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PARTIALLY NEGATIVE CYCLES AND PROJECTIVE EMBEDDINGS OF SURFACES OF GENERAL TYPE

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1. Introduction

Let S denote a nonsingular minimal complex algebraic surface of general type and Γ_1, Γ_2, ..., Γ_n the (-2)-curves on S. Then it is well known that for $m \geq 5$, the pluricanonical system $|mK_S|$ defines a holomorphic map $S \to \mathbf{P}^N$, which is embedding exept Γ_i's (cf. [1] and [2] or [3]). On the other hand, Yang [4] defined the minimally negative cycle W on S and showed that for sufficiently large m, the linear system $|mK_S - W|$ gives a projective embedding of S which is everywhere injective.

In the present paper we will construct some projective embeddings of S which take, in some sense, *middle positions* between the above two.

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2. Minimally negative cycle and partially negative cycles

In the following arguments, we always assume that S has some (-2)-curves Γ_i.

In [4], Yang introduced the notion of the *minimally negative cycle* on S as follows:

DEFINITION 1 (cf. [4, Definition 1.1]). A cycle $D = r_1\Gamma_1 + r_2\Gamma_2 + \cdots + r_n\Gamma_n$ is said to be *negative* if $\Gamma_j D < 0$ $(1 \leq j \leq n)$. A negative cycle W is called the *minimally negative cycle* if, for every negative cycle D, $W \leq D$.

The existance and uniquness of the minimally negative cycle on a surface are shown by the same way as for the foundamental cycle. Moreover we have:

Lemma 1. (cf. [4, Corollary 1.5]) *Let W be the minimally negative cycle. Then we have $\Gamma_j W = -1$ or -2 $(1 \leq j \leq n)$.*

Now we will generalize the above definition.

DEFINITION 2. Let $i_1, i_2, \ldots, i_l$ denote any numbers which are mutually distinct. A cycle $X = X_{i_1,i_2,\ldots,i_l} = s_1\Gamma_1 + s_2\Gamma_2 + \cdots + s_n\Gamma_n$ is called *partially negative cycle* for the suffixes $(i_1, i_2, \ldots, i_l)$ if $\Gamma_{i_j}X = 0$ for every j and $\Gamma_i X < 0$ for every $i \neq i_1, i_2, \ldots, i_l$.

Note that a negative or partially negative cycle is always effective.

Lemma 2. *For any suffixes $(i_1, i_2, \ldots, i_l)$, there exists a partially negative cycle for them.*

Proof. We may assume $(i_1, i_2, \ldots, i_l) = (1, 2, \ldots, l)$ $(1 \leq l \leq n)$. Let $(a_{l+1}, a_{l+2}, \ldots, a_n)$ be negative rational numbers (e.g. $(-1, -1, \ldots, -1)$) and set

$$(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = (0, \ldots, 0, a_{l+1}, \ldots, a_n)M^{-1}$$

where M is the intersection matrix of the (-2)-curves $\Gamma_1, \ldots, \Gamma_n$ (cf. Appendix).

Now let m be a positive integer such that $s_j = m\tilde{s}_j$ is also a nonnegative interger $(1 \leq j \leq n)$. Then we get a cycle $X = s_1\Gamma_1 + \cdots + s_n\Gamma_n$ such that $\Gamma_j X = 0$ $(1 \leq j \leq l)$ and $\Gamma_j X < 0$ $(l+1 \leq j \leq n)$. □

Since the minimally negative cycles are given in [4, p.174], we get the following alternative construction:

Let W be a negative cycle on S (e.g. the minimally negative cycle). For l-variables $x_1, x_2, \cdots, x_l$, we set

$$\tilde{X}(x_1, x_2, \ldots, x_l) := W - x_1\Gamma_{i_1} - x_2\Gamma_{i_2} - \ldots - x_l\Gamma_{i_l}.$$

Let us consider the following linear equations on $(x_1, x_2, \ldots, x_l)$:

$$\Gamma_{i_j}\tilde{X}(x_1, x_2, \ldots, x_l) = 0 \quad (1 \leq j \leq l). \tag{1}$$

