



Title	New Directions in Exact Renormalization Group : Lifshitz-Type Theory, Gradient Flow Equation and Its Supersymmetric Extension
Author(s)	菊地, 健吾
Citation	大阪大学, 2014, 博士論文
Version Type	VoR
URL	https://doi.org/10.18910/50466
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**New Directions in Exact Renormalization
Group: Lifshitz-Type Theory, Gradient Flow
Equation and Its Supersymmetric Extension**

Kengo Kikuchi

*Department of Physics, Graduate School of Science,
Osaka University, Toyonaka, 560-0043, Japan*

A Dissertation for the degree of Doctor of Philosophy

May 9, 2014

Abstract

The purpose of this thesis is to present our study on the new directions in the exact renormalization group (ERG) through the Lifshitz-type theory, the gradient flow equation and its supersymmetric extension. There are two topics in this thesis. The first topic is the study on the restoration of the Lorentz symmetry for a Lifshitz-type scalar theory in the infrared region using non-perturbative methods. We apply the Wegner-Houghton equation, which is one of the exact renormalization group equations, to the Lifshitz-type theory. Analyzing the equation for a $z = 2, d = 3 + 1$ Lifshitz-type scalar model, we find that symmetry violating terms vanish in the infrared region. This shows that the Lifshitz-type scalar model dynamically restores the Lorentz symmetry at low energy. Our result provides a definition of ultraviolet complete renormalizable scalar field theories. These theories can have nontrivial interaction terms of ϕ^n ($n = 4, 6, 8, 10$) even when the Lorentz symmetry is restored at low energy. The second topic is the gradient flow equation and its supersymmetric extension. We explain the expectation value in terms of a gauge field, which is defined by a certain type of diffusion equation called a gradient flow equation, is finite without additional renormalization as a review. And we extend the equation to super Yang-Mills theory. We propose a gradient flow based on superfield formalism. As a result, we construct a supersymmetric extension of the gradient flow equation, which includes only finite terms in the Wess-Zumino gauge. Our result also provide the gradient flow equation of the matter field very naturally. Our studies could be an important step towards the ERG which keeps the symmetry explicitly.

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1 Introduction

The goal of the elementary particle physics is to explain macroscopic phenomena from the fundamental microscopic theory. We know that gauge theories can describe phenomena which involve strong and electroweak interactions. As is well known, however, the theory encounters ultraviolet divergences. It is not so easy to avoid the divergences. For example, the higher-dimensional gauge theory which is typified by physics beyond the standard model is also perturbatively unrenormalizable. In 1940's the theory of perturbative renormalization, originally invented as a prescription to avoid the divergence, contributed to quantum electrodynamics (QED) which has lead to enormous success to date. However the perturbation theory is not enough to understand physics. The non-perturbative approach is also needed to explain strong coupling theories such as quantum chromodynamics (QCD).

We are especially interested in the exact renormalization group (ERG), which is one of the methods to analyze the physical system non-perturbatively. In 1970's, Wilson and Kogut gave a physical meaning to the renormalization method, and constructed the framework of the ERG [1]. The philosophy of the ERG is encoded in the formulation itself. We integrate out the high frequency mode of the action and define the resulting action for the low frequency mode as the Wilson effective action. The ERG equation gives the change of the Wilson effective action as one changes the cut-off scale. The ERG is not just a tool, but a powerful physical approach. One can even say that knowing the renormalization group (RG) flow in the whole theory space is equal to understanding the entire property of the physical system.

There are several different formulations for the ERG which exploit different cut-off functions or calculational methods, e.g. the Wegner-Houghton equation, the Polchinski equation and the Wetterich equation. However because of all of them make a cut-off in a momentum space, they often break the symmetry of the theory explicitly, in particular the gauge symmetry which is very important in the elementary particle physics. If the formulation of the ERG equation which keeps the gauge symmetry explicitly is achieved, the method can give substantial contributions to the elementary particle physics.

The ultimate goal of our study is to formulate a new approach to the ERG which keeps the gauge symmetry explicitly.

To achieve this goal, we focus on the gradient flow equation. The equation was proposed by Martin Lüscher [2] for Yang-Mills theory as a method to give a renormalized physical quantities in an automatic fashion. It is a new approach to renormalization of the gauge theory. The equation is a certain type of diffusion equation and gives a one parameter, which is called flow time, deformation of the gauge field starting from the bare gauge field as the initial condition. He claims that the expectation value of any gauge invariant local operators of the new gauge field, which is the solution of the gradient flow equation, is finite without additional renormalization. It is worth noting that the equation keeps the gauge symmetry explicitly at each any flow time. This nice property gives us a hope that the method may be applied to formulate the ERG which keeps the gauge symmetry explicitly.

Because of the importance studying the gauge theory using the ERG equation, there are various work. For example, there are studies on the ERG for the gauge theory with a momentum cut-off, in which the Ward-Takahashi identity or the quantum master equation is imposed order by order in perturbation theory by fine-tuning of the counterterm. For a review see Ref. [3]. Although they are attractive methods, we would like to formulate the ERG equation which keeps the symmetry explicitly. How to analyze the gauge theory using the ERG without breaking the gauge symmetry still remains an open problem. In this thesis, we do not achieve the formulation, but give a first step on the way to formulate of the new ERG equation to keep the symmetry explicitly.

On the other hand, from the stand point of the study on the relation between the ERG and symmetries, it is also interesting to discuss the theory which is broken the Lorentz symmetry, which is the so called Lifshitz-type theory using the ERG. The Lifshitz-type theory [4, 5] was proposed to control the ultraviolet (UV) divergence in field theory or gravity theory by imposing as an anisotropic scaling for space and time at the Lifshitz fixed point. While it has the advantage of giving a new type of renormalized theory, the broken Lorentz symmetry is the largest problem in applying it to particle physics. If the Lifshitz-type theory indeed explains physical phenomena at our energy

scale properly, the theory should restore the Lorentz symmetry in the infrared (IR) region.

There are various work about the Lifshitz-type theory at low energy. Refs. [17, 18, 19, 20, 21] are related to the Lorentz symmetry in the Lifshitz-type theory in the IR region. Ref. [17] claims that the Lorentz symmetry is not recovered at low energy in the model involving multiple scalar fields. Ref. [18] analyzed the Lorentz violating extension of the standard model. The low energy recovery of the Lorentz invariance is discussed in Refs. [19, 20, 21]. And Refs. [17, 22] classified Lifshitz-type scalar theories clearly.

From the naive power counting, it is expected that the symmetry can be restored in the IR region. However, the restoration of the symmetry should also be examined non-perturbatively. The goal of our work is to study whether the theory defined at the Lifshitz fixed point really flows into the Lorentz invariant theory at low energy using the ERG equation. The ERG equation [1] enables us to analyze theories non-perturbatively. There are similar work, which is studied the Lifshitz-type theory using the ERG equation. Ref. [25] discussed the Lifshitz-type theory using the Wilson-Polchinski ERG equation. Ref. [26] analyzed the Lifshitz-type theory with $z = 3$ in a curved space time, and used the ERG at low energy. The Lifshitz-type theory is also discussed non-perturbatively in a large N limit about the four-fermi model in Refs. [23, 24]. However it is a problem that there is no study using the ERG equation to investigate whether the theory defined at the Lifshitz UV fixed point can lead to the IR region where the Lorentz symmetry is recovered by tracing the entire RG flow at the non-perturbative level.

In our work, we apply the Wegner-Houghton equation, which is one of the ERG equations, to the Lifshitz-type theory, and analyze the RG flow in the theory space. It is found that this theory has a Lorentz symmetrical Gaussian IR fixed point, and we confirm that the method indeed reproduces the previously mentioned naive power counting arguments at the leading order in the perturbation theory. Using numerical analysis, we find that symmetry violating terms in the theory vanish in the IR region. In conclusion, the $z = 2, d = 3 + 1$ Lifshitz-type scalar model restores the Lorentz symmetry in the IR region. Our result provides a definition of ultraviolet complete renor-

malizable scalar field theories. Remarkably, these theories can have nontrivial interaction terms of ϕ^n ($n = 4, 6, 8, 10$) even when the Lorentz symmetry is restored at low energy. The later result is of extreme interest and our notable feature.

In this thesis, we study the new directions in the ERG through the Lifshitz-type theory, the gradient flow equation and its supersymmetric extension. There are two topics in this thesis. The first topic is the study on the dynamical restoration of the broken symmetry in the ERG with momentum cut-off in the so called Lifshitz-type theory. And the second topic is the study on the new method to obtain renormalized quantities in the gauge theory using the gradient flow equation.

In the first topic, we study the Lifshitz-type theory. We apply the ERG to the Lifshitz-type theory, and analyze a scalar model. We find that, starting from the ultraviolet Lifshitz fixed point, the model dynamically restores the Lorentz symmetry at low energy. We also show that the Lifshitz-type scalar model has nontrivial interaction terms $\hat{\lambda}_n \phi^n$ ($n = 4, 6, 8, 10$) even at low energy where the Lorentz symmetry is restored, which means that we have found a UV complete renormalizable theory through the interacting scalar model. This gives a concrete solution to the long-standing problem of triviality of ϕ^4 theory.

In the second topic, we study on the gradient flow equation and the supersymmetric extension of it. The method of the gradient flow is a new approach to a renormalization of the gauge theory. Various applications of the physical observable are studied recently. Ref. [6] give a review of the recent applications. For example, the expectation value of the chiral densities is calculated [7]. More appropriate probes for the translation Ward identities is defined [8]. The methods is also applied in the lattice theory [2, 9, 10, 11, 12, 13, 14], a new scheme of the step scaling, the improve action, and so on. The correctly-normalized conserved energy momentum tensor in the Yang-Mills theory is also examined [15].

We study the extension of the gradient flow to super Yang-Mills theory. We find that there is a natural extension of the gradient flow using superfield formalism and there exists a special gauge fixing term over the flow time direction with which the gradient flow equation keeps the Wess-Zumino gauge

so that we can construct an explicit closed set of equations which has only finite number of terms. Our finding would be an important step towards understanding the ERG which keeps the gauge symmetry explicitly. Also we study the gradient flow of the matter field. Luscher propose the gradient flow of the matter field in Ref. [6], but there is room for further research to derive the equation for matter field. Since the super Yang-Mills theory contains gaugino as a ‘matter’ field, it could derive the equation for the matter field very naturally.

This thesis is organized by two parts. In Part I, we present our study on the restoration of Lorentz symmetry for the Lifshitz-type theory. We apply the ERG equation to the theory, and analyze the RG flow for the Lifshitz-type scalar model. In part II, we give our study on the supersymmetric extension of the gradient flow equation. After reviewing the gradient flow equation for Yang-Mills theory, we extend the equation to the super Yang-Mills theory. We give our notation in Appendix A and C.

Part I

Restoration of Lorentz Symmetry for Lifshitz-Type Scalar Theory

The purpose of this part is to present our study on the restoration of the Lorentz symmetry for a Lifshitz-type scalar theory in the IR region using non-perturbative methods. We apply the Wegner-Houghton equation, which is one of the exact renormalization group equations, to the Lifshitz-type theory. Analyzing the equation for a $z = 2, d = 3 + 1$ Lifshitz-type scalar model, and using some variable transformations, we found that broken symmetry terms vanish in the IR region. This shows that the Lifshitz-type scalar model dynamically restores the Lorentz symmetry at low energy. Our result provides a definition of UV complete renormalizable scalar field theories. These theories can have nontrivial interaction terms of ϕ^n ($n = 4, 6, 8, 10$) even when the Lorentz symmetry is restored at low energy. This part is constituted of our paper [16].

