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SELF-MAPPING DEGREES OF 3-MANIFOLDS

HONGBIN SUN, SHICHENG WANG, JIANCHUN WU and HAO ZHENG

(Received March 3, 2010, revised October 7, 2010)

Abstract

For each closed oriented 3-manifold $M$ in Thurston’s picture, the set of degrees of self-maps on $M$ is given.

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1. Introduction

1.1. Background. Each closed oriented $n$-manifold $M$ is naturally associated with a set of integers, the degrees of all self-maps on $M$, denoted as $D(M) = \{\deg(f) \mid f: M \to M\}$.

Indeed the calculation of $D(M)$ is a classical topic appeared in many literatures. The result is simple and well-known for dimension $n = 1, 2$. For dimension $n > 3$, there are many interesting special results (See [3], [10], [15] for recent ones and references therein), but it is difficult to get general results, since there are no classification results for manifolds of dimension $n > 3$.

The case of dimension 3 becomes attractive in the topic and it is possible to calculate $D(M)$ for any closed oriented 3-manifold $M$. Since Thurston’s geometrization conjecture, which seems to be confirmed, implies that closed oriented 3-manifolds can be classified in a reasonable sense.

2000 Mathematics Subject Classification. 55M25, 57M10.
Thurston’s geometrization conjecture claims that the each Jaco-Shalen-Johanson decomposition piece of a prime 3-manifold supports one of the eight geometries, which are $H^3$, $\overline{PSL}(2, R)$, $H^2 \times E^1$, Sol, Nil, $E^3$, $S^3$ and $S^2 \times E^1$ (for details see [24] and [20]). Call a closed orientable 3-manifold $M$ is geometrizable if each prime factor of $M$ meets Thurston’s geometrization conjecture. All 3-manifolds discussed in this paper are geometrizable.

The following result is known in early 1990’s:

**Theorem 1.0.** Suppose $M$ is a geometrizable 3-manifold. Then $M$ admits a self-map of degree larger than 1 if and only if $M$ is either

(a) covered by a torus bundle over the circle, or
(b) covered by $F \times S^1$ for some compact surface $F$ with $\chi(F) < 0$, or
(c) each prime factor of $M$ is covered by $S^3$ or $S^2 \times E^1$.

Hence for any 3-manifold $M$ not listed in (a)–(c) of Theorem 1.0, $D(M)$ is either $\{0, 1, -1\}$ or $\{0, 1\}$, which depends on whether $M$ admits a self-map of degree $-1$ or not. To determine $D(M)$ for geometrizable 3-manifolds listed in (a)–(c) of Theorem 1.0, let’s have a close look of them.

For short, we often call a 3-manifold supporting Nil geometry a Nil 3-manifold, and so on. Among Thurston’s eight geometries, six of them belong to the list (a)–(c) in Theorem 1.0. 3-manifolds in (a) are exactly those supporting either $E^3$, or Sol or Nil geometries. $E^3$ 3-manifolds, Sol 3-manifolds, and some Nil 3-manifolds are torus bundle or semi-bundles; Nil 3-manifolds which are not torus bundles or semi-bundles are Seifert fibered spaces having Euclidean orbifolds with three singular points. 3-manifolds in (b) are exactly those supporting $H^2 \times E^1$ geometry; 3-manifolds supporting $S^3$ or $S^2 \times E^1$ geometries form a proper subset of (3). Now we divide all 3-manifolds in the list (a)–(c) in Theorem 1.0 into the following five classes:

Class 1. $M$ supporting either $S^3$ or $S^2 \times E^1$ geometries;
Class 2. each prime factor of $M$ supporting either $S^3$ or $S^2 \times E^1$ geometries, but $M$ is not in Class 1;
Class 3. torus bundles and torus semi-bundles;
Class 4. Nil 3-manifolds not in Class 3;
Class 5. $M$ supporting $H^2 \times E^1$ geometry.

$D(M)$ is known recently for $M$ in Class 1 and Class 3. We will calculate $D(M)$ for $M$ in the remaining three classes. For the convenience of the readers, we will present $D(M)$ for $M$ in all those five classes. To do this, we need first to coordinate 3-manifolds in each class, then state the results of $D(M)$ in term of those coordinates. This is carried in the next subsection.
1.2. Main results.

Class 1. According to [13] or [20], the fundamental group of a 3-manifold supporting $S^3$-geometry is among the following eight types: $\mathbb{Z}_p$, $D^*_n$, $T^*_n$, $O^*_q$, $I^*_r$, $T_{8,3r}^*$, $D^*_n,2r^*$ and $\mathbb{Z}_m \times \pi_1(N)$, where $N$ is a 3-manifold supporting $S^3$-geometry, $\pi_1(N)$ belongs to the previous seven ones, and $[\pi_1(N)]$ is coprime to $m$. The cyclic group $\mathbb{Z}_p$ is realized by lens space $L(p, q)$, each group in the remaining types is realized by a unique 3-manifold supporting $S^3$-geometry. Note also the sub-indices of those seven types groups are exactly their orders, and the order of the groups in the last type is $m[\pi_1(N)]$. There are only two closed orientable 3-manifolds supporting $S^2 \times \mathbb{E}^1$ geometry: $S^2 \times S^1$ and $RP^3 \# RP^3$.

Theorem 1.1. (1) $D(M)$ for $M$ supporting $S^3$-geometry are listed below:

<table>
<thead>
<tr>
<th>$\pi_1(M)$</th>
<th>$D(M)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}_p$</td>
<td>${k^2 \mid k \in \mathbb{Z}} + p\mathbb{Z}$</td>
</tr>
<tr>
<td>$D^*_n$</td>
<td>${h^2 \mid h \in \mathbb{Z}; 2 \nmid h \text{ or } h = \text{nor } h = 0} + 4n\mathbb{Z}$</td>
</tr>
<tr>
<td>$T^*_n$</td>
<td>${0, 1, 16} + 24\mathbb{Z}$</td>
</tr>
<tr>
<td>$O^*_q$</td>
<td>${0, 1, 25} + 48\mathbb{Z}$</td>
</tr>
<tr>
<td>$I^*_r$</td>
<td>${0, 1, 49} + 120\mathbb{Z}$</td>
</tr>
<tr>
<td>$T_{8,3r}^*$</td>
<td>${(k^2 \cdot (3^{3r-2p} - 3^q) \mid 3 \nmid k, q \geq p &gt; 0} + 8 \cdot 3^q\mathbb{Z}$</td>
</tr>
<tr>
<td>$D^<em>_n,2r^</em>$</td>
<td>${(k^2 \cdot (3^{3r-2p} - 3^{q+1}) \mid 3 \nmid k, q \geq p &gt; 0} + 8 \cdot 3^q\mathbb{Z}$</td>
</tr>
<tr>
<td>$\mathbb{Z}_m \times \pi_1(N)$</td>
<td>$\left{d \in \mathbb{Z} \left</td>
</tr>
</tbody>
</table>

(2) $D(S^2 \times S^1) = D(RP^3 \# RP^3) = \mathbb{Z}$.

Class 2. We assume that each 3-manifold $P$ supporting $S^3$-geometry has the canonical orientation induced from the canonical orientation on $S^3$. When we change the orientation of $P$, the new oriented 3-manifold is denoted by $\tilde{P}$. Moreover, lens space $L(p, q)$ is orientation reversed homeomorphic to $L(p, p - q)$, so we can write all the lens spaces connected summands as $L(p, q)$. Now we can decompose each 3-manifold in Class 2 as

$$M = (mS^2 \times S^1) \# (m_1P_1 \# n_1\tilde{P}_1) \# \cdots \# (m_sP_s \# n_s\tilde{P}_s) \# \cdots \# L(p, q_{i,1}) \# \cdots \# L(p, q_{i,1}) \# \cdots \# L(p, q_{i,r_i})$$

where all the $P_i$ are 3-manifolds with finite fundamental group different from lens spaces,
all the \( P_i \) are different from each other, and all the positive integer \( p_i \) are different from each other. Define

\[
D_{\text{iso}}(M) = \{ \text{deg}(f) \mid f : M \to M, \ f \text{ induces an isomorphism on } \pi_1(M) \}.
\]

**Theorem 1.2.**  
(1) \( D(M) = D_{\text{iso}}(m_1 P_1 \# n_1 \tilde{P}_1) \cap \cdots \cap D_{\text{iso}}(m_s P_s \# n_s \tilde{P}_s) \cap D_{\text{iso}}(L(p_1, q_{1,1}) \# \cdots \# L(p_1, q_{1,r_1})) \cap \cdots \cap D_{\text{iso}}(L(p_r, q_{1,1}) \# \cdots \# L(p_r, q_{r,r}));
\]

(2) \( D_{\text{iso}}(mP \# n\tilde{P}) = \begin{cases} 
D_{\text{iso}}(P) & \text{if } m \neq n, \\
D_{\text{iso}}(P) \cup (-D_{\text{iso}}(P)) & \text{if } m = n;
\end{cases} \)

(3) \( D_{\text{iso}}(L(p, q_1) \# \cdots \# L(p, q_n)) = H^{-1}(C). \)

The notions \( H \) and \( C \) in Theorem 1.2 (3) is defined as below:

Let \( U_p = \{ \text{all units in ring } \mathbb{Z}_p \}, U^2_p = \{ a^2 \mid a \in U_p \}, \) which is a subgroup of \( U_p \).

We consider the quotient \( U_p/U^2_p = \{ a_1, \ldots, a_m \}, \) every \( a_i \) corresponds with a coset \( A_i \) of \( U^2_p \). For the structure of \( U_p \), see [9] p. 44. Define \( H \) to be the natural projection from \( \{ n \in \mathbb{Z} \mid \text{gcd}(n, p) = 1 \} \) to \( U_p/U^2_p \).

Define \( \tilde{A}_i = \{ L(p, q_i) \mid q_i \in A_i \} \) (with repetition allowed). In \( U_p/U^2_p \), define \( B_i = \{ a_i \mid \# \tilde{A}_i = l \} \) for \( l = 1, 2, \ldots \), there are only finitely many \( l \) such that \( B_i \neq \emptyset \). Let \( C_l = \{ a \in U_p/U^2_p \mid a_i \in B_i, \forall a_i \in B_i \} \) if \( B_i \neq \emptyset \) and \( C_l = U_p/U^2_p \) otherwise. Define \( C = \cap_{l=1}^{\infty} C_l \).

**Class 3.** To simplify notions, for a diffeomorphism \( \phi \) on torus \( T \), we also use \( \phi \) to present its isotopy class and its induced 2 \times 2 matrix on \( \pi_1(T) \) for a given basis.

