus  $8_5$ ,  $8_{10}$ ,  $8_{15}$ ,  $8_{19(n)}$ ,  $8_{20(n)}$ ,  $8_{21(n)}$ ,  $9_{16}$ ,  $9_{24}$  and  $9_{28}$  of the Alexander Briggs table are all composed of two trefoil knots  $3_1$  in the following way (Fig. 1 indicates the composition of  $8_{19}$  out of two  $3_1$ ): First join the trefoil knots  $\kappa$  and  $\kappa'$  together along their arcs  $\widehat{AB}$

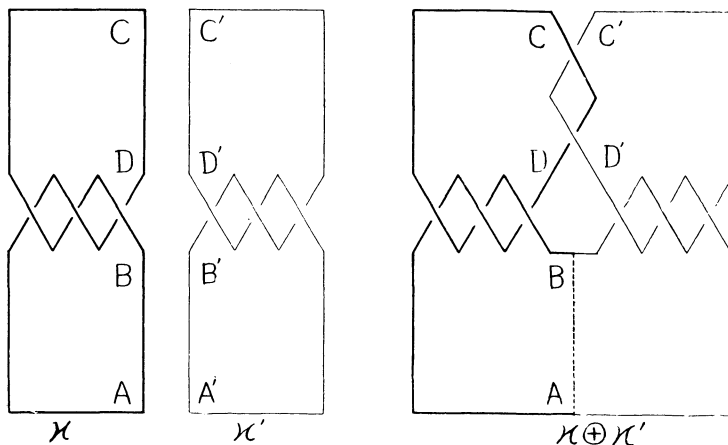


Fig. 1.

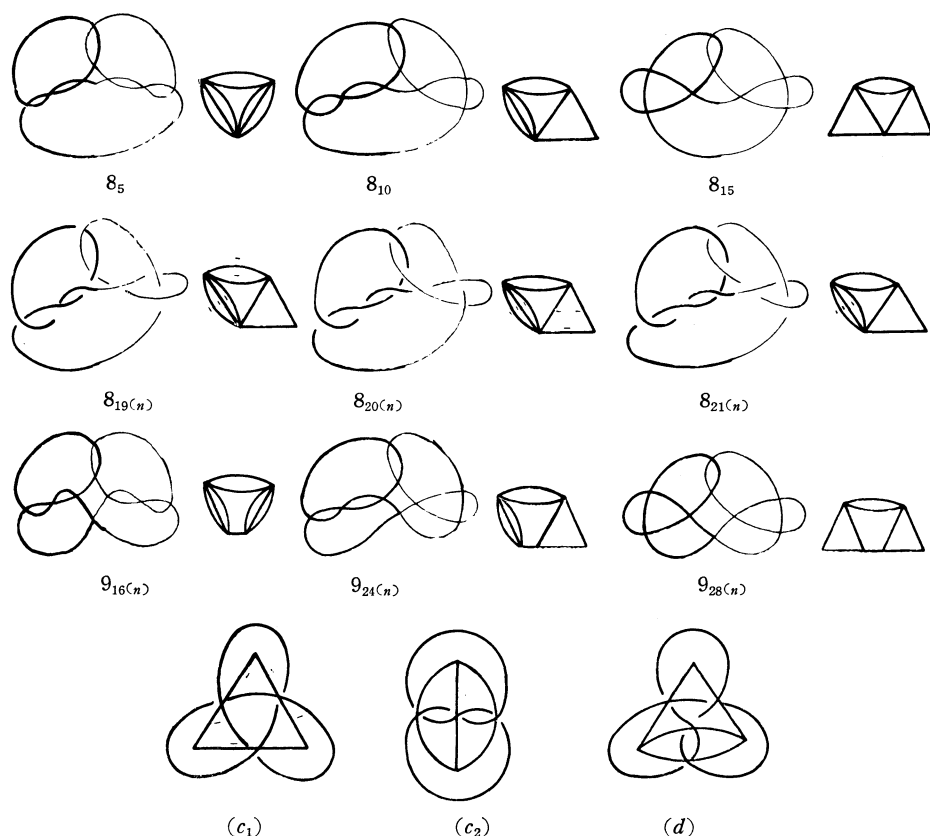


Fig. 2.

and  $\widehat{A'B'}$  as the usual product making and then wind them together in the neighbourhood of the arcs  $\widehat{CD}$  and  $\widehat{C'D'}$ . Likewise  $9_{22}$ ,  $9_{25}$ ,  $9_{30}$ ,  $9_{36}$ ,  $9_{42(n)}$ ,  $9_{43(n)}$ ,  $9_{44(n)}$  and  $9_{45(n)}$  are composites of the trefoil knot  $3_1$  and the knot  $4_1$  (Fig. 3).

Such a composition of knots will best be described if we make use of the graphs of knots [15]: The graph of a knot is a linear graph on a plane, where the vertices represent the alternating, the so-called white or black, domains ([8], p. 9), and the edges the crossing points, of the ordinary regular projection of the knot. To every edge is attached further a sign  $+$ , which may well be understood, or  $-$ , according as the knot passes the crossing point right-handedwise or left-handedwise. Every knot has a pair of graphs, dual to each other. Thus, the knots  $8_5$ – $9_{45}$  are represented by the annexed graphs in Fig. 2 and Fig. 3, where  $(c_1)$  and  $(c_2)$  are the dual forms of a trefoil knot, and  $(d)$  represent the knot  $4_1$  which is self-dual if we disregard its sign.

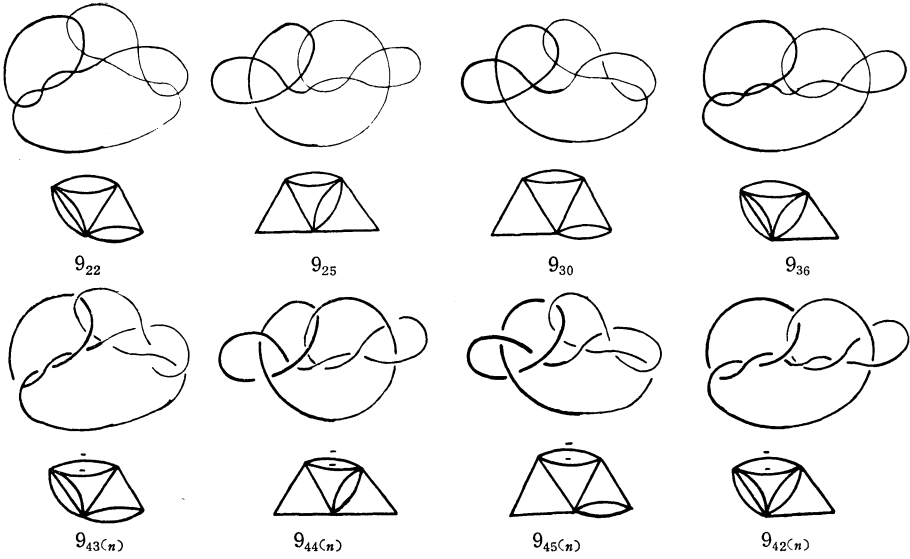


Fig. 3.

In general, if  $(k)$  and  $(k')$  are the graphs of the knots  $\kappa$  and  $\kappa'$ , bring a pair of vertices  $A$  and  $A'$  together to a coincidence and connect another

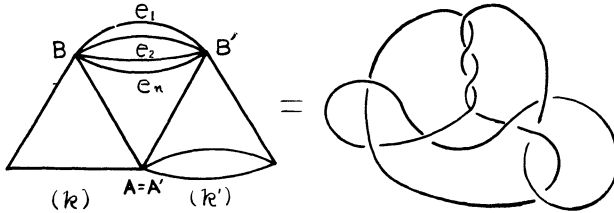


Fig. 4.

pair of vertices  $B$  and  $B'$  with  $n$  edges  $e_1, e_2, \dots, e_n$  attached with one and the same sign (Fig. 4). We shall call the knot whose graph is represented by such a composition a *union* of  $\kappa$  and  $\kappa'$  with the *winding number*  $n$  if  $n$  is even and a *skew union* with the *winding number*  $n$  if  $n$  is odd. A circumstance revealed to the authors that if two knots  $\kappa$  and  $\kappa'$  are symmetrically situated and if we perform the above composition, the Alexander polynomial [2] remains the same irrespective of the winding number and that equals the square of the Alexander polynomial of  $\kappa$  if it is a union: One of the authors conjectured<sup>1)</sup> namely that every knot could be obtained by joining together a pair of (in general linking) trivial knots (=circles) along a common arc, and as the simplest cases made tentatively such joinings with non linking circles as shown in

1) Meanwhile, this conjecture turned out to be true. See Appendix.