2 Lifshitz-Type Theory

Lifshitz-type theory [4, 5] has an anisotropic scaling for space and time at the Lifshitz fixed point. In this theory, we substitute the second-order space differential operator in the kinetic term in the action with the $2z$ order one as follows:

$$\begin{aligned} S &= \int dt d^D x \frac{1}{2} \{ \phi (-\partial_0 \partial_0 + (-\partial_i \partial_i)^z + m^{2z}) \phi \} \\ &= \int_{p, p'} \frac{1}{2} (p_0^2 + p_i^{2z} + m^{2z}) \phi_p \phi_{p'} \delta(p + p'). \end{aligned} \quad (2.1)$$

It is found from this equation that the time dimension is z , while the space dimension is one. The advantage of the Lifshitz-type theory is that the

higher derivative terms in the kinetic terms suppress the UV divergence. This feature broadens the class of perturbatively renormalizable field theories.

As a compensation for these good UV properties, one has to sacrifice the Lorentz symmetry in the UV region. If the Lifshitz-type theory indeed explains physical phenomena at our energy scale properly, the theory should restore the Lorentz symmetry in the IR region. In the next section, we give the extended Wegner-Houghton equation to analyze the restoration of the Lorentz symmetry in the Lifshitz-type theory.

3 Extended Wegner-Houghton Equation for Lifshitz-Type Theory

The usual Wegner-Houghton equation is an ERG equation [27]. We review the derivation of the equation in Appendix B following Refs. [28, 29]. The equation for the effective action S is

$$\Lambda \frac{d}{d\Lambda} S = -\frac{1}{2\delta t} \left\{ \text{tr} \ln \left(\frac{\delta^2 S}{\delta \Omega \delta \Omega} \right) - \frac{\delta S}{\delta \Omega} \left(\frac{\delta^2 S}{\delta \Omega \delta \Omega} \right)^{-1} \frac{\delta S}{\delta \Omega} \right\} - dS + \int_p \Omega_p^i \left(d_\Omega - \gamma + \hat{p}^\mu \frac{\partial'}{\partial \hat{p}^\mu} \right) \frac{\delta}{\delta \Omega_p^i} S, \quad (3.1)$$

where Λ is a cut-off, Ω is a general field, i.e., $\Omega = \phi$ in the case of a real scalar field, γ is an anomalous dimension, and d_Ω is the dimension of the field. The definitions of δt , $\hat{p}$, and ∂' are given in Appendix B. The first term on the R.H.S. is the contribution from shell-mode integrals, and the second and third terms are from the scaling part. When we discuss the Lifshitz-type theory, Eq. (3.1) should be extended as follows:

$$\Lambda \frac{d}{d\Lambda} S = -\frac{1}{2\delta t} \left\{ \text{tr} \ln \left(\frac{\delta^2 S}{\delta \Omega \delta \Omega} \right) - \frac{\delta S}{\delta \Omega} \left(\frac{\delta^2 S}{\delta \Omega \delta \Omega} \right)^{-1} \frac{\delta S}{\delta \Omega} \right\} - (D + z)S + \int_p \Omega_p^i \left(d_\Omega - \gamma + z \hat{p}^0 \frac{\partial'}{\partial \hat{p}^0} + \hat{p}^i \frac{\partial'}{\partial \hat{p}^i} \right) \frac{\delta}{\delta \Omega_p^i} S, \quad (3.2)$$

where D is space dimension and z is time one. This is the extended Wegner-Houghton equation for Lifshitz-type theory.

To solve the equation in the Lifshitz-type theory, we need to know how to perform momentum integrals. Because the Lifshitz-type theory does not have

Lorentz symmetry, it is difficult to understand how to integrate out the shell-mode momentum. There are various discussions on cut-off methods [24, 26]. In this work, we use a cylindrical cut-off as an alternative to a spherical one. See Fig. 1.

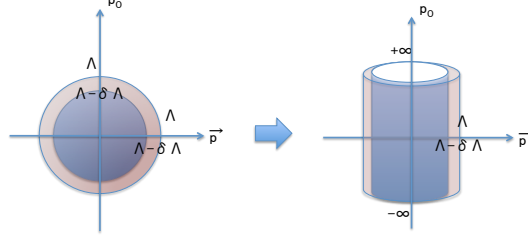


Fig. 1 Cut-off method. The left side of the figure is the usual cut-off. The momentum region is a ball inside a sphere in space and time with the radius $\sqrt{p_0^2 + p_i^2} = \Lambda$. It has a symmetry between space and time. The right side is a cylindrical cutoff; p_0 is integrated out from $-\infty$ to ∞ .

4 Models and Analysis

4.1 $z = 2, d = 3 + 1$ Lifshitz-Type Scalar Model

In general, we need to truncate the action to solve the RG equations concretely. Lifshitz-type scalar theories are classified clearly in Refs. [17, 22]. We adopt an effective action that contains all interactions for which the dimensions of the coupling in units of mass are more than or equal to 0, that is, relevant or marginal operators by naive power counting. We also impose a $\mathbb{Z}_2$ symmetry. The action is given as

$$\begin{aligned}
S = \int dt d^D x \Big\{ & \frac{1}{2} (\partial_0 \phi \partial_0 \phi + \beta_0 \partial_i \partial_i \phi \partial_j \partial_j \phi + m^4 \phi^2) \\
& + \frac{\lambda_4}{4!} \phi^4 + \frac{\lambda_6}{6!} \phi^6 + \frac{\lambda_8}{8!} \phi^8 + \frac{\lambda_{10}}{10!} \phi^{10} \\
& + \frac{1}{2} \left(\alpha_0 \partial_i \phi \partial_i \phi + \frac{\alpha_2}{2!} \phi^2 \partial_i \phi \partial_i \phi + \frac{\alpha_4}{4!} \phi^4 \partial_i \phi \partial_i \phi \right) \Big\}, \quad (4.1)
\end{aligned}$$

where the dimensions for x, t, ϕ , and the parameters in the action are

$$\begin{aligned} [x] &= -1, [t] = -2, [\alpha_0] = 2, [\alpha_2] = 1, [\alpha_4] = 0, [\beta_0] = 0, [\phi] = \frac{1}{2}, \\ [m^4] &= 4, [\lambda_4] = 3, [\lambda_6] = 2, [\lambda_8] = 1, [\lambda_{10}] = 0. \end{aligned} \quad (4.2)$$

In terms of the derivative expansion [30, 31, 32], the action in Eq. (4.1) is the sum of three parts. The first line is the kinetic terms of the free scalar Lifshitz-type theory, and the second and third lines are the local potential approximation terms and first order of the derivative expansion terms, that is,

$$S = S_{\text{Lifshitz}(\text{free})} + S_{\text{LPA}(\text{int})} + S_{\text{Diff}(\text{int})}. \quad (4.3)$$

Note that the term $\partial_i \phi \partial_i \phi$, which is needed for Lorentz symmetry, naturally appears in $S_{\text{Diff}(\text{int})}$. To restore the symmetry in the IR region, interaction terms that break the symmetry should vanish in the IR region.

We obtain the Wegner-Houghton equation for the Lifshitz-type theory as one of the main results of this part. The Wegner-Houghton equation in the present model reads

$$\frac{\delta \alpha_0}{\delta t} = 2\alpha_0 + \frac{\alpha_2}{8\pi^2(\alpha_0 + \beta_0 + m^4)^{1/2}}, \quad (4.4)$$

$$\frac{\delta \beta_0}{\delta t} = 0, \quad (4.5)$$

$$\frac{\delta m^4}{\delta t} = 4m^4 + \frac{1}{8\pi^2(\alpha_0 + \beta_0 + m^4)^{1/2}}(\alpha_2 + \lambda_4), \quad (4.6)$$

$$\begin{aligned} \frac{\delta \lambda_4}{\delta t} &= 3\lambda_4 - \frac{1}{16\pi^2(\alpha_0 + \beta_0 + m^4)^{3/2}}\{3(\alpha_2 + \lambda_4)^2\} \\ &\quad + \frac{1}{8\pi^2(\alpha_0 + \beta_0 + m^4)^{1/2}}(\alpha_4 + \lambda_6), \end{aligned} \quad (4.7)$$

$$\begin{aligned} \frac{\delta \lambda_6}{\delta t} &= 2\lambda_6 + \frac{1}{32\pi^2(\alpha_0 + \beta_0 + m^4)^{5/2}}\{45(\alpha_2 + \lambda_4)^3\} \\ &\quad - \frac{1}{16\pi^2(\alpha_0 + \beta_0 + m^4)^{3/2}}\{15(\alpha_2 + \lambda_4)(\alpha_4 + \lambda_6)\} \\ &\quad + \frac{\lambda_8}{8\pi^2(\alpha_0 + \beta_0 + m^4)^{1/2}}, \end{aligned} \quad (4.8)$$

$$\frac{\delta \lambda_8}{\delta t} = \lambda_8 - \frac{1}{64\pi^2(\alpha_0 + \beta_0 + m^4)^{7/2}}\{1575(\alpha_2 + \lambda_4)^4\}$$

$$\begin{aligned}
& + \frac{1}{16\pi^2(\alpha_0 + \beta_0 + m^4)^{5/2}} \{315(\alpha_2 + \lambda_4)^2(\alpha_4 + \lambda_6)\} \\
& - \frac{1}{16\pi^2(\alpha_0 + \beta_0 + m^4)^{3/2}} \{35(\alpha_4 + \lambda_6)^2 + 28\lambda_8(\alpha_2 + \lambda_4)\} \\
& + \frac{\lambda_{10}}{8\pi^2(\alpha_0 + \beta_0 + m^4)^{1/2}}, \tag{4.9}
\end{aligned}$$

$$\begin{aligned}
\frac{\delta\lambda_{10}}{\delta t} = & \frac{1}{128\pi^2(\alpha_0 + \beta_0 + m^4)^{9/2}} \{99225(\alpha_2 + \lambda_4)^5\} \\
& - \frac{1}{32\pi^2(\alpha_0 + \beta_0 + m^4)^{7/2}} \{23625(\alpha_2 + \lambda_4)^3(\alpha_4 + \lambda_6)\} \\
& + \frac{1}{32\pi^2(\alpha_0 + \beta_0 + m^4)^{5/2}} \{4725(\alpha_2 + \lambda_4)(\alpha_4 + \lambda_6)^2 + 1890(\alpha_2 + \lambda_4)^2\lambda_8\} \\
& - \frac{1}{16\pi^2(\alpha_0 + \beta_0 + m^4)^{3/2}} \{45\lambda_{10}(\alpha_2 + \lambda_4) + 210(\alpha_4 + \lambda_6)\lambda_8\}, \tag{4.10}
\end{aligned}$$

$$\begin{aligned}
\frac{\delta\alpha_2}{\delta t} = & \alpha_2 - \frac{1}{48\pi^2(\alpha_0 + \beta_0 + m^4)^{7/2}} \{5(\alpha_0 + 2\beta_0)^2(\alpha_2 + \lambda_4)^2\} \\
& + \frac{1}{32\pi^2(\alpha_0 + \beta_0 + m^4)^{5/2}} \{4\alpha_2(\alpha_0 + 2\beta_0)(\alpha_2 + \lambda_4) + (3\alpha_0 + 10\beta_0)(\alpha_2 + \lambda_4)^2\} \\
& - \frac{1}{48\pi^2(\alpha_0 + \beta_0 + m^4)^{3/2}} \{2\alpha_2^2 + 15\alpha_2(\alpha_2 + \lambda_4)\} \\
& + \frac{\alpha_4}{8\pi^2(\alpha_0 + \beta_0 + m^4)^{1/2}}, \tag{4.11}
\end{aligned}$$