A torus bundle is \( M_\phi = T \times I/(x, 1) \sim (\phi(x), 0) \) where \( \phi \) is a diffeomorphism of the torus \( T \) and \( I \) is the interval \([0, 1] \). Then the coordinates of \( M_\phi \) is given as below:

(1) \( M_\phi \) admits \( E^2 \) geometry, \( \phi \) conjugates to a matrix of finite order \( n \), where \( n \in \{ 1, 2, 3, 4, 6 \} \);

(2) \( M_\phi \) admits Nil geometry, \( \phi \) conjugates to \( \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \), where \( n \neq 0 \);

(3) \( M_\phi \) admits Sol geometry, \( \phi \) conjugates to \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), where \( |a + d| > 2, ad - bc = 1 \).

A torus semi-bundle \( N_\phi = N \cup_{\phi} N \) is obtained by gluing two copies of \( N \) along their torus boundary \( \partial N \) via a diffeomorphism \( \phi \), where \( N \) is the twisted \( I \)-bundle over the Klein bottle. We have the double covering \( p: S^1 \times S^1 \times I \to N = S^1 \times S^1 \times I/\tau \), where \( \tau \) is an involution such that \( (x, y, z) = (x + \pi, -y, 1 - z) \).

Denote by \( l_0 \) and \( l_\infty \) on \( \partial N \) be the images of the second \( S^1 \) factor and first \( S^1 \) factor on \( S^1 \times S^1 \times \{ 1 \} \). A canonical coordinate is an orientation of \( l_0 \) and \( l_\infty \), hence there are four choices of canonical coordinate on \( \partial N \). Once canonical coordinates on each \( \partial N \) are chosen, \( \phi \) is identified with an element \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) of \( GL_2(\mathbb{Z}) \) given by \( \phi(l_0, l_\infty) = (l_0, l_\infty)_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}. \)
With suitable choice of canonical coordinates of $\partial N$, $N_{\phi}$ has coordinates as below:

1. $N_{\phi}$ admits $E^3$ geometry, $\phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;
2. $N_{\phi}$ admits Nil geometry, $\phi = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}$, or $\begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}$, where $z \neq 0$;
3. $N_{\phi}$ admits Sol geometry, $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $abcd \neq 0$, $ad - bc = 1$.

**Theorem 1.3.** $D(M_{\phi})$ is in the table below for torus bundle $M_{\phi}$, where $\delta(3) = \delta(6) = 1$, $\delta(4) = 0$.

<table>
<thead>
<tr>
<th>$M_{\phi}$</th>
<th>$\phi$</th>
<th>$D(M_{\phi})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E^3$ finite order $k = 1, 2$</td>
<td>$\begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix}$</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>$E^3$ finite order $k = 3, 4, 6$</td>
<td>$(kt + 1)(p^2 - \delta(k)pq + q^2)</td>
<td>t, p, q \in \mathbb{Z}$</td>
</tr>
<tr>
<td>Nil</td>
<td>$\pm \begin{pmatrix} 1 &amp; 0 \ n &amp; 1 \end{pmatrix}$, $n \neq 0$</td>
<td>$\left{l^2</td>
</tr>
<tr>
<td>Sol</td>
<td>$\begin{pmatrix} a &amp; b \ c &amp; d \end{pmatrix}$, $</td>
<td>a + d</td>
</tr>
</tbody>
</table>

(2) $D(N_{\phi})$ is listed in the table below for torus semi-bundle $N_{\phi}$, where $\delta(a, d) = ad/gcd(a, d)^2$.

<table>
<thead>
<tr>
<th>$N_{\phi}$</th>
<th>$\phi$</th>
<th>$D(N_{\phi})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E^3$</td>
<td>$\begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix}$</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>$E^3$</td>
<td>$\begin{pmatrix} 0 &amp; 1 \ 1 &amp; 0 \end{pmatrix}$</td>
<td>$(2l + 1</td>
</tr>
<tr>
<td>Nil</td>
<td>$\begin{pmatrix} 1 &amp; 0 \ z &amp; 1 \end{pmatrix}$, $z \neq 0$</td>
<td>$\left{l^2</td>
</tr>
<tr>
<td>Nil</td>
<td>$\begin{pmatrix} 0 &amp; 1 \ 1 &amp; z \end{pmatrix}$ or $\begin{pmatrix} 1 &amp; 0 \ z &amp; 1 \end{pmatrix}$, $z \neq 0$</td>
<td>$\left{(2l + 1)^2</td>
</tr>
<tr>
<td>Sol</td>
<td>$\begin{pmatrix} a &amp; b \ c &amp; d \end{pmatrix}$, $abcd \neq 0$, $ad - bc = 1$</td>
<td>$\left{(2l + 1)^2</td>
</tr>
</tbody>
</table>

To coordinate 3-manifolds in Class 4 and Class 5, we first recall the well known coordinates of Seifert fibered spaces.

Suppose an oriented 3-manifold $M'$ is a circle bundle with a given section $F$, where $F$ is a compact surface with boundary components $c_1, \ldots, c_n$ with $n > 0$. On each boundary component of $M'$, orient $c_i$ and the circle fiber $h_i$ so that the product of their orientations match with the induced orientation of $M'$ (call such pairs $\{(c_i, h_i)\}$ a section-fiber coordinate system). Now attach $n$ solid tori $S_i$ to the $n$ boundary tori.
of $M'$ such that the meridian of $S_i$ is identified with slope $r_i = c_i^\alpha \beta_i^\beta$, where $\alpha > 0$, $(\alpha, \beta) = 1$. Denote the resulting manifold by $M(\pm g; \beta_1/\alpha_1, \ldots, \beta_n/\alpha_n)$ which has the Seifert fiber structure extended from the circle bundle structure of $M'$, where $g$ is the genus of the section $F$ of $M$, with the sign $+$ if $F$ is orientable and $-$ if $F$ is non-orientable, here ‘genus’ of nonorientable surfaces means the number of $RP^2$ connected summands. Call $e(M) = \sum_{i=1}^n \beta_i/\alpha_i \in \mathbb{Q}$ the Euler number of the Seifert fibration.

**Class 4.** If a Nil manifold $M$ is not a torus bundle or torus semi-bundle, then $M$ has one of the following Seifert fibreing structures: $M(0; \beta_1/2, \beta_2/3, \beta_3/6)$, $M(0; \beta_1/3, \beta_2/3, \beta_3/3)$, or $M(0; \beta_1/2, \beta_2/4, \beta_3/4)$, where $e(M) \in \mathbb{Q} - \{0\}.

**Theorem 1.4.** For 3-manifold $M$ in Class 4, we have

1. $D(M(0; \beta_1/2, \beta_2/3, \beta_3/6)) = \{l^2 \mid l = m^2 + mn + n^2, l \equiv 1 \pmod{6}, m, n \in \mathbb{Z}\}$;
2. $D(M(0; \beta_1/3, \beta_2/3, \beta_3/3)) = \{l^2 \mid l = m^2 + mn + n^2, l \equiv 1 \pmod{3}, m, n \in \mathbb{Z}\}$;
3. $D(M(0; \beta_1/2, \beta_2/4, \beta_3/4)) = \{l^2 \mid l = m^2 + n^2, l \equiv 1 \pmod{4}, m, n \in \mathbb{Z}\}$.

**Class 5.** All manifolds supporting $H^2 \times E^1$ geometry are Seifert fibered spaces $M$ such that $e(M) = 0$ and the Euler characteristic of the orbifold $\chi(O_M) < 0$.

Suppose $M = (g; \beta_{1,1}/\alpha_1, \ldots, \beta_{1,m_1}/\alpha_1, \ldots, \beta_{n,1}/\alpha_n, \ldots, \beta_{n,m_n}/\alpha_n)$, where all the integers $\alpha_i > 1$ are different from each other, and $\sum_{i=1}^n \sum_{j=1}^{m_i} \beta_{i,j}/\alpha_i = 0$.

For each $\alpha_i$ and each $a \in U_{\alpha_i}$, define $\theta_a(\alpha_i) = \#\{\beta_{i,j} \mid p_i(\beta_{i,j}) = a\}$ (with repetition allowed), $p_i$ is the natural projection from $\{n \mid \gcd(n, \alpha_i) = 1\}$ to $U_{\alpha_i}$. Define $B_i(\alpha_i) = \{a \mid \theta_a(\alpha_i) = l\}$ for $l = 1, 2, \ldots$, there are only finitely many $l$ such that $B_i(\alpha_i) \neq \emptyset$. Let $C_i(\alpha_i) = \{b \in U_{\alpha_i} \mid ab \in B_i(\alpha_i), \forall a \in B_i(\alpha_i)\}$ if $B_i(\alpha_i) \neq \emptyset$ and $C_i(\alpha_i) = U_{\alpha_i}$ otherwise. Finally define $C(\alpha_i) = \bigcap_{i=1}^\infty C_i(\alpha_i)$, and $\bar{C}(\alpha_i) = p_i^{-1}(C(\alpha_i))$.

**Theorem 1.5.** $D(M(g; \beta_{1,1}/\alpha_1, \ldots, \beta_{1,m_1}/\alpha_1, \ldots, \beta_{n,1}/\alpha_n, \ldots, \beta_{n,m_n}/\alpha_n)) = \bigcap_{i=1}^m \bar{C}(\alpha_i)$.

1.3. A brief comment of the topic and organization of the paper. Theorem 1.0 was appeared in [25]. The proof of the “only if” part in Theorem 1.0 is based on the results on simplicial volume developed by Gromov, Thurston and Soma (see [21]), and various classical results by others on 3-manifold topology and group theory ([5], [19], [17]). The proof of “if” part in Theorem 1.0 is a sequence elementary constructions, which were essentially known before, for example see [6] and [11] for (3). That graph manifolds admits no self-maps of degrees $> 1$ also follows from a recent work [2].

The table in Theorem 1.1 is quoted from [1], which generalizes the earlier work [7]. The statement below quoted from [7] will be repeatedly used in this paper.

**Proposition 1.6.** For 3-manifold $M$ supporting $S^3$ geometry,

$$D_{\text{to}}(M) = \{k^2 + l |\pi_1(M)|, \text{ where } k \text{ and } |\pi_1(M)| \text{ are co-prime}\.$$
The topic of mapping degrees between (and to) 3-manifolds covered by $S^3$ has been discussed for long times and has many relations with other topics (see [26] for details). We just mention several papers: in very old papers [16] and [14], the degrees of maps between any given pairs of lens spaces are obtained by using equivalent maps between spheres; in [8], $D(M, L(p, q))$ can be computed for any 3-manifold $M$; and in a recent one [12], an algorithm (or formula) is given to the degrees of maps between given pairs of 3-manifolds covered by $S^3$ in term of their Seifert invariants.

