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Author(s)	Sun, Hongbin; Wang, Shicheng; Wu, Jianchun et al.
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SELF-MAPPING DEGREES OF 3-MANIFOLDS

HONGBIN SUN, SHICHENG WANG, JIANCHUN WU and HAO ZHENG

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Abstract

For each closed oriented 3-manifold M in Thurston's picture, the set of degrees of self-maps on M is given.

Contents

1. Introduction

1.1. Background. Each closed oriented n-manifold M is naturally associated with a set of integers, the degrees of all self-maps on M, denoted as $D(M) = \{\deg(f) \mid f\colon M \to M\}$.

Indeed the calculation of $D(M)$ is a classical topic appeared in many literatures. The result is simple and well-known for dimension $n = 1,2$. For dimension $n > 3$, there are many interesting special results (See [3], [10], [15] for recent ones and references therein), but it is difficult to get general results, since there are no classification results for manifolds of dimension $n > 3$.

The case of dimension 3 becomes attractive in the topic and it is possible to calculate $D(M)$ for any closed oriented 3-manifold M. Since Thurston's geometrization conjecture, which seems to be confirmed, implies that closed oriented 3-manifolds can be classified in a reasonable sense.

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Thurston's geometrization conjecture claims that the each Jaco-Shalen-Johanson decomposition piece of a prime 3-manifold supports one of the eight geometries, which are H^3, $\widetilde{PSL}(2, R)$, $H^2 \times E^1$, Sol, Nil, E^3, S^3 and $S^2 \times E^1$ (for details see [24] and [20]). Call a closed orientable 3-manifold M is *geometrizable* if each prime factor of M meets Thurston's geometrization conjecture. All 3-manifolds discussed in this paper are geometrizable.

The following result is known in early 1990's:

Theorem 1.0. *Suppose M is a geometrizable 3-manifold. Then M admits a self-map of degree larger than 1 if and only if M is either*
(a) *covered by a torus bundle over the circle, or*
(b) *covered by $F \times S^1$ for some compact surface F with $\chi(F) < 0$, or*
(c) *each prime factor of M is covered by S^3 or $S^2 \times E^1$.*

Hence for any 3-manifold M not listed in (a)–(c) of Theorem 1.0, $D(M)$ is either $\{0, 1, -1\}$ or $\{0, 1\}$, which depends on whether M admits a self map of degree -1 or not. To determine $D(M)$ for geometrizable 3-manifolds listed in (a)–(c) of Theorem 1.0, let's have a close look of them.

For short, we often call a 3-manifold supporting Nil geometry *a Nil 3-manifold, and so on*. Among Thurston's eight geometries, six of them belong to the list (a)–(c) in Theorem 1.0. 3-manifolds in (a) are exactly those supporting either E^3, or Sol or Nil geometries. E^3 3-manifolds, Sol 3-manifolds, and some Nil 3-manifolds are torus bundle or semi-bundles; Nil 3-manifolds which are not torus bundles or semi-bundles are Seifert fibered spaces having Euclidean orbifolds with three singular points. 3-manifolds in (b) are exactly those supporting $H^2 \times E^1$ geometry; 3-manifolds supporting S^3 or $S^2 \times E^1$ geometries form a proper subset of (3). Now we divide all 3-manifolds in the list (a)–(c) in Theorem 1.0 into the following five classes:
Class 1. M supporting either S^3 or $S^2 \times E^1$ geometries;
Class 2. each prime factor of M supporting either S^3 or $S^2 \times E^1$ geometries, but M is not in Class 1;
Class 3. torus bundles and torus semi-bundles;
Class 4. Nil 3-manifolds not in Class 3;
Class 5. M supporting $H^2 \times E^1$ geometry.

$D(M)$ is known recently for M in Class 1 and Class 3. We will calculate $D(M)$ for M in the remaining three classes. For the convenience of the readers, we will present $D(M)$ for M in all those five classes. To do this, we need first to coordinate 3-manifolds in each class, then state the results of $D(M)$ in term of those coordinates. This is carried in the next subsection.

1.2. Main results.

Class 1. According to [13] or [20], the fundamental group of a 3-manifold supporting S^3-geometry is among the following eight types: $\mathbb{Z}_p$, D^*_{4n}, T^*_{24}, O^*_{48}, I^*_{120}, $T'_{8\cdot 3^q}$, $D'_{n'\cdot 2^q}$ and $\mathbb{Z}_m \times \pi_1(N)$, where N is a 3-manifold supporting S^3-geometry, $\pi_1(N)$ belongs to the previous seven ones, and $|\pi_1(N)|$ is coprime to m. The cyclic group Z_p is realized by lens space $L(p,q)$, each group in the remaining types is realized by a unique 3-manifold supporting S^3-geometry. Note also the sub-indices of those seven types groups are exactly their orders, and the order of the groups in the last type is $m|\pi_1(N)|$. There are only two closed orientable 3-manifolds supporting $S^2 \times \mathbb{E}^1$ geometry: $S^2 \times S^1$ and $RP^3 \# RP^3$.

Theorem 1.1. (1) *$D(M)$ for M supporting S^3-geometry are listed below*:

$\pi_1(M)$	$D(M)$
$\mathbb{Z}_p$	$\{k^2 \mid k \in \mathbb{Z}\} + p\mathbb{Z}$
D^*_{4n}	$\{h^2 \mid h \in \mathbb{Z};\ 2 \nmid h \text{ or } h = n \text{or } h = 0\} + 4n\mathbb{Z}$
T^*_{24}	$\{0, 1, 16\} + 24\mathbb{Z}$
O^*_{48}	$\{0, 1, 25\} + 48\mathbb{Z}$
I^*_{120}	$\{0, 1, 49\} + 120\mathbb{Z}$
$T'_{8\cdot 3^q}$	$\begin{cases}\{k^2 \cdot (3^{2q-2p} - 3^q) \mid 3 \nmid k,\ q \geq p > 0\} + 8 \cdot 3^q\mathbb{Z} & (2 \mid q),\\ \{k^2 \cdot (3^{2q-2p} - 3^{q+1}) \mid 3 \nmid k,\ q \geq p > 0\} + 8 \cdot 3^q\mathbb{Z} & (2 \nmid q)\end{cases}$
$D'_{n'\cdot 2^q}$	$\{k^2 \cdot [1 - (n')^{2^q - 1}]^i \cdot [1 - 2^{(2p-q)(n'-1)}]^j \mid i, j, k, p \in \mathbb{Z},\ q \geq p > 0\} + n'2^q\mathbb{Z}$
$\mathbb{Z}_m \times \pi_1(N)$	$\left\{d \in \mathbb{Z} \;\middle|\; \begin{array}{ll} d = h + \lvert\pi_1(N)\rvert\mathbb{Z}, & h \in D(N), \\ d = k^2 + m\mathbb{Z}, & k \in \mathbb{Z} \end{array}\right\}$

(2) $D(S^2 \times S^1) = D(RP^3 \# RP^3) = \mathbb{Z}$.

Class 2. We assume that each 3-manifold P supporting S^3-geometry has the canonical orientation induced from the canonical orientation on S^3. When we change the orientation of P, the new oriented 3-manifold is denoted by $\bar{P}$. Moreover, lens space $L(p,q)$ is orientation reversed homeomorphic to $L(p, p-q)$, so we can write all the lens spaces connected summands as $L(p,q)$. Now we can decompose each 3-manifold in Class 2 as

$$\begin{aligned} M = {} & (mS^2 \times S^1) \# (m_1 P_1 \# n_1 \bar{P}_1) \# \cdots \# (m_s P_s \# n_s \bar{P}_s) \\ & \# (L(p_1, q_{1,1}) \# \cdots \# L(p_1, q_{1,r_1})) \# \cdots \# (L(p_t, q_{t,1}) \# \cdots \# L(p_t, q_{t,r_t})), \end{aligned}$$

where all the P_i are 3-manifolds with finite fundamental group different from lens spaces, all the P_i are different from each other, and all the positive integer p_i are different from each other. Define

$$D_{\rm iso}(M) = \{\deg(f) \mid f\colon M \to M,\ f \text{ induces an isomorphism on } \pi_1(M)\}.$$

Theorem 1.2. (1) $D(M) = D_{\rm iso}(m_1P_1 \# n_1\bar{P}_1) \cap \cdots \cap D_{\rm iso}(m_sP_s \# n_s\bar{P}_s) \cap D_{\rm iso}(L(p_1, q_{1,1}) \# \cdots \# L(p_1, q_{1,r_1})) \cap \cdots \cap D_{\rm iso}(L(p_t, q_{t,1}) \# \cdots \# L(p_t, q_{t,r_t}))$;

(2) $D_{\rm iso}(mP \# n\bar{P}) = \begin{cases} D_{\rm iso}(P) & \textit{if} \quad m \neq n, \\ D_{\rm iso}(P) \cup (-D_{\rm iso}(P)) & \textit{if} \quad m = n; \end{cases}$

(3) $D_{\rm iso}(L(p, q_1) \# \cdots \# L(p, q_n)) = H^{-1}(C)$.

The notions H and C in Theorem 1.2 (3) is defined as below:

Let $U_p = \{\text{all units in ring } \mathbb{Z}_p\}$, $U_p^2 = \{a^2 \mid a \in U_p\}$, which is a subgroup of U_p. We consider the quotient $U_p/U_p^2 = \{a_1, \ldots, a_m\}$, every a_i corresponds with a coset A_i of U_p^2. For the structure of U_p, see [9] p. 44. Define H to be the natural projection from $\{n \in \mathbb{Z} \mid \gcd(n, p) = 1\}$ to U_p/U_p^2.

Define $\bar{A}_s = \{L(p, q_i) \mid q_i \in A_s\}$ (with repetition allowed). In U_p/U_p^2, define $B_l = \{a_s \mid \#\bar{A}_s = l\}$ for $l = 1, 2, \ldots$, there are only finitely many l such that $B_l \neq \emptyset$. Let $C_l = \{a \in U_p/U_p^2 \mid a_ia \in B_l,\ \forall a_i \in B_l\}$ if $B_l \neq \emptyset$ and $C_l = U_p/U_p^2$ otherwise. Define $C = \bigcap_{l=1}^{\infty} C_l$.

Class 3. To simplify notions, for a diffeomorphism ϕ on torus T, we also use ϕ to present its isotopy class and its induced 2 by 2 matrix on $\pi_1(T)$ for a given basis.

A *torus bundle* is $M_\phi = T \times I/(x, 1) \sim (\phi(x), 0)$ where ϕ is a diffeomorphism of the torus T and I is the interval $[0, 1]$. Then the coordinates of M_ϕ is given as below:
(1) M_ϕ admits E^3 geometry, ϕ conjugates to a matrix of finite order n, where $n \in \{1, 2, 3, 4, 6\}$;
(2) M_ϕ admits Nil geometry, ϕ conjugates to $\pm\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, where $n \neq 0$;
(3) M_ϕ admits Sol geometry, ϕ conjugates to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $|a + d| > 2$, $ad - bc = 1$.