$$\begin{aligned}
\frac{\delta\alpha_4}{\delta t} = & \frac{1}{96\pi^2(\alpha_0 + \beta_0 + m^4)^{9/2}} \{175(\alpha_0 + 2\beta_0)^2(\alpha_2 + \lambda_4)^3\} \\
& - \frac{1}{12\pi^2(\alpha_0 + \beta_0 + m^4)^{7/2}} \{30\alpha_2(\alpha_0 + 2\beta_0)(\alpha_2 + \lambda_4)^2 \\
& + 5(3\alpha_0 + 10\beta_0)(\alpha_2 + \lambda_4)^3 + 5(\alpha_0 + 2\beta_0)^2(\alpha_2 + \lambda_4)(\alpha_4 + \lambda_6)\} \\
& + \frac{1}{32\pi^2(\alpha_0 + \beta_0 + m^4)^{5/2}} \{28\alpha_2^2(\alpha_2 + \lambda_4) + 93\alpha_2(\alpha_2 + \lambda_4)^2 \\
& + 4(3\alpha_0 + 10\beta_0)(\alpha_2 + \lambda_4)(\alpha_4 + \lambda_6) + 8(\alpha_0 + 2\beta_0)(2\alpha_2\alpha_4 + \alpha_4\lambda_4 + \alpha_2\lambda_6)\} \\
& - \frac{1}{48\pi^2(\alpha_0 + \beta_0 + m^4)^{3/2}} (77\alpha_2\alpha_4 + 42\alpha_4\lambda_4 + 27\alpha_2\lambda_6). \tag{4.12}
\end{aligned}$$

As an example, let us discuss the RG flow in the theory subspace, in

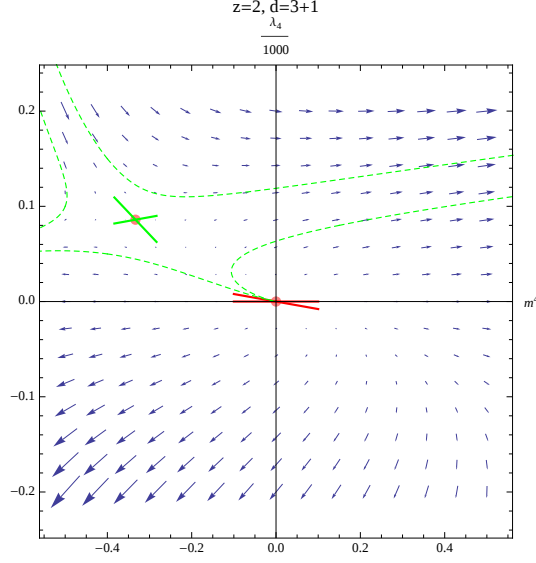


Fig. 2 Flow of m^4, λ_4 in $z = 2, d = 3 + 1$.

which only m^4 and λ_4 are nonzero. Eqs. (4.6) and (4.7) then reduce to

$$\frac{\delta m^4}{\delta t} = 4m^4 + \frac{\lambda_4}{8\pi^2(1+m^4)^{1/2}}, \quad (4.13)$$

$$\frac{\delta \lambda_4}{\delta t} = 3\lambda_4 - \frac{3\lambda_4^2}{16\pi^2(1+m^4)^{3/2}}, \quad (4.14)$$

where we take $\beta_0 = 1$ by rescaling the momentum. All other equations are satisfied trivially. There are two fixed points given as

$$\begin{aligned} m^4 &= 0, \lambda_4 = 0, \\ m^4 &= -\frac{1}{3}, \lambda_4 = \frac{32}{3} \sqrt{\frac{2}{3}} \pi^2. \end{aligned} \quad (4.15)$$

The first is a Gaussian fixed point, and the second is a nontrivial fixed point as seen in Fig. 2. We would like to mention that this flow resembles the one in the ordinary scalar theory with the Lorentz symmetry in three space-time dimensions.

4.2 Transformation of Variables

Our main interest is the restoration of the Lorentz symmetry in the IR region. To discuss the RG flow in the IR region, it is useful to change the variable as

$$h = \frac{1}{\alpha_0}, \quad (4.16)$$

and introduce new variables with hats [20, 21, 23]

$$\hat{t} = h^{-\frac{1}{2}}t, \quad (4.17)$$

$$\hat{x} = x, \quad (4.18)$$

$$\hat{\phi} = h^{-\frac{1}{4}}\phi. \quad (4.19)$$

The action in the model (4.1) with new variables is

$$\begin{aligned} S = \int_{\hat{t}, \hat{x}_i} \left\{ \frac{1}{2} (\hat{\partial}_0 \hat{\phi} \hat{\partial}_0 \hat{\phi} + \hat{\partial}_i \hat{\phi} \hat{\partial}_i \hat{\phi} + \hat{m}^2 \hat{\phi}^2) \right. \\ \left. + \frac{\hat{\lambda}_4}{4!} \hat{\phi}^4 + \frac{\hat{\lambda}_6}{6!} \hat{\phi}^6 + \frac{\hat{\lambda}_8}{8!} \hat{\phi}^8 + \frac{\hat{\lambda}_{10}}{10!} \hat{\phi}^{10} \right. \\ \left. + \frac{1}{2} h \hat{\partial}_i \hat{\phi} \hat{\partial}_j \hat{\phi} \hat{\partial}_j \hat{\phi} + \frac{1}{2} \left(\frac{\hat{\alpha}_2}{2!} \hat{\phi}^2 \hat{\partial}_i \hat{\phi} \hat{\partial}_i \hat{\phi} + \frac{\hat{\alpha}_4}{4!} \hat{\phi}^4 \hat{\partial}_i \hat{\phi} \hat{\partial}_i \hat{\phi} \right) \right\}, \quad (4.20) \end{aligned}$$

where

$$\begin{aligned} \hat{m}^2 &\equiv h m^4, \hat{\lambda}_4 \equiv h^{\frac{3}{2}} \lambda_4, \hat{\lambda}_6 \equiv h^2 \lambda_6, \hat{\lambda}_8 \equiv h^{\frac{5}{2}} \lambda_8, \hat{\lambda}_{10} \equiv h^3 \lambda_{10}, \\ \hat{\alpha}_2 &\equiv h^{\frac{3}{2}} \alpha_2, \hat{\alpha}_4 \equiv h^2 \alpha_4, \end{aligned} \quad (4.21)$$

where we also take $\beta_0 = 1$. The dimensions in the unit of mass of new variables are as follows:

$$\begin{aligned} [\hat{t}] &= -1, [\hat{x}] = -1, [\hat{\phi}] = 1, [\hat{m}^2] = 2, [\hat{\lambda}_4] = 0, [\hat{\lambda}_6] = -2, \\ [\hat{\lambda}_8] &= -4, [\hat{\lambda}_{10}] = -6, [h] = -2, [\hat{\alpha}_2] = -2, [\hat{\alpha}_4] = -4. \end{aligned} \quad (4.22)$$

They are identical to the canonical dimension in the Lorentz theory in four space-time dimensions.

The RG equations (4.4)–(4.12) in terms of the new variables (with hats) are given as

$$\frac{\delta h}{\delta t} = -2h - \frac{\hat{\alpha}_2 h}{8\pi^2(1 + h + \hat{m}^2)^{1/2}}, \quad (4.23)$$

$$\frac{\delta \hat{m}^2}{\delta t} = 2\hat{m}^2 + \frac{1}{8\pi^2(1+h+\hat{m}^2)^{1/2}}(\hat{\alpha}_2 + \hat{\lambda}_4 - \hat{\alpha}_2\hat{m}^2), \quad (4.24)$$

$$\begin{aligned} \frac{\delta \hat{\lambda}_4}{\delta t} &= -\frac{1}{16\pi^2(1+h+\hat{m}^2)^{3/2}}\{3(\hat{\alpha}_2 + \hat{\lambda}_4)^2\} \\ &+ \frac{1}{16\pi^2(1+h+\hat{m}^2)^{1/2}}\{2(\hat{\alpha}_4 + \hat{\lambda}_6) - 3\hat{\lambda}_4\hat{\alpha}_2\}, \end{aligned} \quad (4.25)$$

$$\begin{aligned} \frac{\delta \hat{\lambda}_6}{\delta t} &= -2\hat{\lambda}_6 + \frac{1}{32\pi^2(1+h+\hat{m}^2)^{5/2}}\{45(\hat{\alpha}_2 + \hat{\lambda}_4)^3\} \\ &- \frac{1}{16\pi^2(1+h+\hat{m}^2)^{3/2}}\{15(\hat{\alpha}_2 + \hat{\lambda}_4)(\hat{\alpha}_4 + \hat{\lambda}_6)\} \\ &+ \frac{1}{8\pi^2(1+h+\hat{m}^2)^{1/2}}(\hat{\lambda}_8 - 2\hat{\lambda}_6\hat{\alpha}_2), \end{aligned} \quad (4.26)$$

$$\begin{aligned} \frac{\delta \hat{\lambda}_8}{\delta t} &= -4\hat{\lambda}_8 - \frac{1}{64\pi^2(1+h+\hat{m}^2)^{7/2}}\{1575(\hat{\alpha}_2 + \hat{\lambda}_4)^4\} \\ &+ \frac{1}{16\pi^2(1+h+\hat{m}^2)^{5/2}}\{315(\hat{\alpha}_2 + \hat{\lambda}_4)^2(\hat{\alpha}_4 + \hat{\lambda}_6)\} \\ &- \frac{1}{16\pi^2(1+h+\hat{m}^2)^{3/2}}\{35(\hat{\alpha}_4 + \hat{\lambda}_6)^2 + 28\hat{\lambda}_8(\hat{\alpha}_2 + \hat{\lambda}_4)\} \\ &+ \frac{1}{16\pi^2(1+h+\hat{m}^2)^{1/2}}(2\hat{\lambda}_{10} - 5\hat{\alpha}_2\hat{\lambda}_8), \end{aligned} \quad (4.27)$$

$$\begin{aligned} \frac{\delta \hat{\lambda}_{10}}{\delta t} &= -6\hat{\lambda}_{10} + \frac{1}{128\pi^2(1+h+\hat{m}^2)^{9/2}}\{99225(\hat{\alpha}_2 + \hat{\lambda}_4)^5\} \\ &- \frac{1}{32\pi^2(1+h+\hat{m}^2)^{7/2}}\{23625(\hat{\alpha}_2 + \hat{\lambda}_4)^3(\hat{\alpha}_4 + \hat{\lambda}_6)\} \\ &+ \frac{1}{32\pi^2(1+h+\hat{m}^2)^{5/2}}\{4725(\hat{\alpha}_2 + \hat{\lambda}_4)(\hat{\alpha}_4 + \hat{\lambda}_6)^2 + 1890(\hat{\alpha}_2 + \hat{\lambda}_4)^2\hat{\lambda}_8\} \\ &- \frac{1}{16\pi^2(1+h+\hat{m}^2)^{3/2}}\{45\hat{\lambda}_{10}(\hat{\alpha}_2 + \hat{\lambda}_4) + 210(\hat{\alpha}_4 + \hat{\lambda}_6)\hat{\lambda}_8\} \\ &- \frac{3\hat{\lambda}_{10}\hat{\alpha}_2}{8\pi^2(1+h+\hat{m}^2)^{1/2}}, \end{aligned} \quad (4.28)$$

$$\begin{aligned} \frac{\delta \hat{\alpha}_2}{\delta t} &= -2\hat{\alpha}_2 - \frac{1}{48\pi^2(1+h+\hat{m}^2)^{7/2}}\{5(1+2h)^2(\hat{\alpha}_2 + \hat{\lambda}_4)^2\} \\ &+ \frac{1}{32\pi^2(1+h+\hat{m}^2)^{5/2}}\{4\hat{\alpha}_2(1+2h)(\hat{\alpha}_2 + \hat{\lambda}_4) + (3+10h)(\hat{\alpha}_2 + \hat{\lambda}_4)^2\} \\ &- \frac{1}{48\pi^2(1+h+\hat{m}^2)^{3/2}}\{2\hat{\alpha}_2^2 + 15\hat{\alpha}_2(\hat{\alpha}_2 + \hat{\lambda}_4)\} \end{aligned}$$