Then, since all the components of the inverse matrix of the intersection matrix of Γ_{i_1}, $\Gamma_{i_2}, \ldots, \Gamma_{i_l}$ are nonpositive (see Appendix) and moreover $\Gamma_{i_j}W < 0$ for all j, we get that (1) has a root

$$(x_1, x_2, \ldots, x_l) = (c_1, c_2, \ldots, c_l)$$

where all the c_j's are nonnegative rational numbers. Now let m be a positive integer such that all the mc_j's are also (nonnegative) integers, and we define X as $m\tilde{X}(c_1, c_2, \ldots, c_l)$. Then it is easy to see that $X = X_{i_1,i_2,\ldots i_l}$ is one of the partially negative cycles which we want.

For readers' convenience, we will give a concrete description for small l.Let W be the minimally negative cycle on S:

EXAMPLE 2.1 ($l = 1$). By Lemma 1, we have $\Gamma_1 W = -1$ or -2. So we can define $X = X_1$ by

$$X_1 = \begin{cases} 2W - \Gamma_1 & if \quad \Gamma_1 W = -1 \\ W - \Gamma_1 & if \quad \Gamma_1 W = -2 \end{cases}$$

EXAMPLE 2.2 ($l = 2$). We may assume $\Gamma_1 W \geq \Gamma_2 W$, that is, $(\Gamma_1 W,\ \Gamma_2 W) = (-1, -1)$, $(-1, -2)$ or $(-2, -2)$. Then we can define $X = X_{1,2}$ as follows:

(I) The case of $\Gamma_1 \Gamma_2 = 0$.

$$X_{1,2} = \begin{cases} 2W - \Gamma_1 - \Gamma_2 & if \quad (\Gamma_1 W,\ \Gamma_2 W) = (-1, -1) \\ 2W - \Gamma_1 - 2\Gamma_2 & if \quad (\Gamma_1 W,\ \Gamma_2 W) = (-1, -2) \\ W - \Gamma_1 - \Gamma_2 & if \quad (\Gamma_1 W,\ \Gamma_2 W) = (-2, -2) \end{cases}$$

(II) The case of $\Gamma_1 \Gamma_2 = 1$.

$$X_{1,2} = \begin{cases} W - \Gamma_1 - \Gamma_2 & if \quad (\Gamma_1 W,\ \Gamma_2 W) = (-1, -1) \\ 3W - 4\Gamma_1 - 5\Gamma_2 & if \quad (\Gamma_1 W,\ \Gamma_2 W) = (-1, -2) \\ W - 2\Gamma_1 - 2\Gamma_2 & if \quad (\Gamma_1 W,\ \Gamma_2 W) = (-2, -2) \end{cases}$$

3. Projective embeddings of S

Let $X = X_{i_1, i_2, \ldots,\ i_l}$ denote a partially negative cycle for suffixes $(i_1, i_2, \ldots, i_l)$ on S.

Theorem 1. *For a sufficiently large n, the divisor $nK_S - X$ defines a holomorphic map $S \to \mathbf{P}^N$ which is an embeding except the indicated (-2)curves Γ_{i_1}, Γ_{i_2}, $\ldots$, Γ_{i_l}.*

REMARK 3.1. The meaning of *sufficiently large* is that there exists a number $n_0 = n_0(K_S^2, X^2)$ such that the theorem holds for every n larger than n_0.

REMARK 3.2. The following proof is almost parallel to that of [4, Theorem 2.2] except the last paragraph.

Let us recall a theorem of Reider (cf. [3, THEOREM 1]):

Reider's Theorem *Let S be a smooth complex algebraic surface and let L be a nef divisor on S.*

(i) *If $L^2 \geq 5$ and p is a base point of $|K_S + L|$, then there exists an effective divisor E on S passing through p such that*

either $LE = 0, E^2 = -1$

or $LE = 1, E^2 = 0$.