Theorem 1.3 is proved in [23].

Theorems 1.2, 1.4 and 1.5 will be proved in Sections 3, 4 and 5 respectively in this paper. In Section 2 we will compute $D(M)$ for some concrete 3-manifolds using Theorems 1–5. We will also discuss when $-1 \in D(M)$ and when $-1 \not\in D(M)$ implies that $M$ admits orientation reversing homeomorphisms.

All terminologies not defined are standard, see [5], [20] and [9].

2. Examples of computation, orientation reversing homeomorphisms

**Example 2.1.** Let $M = (P \# \bar{P}) \# (L(7, 1) \# L(7, 2) \# 2L(7, 3))$, where $P$ is the Poincare homology three sphere.

By Theorem 1.2 (2), Proposition 1.6 and the fact $|\tau_1(P)| = 120$, we have $D(P \# \bar{P}) = D_{\text{iso}}(P) \cup (-D_{\text{iso}}(P)) = \{120n + i \mid n \in \mathbb{Z}, \ i = 1, 49, 71, 119\}$.

Now we are going to calculate $D((L(7, 1)\# L(7, 2) \# 2L(7, 3)))$ following the notions of Theorem 1.2 (3). Clearly $U_1 = \{1, 2, 3, 4, 5, 6\}$ and $U_2 = \{1, 2, 4\}$. Then $U_7/U_2^2 = \{a_1, a_2\}$, where $a_1 = 1$ and $a_2 = \bar{3}$; $U_7 = \{A_1 \cup A_2\}$, where $A_1 = U_7^2$, $A_2 = 3U_7^2$; $\# \bar{A}_1 = 2$ and $\# \bar{A}_2 = 2$; $B_2 = \{1, \bar{3}\}$, $B_l = \emptyset$ for $l \not= 2$. Since $U_7/U_2^2 = B_2$, we have $C_2 = B_2$ and also $C_1 = U_7/U_2^2$ for $l \not= 2$; then $C = \bigcap_{l=1}^{\infty} C_l = U_7/U_2^2$. Then for the natural projection $H: \{n \in \mathbb{Z} \mid \gcd(n, p) = 1\} \to U_7/U_2^2$, $H^{-1}(C)$ are all number coprime to 7, hence we have $D_{\text{iso}}((L(7, 1)\# L(7, 2) \# 2L(7, 3))) = \{l \in \mathbb{Z} \mid \gcd(l, 7) = 1\}$ by Theorem 1.2 (3).

Finally by Theorem 1.2 (1), we have $D(M) = \{120n + i \mid n \in \mathbb{Z}, \ i = 1, 49, 71, 119\} \cap \{l \in \mathbb{Z} \mid \gcd(l, 7) = 1\} = \{840n + i \mid n \in \mathbb{Z}, \ i = 1, 71, 121, 169, 191, 239, 241, 289, 311, 359, 361, 409, 431, 479, 481, 529, 551, 599, 601, 649, 671, 719, 769, 839\}$. Note $-1 \not\in D(M)$.

**Example 2.2.** Suppose $M = (2P \# \bar{P}) \# (L(7, 1) \# L(7, 2) \# L(7, 3))$.

Similarly by Theorem 1.2 (2), Proposition 1.6 and $|\tau_1(P)| = 120$, we have $D(2P \# \bar{P}) = D_{\text{iso}}(P) = \{120n + i \mid n \in \mathbb{Z}, \ i = 1, 49\}$.

To calculate $D(L(7, 1) \# L(7, 2) \# L(7, 3))$, we have $U_7, U_7^2, U_7/U_7^2 = \{a_1, a_2\}$, $U_7 = \{A_1, A_2\}$ exactly as last example. But then $\# \bar{A}_1 = 2$ and $\# \bar{A}_2 = 1$; $B_1 = \{\bar{3}\}$, $B_2 = \{1\}$, $B_l = \emptyset$ for $l \not= 1, 2$. Moreover $C_1 = C_2 = \{\bar{1}\}$, and $C_l = U_7/U_7^2$ for $l \not= 1, 2$; then $C = \bigcap_{l=1}^{\infty} C_l = \{\bar{1}\}$, and $H^{-1}(C) = \{7n + i \mid n \in \mathbb{Z}, \ i = 1, 2, 4\}$. Hence we have $D_{\text{iso}}((L(7, 1)\# L(7, 2) \# L(7, 3))) = \{7n + i \mid n \in \mathbb{Z}, \ i = 1, 2, 4\}$ by Theorem 1.2 (3).

By Theorem 1.2 (1), $D(M) = \{120n + i \mid n \in \mathbb{Z}, \ i = 1, 49\} \cap \{7n + i \mid n \in \mathbb{Z}, \ i = 1, 2, 4\} = \{840n + i \mid n \in \mathbb{Z}, \ i = 1, 121, 169, 289, 361, 529\}$. Note $-1 \not\in D(M)$. 

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EXAMPLE 2.3. By Theorem 1.3, for the torus bundle $M_\phi$, $\phi = \left( \begin{array}{cc} 2 & 1 \\ 1 & 1 \end{array} \right)$, among the first 20 integers $> 0$, exactly 1, 4, 5, 9, 11, 16, 19, $20 \in D(M_\phi)$.

EXAMPLE 2.4. For Nil 3-manifold $M = M(0; \beta_1/2, \beta_2/3, \beta_3/6)$, $D(M) = \{l^2 \mid l = m^2 + mn + n^2, l \equiv 1 \mod 6, m, n \in \mathbb{Z}\}$. The numbers in $D(M)$ smaller than 10000 are exactly 1, 49, 169, 361, 625, 961, 1369, 1849, 2401, 3721, 4489, 5329, 6241, 8291, 9409. Since all $l \equiv 1, 4, 5 \mod 6$ and $\frac{l^2}{6}$ is an integer, all $l \leq 10000$ can be presented as $m^2 + mn + n^2$ except $l = 55, 85$ (if $5 \mid m^2 + mn + n^2$, then $5 \mid (2m + n)^2 + 3n^2$, therefore $5 \mid 2m + n$ and $5 \mid n$, it follows that $25 \mid m^2 + mn + n^2$).

EXAMPLE 2.5. For $H^2 \times E^1$ manifold $M = M(2; 1/5, 1/5, -2/5, 1/7, 2/7, -3/7)$, we follow the notions in Theorem 1.5 to calculate $D(M)$.

First we have $U_5 = \{1, 2, 3, 4\}$ with indices $\theta_5(5)$ are $\{2, 0, 1, 0\}$ respectively. Then $B_1(5) = \{3\}, B_2(5) = \{1\}, B_1(5) = \emptyset$ for $l \neq 1, 2$ and $C_1(5) = C_2(5) = \{1\}$. Hence $C(5) = \bigcap_{i=1}^{\infty} C_i(5) = \{1\}$. Hence $\tilde{C}(5) = \{5n + 1 \mid n \in \mathbb{Z}\}$.

Similarly $U_7 = \{1, 2, 3, 4, 5, 6\}$ with indices $\theta_7(7)$ are $\{1, 1, 0, 1, 0, 0\}$ respectively. Then $B_1(7) = C_1(7) = \{1, 2, 4\}, B_1(7) = \emptyset$ and $C(7) = \bigcap_{i=1}^{\infty} C_i(7) = \{1, 2, 4\}$. $\tilde{C}(7) = \{7n + i \mid n \in \mathbb{Z}, i = 1, 2, 4\}$.

Finally $D(M) = \{5n + 1 \mid n \in \mathbb{Z}\} \cap \{7n + i \mid n \in \mathbb{Z}, i = 1, 2, 4\} = \{35n + i \mid n \in \mathbb{Z}, i = 1, 11, 16\}$.

EXAMPLE 2.6. Suppose $M$ is a 3-manifold supporting $S^3$ geometry. By Proposition 1.6, $M$ admits degree $-1$ self mapping if and only if there is integer number $h$, such that $h^2 \equiv -1 \mod \pi_1(M)$. Then we can prove that if $M$ is not a lens space, $-1 \notin D(M)$, (see proof of Proposition 3.10). With some further topological and number theoretical arguments, the following results were proved in [22].

(1) There is a degree $-1$ self map on $L(p, q)$, but no orientation reversing homeomorphism on it if and only if $(p, q)$ satisfies: $p \mid q^2 + 1$, $4 \nmid p$ and all the odd prime factors of $p$ are the $4k + 1$ type.

(2) Every degree $-1$ self map on $L(p, q)$ are homotopic to an orientation reversing homeomorphism if and only if $(p, q)$ satisfies: $q^2 \equiv -1 \mod p, p = 2, p_1^{e_1}, 2p_1^{e_1}$, where $p_1$ is a $4k + 1$ type prime number.

EXAMPLE 2.7. Suppose $M$ is a torus bundle. Then any non-zero degree map is homotopic to a covering ([25] Corollary 0.4). Hence if $-1 \notin D(M)$, then $M$ admits an orientation reversing self homeomorphism.

(1) For the torus bundle $M_\phi$, $\phi = \left( \begin{array}{cc} 2 & 1 \\ 1 & 1 \end{array} \right)$, $-1 \in D(M_\phi)$. Indeed for $\phi = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$, if $|a + d| = 3$, then $-1 \in D(M_\phi)$. Since $p^2 + ((d - a)/b)pr - c/br^2 = -1$ has solution $p = 1 - d, r = b$ when $a + d = 3$, and solution $p = -1 - d, r = b$ when $a + d = -3$.

(2) For the torus bundle $M_\phi$, $\phi = \left( \begin{array}{cc} 2 & 3 \\ 1 & 2 \end{array} \right)$, $-1 \notin D(M_\phi)$. Indeed for $\phi = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$, if $a + d \pm 2$ has prime decomposition $p_1^{e_1} \cdots p_n^{e_n}$ such that $p_i = 4l + 3$ and $e_i = 2m + 1$ for
some $i$, then $-1 \notin D(M_0)$. Since if the equation $p^2 + ((d-a)/b)pr - (c/b)r^2 = -1$ has integer solution, then $(((a+d)^2 - 4)r^2 - 4b^2)/b^2$ should be a square of rational number. That is $((a+d)^2 - 4)r^2 - 4b^2 = s^2$ for some integer $s$. Therefore $(a+d+2)(a+d-2)r^2$ is a sum of two squares. By a fact in elementary number theory, neither $a + d + 2$ nor $a + d - 2$ has $4k + 3$ type prime factor with odd power (see p. 279, [9]).

