Arakawa, T.
Osaka J. Math.
36 (1999), 605 – 614

PARTIALLY NEGATIVE CYCLES AND PROJECTIVE EMBEDDINGS OF SURFACES OF GENERAL TYPE

TATSUYA ARAKAWA

(Received October 14, 1997)

1. Introduction

Let S denote a nonsingular minimal complex algebraic surface of general type and Γ_1, Γ_2, ..., Γ_n the (-2)-curves on S. Then it is well known that for $m \geq 5$, the pluricanonical system $|mK_S|$ defines a holomorphic map $S \to \mathbf{P}^N$, which is embedding exept Γ_i's (cf. [1] and [2] or [3]). On the other hand, Yang [4] defined the minimally negative cycle W on S and showed that for sufficiently large m, the linear system $|mK_S - W|$ gives a projective embedding of S which is everywhere injective.

In the present paper we will construct some projective embeddings of S which take, in some sense, *middle positions* between the above two.

The author would like to express his thanks to Professors Sampei Usui and Kazuhiro Konno for their advises and encouragements. He also thanks to Mr. Yoshiaki Fukuma and the referee for pointing out many mistakes in the earlier version.

2. Minimally negative cycle and partially negative cycles

In the following arguments, we always assume that S has some (-2)-curves Γ_i.

In [4], Yang introduced the notion of the *minimally negative cycle* on S as follows:

DEFINITION 1 (cf. [4, Definition 1.1]). A cycle $D = r_1\Gamma_1 + r_2\Gamma_2 + \cdots + r_n\Gamma_n$ is said to be *negative* if $\Gamma_j D < 0$ $(1 \leq j \leq n)$. A negative cycle W is called the *minimally negative cycle* if, for every negative cycle D, $W \leq D$.

The existance and uniquness of the minimally negative cycle on a surface are shown by the same way as for the foundamental cycle. Moreover we have:

Lemma 1. (cf. [4, Corollary 1.5]) *Let W be the minimally negative cycle. Then we have* $\Gamma_j W = -1$ *or* -2 $(1 \leq j \leq n)$.

Now we will generalize the above definition.

DEFINITION 2. Let $i_1, i_2, \ldots, i_l$ denote any numbers which are mutually distinct. A cycle $X = X_{i_1,i_2,\ldots,i_l} = s_1\Gamma_1 + s_2\Gamma_2 + \cdots + s_n\Gamma_n$ is called *partially negative cycle* for the suffixes $(i_1, i_2, \ldots, i_l)$ if $\Gamma_{i_j}X = 0$ for every j and $\Gamma_i X < 0$ for every $i \neq i_1, i_2, \ldots, i_l$.

Note that a negative or partially negative cycle is always effective.

Lemma 2. *For any suffixes* $(i_1, i_2, \ldots, i_l)$*, there exists a partially negative cycle for them.*

Proof. We may assume $(i_1, i_2, \ldots, i_l) = (1, 2, \ldots, l)$ $(1 \leq l \leq n)$. Let $(a_{l+1}, a_{l+2}, \ldots, a_n)$ be negative rational numbers (e.g. $(-1, -1, \ldots, -1)$) and set

$$(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = (0, \ldots, 0, a_{l+1}, \ldots, a_n)M^{-1}$$

where M is the intersection matrix of the (-2)-curves $\Gamma_1, \ldots, \Gamma_n$ (cf. Appendix).

Now let m be a positive integer such that $s_j = m\tilde{s}_j$ is also a nonnegative interger $(1 \leq j \leq n)$. Then we get a cycle $X = s_1\Gamma_1 + \cdots + s_n\Gamma_n$ such that $\Gamma_j X = 0$ $(1 \leq j \leq l)$ and $\Gamma_j X < 0$ $(l+1 \leq j \leq n)$. □

Since the minimally negative cycles are given in [4, p.174], we get the following alternative construction:

Let W be a negative cycle on S (e.g. the minimally negative cycle). For l-variables $x_1, x_2, \cdots, x_l$, we set

$$\tilde{X}(x_1, x_2, \ldots, x_l) := W - x_1\Gamma_{i_1} - x_2\Gamma_{i_2} - \ldots - x_l\Gamma_{i_l}.$$