A *torus semi-bundle* $N_\phi = N \cup_\phi N$ is obtained by gluing two copies of N along their torus boundary ∂N via a diffeomorphism ϕ, where N is the twisted I-bundle over the Klein bottle. We have the double covering $p\colon S^1 \times S^1 \times I \to N = S^1 \times S^1 \times I/\tau$, where τ is an involution such that $\tau(x, y, z) = (x + \pi, -y, 1 - z)$.

Denote by l_0 and l_∞ on ∂N be the images of the second S^1 factor and first S^1 factor on $S^1 \times S^1 \times \{1\}$. *A canonical coordinate* is an orientation of l_0 and l_∞, hence there are four choices of canonical coordinate on ∂N. Once canonical coordinates on each ∂N are chosen, ϕ is identified with an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $GL_2(\mathbb{Z})$ given by $\phi(l_0, l_\infty) = (l_0, l_\infty)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

With suitable choice of canonical coordinates of ∂N, N_ϕ has coordinates as below:

(1) N_ϕ admits E^3 geometry, $\phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;

(2) N_ϕ admits Nil geometry, $\phi = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}$ or $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, where $z \neq 0$;

(3) N_ϕ admits Sol geometry, $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $abcd \neq 0$, $ad - bc = 1$.

Theorem 1.3. *$D(M_\phi)$ is in the table below for torus bundle M_ϕ, where $\delta(3) = \delta(6) = 1$, $\delta(4) = 0$.*

M_ϕ	ϕ	$D(M_\phi)$
E^3	*finite order* $k = 1, 2$	$\mathbb{Z}$
E^3	*finite order* $k = 3, 4, 6$	$\{(kt+1)(p^2 - \delta(k)pq + q^2) \mid t, p, q \in \mathbb{Z}\}$
Nil	$\pm \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}, n \neq 0$	$\{l^2 \mid l \in \mathbb{Z}\}$
Sol	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \|a+d\| > 2$	$\{p^2 + (d-a)pr/c - br^2/c \mid p, r \in \mathbb{Z}$, *either* $br/c, (d-a)r/c \in \mathbb{Z}$ *or* $(p(d-a) - br)/c \in \mathbb{Z}\}$

(2) *$D(N_\phi)$ is listed in the table below for torus semi-bundle N_ϕ, where $\delta(a, d) = ad/\mathrm{gcd}(a, d)^2$.*

N_ϕ	ϕ	$D(N_\phi)$
E^3	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\mathbb{Z}$
E^3	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\{2l+1 \mid l \in \mathbb{Z}\}$
Nil	$\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, z \neq 0$	$\{l^2 \mid l \in \mathbb{Z}\}$
Nil	$\begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}$ *or* $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, z \neq 0$	$\{(2l+1)^2 \mid l \in \mathbb{Z}\}$
Sol	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, abcd \neq 0, ad - bc = 1$	$\{(2l+1)^2 \mid l \in \mathbb{Z}\}$, *if* $\delta(a, d)$ *is even or* $\{(2l+1)^2 \mid l \in \mathbb{Z}\} \cup \{(2l+1)^2 \cdot \delta(a, d) \mid l \in \mathbb{Z}\}$, *if* $\delta(a, d)$ *is odd*

To coordinate 3-manifolds in Class 4 and Class 5, we first recall the well known coordinates of Seifert fibered spaces.

Suppose an oriented 3-manifold M' is a circle bundle with a given section F, where F is a compact surface with boundary components $c_1, \ldots, c_n$ with $n > 0$. On each boundary component of M', orient c_i and the circle fiber h_i so that the product of their orientations match with the induced orientation of M' (call such pairs $\{(c_i, h_i)\}$ a section-fiber coordinate system). Now attach n solid tori S_i to the n boundary tori of M' such that the meridian of S_i is identified with slope $r_i = c_i^{\alpha_i} h_i^{\beta_i}$ where $\alpha_i > 0$, $(\alpha_i, \beta_i) = 1$. Denote the resulting manifold by $M(\pm g;\ \beta_1/\alpha_1, \ldots, \beta_s/\alpha_s)$ which has the Seifert fiber structure extended from the circle bundle structure of M', where g is the genus of the section F of M, with the sign $+$ if F is orientable and $-$ if F is nonorientable, here 'genus' of nonorientable surfaces means the number of RP^2 connected summands. Call $e(M) = \sum_{i=1}^{s} \beta_i/\alpha_i \in \mathbb{Q}$ the Euler number of the Seifert fiberation.

Class 4. If a Nil manifold M is not a torus bundle or torus semi-bundle, then M has one of the following Seifert fibreing structures: $M(0;\ \beta_1/2,\ \beta_2/3,\ \beta_3/6)$, $M(0;\ \beta_1/3,\ \beta_2/3,\ \beta_3/3)$, or $M(0;\ \beta_1/2,\ \beta_2/4,\ \beta_3/4)$, where $e(M) \in \mathbb{Q} - \{0\}$.

Theorem 1.4. *For* 3*-manifold* M *in* Class 4*, we have*
(1) $D(M(0;\ \beta_1/2,\ \beta_2/3,\ \beta_3/6)) = \{l^2 \mid l = m^2 + mn + n^2,\ l \equiv 1 \bmod 6,\ m, n \in \mathbb{Z}\}$;
(2) $D(M(0;\ \beta_1/3,\ \beta_2/3,\ \beta_3/3)) = \{l^2 \mid l = m^2 + mn + n^2,\ l \equiv 1 \bmod 3,\ m, n \in \mathbb{Z}\}$;
(3) $D(M(0;\ \beta_1/2,\ \beta_2/4,\ \beta_3/4)) = \{l^2 \mid l = m^2 + n^2,\ l \equiv 1 \bmod 4,\ m, n \in \mathbb{Z}\}$.

Class 5. All manifolds supporting $H^2 \times E^1$ geometry are Seifert fibered spaces M such that $e(M) = 0$ and the Euler characteristic of the orbifold $\chi(O_M) < 0$.

Suppose $M = (g;\ \beta_{1,1}/\alpha_1, \ldots, \beta_{1,m_1}/\alpha_1, \ldots, \beta_{n,1}/\alpha_n, \ldots, \beta_{n,m_n}/\alpha_n)$, where all the integers $\alpha_i > 1$ are different from each other, and $\sum_{i=1}^{n} \sum_{j=1}^{m_i} \beta_{i,j}/\alpha_i = 0$.

For each α_i and each $a \in U_{\alpha_i}$, define $\theta_a(\alpha_i) = \#\{\beta_{i,j} \mid p_i(\beta_{i,j}) = a\}$ (with repetition allowed), p_i is the natural projection from $\{n \mid \gcd(n, \alpha_i) = 1\}$ to U_{α_i}. Define $B_l(\alpha_i) = \{a \mid \theta_a(\alpha_i) = l\}$ for $l = 1, 2, \ldots$, there are only finitely many l such that $B_l(\alpha_i) \neq \emptyset$. Let $C_l(\alpha_i) = \{b \in U_{\alpha_i} \mid ab \in B_l(\alpha_i),\ \forall a \in B_l(\alpha_i)\}$ if $B_l(\alpha_i) \neq \emptyset$ and $C_l(\alpha_i) = U_{\alpha_i}$ otherwise. Finally define $C(\alpha_i) = \bigcap_{l=1}^{\infty} C_l(\alpha_i)$, and $\bar{C}(\alpha_i) = p_i^{-1}(C(\alpha_i))$.

Theorem 1.5. $D(M(g;\ \beta_{1,1}/\alpha_1,\ldots,\beta_{1,m_1}/\alpha_1,\ldots,\beta_{n,1}/\alpha_n,\ldots,\beta_{n,m_n}/\alpha_n)) = \bigcap_{i=1}^{n} \bar{C}(\alpha_i)$.

1.3. A brief comment of the topic and organization of the paper. Theorem 1.0 was appeared in [25]. The proof of the "only if" part in Theorem 1.0 is based on the results on simplicial volume developed by Gromov, Thurston and Soma (see [21]), and various classical results by others on 3-manifold topology and group theory ([5], [19], [17]). The proof of "if" part in Theorem 1.0 is a sequence elementary constructions, which were essentially known before, for example see [6] and [11] for (3). That graph manifolds admits no self-maps of degrees > 1 also follows from a recent work [2].

The table in Theorem 1.1 is quoted from [1], which generalizes the earlier work [7]. The statement below quoted from [7] will be repeatedly used in this paper.

Proposition 1.6. *For* 3*-manifold* M *supporting* S^3 *geometry,*

$$D_{\text{iso}}(M) = \{k^2 + l|\pi_1(M)|,\ \text{where } k \text{ and } |\pi_1(M)| \text{ are co-prime}\}.$$

The topic of mapping degrees between (and to) 3-manifolds covered by S^3 has been discussed for long times and has many relations with other topics (see [26] for details). We just mention several papers: in very old papers [16] and [14], the degrees of maps between any given pairs of lens spaces are obtained by using equivalent maps between spheres; in [8], $D(M, L(p,q))$ can be computed for any 3-manifold M; and in a recent one [12], an algorithm (or formula) is given to the degrees of maps between given pairs of 3-manifolds covered by S^3 in term of their Seifert invariants.

Theorem 1.3 is proved in [23].

Theorems 1.2, 1.4 and 1.5 will be proved in Sections 3, 4 and 5 respectively in this paper. In Section 2 we will compute $D(M)$ for some concrete 3-manifolds using Theorems 1–5. We will also discuss when $-1 \in D(M)$ and when $-1 \in D(M)$ implies that M admits orientation reversing homeomorphisms.

All terminologies not defined are standard, see [5], [20] and [9].

2. Examples of computation, orientation reversing homeomorphisms

EXAMPLE 2.1. Let $M = (P \# \bar{P}) \# (L(7,1) \# L(7,2) \# 2L(7,3))$, where P is the Poincare homology three sphere.

By Theorem 1.2 (2), Proposition 1.6 and the fact $|\pi_1(P)| = 120$, we have $D(P \# \bar{P}) = D_{\text{iso}}(P) \cup (-D_{\text{iso}}(P)) = \{120n + i \mid n \in \mathbb{Z},\ i = 1, 49, 71, 119\}$.