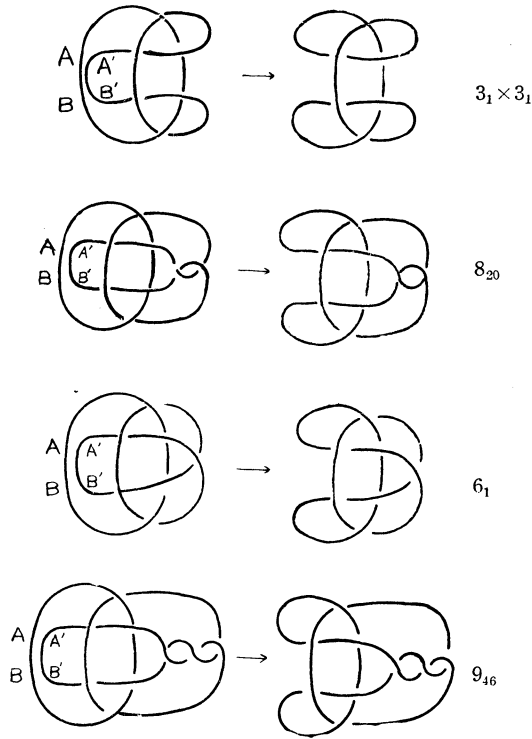


Fig. 5.

Fig. 5. Thus we obtained e. g. the product of trefoil knots, the knot  $8_{20}$  and a knot of 10 double points, all with the Alexander polynomial  $(x^2 - x + 1)^2$  if we gave 0, 2 or 4 windings beforehand, and the knots  $6_1$  and  $9_{16}$  with the Alexander polynomial  $2x^2 - 5x + 2$ , if these were 1 and 3. On writing these compositions by graphs, the other of the authors generalised the fact to symmetric unions and skew unions.

The purpose of the present note is to show among others that every union of two non trivial knots is always non trivial (Theorem 1) and that the Alexander polynomial is independent of the winding number for any symmetric union (skew union) of a knot (Theorems 2 and 3). As an application we give an example of a non trivial knot of eleven crossings with the Alexander polynomial  $\Delta(x) = 1$ , the number of crossings being smaller than those of the examples hitherto known of H. Seifert [13] and of J. H. C. Whitehead [14]. In Appendix is further proved that every knot is a sum of two trivial knots.

### § 1. Union of knots.

In the following we use freely such expressions as arcs, circles, disks,

spheres, mappings, etc. but it should be understood that all figures we shall be talking about are simplicial and mappings are all semi-linear.

Let  $Q$  be a solid cube and let  $\kappa$  be a knot which has two disjoint segments  $\overline{AD}$  and  $\overline{BC}$  in common with the surface  $\dot{Q}$  of  $Q$ , the remaining arcs  $\overline{AB}=\alpha_1$  and  $\overline{CD}=\alpha_2$  of  $\kappa$  lying wholly within  $Q$  except for their endpoints. Similarly for another cube  $Q'$  and a knot  $\kappa'$ , the corresponding points and arcs being primed. It will be convenient to call  $Q$  and  $Q'$  the *cubes of union*.

To define a union of  $\kappa$  and  $\kappa'$  we proceed as follows :

Place  $Q$  and  $Q'$  apart and parallel to each other, one face  $\Sigma$  of  $Q$  facing the face  $\Sigma'$  of  $Q'$  in such a way that  $\Sigma$  and  $\Sigma'$  make together a pair of opposite faces of a rectangular parallelepiped. If  $E$  is the plane containing a face of the last named parallelepiped different from  $\Sigma$  and  $\Sigma'$ , then  $Q$  and  $Q'$  are seen to be lying on  $E$ , when  $E$  is thought of as a ground plane.

By suitable semi-linear transformations of  $Q$  and  $Q'$  onto themselves bring  $\kappa$  and  $\kappa'$  to such positions that the segments  $AD$  and  $BC$  and the segments  $A'D'$  and  $B'C'$  lie respectively on  $\Sigma$  and  $\Sigma'$  and that  $A, B, C, D$  and  $A', B', C', D'$  lie in these orders along straight lines parallel to the ground plane  $E$ , and finally that  $AA', BB', CC'$  and  $DD'$  are perpendicular to  $\Sigma$  and  $\Sigma'$ .

Now let  $\beta$  and  $\beta'$  be disjoint arcs connecting  $B$  with  $C$  and  $B'$  with  $C'$  respectively outside  $Q$  and  $Q'$ , thus  $BC \cup \beta$  and  $B'C' \cup \beta'$  making together a link, such that their projections on  $E$  are each a polygon contained within the projection of the rectangle  $BCC'B'$ , cutting each other alternatively one above the other in the same sense, as indicated in Fig. 6. Speaking more exactly, the projections of  $\beta$  and  $\beta'$  on  $E$  intersect at

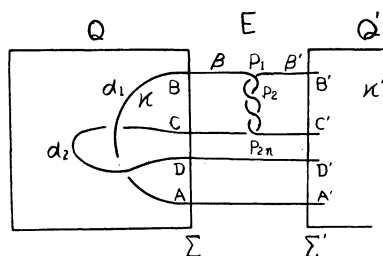


Fig. 6.

$2n$  points  $P_1, P_2, \dots, P_{2n}$  arranged in this order on either of the projections of  $\beta$  and  $\beta'$ , the original  $\beta$  passes above (or always under)  $\beta'$  at every second crossing point  $P_{2i+1}$  beginning with  $P_1$ .

Replace now the segments  $AD$  and  $A'D'$  with the segments  $AA'$  and  $DD'$ , and the segments  $BC$  and  $B'C'$  with  $\beta$  and  $\beta'$  respectively. Then the knot thus obtained will be called a *union* of  $\kappa$  and  $\kappa'$  *joined* along  $AD$  and  $A'D'$  and *winded* along  $BC$  and  $B'C'$  with the *winding number*  $2n$ .  $n$  should be taken  $>0$  if  $\beta$  passes above  $P_1$ , otherwise  $<0$ .

Likewise for a *skew union* with the *winding number*  $2n+1$ .

A union (skew union) depends in general on the locality of its winding

and on its winding number. It is uniquely determined if the winding number is 0, in which case the union is merely the usual product.

REMARK. More general union can be defined if we make windings at several places, not only one. Even in this generalised union Theorem 2 is seen to hold, and most probably Theorem 1 also holds.

First we prove the following

**Lemma 1.** *Let  $\kappa$  be a knot consisting of an arc  $\gamma$  on the surface  $\dot{Q}$  of a solid cube  $Q$  and an arc  $\alpha$  connecting the endpoints of  $\gamma$  within  $Q$ . If  $\kappa = \alpha \cup \gamma$  can be spanned by a disk  $\Delta$ , then  $\kappa$  can also be spanned by a disk  $\Delta'$  wholly contained within  $Q$ , provided that  $\Delta$  lies wholly within  $Q$  in some neighbourhood of  $\gamma$ .*

Proof. On account of the assumption there is a polygonal region  $P$  on  $\dot{Q}$  containing  $\gamma$  in its interior and two disks  $S_1$  and  $S_2$  bounded by the boundary  $\dot{P}$  of  $P$ , of which  $S_1$  lies outside and  $S_2$  inside  $\dot{Q}$ , such that the sphere  $S_1 \cup P$  has no point in common with the disk  $\Delta$  except along  $\gamma$ . By a suitable semi-linear mapping which fixes all points of  $S_2$  map the region outside  $S_1 \cup S_2$  into the region bounded by  $S_2$  and by  $\dot{Q} - P$ . Then the image of the disk  $\Delta$  is just the desired one spanning  $\kappa$  within  $Q$ .