3. $D(M)$ for connected sums

3.1. Relations between $D_{iso}(M_1 \# M_2)$ and $\{D_{iso}(M_1), D_{iso}(M_2)\}$. In this section, we consider the manifolds $M$ in Class 2: $M$ has non-trivial prime decomposition, each connected summand has finite or infinite cyclic fundamental group, and $M$ is not homeomorphic to $RP^3 \# RP^3$. (Note for each geometrizable 3-manifold $P$, $\pi_1(P)$ is finite if and only if $P$ is $S^3$ 3-manifold, and $\pi_1(P)$ is infinite cyclic if and only if $P$ is $S^2 \times E^1$ 3-manifold.) Since each $S^3$ 3-manifold $P$ is covered by $S^3$, we assume $P$ has the canonical orientation induced by the canonical orientation on $S^3$. When we change the orientation of $P$, the new oriented 3-manifold is denoted by $\tilde{P}$. Moreover, lens space $L(p, q)$ is orientation reversing homeomorphic to $L(p, p - q)$, so we can write all the lens spaces connected summands as $L(p, q)$. Now we can decompose the manifold as

$$M = (mS^2 \times S^1) \# (m_1P_1 \# n_1\tilde{P}_1) \# \cdots \# (m_nP_n \# n_n\tilde{P}_n)$$

$$\# (L(p_1, q_{1,1}) \# \cdots \# L(p_1, q_{1,r_1})) \# \cdots \# (L(p_t, q_{t,1}) \# \cdots \# L(p_t, q_{t,r_t})), $$

where all the $P_i$ are 3-manifolds with finite fundamental group different from lens spaces, all the $P_i$ are different with each other, and all the positive integer $p_i$ are different from each other. We will use this convention in this section.

Suppose $F$ (resp. $P$) is a properly embedded surface (resp. an embedded 3-manifold) in a 3-manifold $M$. We use $M \setminus F$ (resp. $M \setminus P$) to denote the resulting manifold obtained by splitting $M$ along $F$ (resp. removing int $P$, the interior of $P$).

The definitions below are quoted from [17]:

**Definition 3.1.** Let $M$, $N$ be 3-manifolds and $B_f = \bigcup_i (B_i^+ \cup B_i^-)$ is a finite collection of disjoint 3-ball pairs in int $M$. A map $f: M \setminus B_f \to N$ is called an almost defined map from $M$ to $N$ if for each $i$, $f|_{\partial B_i^+} = f|_{\partial B_i^-} \circ r_i$ for some orientation reversing homeomorphism $r_i: \partial B_i^+ \to \partial B_i^-$. If identifying $\partial B_i^+$ with $\partial B_i^-$ via $r_i$, we get a quotient closed manifold $M(f)$, and $f$ induces a map $\tilde{f}: M(f) \to N$. We define $\deg(f) = \deg(\tilde{f})$.

**Definition 3.2.** For two almost defined maps $f$ and $g$, we say that $f$ is $B$-equivalent to $g$ if there are almost defined maps $f = f_0, f_1, \ldots, f_n = g$ such that either $f_i$ is homotopic to $f_{i+1}$ rel($\partial B_i \cup \partial B_{f_i}$) or $f_i = f_{i+1}$ on $M \setminus B$ for an union of balls $B$ containing $B_{f_i} \cup B_{f_{i+1}}$. 


Lemma 3.3 ([17] Lemma 3.6, [25] Lemma 1.11). Suppose $f \colon M \to M$ is a map of nonzero degree and $\bigcup S_i^2$ is an union of essential 2-spheres. Then there is an almost defined map $g \colon M \setminus B_g \to M$, B-equivalent to $f$, such that $\deg(g) = \deg(f)$ and $g^{-1}(\bigcup S_i^2)$ is a collection of spheres.

Lemma 3.4 ([25] Corollary 0.2). Suppose $M$ is a geometrizable 3-manifold. Then any nonzero degree proper map $f \colon M \to M$ induces an isomorphism $f_* \colon \pi_1(M) \to \pi_1(M)$ unless $M$ is covered by either a torus bundle over the circle, or $F \times S^1$ for some compact surface $F$, or the $S^3$.

The following lemma is well-known.

Lemma 3.5. Suppose $M$ is a closed orientable 3-manifold, $f \colon M \to M$ is of degree $d \neq 0$. Then $f_* \colon H_2(M, \mathbb{Q}) \to H_2(M, \mathbb{Q})$ is an isomorphism.

Theorem 3.6. Suppose $M = M_1 \# \cdots \# M_n$ is a non-prime manifold which is not homeomorphic to $RP^3 \# RP^3$. Each $\pi_1(M_i)$ is finite or cyclic, and $\pi_1(M_i) \neq 0$. If $f \colon M \to M$ is a map of degree $d \neq 0$, then there exists a permutation $\tau$ of $\{1, \ldots, n\}$, such that there is a map $g_i \colon M_{\tau(i)} \to M_i$ of degree $d$ for each $i$. Moreover, $g_i*$ is an isomorphism on fundamental group.

Proof. Call $M'$ a punctured $M$, if $M' = M \setminus B$, where $B$ is a finitely many disjoint 3-balls in the interior of $M$. We use $\hat{M}_n$ to denote the 3-manifold obtained from $M_n$ by capping off the boundary spheres with 3-balls.

$M$ is obtained by gluing the boundary sphere of $M'_1 = M_1 \setminus \text{int}(B_1)$ to a $n$-punctured 3-sphere. The image of $\partial B_1$ in $M$, which is denoted by $S_1$, is a separating sphere.

By Lemma 3.3, there is an almost defined map $g : M \setminus B_g \to M$, $B$-equivalent to $f$, such that $g^{-1}(\bigcup S_i)$ is a collection of spheres and $\deg(g) = d$. Let $M_g = M \setminus B_g$.

Let $U = M_g \setminus g^{-1}(\bigcup S_i) = \{M'_i \mid j = 1, \ldots, l_i, i = 1, \ldots, n\}$. The components of $g^{-1}(M'_i)$ are denoted by $M'_1, \ldots, M'_i$.

By Lemma 3.4, $f_* \colon \pi_1(M) \to \pi_1(M)$ is an isomorphism. Since $g$ is differ from $f$ just on the 3-balls $B_g$ up to homotopy rel $\partial B_g$, it follows that $g_* \colon \pi_1(M \setminus B_g) = \pi_1(M) \to \pi_1(M)$ is an isomorphism.

Since the prime decomposition of 3-manifold $M$ is unique, and $M_g$ is just a punctured $M$, each component of $U$ is either a punctured non-trivial prime factor of $M$, or a punctured 3-sphere.

By Lemma 3.5, $f_*$ is an injection on $H_2(M, \mathbb{Q})$. If $S_i$ is a separating sphere, then $[S_i] = 0$ in $H_2(M, \mathbb{Q})$. So each component $S'$ of $f^{-1}(S_i)$ is homologous to 0, thus $S'$ separates $M$. By the process of construction of $g$ (see the proof of Lemma 3.4, [17]), which is B-equivalent to $f$, each component $S$ of $g^{-1}(S_i)$ is also a separating sphere in $M_g$. So $\pi_1(M_g)$ is the free product of the $\pi_1(M'_i)$, $i = 1, \ldots, n$, $j = 1, \ldots, l_i$. 


Note \( \pi_1(M) = \pi_1(M_1) \ast \cdots \ast \pi_1(M_n) \), each \( \pi_1(M_i) \) is an indecomposable factor of \( \pi_1(M) \). Since \( g_1 \) is an isomorphism and each punctured 3-sphere has trivial \( \pi_1 \), from the basic fact on free product of groups, it follows that there is at least one punctured prime non-trivial factor in \( g_1^{-1}(M_i) \). Since this is true for each \( i = 1, \ldots, n \) and there are at most \( n \) punctured prime non-trivial factors in \( U \), it follows that there are \( n \) punctured prime non-trivial factors in \( U \). Hence there is exactly one punctured prime non-trivial factor in \( g_1^{-1}(M_i) \), denoted as \( M_{1(i)} \), moreover \( g_1: \pi_1(M_{1(i)}) \to \pi_1(M_i) \) is an isomorphism, where \( \tau \) is a permutation on \( \{1, \ldots, n\} \).

Since \( \pi_1(M_i) = \mathbb{Z} \) if and only if \( M_i = S^2 \times S^1 \), it follows that if \( M_i = S^2 \times S^1 \), then \( M_{1(i)} = S^2 \times S^1 \). Since \( D(S^2 \times S^1) = \mathbb{Z} \), below we assume that \( M_i \neq S^2 \times S^1 \), and to show that there is a map \( g_i: M_{1(i)} \to M_i \) of degree \( d \).

Since the map \( g: M \to M \) has degree \( d \) (see Definition 3.1), then \( g_i = g|_{\bigcup_{j=1}^i M'_j} : \bigcup_{j=1}^i M'_j \to M_i \) is a proper map of degree \( d \), which can extend to a map \( \hat{g}_i : \bigcup_{j=1}^i \hat{M}'_j \to M_i \) of degree \( d \) between closed 3-manifolds. The last map is also defined on \( \left( \bigcup_{j=1}^i M'_j \right) \setminus \partial B_\infty = \left( \bigcup_{j=1}^i \hat{M}'_j \right) \setminus B_\infty \subset \left( \bigcup_{j=1}^i \hat{M}'_j \right) \), where \( \partial B_\infty \subset M(g) \) is the image of \( \partial B_\infty \subset M \).

Now consider the map \( \tilde{g}_i : \left( \bigcup_{j=1}^i \hat{M}'_j \right) \setminus B_\infty \to M_i \). Since \( \pi_2(M_i) = 0 \), we can extend the map \( \tilde{g}_i \) from \( \bigcup_{j=1}^i \hat{M}'_j \setminus B_\infty \) to \( \bigcup_{j=1}^i \hat{M}'_j \). More carefully, for each pair \( B_\infty \), we can make the extension with the property \( \tilde{g}_i|_{B_\infty} = \hat{g}_i|_{B_\infty} \circ \tau_i \), where \( \tau_i : B_\infty \to B_\infty \) is an orientation reversing homeomorphism extending \( \tau_i : \partial B_\infty \to \partial B_\infty \). Now it is easy to see the map \( \hat{g}_i : \bigcup_{j=1}^i \hat{M}'_j \to M_i \) is still of degree \( d \).