Let us consider the following linear equations on $(x_1, x_2, \ldots, x_l)$:

$$\Gamma_{i_j}\tilde{X}(x_1, x_2, \ldots, x_l) = 0 \quad (1 \leq j \leq l). \tag{1}$$

Then, since all the components of the inverse matrix of the intersection matrix of Γ_{i_1}, $\Gamma_{i_2}, \ldots, \Gamma_{i_l}$ are nonpositive (see Appendix) and moreover $\Gamma_{i_j}W < 0$ for all j, we get that (1) has a root

$$(x_1, x_2, \ldots, x_l) = (c_1, c_2, \ldots, c_l)$$

where all the c_j's are nonnegative rational numbers. Now let m be a positive integer such that all the mc_j's are also (nonnegative) integers, and we define X as $m\tilde{X}(c_1, c_2, \ldots, c_l)$. Then it is easy to see that $X = X_{i_1,i_2,\ldots\, i_l}$ is one of the partially negative cycles which we want.

For readers' convenience, we will give a concrete description for small l.Let W be the minimally negative cycle on S:

EXAMPLE 2.1 ($l = 1$). By Lemma 1, we have $\Gamma_1 W = -1$ or -2. So we can define $X = X_1$ by

$$X_1 = \begin{cases} 2W - \Gamma_1 & if \quad \Gamma_1 W = -1 \\ W - \Gamma_1 & if \quad \Gamma_1 W = -2 \end{cases}$$

EXAMPLE 2.2 ($l = 2$). We may assume $\Gamma_1 W \geq \Gamma_2 W$, that is, $(\Gamma_1 W,\ \Gamma_2 W) = (-1, -1)$, $(-1, -2)$ or $(-2, -2)$. Then we can define $X = X_{1,2}$ as follows:

(I) The case of $\Gamma_1 \Gamma_2 = 0$.

$$X_{1,2} = \begin{cases} 2W - \Gamma_1 - \Gamma_2 & if \quad (\Gamma_1 W,\ \Gamma_2 W) = (-1, -1) \\ 2W - \Gamma_1 - 2\Gamma_2 & if \quad (\Gamma_1 W,\ \Gamma_2 W) = (-1, -2) \\ W - \Gamma_1 - \Gamma_2 & if \quad (\Gamma_1 W,\ \Gamma_2 W) = (-2, -2) \end{cases}$$

(II) The case of $\Gamma_1 \Gamma_2 = 1$.

$$X_{1,2} = \begin{cases} W - \Gamma_1 - \Gamma_2 & if \quad (\Gamma_1 W,\ \Gamma_2 W) = (-1, -1) \\ 3W - 4\Gamma_1 - 5\Gamma_2 & if \quad (\Gamma_1 W,\ \Gamma_2 W) = (-1, -2) \\ W - 2\Gamma_1 - 2\Gamma_2 & if \quad (\Gamma_1 W,\ \Gamma_2 W) = (-2, -2) \end{cases}$$

3. Projective embeddings of S

Let $X = X_{i_1, i_2, \ldots,\ i_l}$ denote a partially negative cycle for suffixes $(i_1, i_2, \ldots, i_l)$ on S.

Theorem 1. *For a sufficiently large n, the divisor $nK_S - X$ defines a holomorphic map $S \to \mathbf{P}^N$ which is an embeding except the indicated (-2)curves Γ_{i_1}, Γ_{i_2}, $\ldots$, Γ_{i_l}.*

REMARK 3.1. The meaning of *sufficiently large* is that there exists a number $n_0 = n_0(K_S^2, X^2)$ such that the theorem holds for every n larger than n_0.

REMARK 3.2. The following proof is almost parallel to that of [4, Theorem 2.2] except the last paragraph.

Let us recall a theorem of Reider (cf. [3, THEOREM 1]):

Reider's Theorem *Let S be a smooth complex algebraic surface and let L be a nef divisor on S.*

(i) *If $L^2 \geq 5$ and p is a base point of $|K_S + L|$, then there exists an effective divisor E on S passing through p such that* *either* $LE = 0, E^2 = -1$

or $LE = 1, E^2 = 0$.

(ii) *If* $L^2 \geq 10$ *and points p, q, are not separated by* $|K_S + L|$, *then there exists an effective divisor* E *on* S *passing through* p *and* q *such that*

either $LE = 0$ *and* $E^2 = -1$ *or* -2

or $LE = 1$ *and* $E^2 = -1$ *or* 0

or $LE = 2$ *and* $E^2 = 0$.