Every knot has a representation by bridges (Schubert [10]), the smallest number of its bridges being an invariant. We shall call  $\kappa$  a knot *with  $n$  bridges* if  $\kappa$  has a representation by  $n$  bridges but not less. According to this definition every non trivial knot is a knot with at least two bridges. We shall say that a knot  $\kappa$  has an  *$n$ -bridged form with respect to a cube  $Q$*  if  $\kappa$  is decomposed by  $Q$  into  $2n$  arcs  $\alpha_1, \alpha_2, \dots, \alpha_n$  and  $\beta_1, \beta_2, \dots, \beta_n$  such that  $\alpha_i$  lie wholly inside  $Q$  and  $\beta_i$  lie wholly outside  $Q$  except for their endpoints and that  $\alpha_i (i=1, 2, \dots, n)$  and  $\beta_i (i=1, 2, \dots, n)$  can be spanned by disjoint disks within and without  $Q$  respectively, i. e., if  $\kappa$  makes inside and outside  $Q$   $n$  simple chords in Schubert's terminology ([11], p. 135).

Then we have

**Lemma 2.** *Let  $\kappa''$  be a union of two non trivial knots  $\kappa$  and  $\kappa'$  with the cubes of union  $Q$  and  $Q'$ . Then, if  $\kappa$  is not of the two-bridged form with respect to  $Q$ ,  $\kappa$  is a non trivial knot.*

Proof. Retaining our earlier notations, suppose that the contrary were true and let  $\sigma$  be a disk spanning the union  $\kappa'' = \kappa \oplus \kappa'$ , i. e., let  $\sigma$  be a semi-linear image  $f(K)$  of a circular region  $K$ , the image  $f(\dot{K})$  of

the boundary  $K$  of  $K$  being  $\kappa''$ . By a usual consideration we can modify the mapping  $f$ , if necessary, so that  $\sigma = f(K)$  "intersects" the surface  $\dot{Q}$  of the cube  $Q$  along arcs  $\gamma_1$  and  $\gamma_2$  connecting two pairs of points of  $A$ ,  $B$ ,  $C$  and  $D$  and eventually along polygons, which together with  $\gamma_1$  and  $\gamma_2$  are all disjoint from one another.

Denote the inverse images of  $A, B, C, D, A', B', C', D'$  and  $\gamma_1, \gamma_2$  by  $a, b, c, d, a', b', c', d'$  and  $w_1, w_2$  respectively. We notice in the first place that neither of  $\gamma_1$  and  $\gamma_2$  is a path connecting  $A$  and  $D$  on  $\dot{Q}$ , for if  $\gamma_1$ , say, were such a path, then  $w_1 = f^{-1}(\gamma_1)$  would be a path leading from  $a$  to  $d$  in  $K$ , and the image by  $f$  of the region bounded by  $w_1$  and by the arc  $\widehat{ad}$  of  $K$  containing  $a', b', c', d'$  would be a disk spanning the knot  $\alpha'_1 \cup \beta' \cup \alpha'_2 \cup D'D \cup \gamma_1 \cup AA'$ , which is equivalent to the knot  $\kappa'$ , contrary to the supposition that  $\kappa'$  was a non trivial knot.

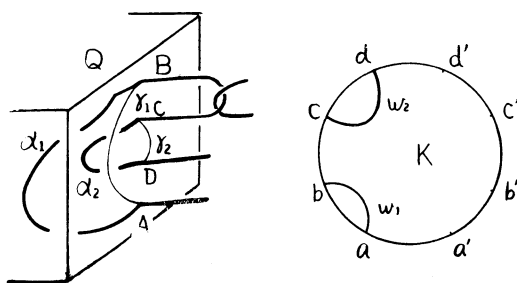


Fig. 7.

We suppose therefore that  $\gamma_1$  connects  $A$  and  $B$  and  $\gamma_2$  connects  $C$  and  $D$  on  $\dot{Q}$ , and consequently that  $w_1$  and  $w_2$  connects  $a$  with  $b$  and  $c$  with  $d$  respectively in  $K$ .

If  $\sigma$  intersects  $\dot{Q}$  besides  $\gamma_1$  and  $\gamma_2$  along some polygons, we shall call each one of the latter an *intersection-polygon*. Now let  $\dot{\Pi}$  be one of the intersection-polygons and let  $\Pi$  be one of the polygonal regions bounded by  $\dot{\Pi}$  on  $\dot{Q}$  such that  $\Pi$  has no point in common with  $\sigma$ . It does not matter whether such an intersection-polygon actually exists or not. Let  $\dot{P}$  be further the inverse image of  $\dot{\Pi}$  in  $K$ :  $f(\dot{P}) = \dot{\Pi}$ , and let  $P$  be the region bounded by  $\dot{P}$  in  $K$ . Then, since  $\Pi \cup f(P)$  is clearly a sphere, we can by a usual consideration (see e.g. [4], [9]) modify the mapping  $f$  in the neighbourhood of  $\dot{P}$  and within  $\dot{P}$ , so that the new semi-linear mapping  $f'$  of  $K$  coincides with  $f$  outside a certain polygon  $\dot{P}'$  containing  $P$  in its interior and that  $f'(P')$ , where  $P'$  denotes the region bounded by  $\dot{P}'$ , runs in the neighbourhood of  $\Pi$  without having any point in common with  $Q$ . Then  $f'(K)$  intersects  $\dot{Q}$  other than  $\gamma_1$  and  $\gamma_2$  along polygons whose number is diminished by one in comparison with  $f(K)$ .

Operating with this modification as long as there remains any intersection-polygon of the above type, we get finally a mapping, which we



denote again by the same letter  $f$ , such that every intersection-polygon, if any, has on either side  $\gamma_1$  or  $\gamma_2$ . Let  $\dot{P}$  be the inverse image of one of these polygons such that there is no inverse image of any of the intersection-polygons within  $\dot{P}$ . If  $P$  denotes the region bounded by  $\dot{P}$ , then  $f(P)$  has points in common with  $\dot{Q}$  solely along  $f(\dot{P})$ .  $f(P)$  lies moreover within  $Q$ , for otherwise the arc  $AA' \cup \alpha'_1 \cup BB' (\subset \kappa'')$  would have point in common with  $f(P)$ , which is impossible. Thus the interior of  $Q$  is divided by  $f(P)$  into two domains  $Q_1$  and  $Q_2$ . If  $T_1$  and  $T_2$  denote two regions of  $\dot{Q}$  divided by  $f(P)$ , of which  $T_1$  contains  $\gamma_1$  and  $T_2$  contains  $\gamma_2$ , then  $Q_1$  and  $Q_2$  are the domains bounded by the sphere  $f(P) \cup T_1$ , which contains  $\alpha_1$ , and the domain bounded by  $f(P) \cup T_2$ , which contains  $\alpha_2$ , in their interiors. Since  $\alpha_1 \cup \gamma_1$  is spanned by the disk which is the image by  $f$  of the region in  $K$  bounded by  $\widehat{ab} \cup w_1$ , so  $\alpha_1 \cup \gamma_1$  can also be spanned by a disk within  $f(P) \cup T_1$ , on account of Lemma 1. Likewise  $\alpha_2 \cup \gamma_2$  can be spanned by a disk within  $f(P) \cup T_2$ . Thus the knot turned out to be equivalent to the knot

$$\gamma_1 \cup (AA' \cup A'D' \cup D'D) \cup \gamma_2 \cup \beta,$$

which is a knot of two-bridged form with respect to  $Q$ , contradicting the original assumption of our Lemma. If there is no intersection-polygon on  $\dot{Q}$ , that is, if  $f(K)$  does not intersect  $\dot{Q}$  except along  $\gamma_1$  and  $\gamma_2$ , then, since  $\alpha_1 \cup \gamma_1$  and  $\alpha_2 \cup \gamma_2$  are spanned within  $Q$  by disks disjoint from each other, namely by disks which are the images of the regions bounded by  $\widehat{ab} \cup w_1$  and by  $\widehat{cd} \cup w_2$  in  $K$ ,  $\kappa$  would be equivalent to a knot of two-bridged form with respect to  $Q$ , contradicting again the assumption.

