Taira HONDA

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1932–1975

Professor Taira HONDA of the Osaka City University was tragically taken away from amidst his intensive scientific activities and from his beloved family on May 15, 1975. At the request of the Editors of this Journal, I am writing these lines to recollect some of the memories of our deceased friend, and to give a brief review of his mathematical work.

Taira HONDA was born in Fukui Prefecture facing the Japan Sea on June 2, 1932. In 1951, he entered the University of Tokyo, from which he graduated in 1955. During the last year at this University, he joined Professor Tsuneo TAMAGAWA's seminar with Yasumasa AKAGAWA and Satoshi ARIMA. He continued his study on the graduate level at the same University, until he joined the Osaka University in 1961, from where he moved to the Osaka City University in 1974. Always frank and lively, he was popular among both his colleagues and students.

In autumn 1955, in the same year when he graduated from the University of Tokyo, an International Symposium on Algebraic Number Theory was held in Tokyo and Nikko. Mathematicians like E. Artin, Chevalley and A. Weil came to our country for this occasion, and significant contributions to the long-standing problems of the theory of complex multiplication by Goro SHIMURA and Yutaka TANIYAMA as well as by A. Weil were reported at this Symposium. These circumstances surely had a strong influence on the mathematical formation of HONDA in addition to the personal direction given to him by TAMAGAWA.

His first published paper*[1] already had remarkable contents. He restudied and generalized the arithmetical theories of Kummer fields and of cyclotomic fields from the stand-point of algebraic geometry, considering them as theories of rational points of abelian varieties over algebraic number fields. For an abelian variety A defined over an algebraic number field of finite degree and for a field k of the same kind, he defined the number $\rho_k(A)$ as the maximal number of $\boldsymbol{Z}$-independent k-rational points in A. It is finite by the well-known Mordell-Weil theorem. He posed the question: how $\rho_k(A)$ depends on k when A is given, and conjectured that $\rho_k(A)/(k:\boldsymbol{Q})$ will have a bound depending only on A. He proved in this paper $\rho_K(A)=(K:k)\rho_k(A)$ in case A is a simple abelian variety of dimension r with the endomorphism algebra isomorphic to a number field of degree $2r$ and K is the composite field of k and the field of "the dual CM-type"

*) Numbers in brackets refer to the bibliography.

(in the terminology of Shimura-Taniyama. CM means "complex multiplication".) This contains as a special case a result obtained by Cassels on rational points on certain elliptic curves, and leads, quite curiously, to results on class numbers of certain quadratic and cubic fields.

The problem of class numbers of algebraic number fields is one of the questions in which HONDA was particularly interested. His papers [2], [3], [4], [7], [10], [17] are all concerned with it. For example in [7], which is directly related with [1], it is proved that there are infinitely many real quadratic fields whose class numbers are multiples of 3. In [10], the natural number n such that class number of $\boldsymbol{Q}(\sqrt[3]{n})$ is a multiple of 3, is completely characterized.

His paper [6] published in 1968, which became his doctoral thesis, is another important paper in continuation of his idea initiated in [1]. To state its main result, we should first recall the definition of "q-Weil number" (so-called by J. Tate). Let p be a prime and $q=p^a$ a power of p. An algebraic integer π is called a q-Weil number, if $\pi^\sigma\bar{\pi}^\sigma=q=p^a$, where bar indicates the complex conjugate, and π^σ is any conjugate of π. Let, on the other hand, Ω be an algebraic closure of the prime field of characteristic p, and k_a be the finite field with $p^a=q$ elements contained in Ω. Tate had established in the meantime an injective map Φ_a of the set of all k_a-isogeny classes of k_a-simple abelian varieties over k_a to the set of all conjugacy classes of q-Weil numbers. HONDA proved in [6] that Φ_a is also surjective, and consequently bijective. This remarkable result was reported at Bourbaki seminar by Tate in 1969 as expose 352. Φ_a is now known as the Tate-Honda map.

Since the year 1966 when he wrote his paper [5], HONDA was attracted by the fruitful idea of utilizing the theory of commutative formal groups to the arithmetical study of abelian varieties. Whereas his complete exposition of the theory was given in 1970 in his paper [9], he wrote his paper [8] concerning the relation between formal groups and zeta functions in 1968. Also his papers [11], [12], [13], [14], [15] relate to this field.

I shall confine myself to explain his idea in the simplest case. Let R be a commutative ring with 1. $R[[x_1, \cdots, x_n]]_0$ means the ring of formal power series with n variables $x_1, \cdots, x_n$ with coefficients from R and with the vanishing absolute terms. $F=F(x,y)\in R[[x,y]]_0$ is called a commutative formal group over R, if $F(x,y)=F(y,x)$, $F(x, F(y,z))=F(F(x,y), z)$ and $F(x,y)\equiv x+y$ mod. deg. 2. Then it is easily shown that the composition $R[[x]]_0\times R[[x]]_0\ni(\phi_1(x), \phi_2(x))\mapsto F(\phi_1(x), \phi_2(x))\in R[[x]]_0$ satisfies the axioms of the commutative group. It is clear that, e.g. the additive group of an elliptic curve can be expressed as a commutative formal group.