Now we are going to calculate $D((L(7,1) \# L(7,2) \# 2L(7,3))$ following the notions of Theorem 1.2 (3). Clearly $U_7 = \{1, 2, 3, 4, 5, 6\}$ and $U_7^2 = \{1, 2, 4\}$. Then $U_7/U_7^2 = \{a_1, a_2\}$, where $a_1 = \bar{1}$ and $a_2 = \bar{3}$; $U_7 = \{A_1 \cup A_2\}$, where $A_1 = U_7^2$, $A_2 = 3U_7^2$; $\#\bar{A}_1 = 2$ and $\#\bar{A}_2 = 2$; $B_2 = \{\bar{1}, \bar{3}\}$, $B_l = \emptyset$ for $l \neq 2$. Since $U_7/U_7^2 = B_2$, we have $C_2 = B_2$ and also $C_l = U_7/U_7^2$ for $l \neq 2$; then $C = \bigcap_{l=1}^{\infty} C_l = U_7/U_7^2$. Then for the natural projection $H\colon \{n \in \mathbb{Z} \mid \gcd(n, p) = 1\} \to U_7/U_7^2$, $H^{-1}(C)$ are all number coprime to 7, hence we have $D_{\text{iso}}((L(7,1) \# L(7,2) \# 2L(7,3)) = \{l \in \mathbb{Z} \mid \gcd(l, 7) = 1\}$ by Theorem 1.2 (3).

Finally by Theorem 1.2 (1), we have $D(M) = \{120n + i \mid n \in \mathbb{Z},\ i = 1, 49, 71, 119\} \cap \{l \in \mathbb{Z} \mid \gcd(l, 7) = 1\} = \{840n + i \mid n \in \mathbb{Z},\ i = 1, 71, 121, 169, 191, 239, 241, 289, 311, 359, 361, 409, 431, 479, 481, 529, 551, 599, 601, 649, 671, 719, 769, 839\}$. Note $-1 \in D(M)$.

EXAMPLE 2.2. Suppose $M = (2P \# \bar{P}) \# (L(7,1) \# L(7,2) \# L(7,3))$.

Similarly by Theorem 1.2 (2), Proposition 1.6 and $|\pi_1(P)| = 120$, we have $D(2P \# \bar{P}) = D_{\text{iso}}(P) = \{120n + i \mid n \in \mathbb{Z},\ i = 1, 49\}$.

To calculate $D(L(7,1) \# L(7,2) \# L(7,3))$, we have U_7, U_7^2, $U_7/U_7^2 = \{a_1, a_2\}$, $U_7 = \{A_1, A_2\}$ exactly as last example. But then $\#\bar{A}_1 = 2$ and $\#\bar{A}_2 = 1$; $B_1 = \{\bar{3}\}$, $B_2 = \{\bar{1}\}$, $B_l = \emptyset$ for $l \neq 1, 2$. Moreover $C_1 = C_2 = \{\bar{1}\}$, and $C_l = U_7/U_7^2$ for $l \neq 1, 2$; then $C = \bigcap_{l=1}^{\infty} C_l = \{\bar{1}\}$, and $H^{-1}(C) = \{7n + i \mid n \in \mathbb{Z},\ i = 1, 2, 4\}$. Hence we have $D_{\text{iso}}(\#(L(7,1) \# L(7,2) \# L(7,3)) = \{7n + i \mid n \in \mathbb{Z},\ i = 1, 2, 4\}$ by Theorem 1.2 (3).

By Theorem 1.2 (1), $D(M) = \{120n + i \mid n \in \mathbb{Z},\ i = 1, 49\} \cap \{7n + i \mid n \in \mathbb{Z},\ i = 1, 2, 4\} = \{840n + i \mid n \in \mathbb{Z},\ i = 1, 121, 169, 289, 361, 529\}$. Note $-1 \notin D(M)$.

EXAMPLE 2.3. By Theorem 1.3, for the torus bundle M_ϕ, $\phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, among the first 20 integers > 0, exactly $1, 4, 5, 9, 11, 16, 19, 20 \in D(M_\phi)$.

EXAMPLE 2.4. For Nil 3-manifold $M = M(0;\ \beta_1/2,\ \beta_2/3,\ \beta_3/6)$, $D(M) = \{l^2 \mid l = m^2 + mn + n^2,\ l \equiv 1 \bmod 6,\ m, n \in \mathbb{Z}\}$. The numbers in $D(M)$ smaller than 10000 are exactly 1, 49, 169, 361, 625, 961, 1369, 1849, 2401, 3721, 4489, 5329, 6241, 8291, 9409. Since all $l = 6k + 1$, $k \in \mathbb{N}$ with $l^2 \leq 10000$ can be presented as $m^2 + mn + n^2$ except $l = 55, 85$ (if $5 \mid m^2 + mn + n^2$, then $5 \mid (2m + n)^2 + 3n^2$, therefore $5 \mid 2m + n$ and $5 \mid n$, it follows that $25 \mid m^2 + mn + n^2$).

EXAMPLE 2.5. For $H^2 \times E^1$ manifold $M = M(2;\ 1/5, 1/5, -2/5, 1/7, 2/7, -3/7)$, we follow the notions in Theorem 1.5 to calculate $D(M)$.

First we have $U_5 = \{1, 2, 3, 4\}$ with indices $\theta_a(5)$ are $\{2, 0, 1, 0\}$ respectively. Then $B_1(5) = \{3\}$, $B_2(5) = \{1\}$, $B_l(5) = \emptyset$ for $l \neq 1, 2$ and $C_1(5) = C_2(5) = \{1\}$. Hence $C(5) = \bigcap_{l=1}^{\infty} C_l(5) = \{1\}$. Hence $\bar{C}(5) = \{5n + 1 \mid n \in \mathbb{Z}\}$.

Similarly $U_7 = \{1, 2, 3, 4, 5, 6\}$ with indices $\theta_a(7)$ are $\{1, 1, 0, 1, 0, 0\}$ respectively. Then $B_1(7) = C_1(7) = \{1, 2, 4\}$. $B_l(7) = \emptyset$ and $C_l(7) = U_7$ for $l \neq 1$. Hence $C(7) = \bigcap_{l=1}^{\infty} C_l(7) = \{1, 2, 4\}$. $\bar{C}(7) = \{7n + i \mid n \in \mathbb{Z},\ i = 1, 2, 4\}$.

Finally $D(M) = \{5n + 1 \mid n \in \mathbb{Z}\} \cap \{7n + i \mid n \in \mathbb{Z},\ i = 1, 2, 4\} = \{35n + i \mid n \in \mathbb{Z},\ i = 1, 11, 16\}$.

EXAMPLE 2.6. Suppose M is a 3-manifold supporting S^3 geometry. By Proposition 1.6, M admits degree -1 self mapping if and only if there is integer number h, such that $h^2 \equiv -1 \bmod \pi_1(M)$. Then we can prove that if M is not a lens space, $-1 \notin D(M)$, (see proof of Proposition 3.10). With some further topological and number theoretical arguments, the following results were proved in [22].

(1) There is a degree -1 self map on $L(p, q)$, but no orientation reversing homeomorphism on it if and only if (p, q) satisfies: $p \nmid q^2 + 1$, $4 \nmid p$ and all the odd prime factors of p are the $4k + 1$ type.

(2) Every degree -1 self map on $L(p, q)$ are homotopic to an orientation reversing homeomorphism if and only if (p,q) satisfies: $q^2 \equiv -1 \bmod p$, $p = 2, p_1^{e_1}, 2p_1^{e_1}$, where p_1 is a $4k + 1$ type prime number.

EXAMPLE 2.7. Suppose M is a torus bundle. Then any non-zero degree map is homotopic to a covering ([25] Corollary 0.4). Hence if $-1 \in D(M)$, then M admits an orientation reversing self homeomorphism.

(1) For the torus bundle M_ϕ, $\phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $-1 \in D(M_\phi)$. Indeed for $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if $|a + d| = 3$, then $-1 \in D(M_\phi)$. Since $p^2 + ((d - a)/b)pr - c/br^2 = -1$ has solution $p = 1 - d$, $r = b$ when $a + d = 3$, and solution $p = -1 - d$, $r = b$ when $a + d = -3$.

(2) For the torus bundle M_ϕ, $\phi = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$, $-1 \notin D(M_\phi)$. Indeed for $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if $a + d \pm 2$ has prime decomposition $p_1^{e_1} \cdots p_n^{e_n}$ such that $p_i = 4l + 3$ and $e_i = 2m + 1$ for

some i, then $-1 \notin D(M_\phi)$. Since if the equation $p^2+((d-a)/b)pr-(c/b)r^2=-1$ has integer solution, then $(((a+d)^2-4)r^2-4b^2)/b^2$ should be a square of rational number. That is $((a+d)^2-4)r^2-4b^2=s^2$ for some integer s. Therefore $(a+d+2)(a+d-2)r^2$ is a sum of two squares. By a fact in elementary number theory, neither $a+d+2$ nor $a+d-2$ has $4k+3$ type prime factor with odd power (see p. 279, [9]).

3. $D(M)$ for connected sums

3.1. Relations between $D_{\mathrm{iso}}(M_1 \# M_2)$ and $\{D_{\mathrm{iso}}(M_1), D_{\mathrm{iso}}(M_2)\}$. In this section, we consider the manifolds M in Class 2: M has non-trivial prime decomposition, each connected summand has finite or infinite cyclic fundamental group, and M is not homeomorphic to $RP^3 \# RP^3$. (Note for each geometrizable 3-manifold P, $\pi_1(P)$ is finite if and only if P is S^3 3-manifold, and $\pi_1(P)$ is infinite cyclic if and only if P is $S^2 \times E^1$ 3-manifold.)

Since each S^3 3-manifold P is covered by S^3, we assume P has the canonical orientation induced by the canonical orientation on S^3. When we change the orientation of P, the new oriented 3-manifold is denoted by $\bar{P}$. Moreover, lens space $L(p,q)$ is orientation reversed homeomorphic to $L(p,p-q)$, so we can write all the lens spaces connected summands as $L(p,q)$. Now we can decompose the manifold as

$$\begin{aligned} M = {} & (mS^2 \times S^1) \# (m_1 P_1 \# n_1 \bar{P}_1) \# \cdots \# (m_s P_s \# n_s \bar{P}_s) \\ & \# (L(p_1, q_{1,1}) \# \cdots \# L(p_1, q_{1,r_1})) \# \cdots \# (L(p_t, q_{t,1}) \# \cdots \# L(p_t, q_{t,r_t})), \end{aligned}$$

where all the P_i are 3-manifolds with finite fundamental group different from lens spaces, all the P_i are different with each other, and all the positive integer p_i are different from each other. We will use this convention in this section.

Suppose F (resp. P) is a properly embedded surface (resp. an embedded 3-manifold) in a 3-manifold M. We use $M \setminus F$ (resp. $M \setminus P$) to denote the resulting manifold obtained by splitting M along F (resp. removing int P, the interior of P).

The definitions below are quoted from [17]:

DEFINITION 3.1. Let M, N be 3-manifolds and $B_f = \bigcup_i (B_i^+ \cup B_i^-)$ is a finite collection of disjoint 3-ball pairs in $\operatorname{int} M$. A map $f \colon M \setminus B_f \to N$ is called an *almost defined map* from M to N if for each i, $f|_{\partial B_i^+} = f|_{\partial B_i^-} \circ r_i$ for some orientation reversing homeomorphism $r_i \colon \partial B_i^+ \to \partial B_i^-$. If identifying ∂B_i^+ with ∂B_i^- via r_i, we get a quotient closed manifold $M(f)$, and f induces a map $\tilde{f} \colon M(f) \to N$. We define $\deg(f) = \deg(\tilde{f})$.