Thus the proof of Lemma 2 is complete.

Next we prove the following

**Lemma 3.** *Let  $\kappa''$  be a union of two non trivial knots  $\kappa$  and  $\kappa'$  with the cubes of union  $Q$  and  $Q'$ . Then, even if  $\kappa$  and  $\kappa'$  are of the two-bridged form with respect to  $Q$  and  $Q'$ ,  $\kappa''$  is a non trivial knot.*

*Proof.* Retaining again our earlier notations, cut  $Q$  by a plane parallel to the face  $\Sigma$  in a square  $\bar{\Sigma}$ . By a semi-linear transformation which maps  $Q$  onto itself bring the knot  $\kappa$  into its normal form  $(\alpha, \beta)$  ([11], p. 140) represented by two disjoint arcs  $\bar{\alpha}_1 = \widehat{AB}$  and  $\bar{\alpha}_2 = \widehat{CD}$  lying on  $\bar{\Sigma}$  with the bridges  $\bar{AA} \cup AD \cup D\bar{D}$  and  $\bar{BB} \cup BC \cup C\bar{C}$ , where  $\bar{AA}$ ,  $\bar{BB}$ ,  $\bar{CC}$ ,  $\bar{D}\bar{D}$  are perpendiculars between  $\Sigma$  and  $\bar{\Sigma}$ . The same for the knot  $\kappa'$  and its cube of union  $Q'$ , and make a union  $\kappa'' = \kappa \oplus \kappa'$  with  $\kappa$  and  $\kappa'$ .

We are now going to compute the fundamental group of the two-fold covering space branched along  $\kappa''$  to show that it is not trivial.

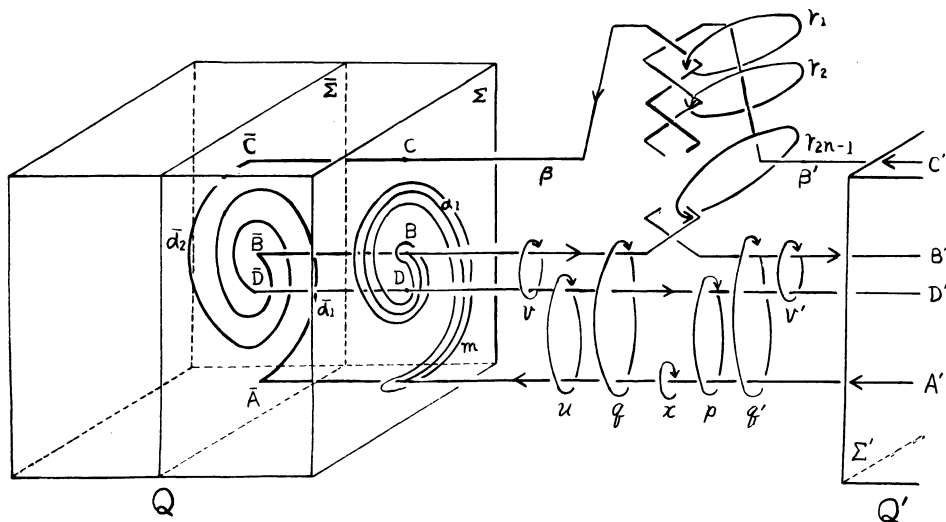


Fig. 8.

To this end consider first the two-fold covering manifold  $Q_2$  of  $Q$  branched along  $\kappa''$ , which is a solid torus ([11], p. 146). Let  $\alpha_i$  be the “projection” of  $\bar{\alpha}_i$  on  $\Sigma$ . If  $m$  is a meridian on the boundary  $\dot{Q}_2$  of  $Q_2$  given by a closed path on  $\Sigma$  including the arc  $\alpha_1$  in its interior and excluding  $\alpha_2$  outside of it, and if  $l$  is a longitude of  $\dot{Q}_2$ , then the fundamental group  $F(Q_2)$  of  $Q_2$  will be given by the presentation

$$\begin{aligned} \text{Generators: } & l, m. \\ \text{Relation: } & m=1, \end{aligned}$$

where  $l$  and  $m$  stand for the elements of the path group corresponding to the path  $l$  and  $m$  respectively.

Similarly the fundamental group  $F(Q'_2)$  of the two-fold covering manifold  $Q'_2$  of  $Q'$  branched along  $\kappa''$  is given by

$$\begin{aligned} \text{Generators: } & l', m'. \\ \text{Relation: } & m'=1, \end{aligned}$$

where  $l'$  and  $m'$  have the similar meaning.

Finally let us compute the fundamental group of the two-fold covering manifold  $R_2$  branched along  $\kappa''$  of the exterior  $R$  of  $Q$  and  $Q'$ , the boundaries of  $Q$  and  $Q'$  being included. Let  $x, p, q$  be closed path in  $R-\kappa''$ , encircling respectively  $AA'$ ,  $AA' \cup DD'$ ,  $AA' \cup DD' \cup BB'$ , the same

for  $q'$ , and let  $r_1, r_2, \dots, r_{2n-1}$  be closed paths in  $R-\kappa''$  passing respectively through where the  $2n$  times windings were performed, thus linking with both  $\beta \cup BC$  and  $\beta' \cup B'C'$ , as shown in Fig. 8. Then the fundamental group of the space  $R-\kappa''$  has the following presentation :

$$\begin{aligned} \text{Generators: } & x, p, q, q', r_1, r_2, \dots, r_{2n-1}. \\ \text{Relations: } & q^{-1}r_1q^{-1} = 1, \\ & r_1q^{-1}r_2q^{-1} = 1, \\ & \dots\dots\dots \\ & r_{2n-1}q^{-1}pq^{-1} = 1, \end{aligned}$$

if the winding number of the union  $2n$  is  $\geq 0$ . This presentation can be transformed into

$$\begin{aligned} \text{Generators: } & x, p, q, q'. \\ \text{Relation: } & p = (qq'^{-1})^n(q^{-1}q')^n. \end{aligned}$$

It should be remarked that this presentation holds true even if  $n$  is negative.

Now, if we introduce on  $\Sigma$  a circular path  $u$  encircling the segment  $AD$  and excluding  $B$  and  $C$  outside of it and a circular path  $v$  encircling  $BD$  and excluding  $A$  and  $C$  outside of it, and similarly  $v'$  on  $\Sigma'$ , then, since

$$p = u, \quad q = vx, \quad q' = v'x,$$

the last obtained presentation can further be transformed into

$$\begin{aligned} \text{Generators: } & x, u, v, v' \\ \text{Relation: } & u = (vv'^{-1})^n(x^{-1}v^{-1}v'x)^n. \end{aligned}$$

Observing the relations

$$xux^{-1} = u^{-1}, \quad xvx^{-1} = v^{-1}, \quad xv'x^{-1} = v'^{-1},$$

which hold in  $R_2$ , we have as a presentation of the fundamental group  $F(R_2)$  the following (cf. [3], [6]) :

$$\begin{aligned} \text{Generators: } & u, v, v' \\ \text{Relations: } & u = (vv'^{-1})^{2n}, \\ & u^{-1} = (v^{-1}v')^{2n}. \end{aligned}$$

We are now in a position to calculate the fundamental group  $F(M_2)$  of the two-fold covering space  $M_2$  of  $S^3$  branched along  $\kappa''$ , which is the sum of the two-fold covering manifolds  $Q_2, R_2$  and  $Q'_2$  considered above.