From the map \( \hat{g}_i : \left( \bigcup_{j=1}^i \hat{M}'_j \right) \to M_i \) one can obviously define a map \( g_i : \#_{j=1}^i \hat{M}'_j \to M_i \) of degree \( d \) between connected 3-manifolds. Since all \( \hat{M}'_j \) are \( S^3 \) except one is \( M_{1(i)} \), we have map \( g_i : M_{1(i)} \to M_i \).

**Definition 3.7.** For closed oriented 3-manifold \( M, M' \), define

\[
D_{\text{iso}}(M, M') = \{ \deg(f) \mid f : M \to M', f \text{ induces isomorphism on fundamental group} \},
\]

\[
D_{\text{iso}}(M) = \{ \deg(f) \mid f : M \to M, f \text{ induces isomorphism on fundamental group} \}.
\]

Under the condition we considered in this section, we have \( D(M) = D_{\text{iso}}(M) \) by Lemma 3.4.

**Lemma 3.8.** Suppose \( f_i : M_i \to M'_i \) is a map of degree \( d \) between closed \( n \)-manifolds, \( n \geq 3 \), \( f_{i*} \) is surjective on \( \pi_1 \), \( i = 1, 2 \). Then there is a map \( f : M_1 \# M_2 \to M'_1 \# M'_2 \) of degree \( d \) and \( f_{A*} \) is surjective on \( \pi_1 \). In particular,

1. \( D_{\text{iso}}(M_1 \# M_2, M'_1 \# M'_2) \supset D_{\text{iso}}(M_1, M'_1) \cap D_{\text{iso}}(M_2, M'_2) \),
2. \( D_{\text{iso}}(M_1 \# M_2) \supset D_{\text{iso}}(M_1) \cap D_{\text{iso}}(M_2) \).

**Proof.** Since \( f_{A*} \) is surjective on \( \pi_1 \), it is known (see [18] for example), we can homotope \( f_i \) such that for some \( n \)-ball \( D'_i \subset M'_i \), \( f_i^{-1}(D_i) \) is an \( n \)-ball \( D_i \subset M_i \). Thus
we get a proper map \( \tilde{f}_i: M_1 \setminus D \to M_1' \setminus D_1' \) of degree \( d \), which also induces a degree \( d \) map from \( \partial D \) to \( \partial D_1' \). Since maps of the same degree between \((n - 1)\)-spheres are homotopic, so after proper homotopy, we can paste \( \tilde{f}_1 \) and \( \tilde{f}_2 \) along the boundary to get map \( f: M_1 \# M_2 \to M_1' \# M_2' \) of degree \( d \) and \( f \) is surjective on \( \pi_1 \).

3.2. \( D(M) \) for connected sums. Suppose

\[
M = (m S^2 \times S^1) \# (m_1 P_1 \# n_1 \tilde{P}_1) \# \cdots \# (m_3 P_3 \# n_3 \tilde{P}_3)
\]

\[
\# (L(p_1, q_{1,1}) \# \cdots \# L(p_1, q_{1,r_1})) \# \cdots \# (L(p_t, q_{t,1}) \# \cdots \# L(p_t, q_{t,r_t}))
\]

where all the \( P_i \) are 3-manifolds with finite fundamental group different from lens spaces, all the \( P_i \) are different with each other, and all the positive integer \( p_i \) are different from each other.

To prove Theorem 1.2, we need only to prove the three propositions below.

**Proposition 3.9.**

\[
D(M) = D_{\text{iso}}(m_1 P_1 \# n_1 \tilde{P}_1) \cap \cdots \cap D_{\text{iso}}(m_3 P_3 \# n_3 \tilde{P}_3) \cap D_{\text{iso}}(L(p_1, q_{1,1}) \# \cdots \# L(p_t, q_{t,1}) \# \cdots \# L(p_t, q_{t,r_t})).
\]

\(^{*}\)

Proof. For every self-mapping degree \( d \) of \( M \), in Theorem 3.6 we have proved that for every oriented connected summand \( P \) of \( M \), it corresponds to an oriented connected summand \( P' \), such that there is a degree \( d \) mapping \( f: P \to P' \), and \( f \) induces isomorphism on fundamental group. By the classification of 3-manifolds with finite fundamental group (see [13], 6.2), \( P \) and \( P' \) are homeomorphism (not considering the orientation) unless they are lens spaces with same fundamental group. Now by Lemma 3.8 (1), we have \( d \in D_{\text{iso}}(m_1 P_1 \# n_1 \tilde{P}_1) \) and \( d \in D_{\text{iso}}(L(p_j, q_{j,1}) \# \cdots \# L(p_j, q_{j,r_j})) \), for \( i = 1, \ldots, s \) and \( j = 1, \ldots, t \). Hence we have proved

\[
D(M) \subset D_{\text{iso}}(m_1 P_1 \# n_1 \tilde{P}_1) \cap \cdots \cap D_{\text{iso}}(m_3 P_3 \# n_3 \tilde{P}_3) \cap D_{\text{iso}}(L(p_1, q_{1,1}) \# \cdots \# L(p_t, q_{t,1}) \# \cdots \# L(p_t, q_{t,r_t})).
\]

(Since \( D(m S^2 \times S^1) = \mathbb{Z} \), we can just forget it in the discussion.)

Apply Lemma 3.8 once more, we finish the proof.

**Proposition 3.10.** If \( P \) is a 3-manifold with finite fundamental group different from lens space, \( D_{\text{iso}}(m P \# n \tilde{P}) = \begin{cases} D_{\text{iso}}(P) & \text{if } m \neq n, \\ D_{\text{iso}}(P) \cup (-D_{\text{iso}}(P)) & \text{if } m = n. \end{cases} \)

Proof. If \( P \) is not a lens space, from the list in [13], we can check that \( 4 \mid |\tau_1(P)| \). By Proposition 1.6, \( D_{\text{iso}}(Q) = \{ k^2 + l |\tau_1(Q)| \mid \gcd(k, |\tau_1(Q)|) = 1 \} \), where \( Q \) is any 3-manifolds with \( S^3 \) geometry. If \( k^2 + l |\tau_1(P)| = -k^2 - l |\tau_1(P)| \), then
\[k^2 + k'^2 = -(l+l')|\pi_1(P)|.\] Since \(4 \mid |\pi_1(P)|\) and \(\gcd(k, |\pi_1(P)|) = \gcd(k', |\pi_1(P)|) = 1\), \(k, k'\) are both odd, thus \(-(l+l')|\pi_1(P)| = k^2 + k'^2 = 4s + 2\), contradicts \(4 \mid |\pi_1(P)|\).

So \(D_{iso}(P) \cap (-D_{iso}(P)) = \emptyset\). (In particular \(-1 \neq D(P)\).)

From the definition we have \(D_{iso}(P) = D_{iso}(\tilde{P})\) and \(D_{iso}(P, \tilde{P}) = D_{iso}(\tilde{P}, P) = -D_{iso}(\tilde{P})\).

If \(m \neq n\), we may assume that \(m > n\). For the self-map \(f\), if some \(P\) corresponds to \(\tilde{P}\), there must also be some \(P\) corresponds to \(P\), so \(\deg(f) \in D_{iso}(P) \cap (-D_{iso}(P))\), it is impossible by the argument in first paragraph. So all the \(P\) correspond to \(P\), and all the \(\tilde{P}\) correspond to \(\tilde{P}\). Since \(D_{iso}(P) = D_{iso}(\tilde{P})\), we have \(D_{iso}(mP \# n\tilde{P}) \subset D_{iso}(P)\).

By Lemma 3.8 and the fact \(D_{iso}(P) = D_{iso}(\tilde{P})\), we have \(D_{iso}(mP \# n\tilde{P}) \subset D_{iso}(P)\).

If \(m = n\), similarly we have either all the \(P\) correspond to \(P\) and all the \(\tilde{P}\) correspond to \(\tilde{P}\); or all the \(P\) correspond to \(\tilde{P}\) and all the \(\tilde{P}\) correspond to \(P\). Since \(D_{iso}(P) = D_{iso}(\tilde{P})\) and \(D_{iso}(P, \tilde{P}) = D_{iso}(\tilde{P}, P) = -D_{iso}(\tilde{P})\), we have \(D_{iso}(mP \# m\tilde{P}) \subset D_{iso}(P) \cup (-D_{iso}(P))\). On the other hand from the argument above, we have \(D_{iso}(P) \subset D_{iso}(mP \# m\tilde{P})\), hence \(D_{iso}(mP \# m\tilde{P}) = D_{iso}(P) \cup (-D_{iso}(P))\). \(\square\)

**Lemma 3.11.** \(D_{iso}(L(p, q), L(p, q')) = \{k^2q^{-1}q' + lp \mid \gcd(k, p) = 1\}\), here \(q^{-1}\) is seen as in group \(U_p = \{\text{all the units in the ring } \mathbb{Z}_p\}\).

Proof. \(L(p, q)\) is the quotient of \(S^3\) by the action of \(\mathbb{Z}_p, (z_1, z_2) \rightarrow (e^{i2\pi/p}z_1, e^{i2\pi/p}z_2)\). Let \(f_{q,q'}: S^3 \rightarrow S^3, f_{q,q'}(z_1, z_2) = (z_1^q / \sqrt{|z_1|^2 + |z_2|^2}, z_2^q / \sqrt{|z_1|^2 + |z_2|^2})\). We can check that this map induces a map \(f_{q,q'}: L(p, q) \rightarrow L(p, q')\) with degree \(qq'\), moreover since \(q, q'\) are coprime with \(p\), \(f_{q,q'}\) is an isomorphism \(\pi_1\). By Proposition 1.6 \(D_{iso}(L(p, q)) = \{k^2 + lp \mid \gcd(k, p) = 1\}\). Compose each self-map on \(L(p, q)\) which induces an isomorphism on \(\pi_1\) with \(f_{q,q'}\), we have \(\{k^2q^{-1}q' + lp \mid \gcd(k, p) = 1\} \subset D_{iso}(L(p, q), L(p, q'))\). On the other hand, for each map \(g: L(p, q) \rightarrow L(p, q')\) of degree \(d\) which induces an isomorphism on \(\pi_1\), then \(f_{q,q'} \circ g\) is a self-map on \(L(p, q)\) which induces an isomorphism on \(\pi_1\), where \(f_{q,q'}: L(p, q') \rightarrow L(p, q)\) is a degree \(qq'\) map. Hence the degree of \(f_{q,q'} \circ g\) is \(qq'd\) which must be in \(\{k^2 + lp \mid \gcd(k, p) = 1\}\), that is \(qq'd = k^2 + lp, \gcd(k, p) = 1, d = k^2q^{-1}q' - 1 + pl = (kq^{-1})^2q^{-1}q' + pl \in \{k^2q^{-1}q' + lp \mid \gcd(k, p) = 1\}\). Hence \(D_{iso}(L(p, q), L(p, q')) = \{k^2q^{-1}q' + lp \mid \gcd(k, p) = 1\}\). \(\square\)

Let \(U_p = \{\text{all units in ring } \mathbb{Z}_p\}, U_p^2 = \{a^2 \mid a \in U_p\}\), which is a subgroup of \(U_p\). Let \(H\) denote the natural projection from \(\{n \in \mathbb{Z} \mid \gcd(n, p) = 1\}\) to \(U_p/U_p^2\).