DEFINITION 3.2. For two almost defined maps f and g, we say that f is *B-equivalent* to g if there are almost defined maps $f = f_0, f_1, \dots, f_n = g$ such that either f_i is homotopic to f_{i+1} rel$(\partial B_{f_i} \cup \partial B_{f_{i+1}})$ or $f_i = f_{i+1}$ on $M \setminus B$ for an union of balls B containing $B_{f_i} \cup B_{f_{i+1}}$.

Lemma 3.3 ([17] Lemma 3.6, [25] Lemma 1.11). *Suppose $f\colon M \to M$ is a map of nonzero degree and $\bigcup S_i^2$ is an union of essential 2-spheres. Then there is an almost defined map $g\colon M \setminus B_g \to M$, B-equivalent to f, such that $\deg(g) = \deg(f)$ and $g^{-1}\big(\bigcup S_i^2\big)$ is a collection of spheres.*

Lemma 3.4 ([25] Corollary 0.2). *Suppose M is a geometrizable 3-manifold. Then any nonzero degree proper map $f\colon M \to M$ induces an isomorphism $f_*\colon \pi_1(M) \to \pi_1(M)$ unless M is covered by either a torus bundle over the circle, or $F \times S^1$ for some compact surface F, or the S^3.*

The following lemma is well-known.

Lemma 3.5. *Suppose M is a closed orientable 3-manifold, $f\colon M \to M$ is of degree $d \neq 0$. Then $f_*\colon H_2(M, \mathbb{Q}) \to H_2(M, \mathbb{Q})$ is an isomorphism.*

Theorem 3.6. *Suppose $M = M_1 \# \cdots \# M_n$ is a non-prime manifold which is not homeomorphic to $RP^3 \# RP^3$. Each $\pi_1(M_i)$ is finite or cyclic, and $\pi_1(M_i) \neq 0$. If $f\colon M \to M$ is a map of degree $d \neq 0$, then there exists a permutation τ of $\{1, \ldots, n\}$, such that there is a map $g_i\colon M_{\tau(i)} \to M_i$ of degree d for each i. Moreover, g_{i*} is an isomorphism on fundamental group.*

Proof. Call M' is a punctured M, if $M' = M \setminus B$, where B is a finitely many disjoint 3-balls in the interior of M. We use $\hat{M}_*$ to denote the 3-manifold obtained from M_* by capping off the boundary spheres with 3-balls.

M is obtained by gluing the boundary sphere of $M_i' = M_i \setminus \mathrm{int}(B_i)$ to a n-punctured 3-sphere. The image of ∂B_i in M, which is denoted by S_i, is a separating sphere.

By Lemma 3.3, there is an almost defined map $g\colon M \setminus B_g \to M$, B-equivalent to f, such that $g^{-1}\big(\bigcup S_i\big)$ is a collection of spheres and $\deg(g) = d$. Let $M_g = M \setminus B_g$.

Let $U = M_g \setminus g^{-1}\big(\bigcup S_i\big) = \{M_i^j \mid j = 1, \ldots, l_i,\ i = 1, \ldots, n\}$. The components of $g^{-1}(M_i')$ are denoted by $M_i^1, \ldots, M_i^{l_i}$.

By Lemma 3.4, $f_*\colon \pi_1(M) \to \pi_1(M)$ is an isomorphism. Since g is differ from f just on the 3-balls B_g up to homotopy rel ∂B_g, it follows that $g_*\colon \pi_1(M \setminus B_g) = \pi_1(M) \to \pi_1(M)$ is an isomorphism.

Since the prime decomposition of 3-manifold M is unique, and M_g is just a punctured M, each component of U is either a punctured non-trivial prime factor of M, or a punctured 3-sphere.

By Lemma 3.5, f_* is an injection on $H_2(M, \mathbb{Q})$. If S_i is a separating sphere, then $[S_i] = 0$ in $H_2(M, \mathbb{Q})$. So each component S' of $f^{-1}(S_i)$ is homologous to 0, thus S' separates M. By the procession of construction of g (see the proof of Lemma 3.4, [17]), which is B-equivalent to f, each component S of $g^{-1}(S_i)$ is also a separating sphere in M_g. So $\pi_1(M_g)$ is the free product of the $\pi_1(M_i^j)$, $i = 1, \ldots, n$, $j = 1, \ldots, l_i$.

Note $\pi_1(M) = \pi_1(M_1) * \cdots * \pi_1(M_n)$, each $\pi_1(M_i)$ is an indecomposable factor of $\pi_1(M)$. Since g_* is an isomorphism and each punctured 3-sphere has trivial π_1, from the basic fact on free product of groups, it follows that there is at least one punctured prime non-trivial factor in $g^{-1}(M_i')$. Since this is true for each $i = 1, \dots, n$ and there are at most n punctured prime non-trivial factors in U, it follows that there are n punctured prime non-trivial factors in U. Hence there is exactly one punctured prime non-trivial factor in $g^{-1}(M_i')$, denoted as $M_{\tau(i)}$, moreover $g_*\colon \pi_1(M_{\tau(i)}) \to \pi_1(M_i)$ is an isomorphism, where τ is a permutation on $\{1, \dots, n\}$.

Since $\pi_1(M_i) = \mathbb{Z}$ if and only if $M_i = S^2 \times S^1$, it follows that if $M_i = S^2 \times S^1$, then $M_{\tau(i)} = S^2 \times S^1$. Since $D(S^2 \times S^1) = Z$, below we assume that $\hat{M}_i' \neq S^2 \times S^1$, and to show that there is a map $g_i : M_{\tau(i)} \to M_i$ of degree d.

Since the map $g\colon M(g) \to M$ has degree d (see Definition 3.1), then $g_i = g|\colon \big(\bigcup_{j=1}^{l_i} M_i^j\big)(g) \to M_i'$ is a proper map of degree d, which can extend to a map $\hat{g}_i\colon \big(\bigcup_{j=1}^{l_i} \hat{M}_i^j\big)(g) \to \hat{M}_i' = M_i$ of degree d between closed 3-manifolds. The last map is also defined on $\big(\bigcup_{j=1}^{l_i} \hat{M}_i^j\big)(g) \setminus \overline{\partial B_g} = \big(\bigcup_{j=1}^{l_i} \hat{M}_i^j\big) \setminus B_g \subset \big(\bigcup_{j=1}^{l_i} \hat{M}_i^j\big)$, where $\overline{\partial B_g} \subset M(g)$ is the image of $\partial B_g \subset M$.

Now consider the map $\hat{g}_i : \big(\bigcup_{j=1}^{l_i} \hat{M}_i^j\big) \setminus B_g \to M_i$. Since $\pi_2(M_i) = 0$, we can extend the map $\hat{g}_i$ from $\bigcup_{j=1}^{l_i} \hat{M}_i^j \setminus B_g$ to $\bigcup_{j=1}^{l_i} \hat{M}_i^j$. More carefully, for each pair $B_k^+, B_k^- \subset \bigcup_{j=1}^{l_i} \hat{M}_i^j$ we can make the extension with the property $\hat{g}_i|_{B_i^+} = \hat{g}_i|_{B_i^-} \circ \hat{r}_i$, where $\hat{r}_i : B_i^+ \to B_i^-$ is an orientation reversing homeomorphism extending $r_i : \partial B_i^+ \to \partial B_i^-$. Now it is easy to see the map $\hat{g}_i : \bigcup_{j=1}^{l_i} \hat{M}_i^j \to M_i$ is still of degree d.

From the map $\hat{g}_i\colon \big(\bigcup_{j=1}^{l_i} \hat{M}_i^j\big) \to M_i$ one can obviously define a map $g_i\colon \#_{j=1}^{l_i} \hat{M}_i^j \to M_i$ of degree d between connected 3-manifolds. Since all $\hat{M}_i^j$ are S^3 except one is $M_{\tau(i)}$, we have map $g_i : M_{\tau(i)} \to M_i$. $\square$

DEFINITION 3.7. For closed oriented 3-manifold M, M', define

$$D_{\text{iso}}(M, M') = \{\deg(f) \mid f : M \to M',\ f \text{ induces isomorphism on fundamental group}\},$$

$$D_{\text{iso}}(M) = \{\deg(f) \mid f : M \to M,\ \ f \text{ induces isomorphism on fundamental group}\}.$$

Under the condition we considered in this section, we have $D(M) = D_{\text{iso}}(M)$ by Lemma 3.4.

Lemma 3.8. *Suppose $f_i : M_i \to M_i'$ is a map of degree d between closed n-manifolds, $n \geq 3$, f_{i*} is surjective on π_1, $i = 1,2$. Then there is a map $f\colon M_1 \# M_2 \to M_1' \# M_2'$ of degree d and f_* is surjective on π_1. In particular,*

(1) $D_{\text{iso}}(M_1 \# M_2, M_1' \# M_2') \supset D_{\text{iso}}(M_1, M_1') \cap D_{\text{iso}}(M_2, M_2')$,

(2) $D_{\text{iso}}(M_1 \# M_2) \supset D_{\text{iso}}(M_1) \cap D_{\text{iso}}(M_2)$.

Proof. Since f_* is surjective on π_1, it is known (see [18] for example), we can homotope f_i such that for some n-ball $D_i' \subset M_i'$, $f_i^{-1}(D_i)$ is an n-ball $D_i \subset M_i$. Thus we get a proper map $\bar{f}_i: M_i \setminus D_i \to M'_i \setminus D'_i$ of degree d, which also induces a degree d map from ∂D_i to $\partial D'_i$. Since maps of the same degree between $(n-1)$-spheres are homotopic, so after proper homotopy, we can paste $\bar{f}_1$ and $\bar{f}_2$ along the boundary to get map $f: M_1 \# M_2 \to M'_1 \# M'_2$ of degree d and f_* is surjective on π_1. □

3.2. $D(M)$ for connected sums. Suppose

$$\begin{aligned} M = {} & (mS^2 \times S^1) \# (m_1 P_1 \# n_1 \bar{P}_1) \# \cdots \# (m_s P_s \# n_s \bar{P}_s) \\ & \# (L(p_1, q_{1,1}) \# \cdots \# L(p_1, q_{1,r_1})) \# \cdots \# (L(p_t, q_{t,1}) \# \cdots \# L(p_t, q_{t,r_t})), \end{aligned}$$

where all the P_i are 3-manifolds with finite fundamental group different from lens spaces, all the P_i are different with each other, and all the positive integer p_i are different from each other.

To prove Theorem 1.2, we need only to prove the three propositions below.