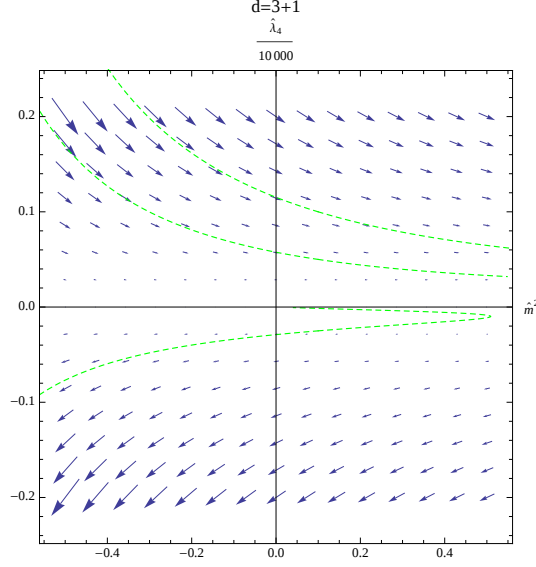


Fig. 3 Flow of $\hat{m}^2, \hat{\lambda}_4$ in $d = 3 + 1$.

$$\begin{aligned}
& + \frac{1}{16\pi^2(1+h+\hat{m}^2)^{1/2}}(2\hat{\alpha}_4 - 3\hat{\alpha}_2^2), \\
\frac{\delta\hat{\alpha}_4}{\delta t} = & -4\hat{\alpha}_4 + \frac{1}{96\pi^2(1+h+\hat{m}^2)^{9/2}}\{175(1+2h)^2(\hat{\alpha}_2 + \hat{\lambda}_4)^3\} \\
& - \frac{1}{12\pi^2(1+h+\hat{m}^2)^{7/2}}\{30\hat{\alpha}_2(1+2h)(\hat{\alpha}_2 + \hat{\lambda}_4)^2 + 5(3+10h)(\hat{\alpha}_2 + \hat{\lambda}_4)^3 \\
& + 5(1+2h)^2(\hat{\alpha}_2 + \hat{\lambda}_4)(\hat{\alpha}_4 + \hat{\lambda}_6)\} \\
& + \frac{1}{32\pi^2(1+h+\hat{m}^2)^{5/2}}\{28\hat{\alpha}_2^2(\hat{\alpha}_2 + \hat{\lambda}_4) + 93\hat{\alpha}_2(\hat{\alpha}_2 + \hat{\lambda}_4)^2 \\
& + 4(3+10h)(\hat{\alpha}_2 + \hat{\lambda}_4)(\hat{\alpha}_4 + \hat{\lambda}_6) + 8(1+2h)(2\hat{\alpha}_2\hat{\alpha}_4 + \hat{\alpha}_4\hat{\lambda}_4 + \hat{\alpha}_2\hat{\lambda}_6)\} \\
& - \frac{1}{48\pi^2(1+h+\hat{m}^2)^{3/2}}(77\hat{\alpha}_2\hat{\alpha}_4 + 42\hat{\alpha}_4\hat{\lambda}_4 + 27\hat{\alpha}_2\hat{\lambda}_6) \\
& - \frac{1}{4\pi^2(1+h+\hat{m}^2)^{1/2}}(\hat{\alpha}_2\hat{\alpha}_4). \tag{4.30}
\end{aligned}$$

If we set $h = 0, \hat{\alpha}_2 = 0$, and $\hat{\alpha}_4 = 0$, these RG equations exactly coincide with the equations in the case of the local potential approximation in the ordinary theory, which has Lorentz symmetry, as expected. As an example, we give the RG flow in the theory subspace, in which only $\hat{m}^2$ and $\hat{\lambda}_4$ are nonzero. (See Fig. 3.)

These RG equations have a Gaussian fixed point at

$$\hat{m}^2 = 0, \hat{\lambda}_4 = 0, \hat{\lambda}_6 = 0, \hat{\lambda}_8 = 0, \hat{\lambda}_{10} = 0, h = 0, \hat{\alpha}_2 = 0, \hat{\alpha}_4 = 0. \quad (4.31)$$

In the neighborhood of the fixed point, the RG equations (4.23)–(4.30) can be approximated as

$$\frac{\delta h}{\delta t} = -2h, \quad (4.32)$$

$$\frac{\delta \hat{m}^2}{\delta t} = 2\hat{m}^2 + \frac{\hat{\alpha}_2 + \hat{\lambda}_4}{8\pi^2}, \quad (4.33)$$

$$\frac{\delta \hat{\lambda}_4}{\delta t} = \frac{\hat{\alpha}_4 + \hat{\lambda}_6}{8\pi^2}, \quad (4.34)$$

$$\frac{\delta \hat{\lambda}_6}{\delta t} = -2\hat{\lambda}_6 + \frac{\hat{\lambda}_8}{8\pi^2}, \quad (4.35)$$

$$\frac{\delta \hat{\lambda}_8}{\delta t} = -4\hat{\lambda}_8 + \frac{\hat{\lambda}_{10}}{8\pi^2}, \quad (4.36)$$

$$\frac{\delta \hat{\lambda}_{10}}{\delta t} = -6\hat{\lambda}_{10}, \quad (4.37)$$

$$\frac{\delta \hat{\alpha}_2}{\delta t} = -2\hat{\alpha}_2 + \frac{\hat{\alpha}_4}{8\pi^2}, \quad (4.38)$$

$$\frac{\delta \hat{\alpha}_4}{\delta t} = -4\hat{\alpha}_4, \quad (4.39)$$

at the linear order in the perturbation theory. From Eq. (4.32), it turns out that the fixed point (4.31) is the IR one. These equations (4.32)–(4.39) tell us that when we increase the energy scale, $\hat{m}^2$ and $\hat{\lambda}_4$ become dominant compared with h , $\hat{\alpha}_2$, and $\hat{\alpha}_4$, which are the coupling constants of terms breaking the Lorentz symmetry. This implies the restoration of the Lorentz symmetry in the neighborhood of the Gaussian fixed point in the IR region. They coincide with the expectation by power counting. However, the purpose of this part is to study the restoration of the Lorentz symmetry non-perturbatively. Thus, we should solve Eqs. (4.4)–(4.12) without linear approximations. In the next section, we perform these calculations by numerical analysis.

5 Numerical Analysis

The terms in the third line in Eq. (4.20) violate the Lorentz symmetry. If they vanish in the IR region, we can say that the Lorentz symmetry is restored

in the IR region. In the following, solving the Wegner-Houghton equations (4.4)–(4.12) with some initial conditions by numerical analysis, we study the RG flow of the Lifshitz-type theory to see if this is the case.

We should choose the initial conditions carefully. To obtain flows of proper physical theories, we looked for initial conditions that satisfy two requirements. First, the coupling constants that violate the Lorentz symmetry should become negligible at low energy. Second, all the coupling constants should approach the Lifshitz fixed point at high energy to obtain UV complete theories. The key problem here is whether such initial conditions exist. Actually, we found flows with typical initial conditions that satisfy the two requirements. In this thesis, as examples, we give results for two initial conditions (case 1 and 2) defined as follows:

Case 1: The initial conditions are

$$m^4 = 1.00 \times 10^{-4}, \lambda_4 = 8.50 \times 10^{-2}, \lambda_6 = 5.00 \times 10^{-1}, \lambda_8 = 5.00 \times 10^{-1}, \\ \lambda_{10} = 0, \alpha_0 = 7.50 \times 10^{-1}, \alpha_2 = 3.50 \times 10^{-1}, \alpha_4 = 0.$$

In terms of hatted coupling constants, they are

$$\hat{m}^2 = 1.33 \times 10^{-4}, \hat{\lambda}_4 = 1.31 \times 10^{-1}, \hat{\lambda}_6 = 8.89 \times 10^{-1}, \hat{\lambda}_8 = 1.03, \\ \hat{\lambda}_{10} = 0, h = 1.33, \hat{\alpha}_2 = 5.39 \times 10^{-1}, \hat{\alpha}_4 = 0,$$

which are written to three significant figures.

Case 2: The initial conditions are

$$m^4 = 1.00 \times 10^{-4}, \lambda_4 = 6.70 \times 10^{-1}, \lambda_6 = 7.80 \times 10^{-1}, \lambda_8 = 7.30 \times 10^{-1}, \\ \lambda_{10} = 1.70 \times 10^{-1}, \alpha_0 = 1.00, \alpha_2 = 4.40 \times 10^{-1}, \alpha_4 = 1.00 \times 10^{-2}.$$

In terms of hatted coupling constants, they are

$$\hat{m}^2 = 1.00 \times 10^{-4}, \hat{\lambda}_4 = 6.70 \times 10^{-1}, \hat{\lambda}_6 = 7.80 \times 10^{-1}, \hat{\lambda}_8 = 7.30 \times 10^{-1}, \\ \hat{\lambda}_{10} = 1.70 \times 10^{-1}, h = 1.00, \hat{\alpha}_2 = 4.40 \times 10^{-1}, \hat{\alpha}_4 = 1.00 \times 10^{-2}.$$

Figs. 4 and 5 show the RG flows of the coupling constants with hatted variables for cases 1 and 2, respectively. The results show that h , $\hat{\alpha}_2$, and $\hat{\alpha}_4$ rapidly become negligible with decreasing energy scale ($t > 0$); therefore, the

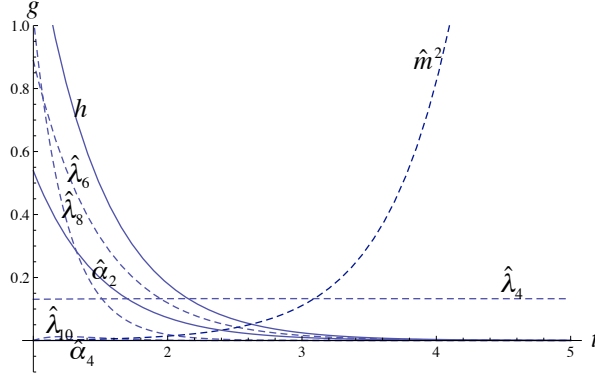


Fig. 4 RG flow of hatted coupling constants against decreasing energy scale t under the initial condition in case 1. The continuous lines show the flows of the coupling constants $h, \hat{\alpha}_2$, and $\hat{\alpha}_4$, which violate the Lorentz symmetry. The dashed lines show the ones of $\hat{m}^2, \hat{\lambda}_4, \hat{\lambda}_6, \hat{\lambda}_8$, and $\hat{\lambda}_{10}$.