Later, we will omit the \(p\), denote them by \(U\) and \(U^2\). We consider the quotient \(U/U^2 = \{a_1, \ldots, a_m\}\), every \(a_i\) corresponds with a coset \(A_i\) of \(U^2\). For the structure of \(U\), see [9] p.44, then we can get the structure of \(U^2\) and \(U/U^2\) easily.

Define \(\bar{A}_i = \{L(p, q_i) \mid q_i \in A_i\}\) (with repetition allowed). In \(U/U^2\), define \(B_i = \{a_i \mid \#\bar{A}_i = l\}\) for \(l = 1, 2, \ldots\), there are only finitely many \(B_i's\) are nonempty. Let \(C_i = \{a \in U/U^2 \mid a_i a \in B_i, \forall a_i \in B_i\}\) if \(B_i \neq \emptyset\) and \(C_i = U/U^2\) otherwise, \(C = \bigcap_{i=1}^{\infty} C_i\).

**Proposition 3.12.** \(D_{iso}(L(p, q_1) \# \cdots \# L(p, q_n)) = H^{-1}(C)\).
Proof. By Lemma 3.11, we have \( D_{iso}(L(p, q), L(p, q')) = \{ k^2 q^{-1} q' + lp \mid \gcd(k, p) = 1 \} \). Therefore \( D_{iso}(L(p, q), L(p, q')) \) will not change if we replace \( L(p, q) \) by \( L(p, s^2 q) \) (resp. \( L(p, q') \) by \( L(p, s^2 q') \)) for any \( s \in U_p \).

Now we consider the relation between two sets \( D_{iso}(L(p, q), L(p, q')) \) and \( D_{iso}(L(p, q_s), L(p, q_s')) \). It is also easy to see if \( (q/q')(q_s/q_s) = s^2 \) in \( U_p \), then \( D_{iso}(L(p, q), L(p, q')) = D_{iso}(L(p, q_s), L(p, q_s)) \). and if \( (q/q')(q_s/q_s) \neq s^2 \) in \( U_p \), then \( D_{iso}(L(p, q), L(p, q')) \cap D_{iso}(L(p, q_s), L(p, q_s)) = \emptyset \).

Let \( f: L(p, q_1) \# \cdots \# L(p, q_n) \rightarrow L(p, q_1) \# \cdots \# L(p, q_n) \) be a map of degree \( d \neq 0 \). Suppose \( f \) sends \( L(p, q_i) \) to \( L(p, q_k) \) and sends \( L(p, q_j) \) to \( L(p, q_l) \) in the sense of Theorem 3.6. Since \( D_{iso}(L(p, q_i), L(p, q_j)) \cap D_{iso}(L(p, q_j), L(p, q_i)) \neq \emptyset \), by last paragraph, we must have \( (q_i/q_j)(q_j/q_i) = s^2 \) in \( U_p \). Hence \( q_i/q_j \) is in \( U^2 \) if and only if \( q_i/q_k \) is in \( U^{2} \); in other words, \( L(p, q_i) \) and \( L(p, q_j) \) are in the same \( \tilde{A}_s \) if and only if \( L(p, q_k) \) and \( L(p, q_i) \) are in the same \( \tilde{A}_i \). Hence \( f \) provides 1-1 self-correspondence on \( \tilde{A}_1, \ldots, \tilde{A}_m \), and if some elements in \( \tilde{A}_s \) corresponds to \( \tilde{A}_t \), there is \( #\tilde{A}_s = #\tilde{A}_t \).

Let \( f: L(p, q_1) \# \cdots \# L(p, q_n) \rightarrow L(p, q_1) \# \cdots \# L(p, q_n) \) be a self-map. For each \( a_i \in U/U^2 \), \( f \) must send \( \tilde{A}_i \) to some \( \tilde{A}_j \) with \( #\tilde{A}_i = #\tilde{A}_j = l \), and both \( a_i, a_j \in B_1 \). Assume \( L(p, q_i) \in \tilde{A}_i, L(p, q_j) \in \tilde{A}_j \), then \( \deg(f) \in \{ k^2 q_i^{-1} q_j + lp \mid \gcd(k, p) = 1 \} \) by Lemma 3.11. By consider in \( U/U^2 \), we have \( H(\deg(f)) = \tilde{q}_i/\tilde{q}_j = a_i/a_j \), that is \( H(\deg(f))a_i = a_j \in B_1 \). Since we choose arbitrary \( a_i \) in \( B_1 \), we have \( H(\deg(f)) \in C_1 \). Also we choose arbitrary \( l \), we have \( H(\deg(f)) \in \bigcap_{i=1}^{\infty} C_i = C \), hence \( \deg(f) \) in \( H^{-1}(C) \).

On the other hand, if \( d \in H^{-1}(C) \), then \( H(d) = c \in C = \bigcap_{i=1}^{\infty} C_i \). For each \( B_i \neq \emptyset \) and each \( a_i \in B_i \), we have \( c a_i = a_j \in B_i \). Then \( A_i \mapsto A_j \) gives 1-1 self-correspondence among \( \{ \tilde{A}_i \mid #\tilde{A}_i = l \} \). We can make further 1-1 correspondence from elements in \( \tilde{A}_i \) to elements in \( \tilde{A}_j \). Since our discussion works for all \( B_i \neq \emptyset \), we have 1-1 self-correspondence on \( \{ L(p, q_1), \ldots, L(p, q_n) \} \) (with repetition allowed). Therefore for each \( L(p, q_i) \in \tilde{A}_i \) and \( L(p, q_j) \in \tilde{A}_j \), we have \( q_i = q_j^{-1} \). Therefore \( d \) have the form \( k^2 q_j q_i^{-1} \) mod \( p \) with \( (k, p) = 1 \). By Lemma 3.11, there is a map \( f_{i,j}: L(p, q_i) \rightarrow L(p, q_j) \) of degree \( d \) which induces an isomorphism on \( \pi_1 \).

By Lemma 3.8, we can construct a self-mapping of degree \( d \) of \( L(p, q_1) \# \cdots \# L(p, q_n) \) which induces an isomorphism on \( \pi_1 \). Hence \( H^{-1}(C) \subset D_{iso}(L(p, q_1) \# \cdots \# L(p, q_n)) \). Thus \( D_{iso}(L(p, q_1) \# \cdots \# L(p, q_n)) = H^{-1}(C) \). \( \square \)

4. \( D(M) \) for Nil manifolds

4.1. Self coverings of Euclidean orbifolds.

DEFINITION 4.1 (20). A 2-orbifold is a Hausdorff, paracompact space which is locally homeomorphic to the quotient space of \( \mathbb{R}^2 \) by a finite group action. Suppose \( O_1 \) and \( O_2 \) are orbifolds and \( f: O_1 \rightarrow O_2 \) is an map. We say \( f \) is an orbifold covering if any point \( p \) in \( O_2 \) has a neighbourhood \( U \) such that \( f^{-1}(U) \) is the disjoint union of
sets \( V_\lambda \), \( \lambda \in \Lambda \), such that \( f|_{V_\lambda} \to U \) is the natural quotient map between two quotients of \( \mathbb{R}^2 \) by finite groups, one of which is a subgroup of the other.

In this paper, we only consider about orbifold with singular points. Here we say a point \( x \) in the orbifold is a \textit{singular point of index} \( q \) if \( x \) has a neighborhood \( U \) homeomorphic to the quotient space of \( \mathbb{R}^2 \) by rotate action of finite cyclic group \( \mathbb{Z}_q \), \( q > 1 \).

An orbifold \( \mathcal{O} \) with singular points \( \{x_1, \ldots, x_s\} \) is homeomorphic to a surface \( F \), but for the sake of the singular points, we would like to distinguish them through denoting \( \mathcal{O} \) by \( F(q_1, \ldots, q_s) \). Here \( q_1, \ldots, q_s \) are indices of singular points. Here the covering map \( f: \mathcal{O}_1 \to \mathcal{O}_2 \) is not the same as the covering map from \( F_1 \) to \( F_2 \).

If \( f: \mathcal{O}_1 \to \mathcal{O}_2 \) is an orbifold covering, the singular points of \( \mathcal{O}_2 \) are \( \{x_1, \ldots, x_s\} \), for any \( y \in \mathcal{O}_2, y \neq x_1 \), define \( \deg(f) = \# f^{-1}(y) \). For any singular point \( x \), let \( f^{-1}(x) = \{a_1, \ldots, a_t\} \). At point \( a_j \), \( f \) is locally equivalent to \( z \to z^{a_j} \) on \( \mathbb{C} \), \( x \) and \( a_j \) correspond to 0. Here we have \( \sum d_j = d, a_j \) is an ordinary point if and only if \( d_j \) equals to the index of \( x \). Define \( D(x) = [d_1, \ldots, d_t] \) to be the \textit{orbifold covering data at singular point} \( x \), and \( \mathcal{D}(f) = \{D(x_1), \ldots, D(x_s)\} \) (with repetition allowed) to be the \textit{orbifold covering data of} \( f \).

The following lemma is easy to verify.

**Lemma 4.2.** If a Nil manifold \( M \) is not a torus bundle or a torus semi-bundle, then \( M \) has one of the following Seifert fibering structures: \( M(0; \beta_1/2, \beta_2/3, \beta_3/6) \), \( M(0; \beta_1/3, \beta_2/3, \beta_3/3) \), or \( M(0; \beta_1/2, \beta_2/4, \beta_3/4) \), where \( e(M) \in \mathbb{Q} - \{0\} \).

Proof. Consider Nil manifold \( M \) as a Seifert fibered space, then its orbifold \( \mathcal{O}(M) \) has zero Euler characteristic. So \( \mathcal{O}(M) \) must be one of following orbifolds: the torus \( T^2 \), the Klein bottle \( K \), \( P^2(2, 2) \), \( S^2(2, 3, 6) \), \( S^2(2, 4, 4) \), \( S^2(3, 3, 3) \) and \( S^2(2, 2, 2, 2) \).