On the torus  $Q_2 \cap R_2$  we have indeed the relations :

$$u = l^\alpha m^{\mu_1}, \quad v = l^\beta m^{\mu_2},$$

where  $\alpha$  and  $\beta$  are the torsion and the crossing-class of the knot  $\kappa''$  in its normal form  $(\alpha, \beta)$  introduced in the beginning of our proof, and where  $\mu_1$  and  $\mu_2$  are some integers.

Similarly we have

$$u' = l'^{\alpha'} m'^{\mu_3}, \quad v' = l'^{-\beta'} m'^{\mu_4}$$

on the torus  $Q'_2 \cap R_2$ .

The fundamental group  $F(M_2)$  assumes consequently (Seifert and Threlfall [12]) the following presentation :

Generators:  $u, v, v', l, l'$ .

$$\begin{aligned} \text{Relations:} \quad u &= l^\alpha, & v &= l^\beta \\ u &= l'^{\alpha'}, & v' &= l'^{-\beta'} \\ u &= (vv'^{-1})^{2n}, & u^{-1} &= (v^{-1}v')^{2n}, \end{aligned}$$

which can be transformed into :

Generators:  $l, l'$ .

$$\text{Relations:} \quad l^\alpha = l'^{\alpha'} = (l^\beta l'^{\beta'})^{2n} = (l'^{\beta'} l^\beta)^{2n}.$$

Since  $\alpha$  and  $\alpha'$  are both  $\geq 3$ , because  $\kappa$  and  $\kappa'$  are non trivial knots,  $F(M_2)$  is seen to be non trivial<sup>2)</sup>. Thus the proof of Lemma 3 is complete.

Combining Lemma 1 and Lemma 2 we have immediately

**Theorem 1.** *Every union of two non trivial knots is a non trivial knot.*

## § 2. Symmetric union and symmetric skew union.

If two knots  $\kappa$  and  $\kappa'$  with their cubes of union  $Q$  and  $Q'$  are situated symmetric to each other, then a union (skew union) of  $\kappa$  and  $\kappa'$  will be called a *symmetric union (skew union)* of  $\kappa$ . It is not assumed that  $\kappa$  is non trivial.

Then one of the theorems to be proved is the following

**Theorem 2.** *If  $\Delta_\kappa(x)$  denotes the Alexander polynomial of a knot  $\kappa$ , then the Alexander polynomial  $\Delta_{\kappa'}(x)$  of every symmetric union  $\kappa'$  of  $\kappa$  is*

2) If we add a relation  $l^\alpha=1$  to the above ones, then we have a presentation

$$\{y, y'; y^\alpha, y'^{\alpha'}, (yy')^{2n}\}$$

of a subgroup of  $F(M_2)$ , where  $y=l^\beta, y'=l'^{\beta'}$ , showing the group to be non trivial.

equal to the square of  $\Delta_\kappa(x)$ :

$$(1) \quad \Delta_{\kappa''}(x) = (\Delta_\kappa(x))^2,$$

and is therefore independent of the winding number and the locality of winding of the union.

Proof. Let  $2n$  be the winding number of the union  $\kappa''$ . If  $n = 0$ , then  $\kappa''(x)$  is the product of  $\kappa$  with its symmetric image  $\kappa'$ , and (1) is immediate.

Suppose now that  $n$  is positive. Then the projection  $\kappa''_E$  of  $\kappa''$  on the ground plane  $E$  assumes the form as shown in Fig. 9.  $\kappa''_E$  decomposes  $E$  into domains, of which those contained within the left-hand square  $Q_E$  are denoted by  $c_1, c_2, \dots, c_m$ , the symmetric image of  $c_i (i = 1, 2, \dots, m)$  by  $c'_i$ , and the remaining ones by  $a_0, a_1, \dots, a_{2n}, b, c_0$  and  $c'_0$ . Since  $\Delta_{\kappa''}(x)$  is independent of the choice of orientation of  $\kappa''$ , we may suppose that  $\kappa''$  is oriented as in Fig. 9. Then the Alexander matrix of  $\kappa''$  will take the following form<sup>3)</sup>:

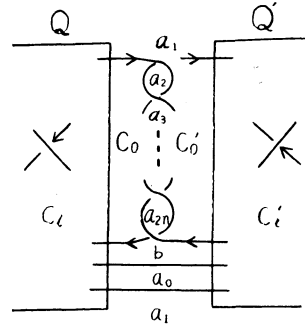


Fig. 9.

$$(2) \quad \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{2n} & b & c_0 & c_1 & \dots & c_m & c'_0 & c'_1 & \dots & c'_m \\ \hline 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \hline 0 & x & 1 & 0 & \dots & 0 & 0 & -x & & & -1 & & & \\ 0 & 0 & 1 & x & \dots & 0 & 0 & -1 & & & -x & & & \\ \vdots & \vdots & & \ddots & & \vdots & \vdots & \vdots & 0 & & \vdots & & 0 & \\ 0 & 0 & 0 & \dots & x & 1 & 0 & -x & & & -1 & & & \\ 0 & 0 & 0 & \dots & 0 & 1 & x & -1 & & & -x & & & \\ \hline * & & 0 & & & b_1 & 0 & c_{ij} & & & 0 & & & \\ & & & & & \vdots & & & & & & & & \\ & & & & & b_{m+1} & & & & & & & & \\ \hline * & & 0 & & & -b_1 & 0 & 0 & & & -c_{ij} & & & \\ & & & & & \vdots & & & & & & & & \\ & & & & & -b_{m+1} & & & & & & & & \end{pmatrix},$$

3) The matrix defined originally by Alexander was not a square one. But some consideration of the presentation of the knot group will convince us of the naturality of using the square matrix of the following shape.

where  $i = 1, 2, \dots, m+1$  and  $j = 0, 1, \dots, m$ .

Subtract the  $(2n+1)$ -th row from the  $(2n+2)$ -th row and then the  $2n$ -th row from the  $(2n+1)$ -th row, etc.; omitting the unnecessary rows and columns from this matrix in order to obtain the Alexander polynomial, we have

$$(3) \quad \left( \begin{array}{c|cc|cc} b & c_0 & c_1 & \cdots & c_m & c'_0 & c'_1 & \cdots & c'_m \\ \hline -x & -nx+n & 0 & \cdots & 0 & nx-n & 0 & \cdots & 0 \\ \hline b_1 & & & & & & & & \\ \vdots & & & & & & & & \\ b_{m+1} & & c_{ij} & & & & 0 & & \\ \hline -b_1 & & & & & & & & \\ \vdots & & & & & & & & \\ -b_{m+1} & & 0 & & & & -c_{ij} & & \end{array} \right).$$

Adding each  $(m+3+i)$ -th column ( $i = 0, 1, \dots, m$ ) to the  $(2+i)$ -th column and then each  $(1+i)$ -th row ( $i = 1, 2, \dots, m+1$ ) to the  $(m+2+i)$ -th row respectively, we have further

$$(4) \quad \left( \begin{array}{c|cc|cc} b & c_0 & c_1 & \cdots & c_m & c'_0 & c'_1 & \cdots & c'_m \\ \hline -x & 0 & 0 & \cdots & 0 & nx-n & 0 & \cdots & 0 \\ \hline b_1 & & & & & & & & \\ \vdots & & & & & & & & \\ b_{m+1} & & c_{ij} & & & & 0 & & \\ \hline 0 & & 0 & & & & & & -c_{ij} \end{array} \right).$$

We have therefore

$$\Delta_{K'}(x) = \pm x^p |c_{ij}|^2,$$

where  $p$  is a suitably chosen integer.