Proof of Theorem 1. Let us begin with an inequality to estimate intersection numbers of X and other divisors. Let E denote an irreducible curve on S. Then, since $D := (K_S^2)E - (K_S E)K_S$ satisfies $DK_S (= XK_S) = 0$, we have $(X^2)(D^2) - (XD)^2 \geq 0$, which implies the following:

$$(XE)^2 \leq X^2 \left(E^2 - \frac{(K_S E)^2}{K^2} \right). \tag{2}$$

Now let L denote the divisor $(n-1)K_S - X$. Then we have $L^2 \geq 10$ if

$$(n-1)^2 \geq \frac{-X^2 + 10}{K_S^2}.$$

Hence, to apply the Reider's theorem, we need to verify that L is nef if n is sufficiently large.

Let E denote an irreducible curve on S as before. If $K_S E = 0$, then we have $XE \leq 0$ and hence $LE \geq 0$ for any n. So we assume $K_S E > 0$ and will show that $(n-1)^2 (K_S E)^2 - (XE)^2 > 0$ (for large n). Since $K_S E > 0$, we have $3(K_S E)^2 + E^2 \geq (K_S E)^2 + E^2 + 2 \geq K_S E + E^2 + 2 \geq 0$ and therefore, by (2),

$$\begin{aligned}
(n-1)^2 (K_S E)^2 - (XE)^2 &\geq (n-1)^2 (K_S E)^2 - X^2 \left(E^2 - \frac{(K_S E)^2}{K_S^2} \right) \\
&= \left((n-1)^2 + \frac{X^2}{K_S^2} \right) (K_S E)^2 - X^2 E^2 \\
&= \left((n-1)^2 + \frac{X^2}{K_S^2} + 3X^2 \right) (K_S E)^2 \\
&\quad - X^2 \big(3(K_S E)^2 + E^2 \big) \geq 0
\end{aligned}$$

for any n such that

$$(n-1)^2 + \frac{X^2}{K_S^2} + 3X^2 \geq 0.$$

Now suppose there exist points p, q on S which are not separated by the linear system $|nK_S - X|$. Then by Reider's theorem (ii), there exists a divisor $E > 0$ on S such that $(LE, E^2) = (0,-1),\ (0,-2),\ (1,0),\ (1,-1)$ or $(2,0)$.

(I) $(LE, E^2) \neq (0,-2)$. If $K_S E = 0$, then E^2 is even and negative, which contradict the above. Hence we have $K_S E > 0$ and therefore

$$\begin{aligned} (XE)^2 &\leq X^2\left(E^2 - \frac{(K_S E)^2}{K_S^2}\right) \\ &\leq X^2(E^2 - \frac{1}{K^2})(K_S E)^2. \end{aligned}$$

Consequently we get that

$$\begin{aligned} LE &= (n-1)K_S E - XE \\ &\geq \left((n-1) - \sqrt{X^2(E^2 - \frac{1}{K_S^2})}\right) K_S E \end{aligned}$$

with $E^2 = 0,\ -1$. Therefore, for large n, we have $LE > 2$, a contradiction.

(II) $(LE, E^2) = (0,-2)$. If $K_S E > 0$, then we get that $LE > 0$ for large n by the same way as in (I). Hence we have $K_S E = 0$, and hence $XE = 0$. Therefore E is nothing but a finite sum of the indicated (-2)-curves $\Gamma_{i_1},\ \Gamma_{i_2},\ \ldots,\ \Gamma_{i_l}$. □

Appendix

Though it may be well known, we will write down the inverse matrices of the intersection matrices of some of (-2)-curves on S. For simplicity, we consider only the cases that the (-2)-curves form a connected subset of S. We denote the (i,j) component of the intersection matrix (resp. the inverse matrix of it) for each types by $(\mathbf{A}_n)_{i,j}$ etc. (resp. $(\mathbf{A}_n^{-1})_{i,j}$ etc.). All the statements in this appendix are easily verified.

1. $\mathbf{A}_n$ (See Figure 1.)