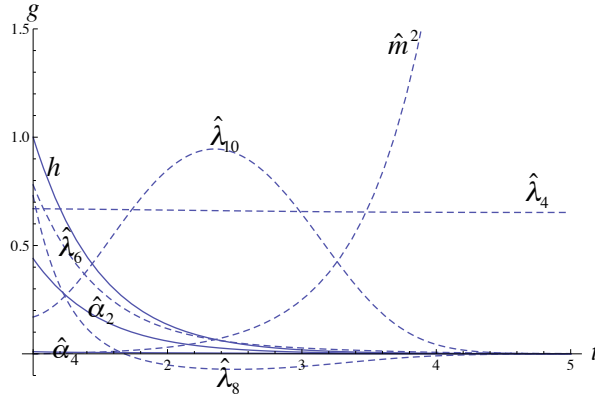


Fig. 5 RG flow of hatted coupling constants against decreasing energy scale t under the initial condition in case 2.

third line terms of Eq. (4.20) turn out to be highly suppressed. This implies that the Lorentz symmetry is restored in the IR region.

On the other hand, Figs. 6 and 7 show the RG flows of the unhatted coupling constants with increasing energy scale ($t < 0$) for cases 1 and 2. The results show that all the coupling constants approach the Lifshitz fixed point with increasing energy. Therefore, we obtain the UV complete renormalizable

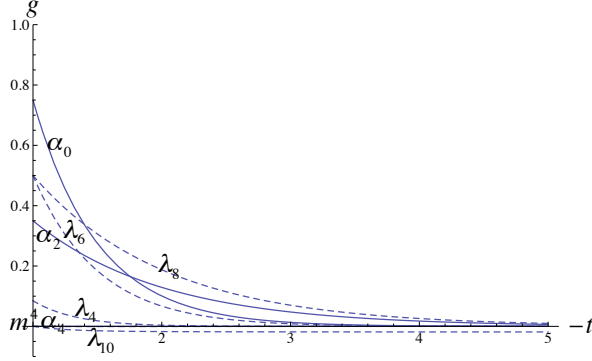


Fig. 6 RG flow of unhatted coupling constants against increasing energy scale $-t$ under the initial condition in case 1. The continuous lines show the flows of the coupling constants α_0, α_2 , and α_4 . The dashed lines show the ones of $m^4, \lambda_4, \lambda_6, \lambda_8$, and λ_{10} .

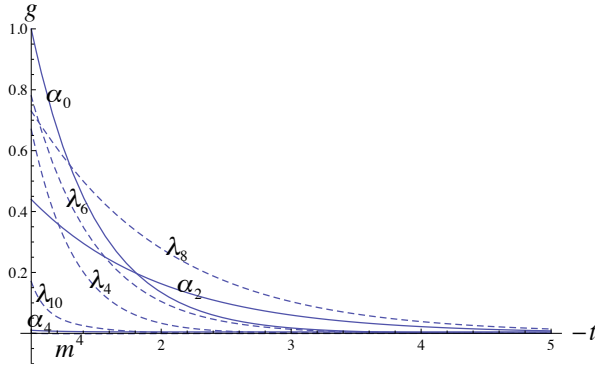


Fig. 7 RG flow of unhatted coupling constants against increasing energy scale $-t$ under the initial condition in case 2.

theories, which have the Lorentz symmetry in the IR region, under proper initial conditions.

Furthermore, case 2 is very interesting. Remarkably, the theory in case 2 has nonzero coupling constants of the interaction term, $\hat{\lambda}_4, \hat{\lambda}_6, \hat{\lambda}_8, \hat{\lambda}_{10}$, even when the Lorentz symmetry is restored at low energy. For example, at the

energy scale $t = 3.5$ in Fig. 5, the coupling constants are

$$\begin{aligned}\hat{m}^2 &= 6.85 \times 10^{-1}, \hat{\lambda}_4 = 6.55 \times 10^{-1}, \hat{\lambda}_6 = 1.46 \times 10^{-2}, \hat{\lambda}_8 = -3.06 \times 10^{-2}, \\ \hat{\lambda}_{10} &= 2.46 \times 10^{-1}, h = 6.72 \times 10^{-3}, \hat{\alpha}_2 = 3.01 \times 10^{-3}, \hat{\alpha}_4 = 2.45 \times 10^{-4},\end{aligned}$$

which are written to three significant figures.

I would like to add comments about a fine-tuning of the initial conditions. Actually, when we choose the initial conditions, it is not so hard to satisfy the first requirement that the coupling constants that violate the Lorentz symmetry should become negligible at low energy. To satisfy the second requirement that all the coupling constants should approach the Lifshitz fixed point at high energy, especially we carefully choose the initial conditions to satisfy $\alpha_4 = 0$ and $\lambda_{10} = 0$. And because of the restriction owing to the transformation of variables, we should satisfy the condition $\alpha_0 > 0$.

Finally, we would like to mention a possibility for other initial conditions. It is possible that there are other interesting initial conditions, and to classify the regions of the flows generally is an interesting future work. The most important thing, however, is that there exists at least one such flow.

6 Short Summary

In the Lifshitz-type theory, higher derivative terms in the kinetic terms suppress the UV divergence. However, there is a problem of broken Lorentz symmetry; therefore, we should examine whether the theory restores the Lorentz symmetry in the IR region. In this part, we applied the Wegner-Houghton equation with the momentum cut-off in cylindrical shape and analyzed the $z = 2, d = 3 + 1$ Lifshitz-type scalar model numerically. We find that the terms that break the Lorentz symmetry vanish at low energy. Remarkably, the Lifshitz-type theory has nontrivial interaction terms $\hat{\lambda}_n \phi^n$ ($n = 4, 6, 8, 10$) even when the Lorentz symmetry is restored at low energy. We find a concrete solution to the long-standing problem of triviality of ϕ^4 theory in $d = 3 + 1$. In summary, $z = 2, d = 3 + 1$ Lifshitz-type scalar theory, at least for the model in this thesis, restores the Lorentz symmetry in the IR region, and we obtain a UV complete renormalizable theory under proper initial conditions.

The truncation method remains as a matter to be discussed further. Given some symmetries, we may improve this analytic method. There is also room for discussion on the cut-off method. In this part, we used the cut-off to respect the spatial symmetry. More pertinent cut-off which keeps the Lorentz symmetry explicitly may exist. It is interesting to challenge such a problem. It would also be interesting to analyze multiple fields including fermions. They are also soluble by this method in principle, although improvements may be needed in this analysis. If the theory can include the gauge field, we could discuss the standard model. The gauge symmetry is incompatible with the ERG, because a cut-off in the momentum space breaks the symmetry explicitly. Therefore it is important to study a renormalization method that keeps the gauge symmetry, which will be the subject of the next part.

Part II

Gradient Flow Equation and Its Supersymmetric Extension

The gauge symmetry guarantees the theoretical consistency such as the unitarity and the renormalizability. However, some UV regularizations do not respect the gauge symmetry. Momentum cut-off or Pauli-Villas regularization are such examples. Since the ERG is formulated using a cut-off in a momentum space, it breaks the gauge symmetry explicitly. How to analyze the gauge theory using the ERG without breaking gauge symmetry explicitly still remains an open problem. In this part, we discuss the method of the gradient flow equation and its supersymmetric expansion. This discussion would be important step towards the understanding the ERG which keeps the symmetry explicitly.

7 Review of Gradient Flow Equation of Yang-Mills Theory

Recently, Martin Lüscher proposed an interesting method [2] to obtain renormalized physical quantities in an automatic fashion.

In this method, the expectation value in terms of new gauge field is finite without additional renormalization. Here the new gauge field is constructed by the solution of a certain type of diffusion equation called a gradient flow equation, whose initial value is the bare gauge field. Lüscher showed this at one loop order as example [2], and later all order proof was given by Lüscher and Weisz [33]. We call the statement the Lüscher-Weisz theorem. This section is the review of Refs. [2, 6, 33].

7.1 Definition of Gradient Flow Equation

The gauge field B_μ is defined by the gradient flow

$$\dot{B}_\mu = D_\nu G_{\nu\mu} + \alpha_0 D_\mu \partial_\nu B_\nu, \quad (7.1)$$

$$B_\mu|_{t=0} = A_\mu. \quad (7.2)$$

where the dot means a differential in terms of the flow time t , A_μ describe a fundamental bare field of SU(N) gauge theory, $G_{\mu\nu}$ and D_μ are defined by

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu], \quad (7.3)$$

$$D_\mu = \partial_\mu + [B_\mu, \cdot] \quad (7.4)$$

respectively. The reason why we call the equation the gradient flow one is that the first term of R.H.S. of Eq. (7.1) is proportional to the gradient of the action,

$$S = \int d^D x \text{Tr}[G_{\mu\nu}(x)G_{\mu\nu}(x)]. \quad (7.5)$$

The second term of the R.H.S. of Eq. (7.1) is the gauge fixing term over the flow time direction. The term is introduced to cause suppression of the increase of the degree of new gauge freedom over the direction. In this thesis, we call this term the α_0 term to avoid needless confusion with the usual gauge fixing term in the Yang-Mills theory.

7.2 Lüscher-Weisz Theorem

Lüscher claims that any expectation value which is described by the gauge field B , which is defined by Eq. (7.1) at positive flow time has a well-defined continuum limit without additional renormalization. Calculating the expectation value of the energy density at one loop order using this method, he showed that it is the case [2]. Soon after that, Lüscher and Weisz proved this claim to all order in perturbation theory [33].

Luscher-Weisz Theorem

In the Yang-Mills theory, the expectation value in terms of the new gauge field, which is defined by the solution of the gradient flow equation, are finite without additional renormalization to all loop order, once the theory in terms of the fundamental gauge field is renormalized in the usual way.

There are various applications owing to the theorem as we shall explain later in Sec. 7.4. The detailed proof of the theorem is given in Ref. [33], and we briefly summarize the points of the proof of the theorem as a review. The proof of the theorem consists of six steps:

1. The theory is reformulated as the field theory in 4+1 dimensions which represent the space coordinate x^μ and the flow time t .
2. The flow equation is realized by introducing the flow action S_{fl} in 4+1 dimensions with the Lagrange-multiplier field $L_\mu(t, x)$.

$$S_{fl} = -2 \int_0^\infty dt \int d^4x \text{tr} \{ L_\mu(t, x) (\partial_t B_\mu - D_\nu G_{\nu\mu} - \alpha_0 D_\mu \partial_\nu B_\nu)(t, x) \}. \quad (7.6)$$

Thus the gradient flow equation is described by the local field theory.

3. If the interactions are local, the divergent part can be canceled by local counterterm which can be localized either in the bulk or at the boundary, that is, the divergent part can be localized in the bulk or at the boundary of the half space.
4. Since the theory has no loop diagrams in the bulk, there are no divergences in the bulk. Therefore divergences are localized at the boundary, if they exist.
5. Since there are divergences only in the term which involves bulk field on the boundary, the divergence terms are proportional to L_μ and $\bar{d}$. Here $\bar{d}$ is an additional ghost field which lives in $D + 1$ dimensions. Owing to the Lorentz symmetry, the BRS symmetry, the ghost number conservation and the dimensional analysis, the possible boundary

counterterms form at l -loop order are described as

$$2g^{2l} \int d^4x \text{tr} \{ z_1 L_\mu(0, x) (A_R)_\mu(x) + z_2 \bar{d}(0, x) c_R(x) \}. \quad (7.7)$$

where A_R is a renormalized fundamental gauge field, c_R is a renormalized ghost field, z_1 and z_2 are coefficients.