(ii) *If* $L^2 \geq 10$ *and points p, q, are not separated by* $|K_S + L|$, *then there exists an effective divisor* E *on* S *passing through* p *and* q *such that*

either $LE = 0$ *and* $E^2 = -1$ *or* -2

or $LE = 1$ *and* $E^2 = -1$ *or* 0

or $LE = 2$ *and* $E^2 = 0$.

Proof of Theorem 1. Let us begin with an inequality to estimate intersection numbers of X and other divisors. Let E denote an irreducible curve on S. Then, since $D := (K_S^2)E - (K_S E)K_S$ satisfies $DK_S(= XK_S) = 0$, we have $(X^2)(D^2) - (XD)^2 \geq 0$, which implies the following:

$$(XE)^2 \leq X^2 \left(E^2 - \frac{(K_S E)^2}{K^2} \right). \tag{2}$$

Now let L denote the divisor $(n-1)K_S - X$. Then we have $L^2 \geq 10$ if

$$(n-1)^2 \geq \frac{-X^2 + 10}{K_S^2}.$$

Hence, to apply the Reider's theorem, we need to verify that L is nef if n is sufficiently large.

Let E denote an irreducible curve on S as before. If $K_S E = 0$, then we have $XE \leq 0$ and hence $LE \geq 0$ for any n. So we assume $K_S E > 0$ and will show that $(n-1)^2(K_S E)^2 - (XE)^2 > 0$ (for large n). Since $K_S E > 0$, we have $3(K_S E)^2 + E^2 \geq (K_S E)^2 + E^2 + 2 \geq K_S E + E^2 + 2 \geq 0$ and therefore, by (2),

$$\begin{aligned}
(n-1)^2(K_S E)^2 - (XE)^2 &\geq (n-1)^2(K_S E)^2 - X^2 \left(E^2 - \frac{(K_S E)^2}{K_S^2} \right) \\
&= \left((n-1)^2 + \frac{X^2}{K_S^2} \right) (K_S E)^2 - X^2 E^2 \\
&= \left((n-1)^2 + \frac{X^2}{K_S^2} + 3X^2 \right) (K_S E)^2 \\
&\qquad - X^2 \big(3(K_S E)^2 + E^2 \big) \geq 0
\end{aligned}$$

for any n such that

$$(n-1)^2 + \frac{X^2}{K_S^2} + 3X^2 \geq 0.$$

Now suppose there exist points p, q on S which are not separated by the linear system $|nK_S - X|$. Then by Reider's theorem (ii), there exists a divisor $E > 0$ on S such that $(LE, E^2) = (0,-1),\ (0,-2),\ (1,0),\ (1,-1)$ or $(2,0)$.

(I) $(LE, E^2) \neq (0,-2)$. If $K_S E = 0$, then E^2 is even and negative, which contradict the above. Hence we have $K_S E > 0$ and therefore

$$\begin{aligned}(XE)^2 &\leq X^2\left(E^2 - \frac{(K_S E)^2}{K_S^2}\right)\\ &\leq X^2(E^2 - \frac{1}{K^2})(K_S E)^2.\end{aligned}$$

Consequently we get that

$$\begin{aligned}LE &= (n-1)K_S E - XE\\ &\geq \left((n-1) - \sqrt{X^2(E^2 - \frac{1}{K_S^2})}\right) K_S E\end{aligned}$$

with $E^2 = 0,\ -1$. Therefore, for large n, we have $LE > 2$, a contradiction.

(II) $(LE, E^2) = (0,-2)$. If $K_S E > 0$, then we get that $LE > 0$ for large n by the same way as in (I). Hence we have $K_S E = 0$, and hence $XE = 0$. Therefore E is nothing but a finite sum of the indicated (-2)-curves $\Gamma_{i_1},\ \Gamma_{i_2},\ \ldots,\ \Gamma_{i_l}$. □

Appendix

Though it may be well known, we will write down the inverse matrices of the intersection matrices of some of (-2)-curves on S. For simplicity, we consider only the cases that the (-2)-curves form a connected subset of S. We denote the (i,j) component of the intersection matrix (resp. the inverse matrix of it) for each types by $(\mathbf{A}_n)_{i,j}$ etc. (resp. $(\mathbf{A}_n^{-1})_{i,j}$ etc.). All the statements in this appendix are easily verified.