By [4] p. 38 and p. 40, we can see that \( M \) has structure of torus bundle if \( \mathcal{O}(M) \) is \( T^2 \) or \( K \), and \( M \) has structure of torus semi-bundle if \( \mathcal{O}(M) \) is \( P^2(2, 2) \) or \( S^2(2, 2, 2, 2) \).

The remaining three cases \( S^2(2, 3, 6) \), \( S^2(2, 4, 4) \) and \( S^2(3, 3, 3) \) correspond to the three cases claimed in the lemma. Clear \( e(M) \in \mathbb{Q} - \{0\} \) since Nil manifolds have non-zero Euler number. \( \square \)

**Proposition 4.3.** Denote the degrees set of self covering of an orbifold \( \mathcal{O} \) by \( D(\mathcal{O}) \). We have:

(1) For \( \mathcal{O} = S^2(2, 3, 6) \), \( D(\mathcal{O}) = \{m^2 + mn + n^2 \mid m, n \in \mathbb{Z}, (m, n) \neq (0, 0)\} \).

Moreover, if \( d \in D(\mathcal{O}) \) is coprime with 6, then

(i) \( d \equiv 1 \mod 6 \);

(ii) this covering map of degree \( d = 6k + 1 \) is realized by an orbifold covering from \( \mathcal{O} \) to \( \mathcal{O} \) with orbifold covering data

\[
\{D(x_1), D(x_2), D(x_3)\} = \{[2, \ldots, 2, 1], [3, \ldots, 3, 1], [6, \ldots, 6, 1]\},
\]
where $x_1$, $x_2$ and $x_3$ are singular points of indices 2, 3 and 6 respectively.

(2) For $\mathcal{O} = S^2(3, 3, 3)$, $D(\mathcal{O}) = \{m^2 + mn + n^2 \mid m, n \in \mathbb{Z}, (m, n) \neq (0, 0)\}$. Moreover, if $d \in D(\mathcal{O})$ is coprime with 3, then

(i) $d \equiv 1 \mod 3$;

(ii) this covering map of degree $d = 6k + 1$ is realized by an orbifold covering from $\mathcal{O}$ to $\mathcal{O}$ with orbifold covering data

$$\{D(x_1), D(x_2), D(x_3)\} = \{[3, \ldots, 3, 1], [3, \ldots, 3, 1], [3, \ldots, 3, 1]\},$$

where $x_1$, $x_2$ and $x_3$ are singular points of indices 3, 3 and 3 respectively.

(3) For $\mathcal{O} = S^2(2, 4, 4)$, $D(\mathcal{O}) = \{m^2 + n^2 \mid m, n \in \mathbb{Z}, (m, n) \neq (0, 0)\}$. Moreover, if $d \in D(\mathcal{O})$ is coprime with 4, then

(i) $d \equiv 1 \mod 4$;

(ii) this covering map of degree $d = 4k + 1$ is realized by an orbifold covering from $\mathcal{O}$ to $\mathcal{O}$ with orbifold covering data

$$\{D(x_1), D(x_2), D(x_3)\} = \{[2, \ldots, 2, 1], [4, \ldots, 4, 1], [4, \ldots, 4, 1]\},$$

where $x_1$, $x_2$ and $x_3$ are singular points of indices 2, 4 and 4 respectively.

Proof. We only prove case (1). The other two cases can be proved similarly.

$S^2(2, 3, 6)$ can be seen as pasting the equilateral triangle as shown in Fig. 1 geometrically.

$\pi_1(S^2(2, 3, 6))$ can be identified with a discrete subgroup $\Gamma$ of $\text{Iso}_+ (\mathbb{E}^2)$, a fundamental domain of $\Gamma$ is shown in Fig. 2. It is as a lattice in $\mathbb{E}^2$ with vertex coordinate $m + ne^{i\pi/3}$, $m, n \in \mathbb{Z}$. 
For the covering $p: T^2 \to S^2(2, 3, 6)$, $T^2$ can be seen as the quotient of a subgroup $\Gamma' \subset \Gamma$ on $\mathbb{E}^2$, with a fundamental domain as Fig. 3. Here $\Gamma'$ is just all the translation elements of $\Gamma$, thus $\Gamma'$ is generated by $z \to z + \sqrt{3}i$ and $z \to z + (\sqrt{3}/2)i + 3/2$.

For every self covering $f: S^2(2, 3, 6) \to S^2(2, 3, 6)$, $f_0: \pi_1(S^2(2, 3, 6)) \to \pi_1(S^2(2, 3, 6))$ is injective. Since $p$ is covering, $f_0 \circ p_0: \pi_1(T^2) \to \pi_1(S^2(2, 3, 6))$ is also injective. So $f_0(p_0(\pi_1(T^2)))$ is a free abelian subgroup of $\pi_1(S^2(2, 3, 6))$.

For every $\gamma \in \Gamma$, which is not translation, it can be represented by $f: z \to e^{2k\pi/n}z + z_0$. $\gcd(k, n) = 1$, $n > 1$. Then $f^n(z) = (e^{2k\pi/n})^n z + \cdots + e^{2k\pi/n + 1})z_0 = z$. So $\gamma$ is a torsion element, thus $\gamma \notin f_0(p_0(\pi_1(T^2)))$ except $\gamma = e$. So $f_0(p_0(\pi_1(T^2))) \subset p_0(\pi_1(T^2))$, thus there exists $\tilde{f}: T^2 \to T^2$ being the lifting of $f$.

$$
\begin{array}{ccc}
T^2 & \xrightarrow{j} & T^2 \\
\downarrow{p} & & \downarrow{p} \\
S^2(2, 3, 6) & \xrightarrow{f} & S^2(2, 3, 6).
\end{array}
$$

Here we have

$$
\deg(f) = \deg(\tilde{f}) = [\pi_1(T^2) : f_0(\pi_1(T^2))] = \frac{\text{area(fundamental domain of } \tilde{f}_0(\pi_1(T^2))\text{)\}}{\text{area(fundamental domain of } \pi_1(T^2)\text{)}}.
$$

here $\tilde{f}_0(\pi_1(T^2)), \pi_1(T^2)$ are all seen as subgroup of $\pi_1(S^2(2, 3, 6))$. 
Clearly, we can choose a fundamental domain of \( f_3(S^2(2, 3, 6)) \) to be an equilateral triangle in \( \mathbb{E}^2 \) with vertices as \( m + ne^{i\pi/3} \), then the fundamental domain of \( \tilde{f}_3(T^2) \) is an equilateral hexagon with vertices as \( m + ne^{i\pi/3} \). The scale of area is the square of the scale of edge length. The scale of edge length must be \( \sqrt{m^2 + mn + n^2} \). So \( \deg(f) = m^2 + mn + n^2 \).

On the other hand, for every \((m, n) \in \mathbb{Z}^2 - \{(0, 0)\}\), choose \( g: \mathbb{E}^2 \to \mathbb{E}^2 \), \( g(z) = (m + ne^{i\pi/3})z \). It is routine to check that for any \( \gamma \in \Gamma \), there is \( \gamma' \in \Gamma \), such that \( g(\gamma(z)) = \gamma'(g(z)) \). So \( g \) induces \( \tilde{g} \), which is self covering on \( S^2(2, 3, 6) \), and \( \deg(\tilde{g}) = m^2 + mn + n^2 \). We have proved the first sentence of Proposition 4.3 (1).

If \( m^2 + mn + n^2 \) is coprime to 6, \( m^2 + mn + n^2 \equiv 1 \) or \( 5 \mod 6 \). Since \( m^2 + mn + n^2 \equiv 4m^2 + 4mn + 4n^2 \equiv (2m + n)^2 \mod 3 \), and any square number must be 0 or 1 \( \mod 3 \), we must have \( m^2 + mn + n^2 \equiv 1 \mod 6 \). We have proved Proposition 4.3 (1) (i).

Assume \( h \) is a self covering of degree \( d = 6k + 1 \), \( x_1, x_2, x_3 \) are the singular points on \( S^2(2, 3, 6) \) with indices 2, 3, 6. For \( x_1 \), \( h^{-1}(x_1) \) must be ordinary points or singular point of index 2. Since the degree \( d = 6k + 1 \), \( h^{-1}(x_1) \) is 3k ordinary points and \( x_1 \). Similarly, for \( x_2 \), \( h^{-1}(x_2) \) is 2k ordinary points and \( x_2 \). Then \( x_1, x_2 \notin h^{-1}(x_3) \), so \( h^{-1}(x_3) \) is k ordinary points and \( x_3 \). Thus the covering map of degree \( d = 6k + 1 \) is realized by a self covering of \( \mathcal{O} \) with orbifold covering data \([2, \ldots, 2, 1], [3, \ldots, 3, 1], [6, \ldots, 6, 1]\). We have proved Proposition 4.3 (1) (ii).
4.2. $D(M)$ for Nil manifolds.

Lemma 4.4. For Nil manifold $M$, $D(M) \subset \{l^2 | l \in \mathbb{Z}\}$.

Proof. Let $f$ be a self map of $M$. By [25, Corollary 0.4], $f$ is either homotopic to a covering map $g: M \to M$, or a homotopy equivalence.

If $f$ is homotopic to a covering, since $M$ has unique Seifert fibering structure up to isomorphism, we can make $g$ to be a fiber preserving map. Denote the orbifold of $M$ by $O_M$. By [20, Lemma 3.5], we have:

\[
\begin{align*}
  e(M) &= e(M) \cdot \frac{l}{m}, \\
  \deg(g) &= l \cdot m,
\end{align*}
\]

where $l$ is the covering degree of $O_M \to O_M$ and $m$ is the covering degree on the fiber direction. Since $e(M) \neq 0$, from equation (4.1) we get $l = m$. Thus $\deg(f) = \deg(g)$ is a square number $l^2$.

If $f$ is a homotopy equivalence, then $\deg(f) = \pm 1$. To finish the proof of the lemma, we need only to show that the degree of $f$ is not $-1$. Otherwise composing a self covering $g$ of degree $n > 1$, then $g \circ f$ is of degree $-n$, which is not a homotopy equivalence, therefore is homotopic to a covering, and must have degree $> 0$ by the last paragraph, a contradiction. \qed

Theorem 4.5. For 3-manifold $M$ in Class 4, we have

1. For $M = M(0; \beta_1/2, \beta_2/3, \beta_3/6)$, $D(M) = \{l^2 | l = m^2 + mn + n^2, l \equiv 1 \mod 6, m, n \in \mathbb{Z}\}$;

2. For $M = M(0; \beta_1/3, \beta_2/3, \beta_3/3)$, $D(M) = \{l^2 | l = m^2 + mn + n^2, l \equiv 1 \mod 3, m, n \in \mathbb{Z}\}$;

3. For $M = M(0; \beta_1/2, \beta_2/4, \beta_3/4)$, $D(M) = \{l^2 | l = m^2 + n^2, l \equiv 1 \mod 4, m, n \in \mathbb{Z}\}$.