6. Since the BRS symmetry of the theory excludes the boundary counterterm, the coefficients z_1 and z_2 have to be 0. There are no more singularities.

Finally, from these discussions, one can see that the theory in 4+1 dimensions does not require further renormalization to all loop orders.

To understand the advantage of the method of the gradient flow, it is particularly helpful to know the example at one loop order. In the next subsection, summarizing the calculation of the expectation value of the energy density, we confirm the theorem is true at one loop order.

7.3 Energy Density

We calculate the expectation value of the energy density perturbatively and confirm that the Luscher's claim is true at one loop order. The detailed calculations are expressed in Ref. [2]. Here we summarize the results. The iterative method to solve the gradient flow equation [33] is given in Appendix D that helps to understand the method of the gradient flow.

The expectation value of the energy density which is defined by $E(x) \equiv \frac{1}{4} G_{\mu\nu}(x) G_{\mu\nu}(x)$ in terms of B fields is given as

$$\begin{aligned} \langle E \rangle &= \frac{1}{2} \langle \partial_\mu B_\nu^a \partial_\mu B_\nu^a - \partial_\mu B_\nu^a \partial_\nu B_\mu^a \rangle + f^{abc} \langle \partial_\mu B_\nu^a B_\mu^b B_\nu^c \rangle \\ &\quad + \frac{1}{4} f^{abe} f^{ade} \langle B_\mu^a B_\nu^b B_\mu^c B_\nu^d \rangle. \end{aligned} \quad (7.8)$$

Perturbative computation through order g_0^4 gives

$$\langle E \rangle = \frac{1}{2} g_0^2 \frac{N^2 - 1}{(8\pi t)^{D/2}} (D - 1) \{ 1 + c_1 g_0^2 + O(g_0^4) \}, \quad (7.9)$$

where

$$c_1 = \frac{(4\pi)^\epsilon (8t)^\epsilon}{16\pi^2} \left\{ N \left(\frac{11}{3\epsilon} + \frac{52}{9} - 3 \ln 3 \right) - N_f \left(\frac{2}{3\epsilon} + \frac{4}{9} - \frac{4}{3} \ln 2 \right) + \mathcal{O}(\epsilon) \right\}. \quad (7.10)$$

On the other hand, the bare coupling g_0 is related to the renormalized coupling g in the $\overline{\text{MS}}$ scheme at scale μ as

$$g_0^2 = g^2 \mu^{2\epsilon} (4\pi e^{-\gamma_E})^{-\epsilon} \left\{ 1 - \frac{1}{\epsilon} b_0 g^2 + \mathcal{O}(g^4) \right\}, \quad (7.11)$$

where

$$b_0 = \frac{1}{16\pi^2} \left\{ \frac{11}{3} N - \frac{2}{3} N_f \right\}. \quad (7.12)$$

Substituting (7.11) into (7.9), we obtain

$$\langle E \rangle = \frac{3(N^2 - 1)g^2}{128\pi^2 t^2} \{1 + \bar{c}_1 g^2 + \mathcal{O}(g^4)\}. \quad (7.13)$$

Here

$$\bar{c}_1 = \frac{1}{16\pi^2} \left\{ N \left(\frac{11}{3} L + \frac{52}{9} - 3 \ln 3 \right) - N_f \left(\frac{2}{3} L + \frac{4}{9} - \frac{4}{3} \ln 2 \right) \right\} \quad (7.14)$$

at $\epsilon = 0$ with $L = \ln 8\mu^2 t + \gamma_E$. Thus one can see that the energy density defined in terms of B field is finite without additional renormalization.

7.4 Applications of Gradient Flow Equation

Various applications of the physical observable are studied recently. We introduce some of them as example. This subsection is a review of Ref. [6].

7.4.1 Chiral Condensate

In lattice QCD, the expectation value of the chiral densities

$$S_t^{rs} \pm P_t^{rs}, \quad r, s \in \{u, d\} \quad (7.15)$$

where $S_t^{rs} = \bar{\chi}_r \chi_s$, $P_t^{rs} = \bar{\chi}_r \gamma_5 \chi_s$, the r and s are flavor labels, is the order parameter for the spontaneous breaking of the chiral symmetry. The bare operators contain the UV divergences. When the lattice theory has a chiral

symmetry, this divergent terms are proportional to the light-quark masses. Because of the arbitrariness of the subtraction for the renormalization, the expectation value loses the meaning as the order parameter.

Luscher proposed that the gradient flow of a matter field [6] is

$$\dot{\chi} = \Delta\chi, \quad \chi|_{t=0} = \psi \quad (7.16)$$

$$\Delta = \not{D}^2 \text{ or simply } \Delta = D_\mu D_\mu, \quad (7.17)$$

where $D_\mu = \partial_\mu + B_\mu$. Using this equation, he calculates the flow time dependent chiral condensate. Because Eq. (7.16) has chiral symmetry, the quark field $\chi(t, x)$ which depends on flow time transforms in the same way as the fundamental quark field $\psi(x)$ under global chiral rotations. He defines the time dependent condensate as

$$\Sigma_t = -\frac{1}{2} \langle S_t^{uu} + S_t^{dd} \rangle, \quad (7.18)$$

Σ_t is also an order parameter for the spontaneous breaking of chiral symmetry. The attractive point is that Σ_t does not have power divergence, so there are no arbitrariness in the subtraction. There are also advantages that the calculation of Σ_t through numerical simulation is straightforward.

7.4.2 Small Flow Time Expansion

The method to use the gradient flow equation has the advantage that there are no divergence at flow time $t > 0$. It is also important to study how exactly the divergence are avoided near the boundary. Through the small flow time expansion, we can examine them. We can describe the asymptotic expansion of gauge invariant local fields $O_t(x)$ in respect to $\phi_k(x)$ which is the field on the boundary as

$$O_t(x) \underset{t \rightarrow 0}{\sim} \sum_k c_k(t) \phi_k(x), \quad (7.19)$$

where $c_k(t)$ is a time dependent coefficient. The coefficients satisfy the RG equation which determines the asymptotic behavior at small flow time as

$$c_k(t) \underset{t \rightarrow 0}{\propto} t^{\frac{1}{2}(d_k - d_0)} g^{\nu_k} \{1 + O(g^2)\}, \quad (7.20)$$

where d_k and d_0 are dimensions of $\phi_k(x)$ and $O_t(x)$ respectively, g is a running coupling of the theory, ν_k is determined by the one loop coefficients of their anomalous dimensions. Judging from (7.20), the expansion of $O_t(x)$ of Eq. (7.19) is dominated by $\phi_k(x)$ with the lowest dimension in the limit $t \rightarrow 0$. For example, the chiral densities (7.15) can be described using the expansion (7.19). The details are explained in Ref. [6] and their references.

In an opposite manner, any gauge invariant local field $\phi(x)$ at the boundary is also expanded in respect to $O_t(x)$ at some positive flow time t . The simplest case is that we restrict the field by their dimension and symmetry. The form is represented as

$$\phi(x) = c(t)O_t(x) + O(t). \quad (7.21)$$

It is important work to determine the coefficient $c(t)$. For example, in Refs. [15] the coefficient in the expansion of the energy-momentum tensor in the pure gauge theory is computed at one loop order perturbatively. Also, in the paper, using the method of the gradient flow equation, the relation between the small flow time behavior of certain gauge invariant local products and the correctly-normalized conserved energy-momentum tensor in the Yang-Mills theory is given. In Refs. [8], using the Ward identities that derive from the conservation of the energy-momentum tensor in the continuum theory, the coefficients is determined non-perturbatively. In the paper, they also use the gradient flow in order to define more appropriate probes for the translation Ward identities.

7.4.3 Step Scaling and Improved Action

Since the gradient flow gives UV finite quantity, one can define a new renormalization scheme for the running coupling constant. In Refs. [2, 9, 10, 11, 12, 13], the running coupling constant is defined by the renormalization condition:

$$g^2(L) = \text{constant} \times \{t^2 \langle E_t \rangle\}_{\sqrt{8t}=\frac{1}{3}L}. \quad (7.22)$$

Using the step scaling, the RG evolution of this running coupling constant can be studied non-perturbatively.

On the other hand, the method of the gradient flow is useful to construct improved actions. The parameter of the improved theory can be tuned by matching lattices with different spacings using sufficient number of observables as inputs. Since there are a lot of candidates of observables such as

$$\text{tr}\{G_{\mu\nu}G_{\mu\nu}\}, \quad \bar{\chi}\chi, \quad \bar{\chi}\sigma_{\mu\nu}G_{\mu\nu}\chi, \quad (\bar{\chi}\Gamma\chi)(\bar{\chi}\Gamma\chi), \quad (7.23)$$

the method of the gradient flow is very useful.

8 Supersymmetric Gradient Flow Equation

As we have seen in the previous section, the gradient flow equation has spurred a great deal of research. There are a lot of applications using this new renormalization, but there is also room for theoretical study. One of them is to find out what physical system this method can be applied. The equation is very attractive, therefore it is worth extending the equation to other theory, for example, to the QCD with matter field. Luscher proposed the gradient flow of the matter field as Eq. (7.16). However the quark part of this equation is no longer defined from the gradient of the action and there is an arbitrariness of Δ as in Eq. (7.17). It would be important to study the theoretical basis how to defined the gradient flow equation with matter fields in the gauge theory. One of the interesting systems is the super Yang-Mills theory. This theory has a gaugino, which is a matter field in the adjoint representation. And the gaugino is closely related to the gauge fields by supersymmetry (SUSY). Therefore we study the supersymmetric extension of the gradient flow equation. Since the supersymmetric Yang-Mills theory is an attractive of its own sake, it would be useful to study the gradient flow equation for this system. It may also give us a hint to understand the theoretical basis of the gradient flow equation including matter fields.

We give a short summary of SUSY in Appendix E as a review of Ref. [34].

8.1 Our Proposal for Supersymmetric Gradient Flow Equation

The purpose of this subsection is to explain our proposal for the gradient flow equation in super Yang-Mills theory. We success the expansion of the gradient flow equation to super Yang-Mills theory. At first, we summarize how to derive the gradient flow in Yang-Mills theory as follows:

1. Starting from the Yang-Mills action S_{YM} , we make a variation over the $A_\mu(x)$ field.
2. We replace the $A_\mu(x)$ field with the new gauge field $B_\mu(t, x)$, and impose the initial condition $B_\mu(0, x) = A_\mu(x)$
3. We add a new gauge fixing term to suppress the increase of the degree of new gauge freedom in the flow time direction. It has to be proportional to the gauge transformation, because physical quantities does not depend on the term.
4. We regard the sum of them as R.H.S. of the gradient flow equation.
5. We regard the derivative of $B_\mu(t, x)$ with respect to t as L.H.S. of the gradient flow equation.

Thus, we obtain the gradient flow equation in Yang-Mills theory as Eqs. (7.1) and (7.2).