Proposition 3.9.

$$(*) \quad \begin{aligned} D(M) = {} & D_{\text{iso}}(m_1 P_1 \# n_1 \bar{P}_1) \cap \cdots \cap D_{\text{iso}}(m_s P_s \# n_s \bar{P}_s) \cap D_{\text{iso}}(L(p_1, q_{1,1}) \# \cdots \\ & \# L(p_1, q_{1,r_1})) \cap \cdots \cap D_{\text{iso}}(L(p_t, q_{t,1}) \# \cdots \# L(p_t, q_{t,r_t})). \end{aligned}$$

Proof. For every self-mapping degree d of M, in Theorem 3.6 we have proved that for every oriented connected summand P of M, it corresponds to an oriented connected summand P', such that there is a degree d mapping $f: P \to P'$, and f induces isomorphism on fundamental group. By the classification of 3-manifolds with finite fundamental group (see [13], 6.2), P and P' are homeomorphism (not considering the orientation) unless they are lens spaces with same fundamental group. Now by Lemma 3.8 (1), we have $d \in D_{\text{iso}}(m_i P_i \# n_i \bar{P}_i)$ and $d \in D_{\text{iso}}(L(p_j, q_{j,1}) \# \cdots \# L(p_j, q_{j,r_j}))$, for $i = 1, \ldots, s$ and $j = 1, \ldots, t$. Hence we have proved

$$\begin{aligned} D(M) \subset {} & D_{\text{iso}}(m_1 P_1 \# n_1 \bar{P}_1) \cap \cdots \cap D_{\text{iso}}(m_s P_s \# n_s \bar{P}_s) \cap D_{\text{iso}}(L(p_1, q_{1,1}) \# \cdots \\ & \# L(p_1, q_{1,r_1})) \cap \cdots \cap D_{\text{iso}}(L(p_t, q_{t,1}) \# \cdots \# L(p_t, q_{t,r_t})). \end{aligned}$$

(Since $D(mS^2 \times S^1) = \mathbb{Z}$, we can just forget it in the discussion.)

Apply Lemma 3.8 once more, we finish the proof. □

Proposition 3.10. *If P is a 3-manifold with finite fundamental group different from lens space,* $D_{\text{iso}}(mP \# n\bar{P}) = \begin{cases} D_{\text{iso}}(P) & \text{if} \quad m \neq n, \\ D_{\text{iso}}(P) \cup (-D_{\text{iso}}(P)) & \text{if} \quad m = n. \end{cases}$

Proof. If P is not a lens space, from the list in [13], we can check that $4 \mid |\pi_1(P)|$. By Proposition 1.6, $D_{\text{iso}}(Q) = \{k^2 + l|\pi_1(Q)| \mid \gcd(k, |\pi_1(Q)|) = 1\}$, where Q is any 3-manifolds with S^3 geometry. If $k^2 + l|\pi_1(P)| = -k'^2 - l'|\pi_1(P)|$, then

$k^2+k'^2=-(l+l')|\pi_1(P)|$. Since $4 \mid |\pi_1(P)|$ and $\gcd(k,|\pi_1(P)|)=\gcd(k',|\pi_1(P)|)=1$, k,k' are both odd, thus $-(l+l')|\pi_1(P)|=k^2+k'^2=4s+2$, contradicts with $4 \mid |\pi_1(P)|$. So $D_{\rm iso}(P)\cap(-D_{\rm iso}(P))=\emptyset$. (In particular $-1\neq D(P)$.)

From the definition we have $D_{\rm iso}(P)=D_{\rm iso}(\bar P)$ and $D_{\rm iso}(P,\bar P)=D_{\rm iso}(\bar P,P)=-D_{\rm iso}(\bar P)$.

If $m\neq n$, we may assume that $m>n$. For the self-map f, if some P corresponds to $\bar P$, there must also be some P corresponds to P, so $\deg(f)\in D_{\rm iso}(P)\cap(-D_{\rm iso}(P))$, it is impossible by the argument in first paragraph. So all the P correspond to P, and all the $\bar P$ correspond to $\bar P$. Since $D_{\rm iso}(P)=D_{\rm iso}(\bar P)$, we have $D_{\rm iso}(mP\#n\bar P)\subset D_{\rm iso}(P)$. By Lemma 3.8 and the fact $D_{\rm iso}(P)=D_{\rm iso}(\bar P)$, we have $D_{\rm iso}(mP\#n\bar P)=D_{\rm iso}(P)$.

If $m=n$, similarly we have either all the P correspond to P and all the $\bar P$ correspond to $\bar P$; or all the P correspond to $\bar P$ and all the $\bar P$ correspond to P. Since $D_{\rm iso}(P)=D_{\rm iso}(\bar P)$ and $D_{\rm iso}(P,\bar P)=D_{\rm iso}(\bar P,P)=-D_{\rm iso}(\bar P)$, we have $D_{\rm iso}(mP\#m\bar P)\subset D_{\rm iso}(P)\cup(-D_{\rm iso}(P))$. On the other hand from the argument above, we have $D_{\rm iso}(P),-D_{\rm iso}(P)\subset D_{\rm iso}(mP\#m\bar P)$, hence $D_{\rm iso}(mP\#m\bar P)=D_{\rm iso}(P)\cup(-D_{\rm iso}(P))$. □

Lemma 3.11. $D_{\rm iso}(L(p,q),L(p,q'))=\{k^2q^{-1}q'+lp \mid \gcd(k,p)=1\}$, *here* q^{-1} *is seen as in group* $U_p=\{$*all the units in the ring* $\mathbb{Z}_p\}$.

Proof. $L(p,q)$ is the quotient of S^3 by the action of $\mathbb{Z}_p$, $(z_1,z_2)\to(e^{i2\pi/p}z_1,e^{i2q\pi/p}z_2)$. Let $\tilde f_{q,q'}\colon S^3\to S^3$, $\tilde f_{q,q'}(z_1,z_2)=(z_1^q/\sqrt{|z_1|^{2q}+|z_2|^{2q'}},\, z_2^{q'}/\sqrt{|z_1|^{2q}+|z_2|^{2q'}})$. We can check that this map induces a map $f_{q,q'}\colon L(p,q)\to L(p,q')$ with degree qq', moreover since q,q' are coprime with p, $f_{q,q'*}$ is an isomorphism π_1. By Proposition 1.6 $D_{\rm iso}(L(p,q))=\{k^2+lp \mid \gcd(k,p)=1\}$. Compose each self-map on $L(p,q)$ which induces an isomorphism on π_1 with $f_{q,q'}$, we have $\{k^2q^{-1}q'+lp \mid \gcd(k,p)=1\}\subset D_{\rm iso}(L(p,q),L(p,q'))$. On the other hand, for each map $g\colon L(p,q)\to L(p,q')$ of degree d which induces an isomorphism on π_1, then $f_{q',q}\circ g$ is a self-map on $L(p,q)$ which induces an isomorphism on π_1, where $f_{q',q}\colon L(p,q')\to L(p,q)$ is a degree qq' map. Hence the degree of $f_{q',q}\circ g$ is $qq'd$ which must be in $\{k^2+lp \mid \gcd(k,p)=1\}$, that is $qq'd=k^2+lp$, $\gcd(k,p)=1$, then $d=k^2q^{-1}q'^{-1}+pl=(k/q^{-1})^2q^{-1}q'+pl\in\{k^2q^{-1}q'+lp \mid \gcd(k,p)=1\}$. Hence $D_{\rm iso}(L(p,q),L(p,q'))=\{k^2q^{-1}q'+lp \mid \gcd(k,p)=1\}$. □

Let $U_p=\{\text{all units in ring } \mathbb{Z}_p\}$, $U_p^2=\{a^2 \mid a\in U_p\}$, which is a subgroup of U_p. Let H denote the natural projection from $\{n\in\mathbb{Z} \mid \gcd(n,p)=1\}$ to U_p/U_p^2.

Later, we will omit the p, denote them by U and U^2. We consider the quotient $U/U^2=\{a_1,\dots,a_m\}$, every a_i corresponds with a coset A_i of U^2. For the structure of U, see [9] p. 44, then we can get the structure of U^2 and U/U^2 easily.

Define $\bar A_s=\{L(p,q_i) \mid q_i\in A_s\}$ (with repetition allowed). In U/U^2, define $B_l=\{a_s \mid \#\bar A_s=l\}$ for $l=1,2,\ldots$, there are only finitely many B_l's are nonempty. Let $C_l=\{a\in U/U^2 \mid a_ia\in B_l,\ \forall a_i\in B_l\}$ if $B_l\neq\emptyset$ and $C_l=U/U^2$ otherwise, $C=\bigcap_{l=1}^{\infty}C_l$.

Proposition 3.12. $D_{\rm iso}(L(p,q_1)\#\cdots\#L(p,q_n))=H^{-1}(C)$.

Proof. By Lemma 3.11, we have $D_{\rm iso}(L(p, q), L(p, q')) = \{k^2q^{-1}q' + lp \mid \gcd(k, p) = 1\}$. Therefore $D_{\rm iso}(L(p, q), L(p, q'))$ will not change if we replace $L(p, q)$ by $L(p, s^2q)$ (resp. $L(p, q')$ by $L(p, s^2q')$) for any s in U_p.

Now we consider the relation between two sets $D_{\rm iso}(L(p, q), L(p, q'))$ and $D_{\rm iso}(L(p, q_*), L(p, q'_*))$. It is also easy to see if $(q/q')(q'_*/q_*) = s^2$ in U_p, then $D_{\rm iso}(L(p, q), L(p, q')) = D_{\rm iso}(L(p, q_*), L(p, q_*))$, and if $(q/q)(q'_*/q_*) \neq s^2$ in U_p, then $D_{\rm iso}(L(p, q), L(p, q')) \cap D_{\rm iso}(L(p, q_*), L(p, q'_*)) = \emptyset$.

Let $f\colon L(p, q_1) \# \cdots \# L(p, q_n) \to L(p, q_1) \# \cdots \# L(p, q_n)$ be a map of degree $d \neq 0$. Suppose f sends $L(p, q_i)$ to $L(p, q_k)$ and sends $L(p, q_j)$ to $L(p, q_l)$ in the sense of Theorem 3.6. Since $D_{\rm iso}(L(p, q_i), L(p, q_k)) \cap D_{\rm iso}(L(p, q_j), L(p, q_l)) \neq \emptyset$, by last paragraph, we must have $(q_i/q_k)(q_l/q_j) = s^2$ in U_p. Hence q_i/q_j is in U^2 if and only if q_l/q_k is in U^2; in other words, $L(p, q_i)$ and $L(p, q_j)$ are in the same $\bar{A}_s$ if and only if $L(p, q_k)$ and $L(p, q_l)$ are in the same $\bar{A}_t$. Hence f provides 1-1 self-correspondence on $\bar{A}_1, \ldots, \bar{A}_m$, and if some elements in $\bar{A}_s$ corresponds to $\bar{A}_t$, there is $\#\bar{A}_s = \#\bar{A}_t$.