On the other hand, since the Alexander matrix of  $\kappa$  takes the form

$$(5) \quad \left( \begin{array}{c|ccc|ccc} a_0 & a_1 & c_0 & \cdots & c_m & & & \\ \hline 1 & 0 & 0 & \cdots & 0 & & & \\ 0 & 1 & 0 & \cdots & 0 & & & \\ \hline * & & & & c_{ij} & & & \end{array} \right),$$

where  $i = 1, 2, \dots, m+1$  and  $j = 0, 1, \dots, m$ , we have

$$\Delta_{\kappa}(x) = \pm x^{p'} |c_{ij}|,$$

where  $p'$  is a suitably chosen integer. We have therefore

$$\Delta_{\kappa''}(x) = (\Delta_{\kappa}(x))^2,$$

which was to be proved.

Since our method yields the same result even if  $n$  is negative, the proof of Theorem 2 is thus complete.

Concerning a symmetric skew union we have only the following weaker result:

**Theorem 3.** *If  $\kappa_1$  and  $\kappa_2$  are symmetric skew unions of a knot  $\kappa$  with the same place of joinings and windings, their Alexander polynomials  $\Delta_{\kappa_1}(x)$  and  $\Delta_{\kappa_2}(x)$  coincide:*

$$\Delta_{\kappa_1}(x) = \Delta_{\kappa_2}(x),$$

*even if the winding numbers are different from each other.*

Proof. Let  $\kappa''$  be a symmetric skew union of  $\kappa$  with the winding number  $2n+1$ . Suppose first that  $2n+1$  is positive, and let the projection  $\kappa''_E$  of  $\kappa''$  on the ground plane  $E$  assume the form as shown in Fig. 10.  $\kappa''_E$  decomposes  $E$  into domains as indicated in the figure. Then we are only to prove that the determinant of the matrix

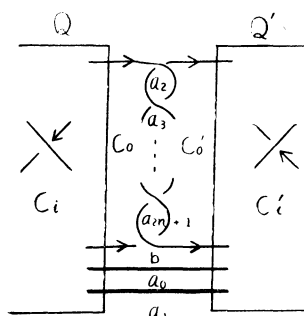


Fig. 10.

$$\begin{pmatrix} b & c_0 & c_1 & \dots & c_m & c'_0 & c'_1 & \dots & c'_m \\ x & -(n+1)x+n & 0 & \dots & 0 & nx-(n+1) & 0 & \dots & 0 \\ \hline b_1 & & & & & & & & \\ \vdots & & c_{ij} & & & & & & 0 \\ b_{m+1} & & & & & & & & \\ \hline -b_1 & & & & & & & & \\ \vdots & & 0 & & & & & & -c_{ij} \\ -b_{m+1} & & & & & & & & \end{pmatrix}$$

corresponding to (3) of the proof of Theorem 2 is independent of  $n$ . The matrix corresponding to (4) takes the form

$$(6) \left( \begin{array}{c|ccc|ccc} b & c_0 & c_1 & \cdots & c_m & c_0' & c_1' & \cdots & c_m' \\ \hline x & -x-1 & 0 & \cdots & 0 & -(n+1)x+n & 0 & \cdots & 0 \\ \hline b_1 & & & & & & & & \\ \vdots & & & & & & & & \\ b_{m+1} & & c_{ij} & & & & 0 & & \\ \hline 0 & & 0 & & & & -c_{ij} & & \end{array} \right).$$

Adding the first column to the second and then the second column multiplied by  $x$  to the first, we have

$$\left( \begin{array}{c|ccc|ccc} 0 & -1 & 0 & \cdots & 0 & -(n+1)x+n & 0 & \cdots & 0 \\ \hline b_1' & & & & & & & & \\ \vdots & & & & & & & & \\ b_{m+1}' & & c'_{ij} & & & & 0 & & \\ \hline 0 & & 0 & & & & -c_{ij} & & \end{array} \right),$$

the determinant of which is seen to be independent of  $n$ . Since our method yields the same result even if  $2n+1$  is negative, the proof of Theorem 3 is thus complete.

**Theorem 4.** *If  $\kappa'$  is a symmetric union or a symmetric skew union of a knot  $\kappa$ , then*

$$T_2(\kappa') = (T_2(\kappa))^2,$$

where  $T_2(\kappa)$  denotes the product of torsion numbers of the two-fold branched covering space of  $\kappa$ .

Proof. Since  $T_2(\kappa) = |\Delta_\kappa(-1)|$  (cf. [5], [6]) for every knot  $\kappa$ , we are only to prove that

$$|\Delta_{\kappa'}(-1)| = |\Delta_\kappa(-1)|^2.$$

If  $\kappa'$  is a symmetric union, this equality follows immediately from Theorem 2. Therefore let  $\kappa'$  be a symmetric skew union. Then from (5) we have

$$|\Delta_\kappa(-1)| = \pm |c_{ij}(-1)|$$



and from (6), where we put  $c'_{ij}$  instead of  $c_{ij}$ ,

$$|\Delta_{\kappa'}(-1)| = |c'_{ij}(-1)|^2.$$

It is easy to see that for each  $i$  ( $i=1, 2, \dots, m-1$ ) (see Fig. 11)

$$\begin{aligned} (c_{i,0}(-1), c_{i,i}(-1), \dots, c_{i,m}(-1)) \\ = \pm (c'_{i,0}(-1), c'_{i,i}(-1), \dots, c'_{i,m}(-1)). \end{aligned}$$

Therefore

$$|\Delta_{\kappa'}(-1)| = |\Delta_{\kappa}(-1)|^2$$

and the proof is complete.

### §3. Union of knots with the same Alexander polynomial.

We have mentioned in Introduction that the knots  $8_5, 8_{10}, 8_{15}, 8_{19}, 8_{20}, 8_{21}, 9_{16}, 9_{24}$  and  $9_{28}$  of Alexander and Briggs table are unions of two trefoil knots. Now, if we decompose the Alexander polynomials of these knots into factors, we find a remarkable fact that they have a factor  $x^2-x+1$  in common:

Knot	Alexander polynomial	Factorization <sup>4)</sup>
$8_5$	$1-3+4-5$	$= (1-1)(1-2+1)$
$8_{10}$	$1-3+6-7$	$= (1-1)^3$
$8_{15}$	$3-8+11$	$= (1-1)(3-5)$
$8_{19}$	$1-1+0+1$	$= (1-1)(1+0-1)$
$8_{20}$	$1-2+3$	$= (1-1)^2$
$8_{21}$	$1-4+5$	$= (1-1)(1-3)$
$9_{16}$	$2-5+8-9$	$= (1-1)(2-3+3)$
$9_{24}$	$1-5+10-13$	$= (1-1)^2(1-3)$
$9_{28}$	$1-5+12-15$	$= (1-1)(1-4+7)$ .

But this is not a mere contingency. We have in fact

**Theorem 5.** *If  $\kappa$  and  $\kappa'$  are knots with the same Alexander polynomial  $\Delta(x)$ , then every union of  $\kappa$  and  $\kappa'$  has a factor  $\Delta(x)$  in its Alexander polynomial.*

Proof. Since the theorem is clear in the case  $2n=0$ , first let  $2n > 0$ .

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4)  $(1-1)(1-2+1)$  means e. g.  $(x^2-x+1)(x^4-2x^3+x^2-2x+1)$ .

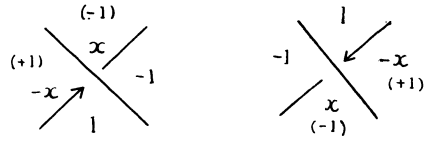


Fig. 11.



$a_0$	$a_1$	$a_2$	$\cdots$	$a_{2n}$	$b$	$c_0$	$\cdots$	$c'_0$	$\cdots$	
1	0	0			0	0		0		
0	1	0			0	0		0		
*		*		*	*	0		*	0	
*		0		0	*	$\pm x^p \Delta(x)$		0		
0		0		0		1			0	
							$\ddots$			1
*		0		*		0		$\pm x^q \Delta(x)$		
								1	0	
									0	1

the determinant of which, by deleting the first two rows and columns and by expanding along the last  $2(m+1)$  rows, shows immediately that it has indeed a factor  $\Delta(x) = \pm x^s |c_{ij}| = \pm x^r |c'_{ij}|$ .