To obtain the gradient flow equation to super Yang-Mills theory, we replace the statement partly as follows:

- Yang-Mills action $S_{\text{YM}} \rightarrow$ Super Yang-Mills action S_{SYM} .
- Gauge field $A_\mu(x) \rightarrow$ Superfield V .
- New gauge field $B_\mu(t, x) \rightarrow$ New superfield v .
- Gauge transformation $\rightarrow$ Super gauge transformation.

Thus we propose a general form of the supersymmetric extension of the gradient flow equation.

Supersymmetric Gradient Flow Equation

$$\frac{\partial v_a}{\partial t} = \frac{\delta S_{\text{SYM}}}{\delta v^a} + \alpha_0 \delta v_a. \quad (8.1)$$

where $V = v_a T^a$ and T^a is a representation matrix. Because Eq. (8.1), however, have infinite number of terms, it is very difficult to solve it non-perturbatively. In order to obtain the flow equation with finite number of terms, we choose the Wess-Zumino (WZ) gauge. However, generally the time evolution from the flow equation can carry the system away from the WZ gauge. Therefore, the most important question is whether there exists the special α_0 term, which keeps the WZ gauge. As a result, we find that such a α_0 term exists:

Supersymmetric Gradient Flow Equation under Wess-Zumino Gauge

$$\frac{\partial v}{\partial t} = \frac{e^{L_v} - 1}{L_v} \cdot (D^\alpha w_\alpha + \{e^{-v} D^\alpha e^v, w_\alpha\}) + h. c. + \delta v, \quad (8.2)$$

where

$$\delta v = \phi + \phi^\dagger + \frac{1}{2}[v, \phi - \phi^\dagger] + \frac{1}{12}[v, [v, \phi + \phi^\dagger]], \quad (8.3)$$

$$\phi = \bar{D}^2(D^2 v + [D^2 v, v]). \quad (8.4)$$

Using this new equation, we find out that, in this case, an extra term is required in addition to the gradient flow equation of the matter field which is proposed by Luscher.

9 Pure Abelian Supersymmetric Theory

At first, we consider a supersymmetric pure Abelian gauge theory to simplify the discussion. Because this theory does not have an interaction, the theory also does not have divergences in the first place, but it is useful to understand the basic structure as a toy model.

9.1 Derivation of Gradient Flow Equation of Pure Abelian Supersymmetric Theory

From the discussion in Sec. 8.1, we obtain the gradient flow Equation of the pure Abelian supersymmetric theory. The free vector field action which is invariant under the supersymmetric gauge transformation is

$$\begin{aligned} S &= \frac{1}{4} \int d^4x (W^\alpha W_\alpha|_{\theta\theta} + \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}|_{\bar{\theta}\bar{\theta}}) \\ &= \frac{1}{4} \int d^8z (D^\alpha W_\alpha + \bar{D}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}) V \end{aligned} \quad (9.1)$$

where V is vector multiplet, $V = \{C, X, \bar{X}, M, M^*, V_m, \Lambda, \bar{\Lambda}, D\}$. W and $\bar{W}$ are defined by

$$W_\alpha = -\bar{D}\bar{D}D_\alpha V, \quad (9.2)$$

$$\bar{W}_{\dot{\alpha}} = -DD\bar{D}_{\dot{\alpha}} V. \quad (9.3)$$

Making variation of the action S over V , we obtain

$$\frac{\delta S}{\delta V} = D^\alpha W_\alpha. \quad (9.4)$$

We used here the relation equation,

$$D^\alpha W_\alpha = \bar{D}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}. \quad (9.5)$$

Then we proposed the extended gradient flow equation of the pure supersymmetric theory as

$$\dot{v} = D^\alpha w_\alpha + \alpha_0 (D^2 \bar{D}^2 + \bar{D}^2 D^2) v, \quad (9.6)$$

$$v|_{t=0} = V, \quad w_\alpha|_{t=0} = W_\alpha. \quad (9.7)$$

where v is vector multiplet depending on the flow time, $v = \{c, \chi, \bar{\chi}, m, m^*, v_m, \lambda, \bar{\lambda}, d\}$. The α_0 term, which is the second term of the R.H.S. of Eq. (9.6), is introduced to suppress the new gauge degrees of freedom under the evolution in the flow time. The α_0 term may not be unique, but we here only show that the form Eq. (9.6) is adequate. We postpone to explain how to determine the α_0 term to Sec. 10.2.2.

9.2 Gradient Flow Equation of Pure Yang-Mills Theory for Each Component of Vector Multiplet

Describing the extended gradient flow equation in the coordinate of super-space which are labeled $(x, \theta, \bar{\theta})$, we find out the each dependence of the component of vector multiplet on the flow time.

$$\begin{aligned}
v(x, \theta, \bar{\theta}) = & c + i\theta\chi - i\bar{\theta}\bar{\chi} + \frac{i}{2}\theta\theta m - \frac{i}{2}\bar{\theta}\bar{\theta}m^* \\
& - \theta\sigma^m\bar{\theta}v_m + i\theta\theta\bar{\theta}[\bar{\lambda} + \frac{i}{2}\bar{\sigma}^m\partial_m\chi] \\
& - i\bar{\theta}\bar{\theta}\theta[\lambda + \frac{i}{2}\sigma^m\partial_m\bar{\chi}] + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}[d + \frac{1}{2}\square c]
\end{aligned} \tag{9.8}$$

Using (9.8), we calculate each terms of the gradient flow equation, we obtain

$$\begin{aligned}
D^\alpha w_\alpha = & -2d + 2\theta\sigma^m\partial_m\bar{\lambda} - 2\bar{\theta}\bar{\sigma}^m\partial_m\lambda + 2(\theta\sigma^k\bar{\theta})\partial^m v_{km} \\
& - i\bar{\theta}\bar{\theta}\theta\square\lambda + i\theta\theta\bar{\theta}\square\bar{\lambda} + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\square d,
\end{aligned} \tag{9.9}$$

$$\begin{aligned}
(D^2\bar{D}^2 + \bar{D}^2D^2)v = & 16(d + \square c) - 16\theta(\sigma^m\partial_m\bar{\lambda} - i\square\chi) + 16\bar{\theta}(\bar{\sigma}^m\partial_m\lambda - i\square\bar{\chi}) \\
& + 8i\theta\theta\square m - 8i\bar{\theta}\bar{\theta}\square m^* - 16(\theta\sigma^m\bar{\theta})\partial_m\partial^k v_k \\
& + 8i\theta\theta\bar{\theta}(\square\bar{\lambda} + i\bar{\sigma}^m\partial_m\square\chi) - 8i\bar{\theta}\bar{\theta}\theta(\square\lambda + i\sigma^m\partial_m\square\bar{\chi}) \\
& + 4\theta\theta\bar{\theta}\bar{\theta}(\square d + \square\square c).
\end{aligned} \tag{9.10}$$

Substituting (9.9) and (9.10) into (9.6), finally, we obtain the flow equations for the each component of the vector multiplet as

$$\dot{c} = 16\alpha_0\square c - 2(1 - 8\alpha_0)d, \tag{9.11}$$

$$\dot{\chi} = 16\alpha_0\square\chi - 2i(1 - 8\alpha_0)\sigma^m\partial_m\bar{\lambda}, \tag{9.12}$$

$$\dot{\bar{\chi}} = 16\alpha_0\square\bar{\chi} - 2i(1 - 8\alpha_0)\bar{\sigma}^m\partial_m\lambda, \tag{9.13}$$

$$\dot{m} = 16\alpha_0\square m, \tag{9.14}$$

$$\dot{m}^* = 16\alpha_0\square m^*, \tag{9.15}$$

$$\dot{v}_m = 2\square v_m - 2(1 - 8\alpha_0)\partial_m\partial^k v_k, \tag{9.16}$$

$$\dot{\bar{\lambda}} = 2\square\bar{\lambda}, \tag{9.17}$$

$$\dot{\lambda} = 2\square\lambda, \tag{9.18}$$

$$\dot{d} = 2\square d. \tag{9.19}$$

Taking α_0 as

$$\alpha_0 = \frac{1}{8}, \quad (9.20)$$

we obtain

$$\dot{c} = 2\Box c, \quad (9.21)$$

$$\dot{\chi} = 2\Box \chi, \quad (9.22)$$

$$\dot{\bar{\chi}} = 2\Box \bar{\chi}, \quad (9.23)$$

$$\dot{m} = 2\Box m, \quad (9.24)$$

$$\dot{m}^* = 2\Box m^*, \quad (9.25)$$

$$\dot{v}_m = 2\Box v_m, \quad (9.26)$$

$$\dot{\bar{\lambda}} = 2\Box \bar{\lambda}, \quad (9.27)$$

$$\dot{\lambda} = 2\Box \lambda, \quad (9.28)$$

$$\dot{d} = 2\Box d. \quad (9.29)$$

One can see that each component of the vector multiplet evolves separately in time.

9.3 Flow Time Dependence of Super Gauge Transformation

When we demand that the gradient flow equation (9.6) is invariant under the super gauge transformation,

$$v' = v + \phi + \phi^\dagger, \quad (9.30)$$

at each time, ϕ have to satisfy the equation as

$$\dot{\phi} = \alpha_0 \bar{D}^2 D^2 \phi, \quad (9.31)$$

$$\phi|_{t=0} = \Phi, \quad (9.32)$$

where Φ is a chiral field,

$$\bar{D}\Phi = 0. \quad (9.33)$$

The chirality of the ϕ at each flow time is guaranteed by Eq. (9.31).

10 Super Yang-Mills Theory

Following the toy model example in Sec. 9, we extend the gradient flow equation to the case of super Yang-Mills theory. In the general gauge, the flow equation contains infinite number of commutators so that it is very difficult to solve it non-perturbatively. In order to obtain the flow equation with finite number of terms, we choose the WZ gauge. However, generally the time evolution from the flow equation can carry the system away from the WZ gauge. Therefore, the most important question is whether there exists a special α_0 term, which keeps the WZ gauge. As a result, we find that such a α_0 term exists.

10.1 Derivation of Gradient Flow Equation for Super Yang-Mills Theory

The gradient flow for super Yang-Mills theory is similar to the one for pure Abelian supersymmetric theory in Sec. 9. The gauge fixing term is positive and described by super gauge transformation δv . Then the most general form of the supersymmetric gradient flow equation is

$$\frac{\partial v_a}{\partial t} = \frac{\delta S_{\text{SYM}}}{\delta v^a} + \alpha_0 \delta v_a. \quad (10.1)$$

In what follows we call the first term of R.H.S. as the gauge covariant term, the second term of R.H.S. as the α_0 term.

At first, we derive the gauge covariant term of the super Yang-Mills theory. The action of the super Yang-Mills theory is

$$S = \int d^4x \text{Tr}[W^\alpha W_\alpha|_{\theta\theta} + \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}|_{\bar{\theta}\bar{\theta}}]. \quad (10.2)$$

The action is rewritten as

$$S = \int d^8z \text{Tr}[W^\alpha e^{-V} D_\alpha e^V] + h.c., \quad (10.3)$$

where $d^8z \equiv d^4x d^2\theta d^2\bar{\theta}$, $\frac{\delta v^a(z)}{\delta v^b(z')} \equiv \delta_b^a \delta^8(z-z') \equiv \delta_b^a \delta^4(x-x') \delta^2(\theta-\theta') \delta^2(\bar{\theta}-\bar{\theta}')$. W_α is defined as

$$W_\alpha = -\bar{D}^2(e^{-V} D_\alpha e^V). \quad (10.4)$$

The variation of S over V is

$$\frac{\delta S}{\delta V} \equiv \sum_a T^a \frac{\delta S}{\delta v^a} \quad (10.5)$$

$$= \frac{e^{L_V} - 1}{L_V} \cdot (D^\alpha W_\alpha + \{e^{-V} D^\alpha e^V, W_\alpha\}) + h.c., \quad (10.6)$$

where

$$L_V \cdot \equiv [V, \cdot]. \quad (10.7)$$

The detailed derivation of Eq. (10.6) is given in Appendix F. In the next step, we would like to determine the form of the α_0 term. As mentioned above, however, the gradient flow equation contains infinite number of commutators in the general gauge. In order to obtain the flow equation with finite number of terms, we choose the WZ gauge.