Let $f\colon L(p, q_1) \# \cdots \# L(p, q_n) \to L(p, q_1) \# \cdots \# L(p, q_n)$ be a self-map. For each $a_i \in U/U^2$, f must send $\bar{A}_i$ to some $\bar{A}_j$ with $\#\bar{A}_i = \#\bar{A}_j = l$, and both $a_i, a_j \in B_l$. Assume $L(p, q_i) \in \bar{A}_i$, $L(p, q_j) \in \bar{A}_j$, then $\deg(f) \in \{k^2q_i^{-1}q_j + lp \mid \gcd(k, p) = 1\}$ by Lemma 3.11. By consider in U/U^2, we have $H(\deg(f)) = \bar{q}_j/\bar{q}_i = a_j/a_i$, that is $H(\deg(f))a_i = a_j \in B_l$. Since we choose arbitrary a_i in B_l, we have $H(\deg(f)) \in C_l$. Also we choose arbitrary l, we have $H(\deg(f)) \in \bigcap_{l=1}^{\infty} C_l = C$, hence $\deg(f) \in H^{-1}(C)$.

On the other hand, if $d \in H^{-1}(C)$, then $H(d) = c \in C = \bigcap_{l=1}^{\infty} C_l$. For each $B_l \neq \emptyset$ and each $a_i \in B_l$, we have $ca_i = a_j \in B_l$. Then $A_i \mapsto A_j$ gives 1-1 self-correspondence among $\{\bar{A}_i \mid \#\bar{A}_i = l\}$. We can make further 1-1 correspondence from elements in $\bar{A}_i$ to elements in $\bar{A}_j$. Since our discussion works for all $B_l \neq \emptyset$, we have 1-1 self-correspondence on $\{L(p, q_1), \ldots, L(p, q_n)\}$ (with repetition allowed). Therefore for each $L(p, q_i) \in \bar{A}_i$ and $L(p, q_j) \in \bar{A}_j$, $c = \bar{q}_j\bar{q}_i^{-1}$. Therefore d have the form $k^2q_jq_i^{-1}$ mod p with $(k, p) = 1$. By Lemma 3.11, there is a map $f_{i,j}\colon L(p, q_i) \to L(p, q_j)$ of degree d which induces an isomorphism on π_1.

By Lemma 3.8, we can construct a self-mapping of degree d of $L(p, q_1) \# \cdots \# L(p, q_n)$ which induces an isomorphism on π_1. Hence $H^{-1}(C) \subset D_{\rm iso}(L(p, q_1) \# \cdots \# L(p, q_n))$. Thus $D_{\rm iso}(L(p, q_1) \# \cdots \# L(p, q_n)) = H^{-1}(C)$. □

4. $D(M)$ for Nil manifolds

4.1. Self coverings of Euclidean orbifolds.

DEFINITION 4.1 ([20]). A 2-*orbifold* is a Hausdorff, paracompact space which is locally homeomorphic to the quotient space of $\mathbb{R}^2$ by a finite group action. Suppose $\mathcal{O}_1$ and $\mathcal{O}_2$ are orbifolds and $f\colon \mathcal{O}_1 \to \mathcal{O}_2$ is an map. We say f is an *orbifold covering* if any point p in $\mathcal{O}_2$ has a neighbourhood U such that $f^{-1}(U)$ is the disjoint union of sets $V_\lambda, \lambda \in \Lambda$, such that $f|: V_\lambda \to U$ is the natural quotient map between two quotients of $\mathbb{R}^2$ by finite groups, one of which is a subgroup of the other.

In this paper, we only consider about orbifold with singular points. Here we say a point x in the orbifold is a *singular point of index q* if x has a neighborhood U homeomorphic to the quotient space of $\mathbb{R}^2$ by rotate action of finite cyclic group $\mathbb{Z}_q$, $q > 1$.

An orbifold $\mathcal{O}$ with singular points $\{x_1,\dots,x_s\}$ is homeomorphic to a surface F, but for the sake of the singular points, we would like to distinguish them through denoting $\mathcal{O}$ by $F(q_1,\dots,q_s)$. Here $q_1,\dots,q_s$ are indices of singular points. Here the covering map $f: \mathcal{O}_1 \to \mathcal{O}_2$ is not the same as the covering map from F_1 to F_2.

If $f: \mathcal{O}_1 \to \mathcal{O}_2$ is an orbifold covering, the singular points of $\mathcal{O}_2$ are $\{x_1,\dots,x_s\}$, for any $y \in \mathcal{O}_2, y \neq x_i$, define $\deg(f) = \# f^{-1}(y)$. For any singular point x, let $f^{-1}(x) = \{a_1,\dots,a_i\}$. At point a_j, f is locally equivalent to $z \to z^{d_j}$ on $\mathbb{C}$, x and a_j correspond to 0. Here we have $\sum d_j = d$, a_j is an ordinary point if and only if d_j equals to the index of x. Define $D(x) = [d_1,\cdots,d_i]$ to be the *orbifold covering data at singular point x*, and $\mathfrak{D}(f) = \{D(x_1),\dots,D(x_s)\}$ (with repetition allowed) to be the *orbifold covering data of f*.

The following lemma is easy to verify.

Lemma 4.2. *If a Nil manifold M is not a torus bundle or a torus semi-bundle, then M has one of the following Seifert fibreing structures*: $M(0;\beta_1/2,\beta_2/3,\beta_3/6)$, $M(0;\beta_1/3,\beta_2/3,\beta_3/3)$, *or* $M(0;\beta_1/2,\beta_2/4,\beta_3/4)$, *where* $e(M) \in \mathbb{Q} - \{0\}$.

Proof. Consider Nil manifold M as a Seifert fibered space, then its orbifold $\mathcal{O}(M)$ has zero Euler characteristic. So $\mathcal{O}(M)$ must be one of following orbifolds: the torus T^2, the Klein bottle K, $P^2(2,2)$, $S^2(2,3,6)$, $S^2(2,4,4)$, $S^2(3,3,3)$ and $S^2(2,2,2,2)$.

By [4] p. 38 and p. 40, we can see that M has structure of torus bundle if $\mathcal{O}(M)$ is T^2 or K, and M has structure of torus semi-bundle if $\mathcal{O}(M)$ is $P^2(2,2)$ or $S^2(2,2,2,2)$.

The remaining three cases $S^2(2,3,6)$, $S^2(2,4,4)$ and $S^2(3,3,3)$ correspond to the three cases claimed in the lemma. Clear $e(M) \in \mathbb{Q} - \{0\}$ since Nil manifolds have non-zero Euler number. □

Proposition 4.3. *Denote the degrees set of self covering of an orbifold $\mathcal{O}$ by $D(\mathcal{O})$. We have*:

(1) *For* $\mathcal{O} = S^2(2,3,6)$, $D(\mathcal{O}) = \{m^2 + mn + n^2 \mid m, n \in \mathbb{Z}, (m,n) \neq (0,0)\}$.
Moreover, if $d \in D(\mathcal{O})$ is coprime with 6, then
(i) $d \equiv 1 \bmod 6$;
(ii) *this covering map of degree $d = 6k+1$ is realized by an orbifold covering from $\mathcal{O}$ to $\mathcal{O}$ with orbifold covering data*

$$\{D(x_1), D(x_2), D(x_3)\} = \{[\underbrace{2,\dots,2}_{3k},1], [\underbrace{3,\dots,3}_{2k},1], [\underbrace{6,\dots,6}_{k},1]\},$$

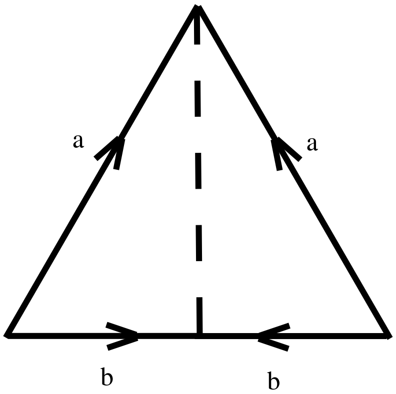

Fig. 1.

where x_1, x_2 *and* x_3 *are singular points of indices* 2, 3 *and* 6 *respectively.*

(2) *For* $\mathcal{O} = S^2(3,3,3)$, $D(\mathcal{O}) = \{m^2 + mn + n^2 \mid m, n \in \mathbb{Z}, (m,n) \neq (0,0)\}$. *Moreover, if* $d \in D(\mathcal{O})$ *is coprime with* 3, *then*

(i) $d \equiv 1 \bmod 3$;

(ii) *this covering map of degree* $d = 6k+1$ *is realized by an orbifold covering from* $\mathcal{O}$ *to* $\mathcal{O}$ *with orbifold covering data*

$$\{D(x_1), D(x_2), D(x_3)\} = \{[\underbrace{3, \ldots, 3}_{k}, 1], [\underbrace{3, \ldots, 3}_{k}, 1], [\underbrace{3, \ldots, 3}_{k}, 1]\},$$

where x_1, x_2 *and* x_3 *are singular points of indices* 3, 3 *and* 3 *respectively.*

(3) *For* $\mathcal{O} = S^2(2,4,4)$, $D(\mathcal{O}) = \{m^2 + n^2 \mid m, n \in \mathbb{Z}, (m,n) \neq (0,0)\}$. *Moreover, if* $d \in D(\mathcal{O})$ *is coprime with* 4, *then*

(i) $d \equiv 1 \bmod 4$;

(ii) *this covering map of degree* $d = 4k+1$ *is realized by an orbifold covering from* $\mathcal{O}$ *to* $\mathcal{O}$ *with orbifold covering data*

$$\{D(x_1), D(x_2), D(x_3)\} = \{[\underbrace{2, \ldots, 2}_{2k}, 1], [\underbrace{4, \ldots, 4}_{k}, 1], [\underbrace{4, \ldots, 4}_{k}, 1]\},$$

where x_1, x_2 *and* x_3 *are singular points of indices* 2, 4 *and* 4 *respectively.*

Proof. We only prove case (1). The other two cases can be proved similarly.

$S^2(2,3,6)$ can be seen as pasting the equilateral triangle as shown in Fig. 1 geometrically.

$\pi_1(S^2(2,3,6))$ can be identified with a discrete subgroup Γ of $\mathrm{Iso}_+(\mathbb{E}^2)$, a fundamental domain of Γ is shown in Fig. 2. It is as a lattice in $\mathbb{E}^2$ with vertex coordinate $m + ne^{i\pi/3}$, $m, n \in \mathbb{Z}$.