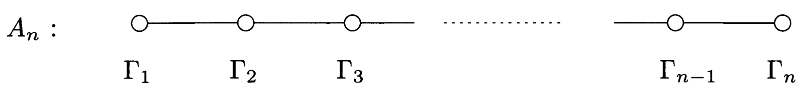

Figure 1:

The intersection matrix is as follows:

(i) $i = 1$

$$(\mathbf{A}_n)_{1,j} = \begin{cases} -2 & j = 1 \\ 1 & j = 2 \\ 0 & 3 \le j \le n \end{cases}$$

(ii) $2 \le i \le n-1$

$$(\mathbf{A}_n)_{i,j} = \begin{cases} 0 & 1 \le j \le i-2 \\ 1 & j = i-1 \\ -2 & j = i \\ 1 & j = i+1 \\ 0 & i+2 \le j \le n \end{cases}$$

(iii) $i = n$

$$(\mathbf{A}_n)_{n,j} = \begin{cases} 0 & 1 \le j \le n-2 \\ 1 & j = n-1 \\ -2 & j = n \end{cases}$$

Its inverse is given as follows:

$$(\mathbf{A}_n^{-1})_{i,j} = \begin{cases} -\dfrac{i(n-j+1)}{n+1} & 1 \le i \le j \\ -\dfrac{j(n-i+1)}{n+1} & j \le i \le n \end{cases}$$

2. $\mathbf{D}_n$ (See Figure 2.)

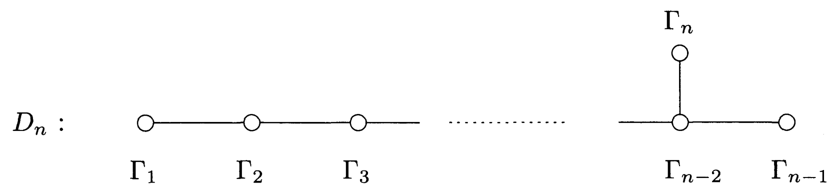

Figure 2:

The intersection matrix is as follows:

(i) $i = 1$

$$(\mathbf{D}_n)_{1,j} = \begin{cases} -2 & j = 1 \\ 1 & j = 2 \\ 0 & 3 \le j \le n \end{cases}$$

(ii) $2 \leq i \leq n-3$

$$(\mathbf{D}_n)_{i,j} = \begin{cases} 0 & 1 \leq j \leq i-2 \\ 1 & j = i-1 \\ -2 & j = i \\ 1 & j = i+1 \\ 0 & i+2 \leq j \leq n \end{cases}$$

(iii) $i = n-2$

$$(\mathbf{D}_n)_{n-2,j} = \begin{cases} 0 & 1 \leq j \leq n-4 \\ 1 & j = n-3 \\ -2 & j = n-2 \\ 1 & j = n-1, n \end{cases}$$

(iv) $i = n-1$

$$(\mathbf{D}_n)_{n-1,j} = \begin{cases} 0 & 1 \leq j \leq n-3 \\ 1 & j = n-2 \\ -2 & j = n-1 \\ 0 & j = n \end{cases}$$

(v) $i = n$

$$(\mathbf{D}_n)_{n,j} = \begin{cases} 0 & 1 \leq j \leq n-3 \\ 1 & j = n-2 \\ 0 & j = n-1 \\ -2 & j = n \end{cases}$$

Its inverse is given as follows:

(i) $1 \leq j \leq n-2$

$$(\mathbf{D}_n^{-1})_{i,j} = \begin{cases} -i & 1 \leq i \leq j \\ -j & j \leq i \leq n-2 \\ -\frac{j}{2} & j = n-1, n \end{cases}$$

(ii) $j = n-1$

$$(\mathbf{D}_n^{-1})_{i,n-1} = \begin{cases} -\frac{i}{2} & 1 \leq i \leq n-2 \\ -\frac{n}{4} & i = n-1 \\ -\frac{n-2}{4} & i = n \end{cases}$$

(iii) $j = n$

$$(\mathbf{D}_n^{-1})_{i,n} = \begin{cases} -\dfrac{i}{2} & 1 \le i \le n-2 \\ -\dfrac{n-2}{4} & i = n-1 \\ -\dfrac{n}{4} & i = n \end{cases}$$

3. $\mathbf{E}_6$, $\mathbf{E}_7$ and $\mathbf{E}_8$ (See Figure 3, 4 and 5.)