Similarly for the case  $2n < 0$  and the proof is complete.

**§ 4. Knots with the Alexander polynomial equal to unity.**

H. Seifert [13] and J. H. C. Whitehead [14] determined some types of knots whose Alexander polynomials are equal to 1. Here another type of such knots will be given.

The basic idea of the construction is this: If the Alexander polynomial of a knot  $\kappa$  is unity, then by Theorem 2 that of every symmetric union of  $\kappa$  is also unity; therefore, if we take as  $\kappa$  a trivial knot, we would obtain a knot of the above property. The main difficulty is, how to determine the place of joining and winding in order that the union be a non trivial knot.

Let  $\kappa$  be a trivial knot given in its graphical representation ( $k$ ) by  $2p+1$  sided polygon  $A_0A_1 \cdots A_{2p}$  of  $p$  consecutive sides  $A_0A_1, A_1A_2, \dots, A_{p-1}A_p$  with negative signs. Let  $A'_0A'_1 \cdots A'_{2p}$  be another polygon of consecutive sides  $A'_pA'_{p+1}, A'_{p+1}A'_{p+2}, \dots, A'_{2p-1}A'_{2p}$  with negative signs, symmetric to  $A_0A_1 \cdots A_{2p}$  with respect to a straight line. Bring  $A_0$  and  $A'_0$  to a coincidence and connect  $A_p$  and  $A'_p$  with  $2n$  arcs  $e_1, e_2, \dots, e_{2n}$ . The knot represented by this graph, which is a symmetric union of  $\kappa$  with the winding number  $2n$ , will be denoted by  $\kappa(p, 2n)$ .  $n$  should be taken negative if the signs of  $e_i$  are negative.

Then we have

**Theorem 6.**  $\kappa'' = \kappa(p, 2n)$  is a non trivial knot with the Alexander polynomial  $\Delta_{\kappa''}(x) = 1$ , whenever  $p \geq 2$  and  $n \neq 0$ . Especially  $\kappa(2, 2)$  is a knot of eleven crossings with the Alexander polynomial equal to unity.

Proof. As will be seen from the structure of  $\kappa(p, 2n)$  it suffices to prove the theorem only for the case  $n > 0$ . It is also immaterial whether  $p$  is even or odd, but first let  $p$  be an odd number. Then, passing over from the original graph to its dual form, we see that  $\kappa(p, 2n)$  is a union of knots  $\kappa_1$  and  $\kappa_2$  whose graphs  $(k_1)$  and  $(k_2)$  consist respectively of:

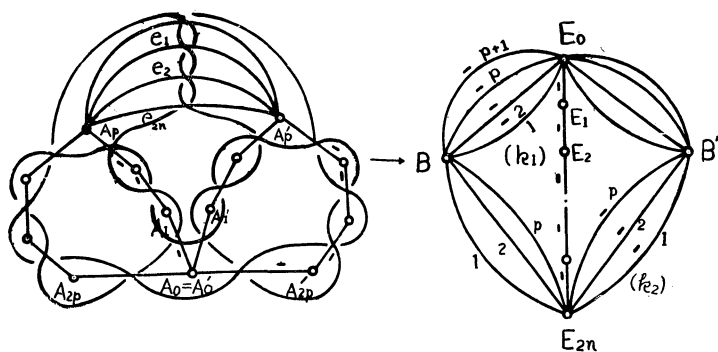


Fig. 12.

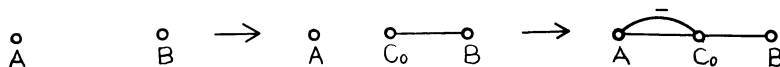
$(k_1)$ : vertices  $E_0, E_1, \dots, E_{2n}$  and  $B$  and  $2n$  consecutive edges with negative signs  $E_0E_1, E_1E_2, \dots, E_{2n-1}E_{2n}$ , a bundle of  $p+1$  edges with negative signs connecting  $B$  and  $E_0$  and a bundle of  $p$  edges connecting  $B$  and  $E_{2n}$ .

$(k_2)$ : vertices  $E_{2n}$  and  $B'$  and a bundle of  $p$  edges with negative signs connecting these vertices.

Now the knot  $\kappa_2$  represented by  $(k_2)$  is alternating and so by Bankwitz's theorem ([3], see [8], p. 34) a non trivial knot.

In order to show that  $\kappa_1$  is also non trivial, we proceed as follows<sup>5)</sup>:

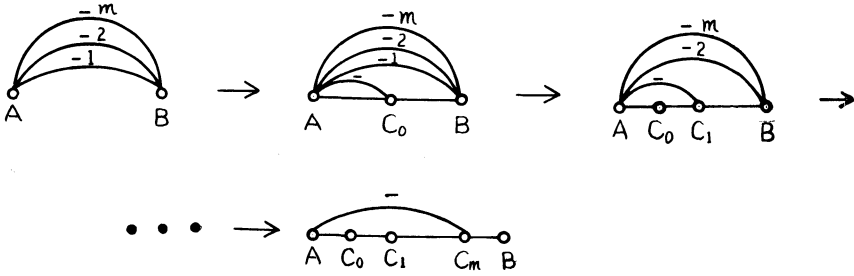
(1) A graph can be carried over to an equivalent one, if we insert between two vertices  $A$  and  $B$  of the graph two edges  $AC_0$  and  $C_0B$  with positive signs (which are understood) and an edge  $AC_0$  with a negative sign:



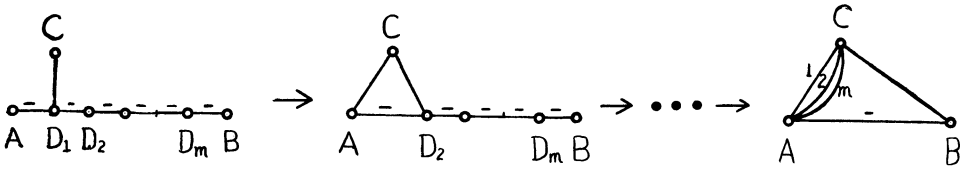
(2) By inserting between the vertices  $A$  and  $B$  of a graph three

5) For the reduction of graphs see [15].

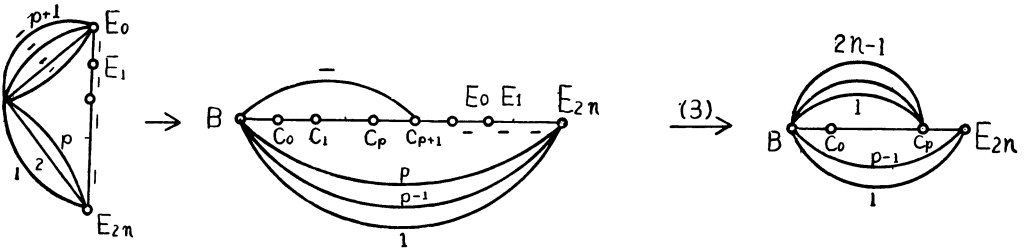
edges of the type of (1), a bundle of  $m$  edges with negative signs connecting  $A$  and  $B$  can be substituted by  $m+2$  edges  $AC_0, C_0C_1, \dots, C_mB$  and by an edge  $AC_m$  with a negative sign:



(3) If two vertices  $A$  and  $B$  of a graph is connected by  $m+1$  consecutive edges  $AD_1, D_1D_2, \dots, D_mB$  with negative signs and another vertex  $C$  is further connected with  $D_1$  by an edge  $CD_1$ , then it will be shown by repetition of a fundamental operation on graphs [15] that these edges can be substituted by a bundle of  $m$  edges connecting  $A$  and  $C$ , by the edge  $BC$  and by the edge  $AB$  with a negative sign.