1. $\mathbf{A}_n$ (See Figure 1.)

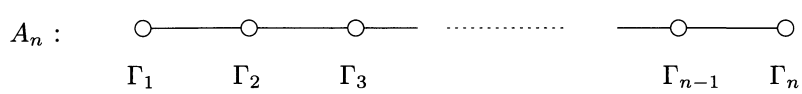

Figure 1:

The intersection matrix is as follows:

(i) $i = 1$

$$(\mathbf{A}_n)_{1,j} = \begin{cases} -2 & j = 1 \\ 1 & j = 2 \\ 0 & 3 \leq j \leq n \end{cases}$$

(ii) $2 \leq i \leq n-1$

$$(\mathbf{A}_n)_{i,j} = \begin{cases} 0 & 1 \leq j \leq i-2 \\ 1 & j = i-1 \\ -2 & j = i \\ 1 & j = i+1 \\ 0 & i+2 \leq j \leq n \end{cases}$$

(iii) $i = n$

$$(\mathbf{A}_n)_{n,j} = \begin{cases} 0 & 1 \leq j \leq n-2 \\ 1 & j = n-1 \\ -2 & j = n \end{cases}$$

Its inverse is given as follows:

$$(\mathbf{A}_n^{-1})_{i,j} = \begin{cases} -\dfrac{i(n-j+1)}{n+1} & 1 \leq i \leq j \\ -\dfrac{j(n-i+1)}{n+1} & j \leq i \leq n \end{cases}$$

2. $\mathbf{D}_n$ (See Figure 2.)

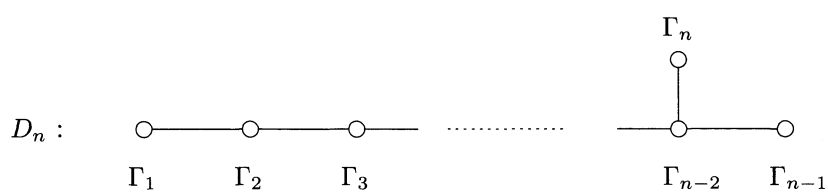

Figure 2:

The intersection matrix is as follows:

(i) $i = 1$

$$(\mathbf{D}_n)_{1,j} = \begin{cases} -2 & j = 1 \\ 1 & j = 2 \\ 0 & 3 \leq j \leq n \end{cases}$$

(ii) $2 \le i \le n-3$

$$(\mathbf{D}_n)_{i,j} = \begin{cases} 0 & 1 \le j \le i-2 \\ 1 & j = i-1 \\ -2 & j = i \\ 1 & j = i+1 \\ 0 & i+2 \le j \le n \end{cases}$$

(iii) $i = n-2$

$$(\mathbf{D}_n)_{n-2,j} = \begin{cases} 0 & 1 \le j \le n-4 \\ 1 & j = n-3 \\ -2 & j = n-2 \\ 1 & j = n-1, n \end{cases}$$

(iv) $i = n-1$

$$(\mathbf{D}_n)_{n-1,j} = \begin{cases} 0 & 1 \le j \le n-3 \\ 1 & j = n-2 \\ -2 & j = n-1 \\ 0 & j = n \end{cases}$$

(v) $i = n$

$$(\mathbf{D}_n)_{n,j} = \begin{cases} 0 & 1 \le j \le n-3 \\ 1 & j = n-2 \\ 0 & j = n-1 \\ -2 & j = n \end{cases}$$

Its inverse is given as follows:

(i) $1 \le j \le n-2$

$$(\mathbf{D}_n^{-1})_{i,j} = \begin{cases} -i & 1 \le i \le j \\ -j & j \le i \le n-2 \\ -\dfrac{j}{2} & j = n-1, n \end{cases}$$

(ii) $j = n-1$

$$(\mathbf{D}_n^{-1})_{i,n-1} = \begin{cases} -\dfrac{i}{2} & 1 \le i \le n-2 \\ -\dfrac{n}{4} & i = n-1 \\ -\dfrac{n-2}{4} & i = n \end{cases}$$

(iii) $j = n$

$$(\mathbf{D}_n^{-1})_{i,n} = \begin{cases} -\dfrac{i}{2} & 1 \le i \le n-2 \\ -\dfrac{n-2}{4} & i = n-1 \\ -\dfrac{n}{4} & i = n \end{cases}$$

3. $\mathbf{E}_6$, $\mathbf{E}_7$ and $\mathbf{E}_8$ (See Figure 3, 4 and 5.)