Proof. We will just prove Case (1). The proof of Cases (2) and (3) are exactly as that of Case (1). Below $M = M(0; \beta_1/2, \beta_2/3, \beta_3/6)$.

First we show that $D(M) \subset \{l^2 | l = m^2 + mn + n^2, l \equiv 1 \mod 6, m, n \in \mathbb{Z}\}$.

Since the orbifold $O_M = S^2(2, 3, 6)$, by Proposition 4.3 (1), we have $l = m^2 + mn + n^2$. Below we show that $l = 6k + 1$.

Let $N$ be the regular neighborhood of 3 singular fibers. To define the Seifert invariants, a section $F$ of $M \setminus N$ is chosen, and moreover $\partial F$ and fibers on each component of $\partial(M \setminus N)$ are oriented.

Consider the covering $g_1: M \setminus g^{-1}(N) \to M \setminus N$. Let $\tilde{F}$ be a component $g^{-1}(F)$. It is easy to verify that $\tilde{F}$ is a section of $M \setminus g^{-1}(N)$. Now we lift the orientations on $\partial F$ and the fibers on $\partial(M \setminus N)$ to those on $\partial(M \setminus g^{-1}(N))$, we get a coordinate system on $\partial(M \setminus g^{-1}(N))$. Therefore we have a coordinate preserving covering

\[
g: (M, M \setminus g^{-1}(N), g^{-1}(N)) \to (M, M \setminus N, N).
\]
Suppose $V'$ is a tubular neighborhood of some singular fiber $L'$. The meridian of $V'$ can be represented by $(c')^\alpha(h')^\beta$ ($\alpha > 0$), where $(c', h')$ is the section-fiber coordinate of $\partial V'$.

Suppose $V$ is a component of $g^{-1}(V')$ and the meridian of $V$ is represented as $c^\alpha h^\beta$ ($\alpha > 0$), where $(c, h)$ is the lift of $(c', h')$. Since $g: V \to V'$ is a covering of solid torus, so $g$ must send meridian to meridian homeomorphically, thus $g(c^\alpha h^\beta) = (c')^\alpha(h')^\beta$. See Fig. 4.

Since $g$ has the fiber direction covering degree $m = l$, $g(h) = (h')^l$. Since $c, c'$ are the boundaries of sections and $g$ send $c$ to $c'$, we have $g(c) = (c')^l$. Then $g(c^\alpha h^\beta) = (c')^\alpha(h')^\beta = (c')^\alpha(h')^\beta$. Hence we get $\beta \cdot l = \beta'$.

Let $V'$ be a tubular neighborhood of singular fiber whose meridian can be represented as $(c')^\beta(h')^\beta$. By the arguments above, the meridian of the preimage $V$ can be represent by $c^\alpha h^\beta$.

Since $\beta'$ is coprime with 6. By $\beta \cdot l = \beta'$, so $l$ is coprime with 6. Still by Proposition 4.3 (1), we have $l = 6k + 1$.

Then we show $\{l^2 \mid m^2 + mn + n^2, l \equiv 1 \mod 6, m, n \in \mathbb{Z}\} \subset D(M)$.

Suppose $l = m^2 + mn + n^2$ and $l = 6k + 1$, denote the quotient manifold of $\mathbb{Z}_l$ free action on $M$ by $M_l$. Then $M_l$ has the Seifert fibering structure $M(0; l \cdot \beta_1/2, l \cdot \beta_2/3, l \cdot \beta_3/6)$. We have the covering $g_l: M \to M_l$ of degree $l$.

**Claim.** there exists a map $f_l: M_l \to M$ of degree $l$.

Let $D = D_1 \cup D_2 \cup D_3 \subset S^2(2, 3, 6)$ be the regular neighborhood discs of 3 singular points of indices 2, 3, and 6 respectively. By Proposition 4.3 (1), there exists a branched covering map $\tilde{f}_l: S^2(2, 3, 6) \to S^2(2, 3, 6)$ of degree $l$ such that

1. $\tilde{f}_l$ induce a covering map $\tilde{f}_l: S^2 \setminus \tilde{f}_l^{-1}(D) \to S^2 \setminus D$;
2. $\tilde{f}_l^{-1}(D_i)$ consists of $(3k + 1)$ discs with orbifold covering data $[2, \ldots, 2, 1]$ for $i = 1$, and $(2k + 1)$ discs with orbifold covering data $[3, \ldots, 3, 1]$ for $i = 2$, and $(k + 1)$ discs with orbifold covering data $[6, \ldots, 6, 1]$ for $i = 3$. 

Fig. 4.
Clearly \( f_l^{-1}(D) \) consists of \((3k + 1) + (2k + 1) + (k + 1) = 6k + 3 \) disks.

Then we have the covering map \( f_l \times \text{id} \colon (S^2 \setminus f_l^{-1}(D)) \times S^1 \to (S^2 \setminus D) \times S^1 \) of degree \( l \), which can be extended to a covering map \( f_l \colon M' \to M \), where \( M' \) has the Seifert structure \( M(0; \beta_1, \ldots, \beta_l, \beta_2, \ldots, \beta_k, \beta_3/3, \beta_3, \beta_3/6) \). Clearly \( M' \) is isomorphic to \( M_l \).

Now the covering \( f_l \circ g_l \colon M \to M_l \to M \) has degree \( l^2 \).

We finish the proof of Case (1).

\[ \Box \]

5. \( D(M) \) for \( H^2 \times E^1 \) manifolds

In this case, all the manifolds are Seifert fibered spaces \( M \) such that the Euler number \( e(M) = 0 \) and the Euler characteristic of the orbifold \( \chi(OM) < 0 \).

Suppose \( M = (g; \beta_{1,1}/\alpha_1, \ldots, \beta_{1,m_1}/\alpha_1, \ldots, \beta_{n,1}/\alpha_n, \ldots, \beta_{n,m_n}/\alpha_n) \), where all the integers \( \alpha_i > 1 \) are different from each other, and \( \sum_{i=1}^n \sum_{j=1}^{m_i} \beta_{i,j}/\alpha_i = 0 \).

For every \( \alpha_i \), consider \( U_{\alpha_i} \). For every \( a \in U_{\alpha_i} \), define \( \theta_a(\alpha_i) = \# \{ \beta_{i,j} | p_l(\beta_{i,j}) = a \} \) (with repetition allowed), where \( p_l \) is the natural projection from \( \{ n \mid \text{gcd}(n, \alpha_i) = 1 \} \) to \( U_{\alpha_i} \). Define \( B_l(\alpha_i) = \{ a \mid \theta_a(\alpha_i) = l \} \) for \( l = 0, 1, \ldots \), there are only finitely many \( B_l(\alpha_i) \) nonempty. Let \( C_l(\alpha_i) = \{ b \in U_{\alpha_i} \mid ab \in B_l(\alpha_i), \forall a \in B_l(\alpha_i) \} \) if \( B_l(\alpha_i) \neq \emptyset \) and \( C_l(\alpha_i) = U_{\alpha_i} \) otherwise. Finally define \( C(\alpha_i) = \bigcap_{l=1}^{\infty} C_l(\alpha_i) \), and \( \tilde{C}(\alpha_i) = p_l^{-1}(C_l(\alpha_i)) \).

**Theorem 5.1.**

\[
D \left( M \left( g; \frac{\beta_{1,1}}{\alpha_1}, \ldots, \frac{\beta_{1,m_1}}{\alpha_1}, \ldots, \frac{\beta_{n,1}}{\alpha_n}, \ldots, \frac{\beta_{n,m_n}}{\alpha_n} \right) \right) = \bigcap_{l=1}^{\infty} \tilde{C}(\alpha_i).
\]

Proof. Suppose \( f \) is a non-zero degree self-mapping of \( M \). By [25, Corollary 0.4], \( f \) is homotopic to a covering map \( g \colon M \to M \). Since \( M \) has the unique Seifert structure, we can isotope \( g \) to a fiber preserving map. Denote the orbifold of \( M \) by \( O_M \). Then \( g \) induces a self-covering \( \tilde{g} \) on \( O_M \), since \( \chi(OM) < 0 \), then \( \tilde{g} \) must be 1-sheet, thus isomorphism of \( OM \).

So \( g \) is a degree \( d \) covering on the fiber direction. Or equivalently, by the action of \( \mathbb{Z}_d \) on each fiber, the quotient of \( M \) is also \( M \). Thus \( d \in D(M) \) if and only if

\[
M' = M \left( g; d\frac{\beta_{1,1}}{\alpha_1}, \ldots, d\frac{\beta_{1,m_1}}{\alpha_1}, \ldots, d\frac{\beta_{n,1}}{\alpha_n}, \ldots, d\frac{\beta_{n,m_n}}{\alpha_n} \right)
\]

is homeomorphic to \( M \).

By the uniqueness of Seifert structure ([20] Theorem 3.9) and the fact \( e(M) = 0 \), we have that \( M \) is homeomorphic to \( M' \) if and only if \( (\beta_{1,1}, \ldots, \beta_{1,m_1}) = (d\beta_{1,1}, \ldots, d\beta_{1,m_1}) \) under a permutation, all the numbers are seen as in \( U(\alpha_i) \).

For every \( a \in U(\alpha_i) \), if \( (\beta_{1,1}, \ldots, \beta_{1,m_1}) = (d\beta_{1,1}, \ldots, d\beta_{1,m_1}) \) holds, we must have \( \theta_a(\alpha_i) = \theta_{d\alpha}(\alpha_i) \), thus \( p_l(d) \in C_{d\alpha}(\alpha_i) \). For \( a \) is an arbitrary element in \( U(\alpha_i) \), we have
\( p_i(d) \in C(\alpha_i) \), thus \( d \in \tilde{C}(\alpha_i) \). Since \( \alpha_i \) is also chosen arbitrarily, \( d \in \bigcap_{i=1}^{n} \tilde{C}(\alpha_i) \), thus \( D(M) \subset \bigcap_{i=1}^{n} \tilde{C}(\alpha_i) \).

For any \( d \in \bigcap_{i=1}^{n} \tilde{C}(\alpha_i) \), \( M \) is homeomorphic to \( M' \), so \( D(M) \supset \bigcap_{i=1}^{n} \tilde{C}(\alpha_i) \)

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