E_6 :

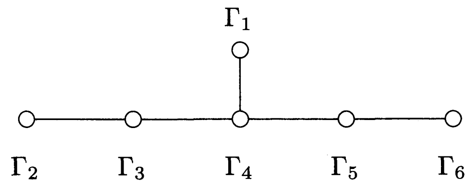

Figure 3:

E_7 :

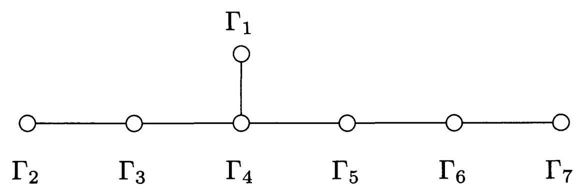

Figure 4:

E_8 :

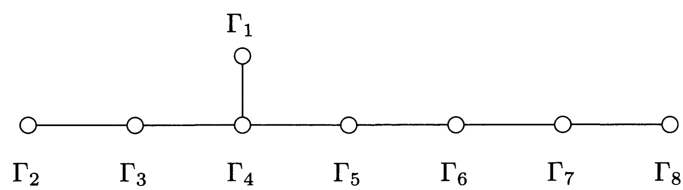

Figure 5:

We have the following;

$\mathbf{E}_6$:

$$\begin{pmatrix} -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}^{-1} = -\frac{1}{3}\begin{pmatrix} 6 & 3 & 6 & 9 & 6 & 3 \\ 3 & 4 & 5 & 6 & 4 & 2 \\ 6 & 5 & 10 & 12 & 8 & 4 \\ 9 & 6 & 12 & 18 & 12 & 6 \\ 6 & 4 & 8 & 12 & 10 & 5 \\ 3 & 2 & 4 & 6 & 5 & 4 \end{pmatrix}$$

$\mathbf{E}_7$:

$$\begin{pmatrix} -2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}^{-1}$$

$$= -\frac{1}{2}\begin{pmatrix} 7 & 4 & 8 & 12 & 9 & 6 & 3 \\ 4 & 4 & 6 & 8 & 6 & 4 & 2 \\ 8 & 6 & 12 & 16 & 12 & 8 & 4 \\ 12 & 8 & 16 & 24 & 18 & 12 & 6 \\ 9 & 6 & 12 & 18 & 15 & 10 & 5 \\ 6 & 4 & 8 & 12 & 10 & 8 & 4 \\ 3 & 2 & 4 & 6 & 5 & 4 & 3 \end{pmatrix}$$

$\mathbf{E}_8$:

$$\begin{pmatrix} -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}^{-1}$$

$$
= - \begin{pmatrix}
8 & 5 & 10 & 15 & 12 & 9 & 6 & 3 \\
5 & 4 & 7 & 10 & 8 & 6 & 4 & 2 \\
10 & 7 & 14 & 20 & 8 & 6 & 4 & 2 \\
15 & 10 & 20 & 30 & 24 & 18 & 12 & 6 \\
12 & 8 & 16 & 24 & 20 & 15 & 10 & 5 \\
9 & 6 & 12 & 18 & 15 & 12 & 8 & 4 \\
6 & 4 & 8 & 12 & 10 & 8 & 6 & 3 \\
3 & 2 & 4 & 6 & 5 & 4 & 3 & 2
\end{pmatrix}
$$

References

[1] E. Bombieri: *Canonical models of surfaces of general type*, Pub.Math.IHES **42** (1973), 171–219.

[2] K. Kodaira: *Pluricanonical systems on algebraic surfaces of general type*, J. Math.Soc.Japan **20** (1968), 170–192.

[3] I. Reider: *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann. Math. **127** (1988), 309–316.

[4] J. Yang: *A projective embedding of surfaces of general type*, Algebraic Geometry and Algebraic Number Theory, (Feng Ke-Qin and Li Ke-Zheng eds.), Nankai Series in Pure, Applied Mathematics and Theoretical Physics vol. **3**, World Scientific, Singapore-New Jersey-London-Hong Kong, 1992.

Note added in proof. Recently, the auther was intormed by Professor Sigehumi Mori that Theorem 1 is also obtained as a conseguence of J.Lipman's classical result (cf. Inst. Hutes. É tudes. Sci. publ. 195-279). The author would like to express his gratitude to Professor Mori for this instruction.

Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka Osaka 560
Japan
e-mail: arakawa@math.sci.osaka-u.ac.jp