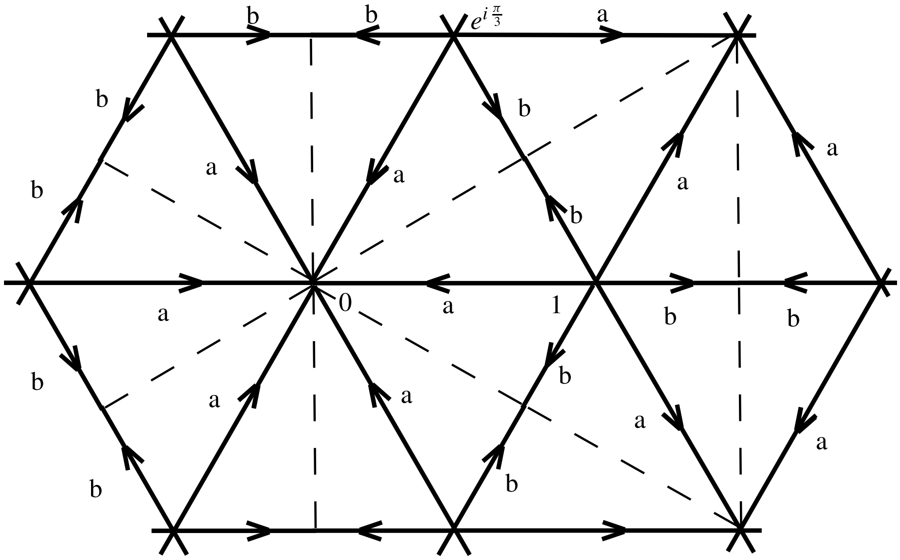

Fig. 2.

For the covering p: $T^2 \to S^2(2,3,6)$, T^2 can be seen as the quotient of a subgroup $\Gamma' \subset \Gamma$ on $\mathbb{E}^2$, with a fundamental domain as Fig. 3. Here Γ' is just all the translation elements of Γ, thus Γ' is generated by $z \to z + \sqrt{3}i$ and $z \to z + (\sqrt{3}/2)i + 3/2$.

For every self covering f: $S^2(2,3,6) \to S^2(2,3,6)$, f_*: $\pi_1(S^2(2,3,6)) \to \pi_1(S^2(2,3,6))$ is injective. Since p is covering, $f_* \circ p_* : \pi_1(T^2) \to \pi_1(S^2(2, 3, 6))$ is also injective. So $f_*(p_*(\pi_1(T^2)))$ is a free abelian subgroup of $\pi_1(S^2(2, 3, 6))$.

For every $\gamma \in \Gamma$, which is not translation, it can be represented by f: $z \to e^{i2k\pi/n}z + z_0$, $\gcd(k, n) = 1$, $n > 1$. Then $f^n(z) = (e^{i2k\pi/n})^n z + (e^{i2k(n-1)\pi/n} + \cdots e^{i2k\pi/n} + 1)z_0 = z$. So γ is a torsion element, thus $\gamma \notin f_*(p_*(\pi_1(T^2)))$ except $\gamma = e$. So $f_*(p_*(\pi_1(T^2))) \subset p_*(\pi_1(T^2))$, thus there exists $\tilde{f}$: $T^2 \to T^2$ being the lifting of f.

$$\begin{tikzcd}
T^2 \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & T^2 \arrow[d, "p"] \\
S^2(2, 3, 6) \arrow[r, "f"] & S^2(2, 3, 6).
\end{tikzcd}$$

Here we have

$$\deg(f) = \deg(\tilde{f}) = [\pi_1(T^2) : \tilde{f}_*(\pi_1(T^2))] = \frac{\text{area(fundamental domain of } \tilde{f}_*(\pi_1(T^2)))}{\text{area(fundamental domain of } \pi_1(T^2))},$$

here $\tilde{f}_*(\pi_1(T^2))$, $\pi_1(T^2)$ are all seen as subgroup of $\pi_1(S^2(2, 3, 6))$.

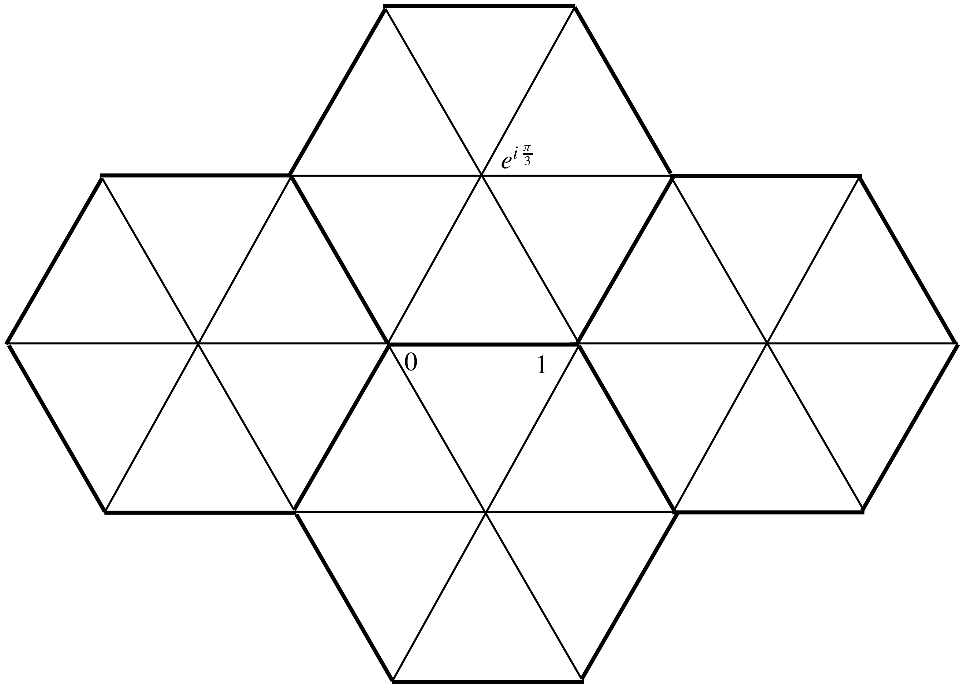

Fig. 3.

Clearly, we can choose a fundamental domain of $f_*(\pi_1(S^2(2,3,6)))$ to be an equilateral triangle in $\mathbb{E}^2$ with vertices as $m + ne^{i\pi/3}$, then the fundamental domain of $\tilde{f}_*(\pi_1(T^2))$ is an equilateral hexagon with vertices as $m + ne^{i\pi/3}$. The scale of area is the square of the scale of edge length. The scale of edge length must be $|m+ne^{i\pi/3}| = \sqrt{m^2 + mn + n^2}$. So $\deg(f) = m^2 + mn + n^2$.

On the other hand, for every $(m, n) \in \mathbb{Z}^2 - \{(0, 0)\}$, choose $g\colon \mathbb{E}^2 \to \mathbb{E}^2$, $g(z) = (m + ne^{i\pi/3})z$. It is routine to check that for any $\gamma \in \Gamma$, there is $\gamma' \in \Gamma$, such that $g(\gamma(z)) = \gamma'(g(z))$. So g induces $\bar{g}$, which is self covering on $S^2(2,3,6)$, and $\deg(\bar{g}) = m^2 + mn + n^2$. We have proved the first sentence of Proposition 4.3 (1).

If m^2+mn+n^2 is coprime to 6, $m^2+mn+n^2 \equiv 1$ or 5 mod 6. Since $m^2+mn+n^2 \equiv 4m^2 + 4mn + 4n^2 \equiv (2m + n)^2$ mod 3, and any square number must be 0 or 1 mod 3, we must have $m^2 + mn + n^2 \equiv 1$ mod 6. We have proved Proposition 4.3 (1) (i).

Assume h is a self covering of degree $d = 6k+1$, x_1, x_2, x_3 are the singular points on $S^2(2, 3, 6)$ with indices 2, 3, 6. For x_1, $h^{-1}(x_1)$ must be ordinary points or singular point of index 2. Since the degree $d = 6k + 1$, $h^{-1}(x_1)$ is $3k$ ordinary points and x_1. Similarly, for x_2, $h^{-1}(x_2)$ is $2k$ ordinary points and x_2. Then $x_1, x_2 \notin h^{-1}(x_3)$, so $h^{-1}(x_3)$ is k ordinary points and x_3. Thus the covering map of degree $d = 6k + 1$ is realized by a self covering of $\mathcal{O}$ with orbifold covering data $\{[2, \dots, 2, 1], [3, \dots, 3, 1], [6, \dots, 6, 1]\}$. We have proved Proposition 4.3 (1) (ii). $\square$

4.2. $D(M)$ for Nil manifolds.

Lemma 4.4. *For Nil manifold M, $D(M) \subset \{l^2 | l \in \mathbb{Z}\}$.*

Proof. Let f be a self map of M. By [25, Corollary 0.4], f is either homotopic to a covering map $g\colon M \to M$, or a homotopy equivalence.

If f is homotopic to a covering, since M has unique Seifert fibering structure up to isomorphism, we can make g to be a fiber preserving map. Denote the orbifold of M by O_M. By [20, Lemma 3.5], we have:

$$\begin{cases} e(M) = e(M) \cdot \dfrac{l}{m}, \\ \deg(g) = l \cdot m, \end{cases} \tag{4.1}$$

where l is the covering degree of $O_M \to O_M$ and m is the covering degree on the fiber direction. Since $e(M) \neq 0$, from equation (4.1) we get $l = m$. Thus $\deg(f) = \deg(g)$ is a square number l^2.

If f is a homotopy equivalence, then $\deg(f) = \pm 1$. To finish the proof of the lemma, we need only to show that the degree of f is not -1. Otherwise composing a self covering g of degree $n > 1$, then $g \circ f$ is of degree $-n$, which is not a homotopy equivalence, therefore is homotopic to a covering, and must have degree > 0 by the last paragraph, a contradiction. □

Theorem 4.5. *For* 3*-manifold M in* Class 4, *we have*
(1) *For $M = M(0;\ \beta_1/2,\ \beta_2/3,\ \beta_3/6)$, $D(M) = \{l^2 \mid l = m^2 + mn + n^2,\ l \equiv 1 \bmod 6,\ m, n \in \mathbb{Z}\}$;*
(2) *For $M = M(0;\ \beta_1/3,\ \beta_2/3,\ \beta_3/3)$, $D(M) = \{l^2 \mid l = m^2 + mn + n^2,\ l \equiv 1 \bmod 3,\ m, n \in \mathbb{Z}\}$;*
(3) *For $M = M(0;\ \beta_1/2, \beta_2/4, \beta_3/4)$, $D(M) = \{l^2 \mid l = m^2 + n^2,\ l \equiv 1 \bmod 4,\ m, n \in \mathbb{Z}\}$.*

Proof. We will just prove Case (1). The proof of Cases (2) and (3) are exactly as that of Case (1). Below $M = M(0;\ \beta_1/2,\ \beta_2/3,\ \beta_3/6)$.

First we show that $D(M) \subset \{l^2 \mid l = m^2 + mn + n^2,\ l \equiv 1 \bmod 6,\ m, n \in \mathbb{Z}\}$.

Since the orbifold $O_M = S^2(2, 3, 6)$, by Proposition 4.3 (1), we have $l = m^2 + mn + n^2$. Below we show that $l = 6k + 1$.