(4) Applications of (2) and (3) shows finally the following reduction of  $(k_1)$  to an alternating knot:



Again by Bankwitz's theorem the knot  $\kappa_1$  is seen to be a non trivial one.

Hitherto we have assumed that  $p$  was an odd number, but if  $p$  is

even, a similar decomposition of  $(k'')$  into knots will be obtained if we exchange  $E_0, E_1, \dots, E_{2n}, B, B', p$  and  $p+1$  seriatim by  $E_{2n}, E_{2n-1}, \dots, E_0, B', B, p+1$  and  $p$  in the definitions of  $(k_1)$  and  $(k_2)$ .

The knot  $\kappa''$  is thus seen to be a union of two non trivial knots  $\kappa_1$  and  $\kappa_2$ , and consequently it is by Theorem 1 non trivial.

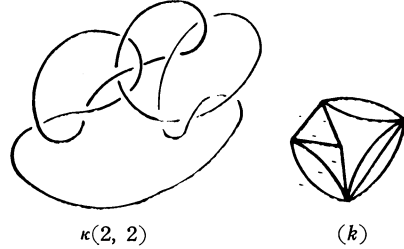


Fig. 13.

By a similar reduction of graph it can be proved without difficulty that if  $p=2$  and  $n=1$ ,  $\kappa(2, 2)$  is a knot of 11 crossings with the graph  $(k)$  as shown in Fig. 13.

REMARK. The non triviality of knots  $\kappa_1$  and  $\kappa_2$  can also be proved directly and without difficulty by means of the calculation of the determinant of knot.

APPENDIX

The sum of knots modulo 2.

Let  $\kappa_1$  and  $\kappa_2$  be two knots having a single arc  $\gamma$  as their meet:  $\kappa_1 \cap \kappa_2 = \gamma$ ; then the sum modulo 2 of  $\kappa_1$  and  $\kappa_2$ , that is the point set  $\kappa_1 \cup \kappa_2 - (\gamma)$ , where  $(\gamma)$  denotes the open arc of  $\gamma$ , will be called the sum of  $\kappa_1$  and  $\kappa_2$ , and denoted by  $\kappa_1(+)\kappa_2$ . We are going to prove that any knot can always be represented as a sum of two trivial knots.

A knot  $\kappa$  will be said to be decomposable into trivial knots by a pair of different points  $A$  and  $B$  on  $\kappa$ , if there is an arc  $\gamma$ , a decomposition-arc, connecting  $A$  and  $B$  outside  $\kappa$  such that  $\kappa$  is represented as the sum of knots  $\alpha \cup \gamma$  and  $\beta \cup \gamma$ , where  $\alpha$  and  $\beta$  are two arcs of  $\kappa$  of endpoints  $A$  and  $B$ .

According to this definition, if a knot  $\kappa$  is the sum of two trivial knots  $\kappa_1$  and  $\kappa_2$ , then there must be a pair of points  $A$  and  $B$  on  $\kappa$  such that  $\kappa$  is decomposable into trivial knots by  $A$  and  $B$ ; for the endpoints of the common arc  $\kappa_1 \cap \kappa_2$  have this property.

Then we have

**Lemma 1.** *If a knot  $\kappa$  is decomposable into trivial knots by two points  $A$  and  $B$  on  $\kappa$ , then  $\kappa$  can also be decomposable into trivial knots by any two points  $A'$  and  $B'$  of  $\kappa$ .*

Proof. Let  $\widehat{AA'}$  be the arc of  $\kappa$  which does not contain  $B$ . Then, if  $\gamma$  is a decomposition-arc of endpoints  $A$  and  $B$ , an arc  $\gamma_1$  connecting

$A'$  and  $B$  sufficiently near  $\widehat{AA'} \cup \gamma_1$ , will be found as a decomposition-arc of endpoints  $A'$  and  $B$ . By the same reason there can be found further a decomposition-arc of endpoints  $A'$  and  $B'$  sufficiently near  $\gamma_2$ , q.e.d.

**Lemma 2.** *Let  $Q$  be a solid cube that has two segments in common with a knot  $\kappa$  and let  $\gamma$  be an arc that connects two points  $X$  and  $Y$  of  $\kappa$  lying outside  $Q$  and that has no points in common with  $\kappa$  except for  $X$  and  $Y$ . Then  $X$  and  $Y$  can be connected wholly outside  $Q$  by an arc that is isotopic to  $\gamma$  in the complementary  $E^3 - \kappa$  of  $\kappa$ .*

Proof.  $Q \cap \kappa$  can be thought of as consisting of two perpendiculars  $AB$  and  $CD$  between a pair of opposite faces  $\Sigma$  and  $\Sigma'$  of  $Q$ . Suppose  $\gamma$  intersect  $\Sigma$  in a point  $P$ . Then we can substitute a sufficiently small arc  $\beta$  of  $\gamma$  containing  $P$  in its interior by an arc  $\beta'$  that is isotopic to  $\beta$  in  $E^3 - \kappa$  and that has no point in common with  $\Sigma$ , so that the newly obtained arc  $\gamma'$  intersects  $\Sigma$  in points less by one in number as compared with  $\gamma$ . Proceeding in the same way with all intersection points of  $\gamma$  and  $\Sigma$ , which we may suppose finite in number, we can obtain finally an arc  $\gamma''$  that is isotopic to  $\gamma$  in  $E^3 - \kappa$  and that has no points in common with  $\Sigma$ . Then by a homeomorphism of  $E^3$  leaving all points of  $\kappa$  and  $\Sigma$  fixed "drive away"  $\gamma''$  out of  $Q$  in the direction from  $\Sigma$  to  $\Sigma'$  so that the image of  $\gamma''$  becomes the desired arc lying wholly outside  $Q$ .

We are now in a position to prove

**Theorem.** *Every knot can be represented as the sum of two trivial knots.*

Proof. We prove the theorem by induction on the unknotting number of a knot [7], that is the minimum number of cuts in order to change the given knot to a trivial one.

First suppose that the theorem holds true if the unknotting number of a knot is  $n-1$ , and let  $\kappa$  be a knot of unknotting number  $n (\geq 1)$ . Then there are by the definition of the unknotting number an equivalent of  $\kappa$ , which we denote again by the same letter  $\kappa$ , and a solid cube  $Q$  having two segments  $AB$  and  $CD$  in common with  $\kappa$  such that if  $AEB$  is a triangle within  $Q$  having a point in common with  $CD$  and if we substitute the arc  $AB$  of  $\kappa$  by  $AE \cup EB$ , the newly obtained knot  $\kappa'$  becomes a knot of the unknotting number  $n-1$ . By our assumption  $\kappa'$  is the sum of two trivial knots, and so  $\kappa'$  is decomposable into trivial knots by two points  $X$  and  $Y$  on  $\kappa'$ . Then by Lemma 1  $X$  and  $Y$  can be taken outside  $Q$ . Let  $\gamma$  be a decomposition-arc of endpoints  $X$  and  $Y$ : by Lemma 2  $\gamma$  can be so chosen that it lies wholly outside  $Q$ . If  $\alpha$  and  $\beta$  are arcs of  $\kappa'$  of endpoints  $X$  and  $Y$ , of which  $\beta$  contains the

segment  $CD$ , then  $\alpha \cup \gamma$  and  $\beta \cup \gamma$  are trivial knots. But since  $\gamma$  has no points in common with  $Q$ ,  $(\alpha(+)\Delta AEB) \cup \gamma$  is also a trivial knot. Thus  $\kappa$  turns out to be the sum of two trivial knots  $(\alpha(+)\Delta AEB) \cup \gamma$  and  $\beta \cup \gamma$ .

Since the theorem is true if  $n=0$ , that is, if the knot is itself trivial, the theorem is thus proved to be true for any knot.

(Received August 1, 1957)

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