E_6 :

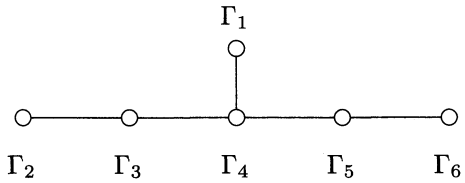

Figure 3:

E_7 :

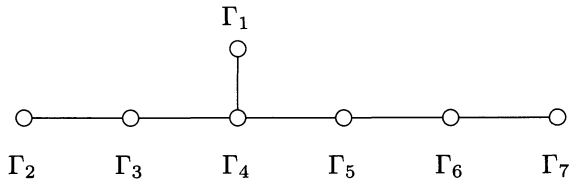

Figure 4:

E_8 :

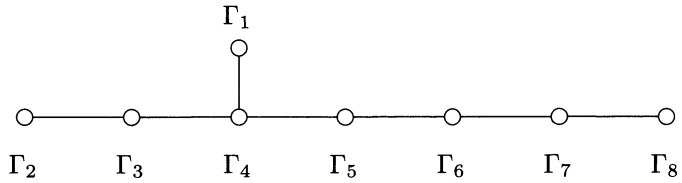

Figure 5:

We have the following;

$\mathbf{E}_6$:

$$\begin{pmatrix} -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}^{-1} = -\frac{1}{3}\begin{pmatrix} 6 & 3 & 6 & 9 & 6 & 3 \\ 3 & 4 & 5 & 6 & 4 & 2 \\ 6 & 5 & 10 & 12 & 8 & 4 \\ 9 & 6 & 12 & 18 & 12 & 6 \\ 6 & 4 & 8 & 12 & 10 & 5 \\ 3 & 2 & 4 & 6 & 5 & 4 \end{pmatrix}$$

$\mathbf{E}_7$:

$$\begin{pmatrix} -2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}^{-1}$$

$$= -\frac{1}{2}\begin{pmatrix} 7 & 4 & 8 & 12 & 9 & 6 & 3 \\ 4 & 4 & 6 & 8 & 6 & 4 & 2 \\ 8 & 6 & 12 & 16 & 12 & 8 & 4 \\ 12 & 8 & 16 & 24 & 18 & 12 & 6 \\ 9 & 6 & 12 & 18 & 15 & 10 & 5 \\ 6 & 4 & 8 & 12 & 10 & 8 & 4 \\ 3 & 2 & 4 & 6 & 5 & 4 & 3 \end{pmatrix}$$

$\mathbf{E}_8$:

$$\begin{pmatrix} -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}^{-1}$$

$$= -\begin{pmatrix} 8 & 5 & 10 & 15 & 12 & 9 & 6 & 3 \\ 5 & 4 & 7 & 10 & 8 & 6 & 4 & 2 \\ 10 & 7 & 14 & 20 & 8 & 6 & 4 & 2 \\ 15 & 10 & 20 & 30 & 24 & 18 & 12 & 6 \\ 12 & 8 & 16 & 24 & 20 & 15 & 10 & 5 \\ 9 & 6 & 12 & 18 & 15 & 12 & 8 & 4 \\ 6 & 4 & 8 & 12 & 10 & 8 & 6 & 3 \\ 3 & 2 & 4 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}$$

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Note added in proof. Recently, the auther was intormed by Professor Sigehumi Mori that Theorem 1 is also obtained as a conseguence of J.Lipman's classical result (cf. Inst. Hutes. É tudes. Sci. publ. 195-279). The author would like to express his gratitude to Professor Mori for this instruction.

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