Let N be the regular neighborhood of 3 singular fibers. To define the Seifert invariants, a section F of $M \setminus N$ is chosen, and moreover ∂F and fibers on each component of $\partial(M \setminus N)$ are oriented.

Consider the covering $g|\colon M \setminus g^{-1}(N) \to M \setminus N$. Let $\tilde{F}$ be a component $g^{-1}(F)$. It is easy to verify that $\tilde{F}$ is a section of $M \setminus g^{-1}(N)$. Now we lift the orientations on ∂F and the fibers on $\partial(M \setminus N)$ to those on $\partial(M \setminus g^{-1}(N))$, we get a coordinate system on $\partial(M \setminus g^{-1}(N))$. Therefore we have a coordinate preserving covering

$$g\colon (M,\ M \setminus g^{-1}(N),\ g^{-1}(N)) \to (M,\ M \setminus N,\ N).$$

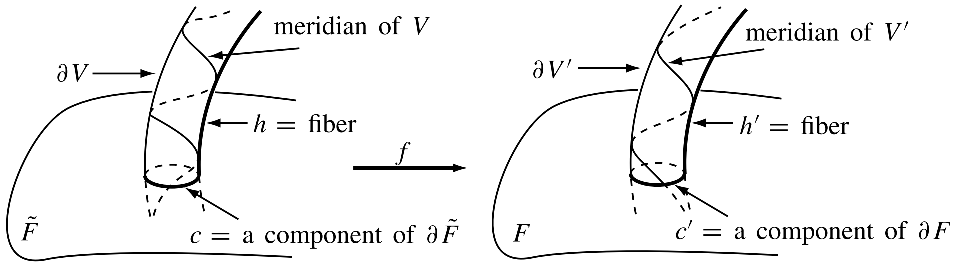

Fig. 4.

Suppose V' is a tubular neighborhood of some singular fiber L'. The meridian of V' can be represented by $(c')^{\alpha'}(h')^{\beta'}$ $(\alpha' > 0)$, where (c', h') is the section- fiber coordinate of $\partial V'$.

Suppose V is a component of $g^{-1}(V')$ and the meridian of V is represented as $c^{\alpha}h^{\beta}$ $(\alpha > 0)$, where (c, h) is the lift of (c', h'). Since $g|\colon V \to V'$ is a covering of solid torus, so g must send meridian to meridian homeomorphically, thus $g(c^{\alpha}h^{\beta}) = (c')^{\alpha'}(h')^{\beta'}$. See Fig. 4.

Since g has the fiber direction covering degree $m = l$, $g(h) = (h')^l$. Since c, c' are the boundaries of sections and g send c to c', we have $g(c) = (c')^s$. Then $g(c^{\alpha}h^{\beta}) = (c')^{\alpha\cdot s}(h')^{\beta\cdot l} = (c')^{\alpha'}(h')^{\beta'}$. Hence we get $\beta \cdot l = \beta'$.

Let V' be a tubular neighborhood of singular fiber whose meridian can be represented as $(c')^6(h')^{\beta'}$. By the arguments above, the meridian of the preimage V can be represent by $c^{\alpha}h^{\beta}$.

Since β' is coprime with 6. By $\beta \cdot l = \beta'$, so l is coprime with 6. Still by Proposition 4.3 (1), we have $l = 6k + 1$.

Then we show $\{l^2 \mid l = m^2 + mn + n^2,\ l \equiv 1 \bmod 6,\ m, n \in \mathbb{Z}\} \subset D(M)$.

Suppose $l = m^2 + mn + n^2$ and $l = 6k + 1$, denote the quotient manifold of $\mathbb{Z}_l$ free action on M by M_l. Then M_l has the Seifert fibering structure $M(0;\ l \cdot \beta_1/2,\ l \cdot \beta_2/3,\ l \cdot \beta_3/6)$. We have the covering $g_l\colon M \to M_l$ of degree l.

Claim. *there exists a map $f_l\colon M_l \to M$ of degree l.*

Let $D = D_1 \cup D_2 \cup D_3 \subset S^2(2, 3, 6)$ be the regular neighborhood discs of 3 singular points of indices 2, 3, and 6 respectively. By Proposition 4.3 (1), there exists a branched covering map $\bar{f}_l\colon S^2(2, 3, 6) \to S^2(2, 3, 6)$ of degree l such that

(1) $\bar{f}_l$ induce a covering map $\bar{f}_l|\colon S^2 \setminus \bar{f}_l^{-1}(D) \to S^2 \setminus D$;

(2) $\bar{f}_l^{-1}(D_i)$ consists of $(3k + 1)$ discs with orbifold covering data $[\underbrace{2, \dots, 2}_{3k}, 1]$ for $i = 1$, and $(2k + 1)$ discs with orbifold covering data $[\underbrace{3, \dots, 3}_{2k}, 1]$ for $i = 2$, and $(k + 1)$ discs with orbifold covering data $[\underbrace{6, \dots, 6}_{k}, 1]$ for $i = 3$.

Clearly $\bar{f}_l^{-1}(D)$ consists of $(3k+1)+(2k+1)+(k+1)=6k+3$ disks.

Then we have the covering map $\bar{f}_l \times id\colon (S^2 \setminus f_l^{-1}(D)) \times S^1 \to (S^2 \setminus D) \times S^1$ of degree l, which can be extends to a covering map $f_l\colon M' \to M$, where M' has the Seifert structure $M(0;\ \underbrace{\beta_1, \ldots, \beta_1}_{3k}, \beta_1/2, \underbrace{\beta_2, \ldots, \beta_2}_{2k}, \beta_2/3, \underbrace{\beta_3, \ldots, \beta_3}_{k}, \beta_3/6)$. Clearly M' is isomorphic to M_l.

Now the covering $f_l \circ g_l\colon M \to M_l \to M$ has degree l^2.

We finish the proof of Case (1). □

5. $D(M)$ for $H^2 \times E^1$ manifolds

In this case, all the manifolds are Seifert fibered spaces M such that the Euler number $e(M)=0$ and the Euler characteristic of the orbifold $\chi(O_M)<0$.

Suppose $M=(g;\ \beta_{1,1}/\alpha_1, \ldots, \beta_{1,m_1}/\alpha_1, \ldots, \beta_{n,1}/\alpha_n, \ldots, \beta_{n,m_n}/\alpha_n)$, where all the integers $\alpha_i>1$ are different from each other, and $\sum_{i=1}^{n}\sum_{j=1}^{m_i}\beta_{i,j}/\alpha_i=0$.

For every α_i, consider U_{α_i}. For every $a \in U_{\alpha_i}$, define $\theta_a(\alpha_i)=\#\{\beta_{i,j} \mid p_i(\beta_{i,j})=a\}$ (with repetition allowed), where p_i is the natural projection from $\{n \mid \gcd(n,\alpha_i)=1\}$ to U_{α_i}. Define $B_l(\alpha_i)=\{a \mid \theta_a(\alpha_i)=l\}$ for $l=0,1,\ldots,$ there are only finitely many $B_l(\alpha_i)$ nonempty. Let $C_l(\alpha_i)=\{b \in U_{\alpha_i} \mid ab \in B_l(\alpha_i),\ \forall a \in B_l(\alpha_i)\}$ if $B_l(\alpha_i)\neq\emptyset$ and $C_l(\alpha_i)=U_{\alpha_i}$ otherwise. Finally define $C(\alpha_i)=\bigcap_{l=1}^{\infty}C_l(\alpha_i)$, and $\bar{C}(\alpha_i)=p_i^{-1}(C(\alpha_i))$.

Theorem 5.1.

$$D\left(M\left(g;\ \frac{\beta_{1,1}}{\alpha_1}, \ldots, \frac{\beta_{1,m_1}}{\alpha_1}, \ldots, \frac{\beta_{n,1}}{\alpha_n}, \ldots, \frac{\beta_{n,m_n}}{\alpha_n}\right)\right)=\bigcap_{i=1}^{n}\bar{C}(\alpha_i).$$

Proof. Suppose f is a non-zero degree self-mapping of M. By [25, Corollary 0.4], f is homotopic to a covering map $g\colon M \to M$. Since M has the unique Seifert structure, we can isotope g to a fiber preserving map. Denote the orbifold of M by O_M. Then g induces a self-covering $\bar{g}$ on O_M, since $\chi(O_M)<0$, then $\bar{g}$ must be 1-sheet, thus isomorphism of O_M.

So g is a degree d covering on the fiber direction. Or equivalently, by the action of $\mathbb{Z}_d$ on each fiber, the quotient of M is also M. Thus $d \in D(M)$ if and only if

$$M'=M\left(g\colon d\frac{\beta_{1,1}}{\alpha_1}, \ldots, d\frac{\beta_{1,m_1}}{\alpha_1}, \ldots, d\frac{\beta_{n,1}}{\alpha_n}, \ldots, d\frac{\beta_{n,m_n}}{\alpha_n}\right)$$

is homeomorphic to M.

By the uniqueness of Seifert structure ([20] Theorem 3.9) and the fact $e(M)=0$, we have that M is homeomorphism to M' if and only if $(\beta_{i,1}, \ldots, \beta_{i,m_i})=(d\beta_{i,1}, \ldots, d\beta_{i,m_i})$ under a permutation, all the numbers are seen as in $U(\alpha_i)$.

For every $a \in U(\alpha_i)$, if $(\beta_{i,1}, \ldots, \beta_{i,m_i})=(d\beta_{i,1}, \ldots, d\beta_{i,m_i})$ holds, we must have $\theta_a(\alpha_i)=\theta_{da}(\alpha_i)$, thus $p_i(d) \in C_{\theta_a}(\alpha_i)$. For a is an arbitrary element in $U(\alpha_i)$, we have

$p_i(d) \in C(\alpha_i)$, thus $d \in \bar{C}(\alpha_i)$. Since α_i is also chosen arbitrarily, $d \in \bigcap_{i=1}^n \bar{C}(\alpha_i)$, thus $D(M) \subset \bigcap_{i=1}^n \bar{C}(\alpha_i)$.

For any $d \in \bigcap_{i=1}^n \bar{C}(\alpha_i)$, M is homeomorphic to M', so $D(M) \supset \bigcap_{i=1}^n \bar{C}(\alpha_i)$ $\square$

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Hongbin Sun
School of Mathematical Sciences
Peking University
Beijing 100871
P.R. China
e-mail: hongbin.sun2331@gmail.com

Shicheng Wang
School of Mathematical Sciences
Peking University
Beijing 100871
P.R. China
e-mail: wangsc@math.pku.edu.cn

Jianchun Wu
School of Mathematical Sciences
Peking University
Beijing 100871
P.R. China
e-mail: wujianchun@math.pku.edu.cn

Hao Zheng
School of Mathematical Sciences
Peking University
Beijing 100871
P.R. China
e-mail: zhenghao@mail.sysu.edu.cn