If F and G are two formal groups, and there exists an invertible element $\varphi(x)$ of $R[[x]]_0$, such that $F(x,y)=\phi^{-1}(G(\phi(x), \phi(y)))$, F and G are said to be isomorphic, and if moreover $\varphi(x)\equiv x$ mod. deg. 2, F, G are called strongly isomorphic. If R is a $\boldsymbol{Q}$-algebra, it is shown that every commutative formal group is strongly isomorphic to $F(x,y)=x+y$.

The above classical case of dimensional 1 had been already treated by M. Lazard in 1955. HONDA's paper [9] deals with the general case of arbitrary dimensions over $\mathfrak{p}$-adic integer rings with applications to the case $R=\boldsymbol{Z}$. He succeeded in particular in giving a complete classification (in the sense of strong isomorphism) of 1-dimensional commutative formal groups over Z. A remarkable result is that such formal groups are closely related to zeta-functions which have Euler product expressions. As an interesting concrete example, I shall mention the following proposition (Theorem 4 in [8]):

Let D be the discriminant of a quadratic number field $K=\boldsymbol{Q}(\sqrt{D})$. Then the Dirichlet L-function $\sum_{n=1}^{\infty}\left(\frac{D}{n}\right)n^{-s}$ is related with the following formal group $G(x,y)$ over Z: if we put $g(x)=\sum_{n=1}^{\infty}\left(\frac{D}{n}\right)n^{-1}x^{n}$, we have $G(x,y)=g^{-1}(g(x)+g(y))$ $\in\boldsymbol{Z}[[x,y]]_0$ and G is strongly isomorphic with $F(x,y)=x+y+\sqrt{D}\,xy$ over the ring of integers of K.

In his papers [12], [13], HONDA looked for commutative formal groups over $\boldsymbol{Q}$, which are "non-algebraic" (i.e. not directly connected with abelian varieties), and whose coefficients are integral at almost all primes of $\boldsymbol{Q}$, succeeded in finding some of them related to differential equations of Fuchsian type and to generalized hyper-geometric functions, and dreamed of a new field of number theory. His results on commutative formal groups over Z are quoted by S. P. Novikov and Morava in their papers in algebraic topology.

Though very incomplete, I hope to have shown in the above review, how interesting and how intriguing HONDA's work is. He has well founded his arithmetical theory of commutative formal groups, but he has left many things for the younger generation to explore. In deploring once again his premature death, I wish to express our thanks for his pioneering work and hope and believe that it will be pursued in future by his successors.

S. IYANAGA

Mathematical works of T. Honda

[1] Isogenies, rational points and section points of group varieties, Japanese J. Math. 30 (1960), 84–101.

[2] On the absolute ideal class groups of relatively meta-cyclic number fields of a certain type, Nagoya Math. J. 17 (1960), 171-179.

[3] On absolute class fields of certain algebraic number fields, J. Reine Angew. Math. 203 (1960), 80-89.

[4] 代数体の類数公式について (On class number formulas of algebraic number fields), Sūgaku 16 (1964), 129-138.

[5] On the Jacobian variety of the algebraic curve $y^2=1-x^l$ over a field of characteristic $p>0$, Osaka J. Math. 3 (1966), 189-194.

[6] Isogeny classes of abelian varieties over finite fields, J. Math. Soc. Japan 20 (1968), 83-95.

[7] On real quadratic fields whose class numbers are multiples of 3, J. Reine Angew. Math. 233 (1968), 101-102.

[8] Formal groups and zeta functions, Osaka J. Math. 5 (1968), 199-213.

[9] On the theory of commutative formal groups, J. Math. Soc. Japan 22 (1970), 213-246.

[10] Pure cubic fields whose class numbers are multiples of three, J. Number Theory 3 (1971), 7-12.

[11] 可換形式群について (On commutative formal groups), Sūgaku 23 (1971), 205-213.

[12] Differential equations and formal groups, U.S.-Japan Seminar on modern methods in number theory, Tokyo, 1971.

[13] Formal groups obtained from generalized hypergeometric functions, Osaka J. Math. 9 (1972), 447–462.

[14] Invariant differentials and L-functions.—Reciprocity law for quadratic fields and elliptic curves over $\boldsymbol{Q}$—Rend. Sem. Math. Univ. Padova 49 (1973), 323-335.

[15] On the formal structure of the Jacobian variety of the Fermat curve over a p-adic integer ring, Symposia Mathematica 11 (1973), 271-284.

[16] Zeta-functions of elliptic curves of 2-power conductor, (with Isao Miyawaki), J. Math. Soc. Japan 26 (1974), 362-373.

[17] A few remarks on class numbers of imaginary quadratic number fields, Osaka J. Math. 12 (1975), 19-21.

[18] Two congruence properties of Legendre polynomials, to appear.