10.2 Gradient Flow Equation in Super Yang-Mills Theory under Wess-Zumino Gauge

In this subsection, we determine the form of the gradient flow equation for super Yang-Mills theory under the WZ gauge .

10.2.1 Determination of Gauge Covariant Term

We define A as

$$A = D^\alpha w_\alpha + \{e^{-v} D^\alpha e^v, w_\alpha\}, \quad (10.8)$$

Useful formulae expand Eq. (10.8) in component fields are given in Appendix G. The gauge covariant term is given as

$$\begin{aligned} & \left(\frac{e^{L_v} - 1}{L_v} \cdot A \right) + \left(\frac{e^{L_v} - 1}{L_v} \cdot A \right)^\dagger \\ &= A + A^\dagger + \frac{1}{2!} [v, A - A^\dagger] + \frac{1}{3!} [v, [v, A + A^\dagger]] + O(v^3), \end{aligned} \quad (10.9)$$

where A is represented in $(x, \theta, \bar{\theta})$ coordinates by

$$\begin{aligned}
A(x, \theta, \bar{\theta}) = & -8d + 8\theta\sigma^m\mathcal{D}_m\bar{\lambda} - 8\bar{\theta}\bar{\sigma}^m\mathcal{D}_m\lambda \\
& + 4(\bar{\theta}\bar{\sigma}^m\theta)[v_m, d] + 4(\theta\sigma^k\bar{\sigma}^m\sigma^l\bar{\theta})\mathcal{D}_lv_{mk} + 8[\bar{\theta}\bar{\lambda}, \theta\lambda] \\
& - 8i\theta\theta(\bar{\theta}\bar{\sigma}^l\sigma^m\mathcal{D}_l\mathcal{D}_m\bar{\lambda}) + 8i\theta\theta[\bar{\theta}\bar{\lambda}, d] \\
& + 4i\theta\theta(\bar{\theta}\bar{\sigma}^k\sigma^m\partial_k\mathcal{D}_m\bar{\lambda}) + 4i\bar{\theta}\bar{\theta}(\theta\sigma^k\bar{\sigma}^m\partial_k\mathcal{D}_m\lambda) \\
& + \theta\theta\bar{\theta}\bar{\theta}(2\Box d + 2i\partial^m[v_m, d] + i\text{Tr}[\bar{\sigma}^m\sigma^l\bar{\sigma}^n\sigma^k]\partial_n\mathcal{D}_lv_{mk} \\
& - 2i\partial_m\{\bar{\lambda}_{\dot{\alpha}}, (\bar{\sigma}^m\lambda)^{\dot{\alpha}}\}).
\end{aligned} \tag{10.10}$$

On the other hand, $A^\dagger$ is represented in $(x, \theta, \bar{\theta})$ coordinates by

$$\begin{aligned}
A^\dagger(x, \theta, \bar{\theta}) = & -8d + 8\theta\sigma^m\mathcal{D}_m\bar{\lambda} - 8\bar{\theta}\bar{\sigma}^m\mathcal{D}_m\lambda \\
& - 4(\bar{\theta}\bar{\sigma}^m\theta)[v_m, d] + 4(\theta\sigma^l\bar{\sigma}^m\sigma^k\bar{\theta})\mathcal{D}_lv_{mk} + 8[\bar{\theta}\bar{\lambda}, \theta\lambda] \\
& - 4i\theta\theta(\bar{\theta}\bar{\sigma}^k\sigma^m\partial_k\mathcal{D}_m\bar{\lambda}) - 4i\bar{\theta}\bar{\theta}(\theta\sigma^k\bar{\sigma}^m\partial_k\mathcal{D}_m\lambda) \\
& + 8i\bar{\theta}\bar{\theta}(\theta\sigma^l\bar{\sigma}^m\mathcal{D}_l\mathcal{D}_m\lambda) + 8i\bar{\theta}\bar{\theta}[\theta\lambda, d] \\
& + \theta\theta\bar{\theta}\bar{\theta}(2\Box d + 2i\partial^m[v_m, d] - i\text{Tr}[\bar{\sigma}^m\sigma^k\bar{\sigma}^n\sigma^l]\partial_n\mathcal{D}_lv_{mk} \\
& + 2i\partial_m\{(\bar{\lambda}\bar{\sigma}^m)^\alpha, \lambda_\alpha\}).
\end{aligned} \tag{10.11}$$

Finally, we get the gauge covariant term in $(x, \theta, \bar{\theta})$ coordinates as follows.

$$\begin{aligned}
& \left(\frac{e^{L_v} - 1}{L_v} \cdot (D^\alpha w_\alpha + \{e^{-v} D^\alpha e^v\}) \right) + h.c. \\
= & -16d + 16\theta\sigma^m\mathcal{D}_m\bar{\lambda} - 16\bar{\theta}\bar{\sigma}^m\mathcal{D}_m\lambda \\
& + 16\theta\sigma^m\bar{\theta}\mathcal{D}^k v_{mk} + 16[\bar{\theta}\bar{\lambda}, \theta\lambda] \\
& - 8i\theta\theta(\bar{\theta}\bar{\sigma}^l\sigma^m\mathcal{D}_l\mathcal{D}_m\bar{\lambda}) + 8i\theta\theta[\bar{\theta}\bar{\lambda}, d] \\
& + 8i\bar{\theta}\bar{\theta}(\theta\sigma^l\bar{\sigma}^m\mathcal{D}_l\mathcal{D}_m\lambda) + 8i\bar{\theta}\bar{\theta}[\theta\lambda, d] \\
& + \theta\theta\bar{\theta}\bar{\theta}(4\Box d + 4i\partial^m[v_m, d] \\
& + i\text{Tr}[\bar{\sigma}^m\sigma^l\bar{\sigma}^n\sigma^k - \bar{\sigma}^m\sigma^k\bar{\sigma}^n\sigma^l]\mathcal{D}_n\mathcal{D}_lv_{mk} \\
& - 2i\partial_m\{\bar{\lambda}_{\dot{\alpha}}, (\bar{\sigma}^m\lambda)^{\dot{\alpha}}\} + 2i\partial_m\{(\bar{\lambda}\bar{\sigma}^m)^\alpha, \lambda_\alpha\} \\
& - \frac{2}{3}[v_m, [v^m, d]]).
\end{aligned} \tag{10.12}$$

10.2.2 Determination of α_0 Term

As mentioned above, we need to add the α_0 term with (10.12) to suppress the gauge degrees of freedom under the flow time evolution. Here we discuss how to determine the form of the α_0 term. The gauge transformation is defined by

$$\delta V = L_{V/2} \cdot [(\Phi - \Phi^\dagger) + \coth(L_{V/2}) \cdot (\Phi + \Phi^\dagger)], \quad (10.13)$$

where Φ and $\Phi^\dagger$ are a chiral superfield and an anti-chiral superfield respectively. Under the WZ gauge, the gauge transformation is expressed by a finite number of terms as

$$\delta V = \Phi + \Phi^\dagger + \frac{1}{2}[V, \Phi - \Phi^\dagger] + \frac{1}{12}[V, [V, \Phi + \Phi^\dagger]]. \quad (10.14)$$

We try to find out the special form of the α_0 term so that the gradient flow equation is consistent within the WZ gauge. This means the α_0 term has to satisfy the following requirements.

- It is positive.
- The mass dimension is two.
- It is described by super gauge transformation δV .
- The flow of the vector field keeps the WZ gauge at any flow time.

As a result, we found out that there exists at least one example of the α_0 term which satisfies these conditions. The form is δV , which consists of Φ defined by

$$\Phi = \bar{D}^2(D^2V + [D^2V, V]). \quad (10.15)$$

Using this, we obtain δv in terms of $(x, \theta, \bar{\theta})$ coordinates as

$$\begin{aligned}
\delta v(x, \theta, \bar{\theta}) &= \phi + \phi^\dagger + \frac{1}{2}[v, \phi - \phi^\dagger] + \frac{1}{12}[v, [v, \phi + \phi^\dagger]] \\
&= 16d - 16\theta\sigma^m\mathcal{D}_m\bar{\lambda} + 16\bar{\theta}\bar{\sigma}^m\mathcal{D}_m\lambda - 16\theta\sigma^k\bar{\theta}\mathcal{D}_k\partial_mv^m \\
&\quad - 8i\theta\theta\bar{\theta}\bar{\sigma}^k\sigma^m\mathcal{D}_k\mathcal{D}_m\bar{\lambda} - 8\theta\theta\bar{\theta}_{\dot{\alpha}}[\bar{\lambda}^{\dot{\alpha}}, \partial_mv^m] \\
&\quad - 8i\bar{\theta}\bar{\theta}\theta\sigma^k\bar{\sigma}^m\mathcal{D}_k\mathcal{D}_m\lambda - 8\bar{\theta}\bar{\theta}\theta^\alpha[\lambda_\alpha, \partial_mv^m] \\
&\quad + 4\theta\theta\bar{\theta}\bar{\theta}(\square d + i\{\bar{\lambda}_{\dot{\alpha}}, (\bar{\sigma}^m\mathcal{D}_m\lambda)^{\dot{\alpha}}\} - i\{\lambda^\alpha, (\sigma^m\mathcal{D}_m\bar{\lambda})_\alpha\} \\
&\quad + i[d, \partial_mv^m] + i[v^m, \partial_md] - \frac{1}{6}[v_m, [v^m, d]]). \tag{10.16}
\end{aligned}$$

10.3 Gradient Flow Equation of Super Yang-Mills Theory for Each Component of Vector Multiplet

We substitute (10.12) and (10.16) into (10.1) under the WZ gauge. Because a physical quantity does not depend on the form of the α_0 term, we choose a particular value $\alpha_0 = 1$. Then we obtain

$$\begin{aligned}
&\left(\frac{e^{L_v} - 1}{L_v} \cdot (D^\alpha w_\alpha + \{e^{-v} D^\alpha e^v, w_\alpha\})\right) + h.c. + 1 \cdot \delta v \\
&= 16\theta\sigma^m\bar{\theta}\mathcal{D}^k v_{mk} + 16[\bar{\theta}\bar{\lambda}, \theta\lambda] - 16\theta\sigma^k\bar{\theta}\mathcal{D}_k\partial_mv^m \\
&\quad - 16i\theta\theta\bar{\theta}\bar{\sigma}^k\sigma^m\mathcal{D}_k\mathcal{D}_m\bar{\lambda} + 8i\theta\theta[\bar{\theta}\bar{\lambda}, d + i\partial_mv^m] \\
&\quad + 16i\bar{\theta}\bar{\theta}\theta\sigma^k\bar{\sigma}^m\mathcal{D}_k\mathcal{D}_m\lambda + 8i\bar{\theta}\bar{\theta}[\theta\lambda, d - i\partial_mv^m] \\
&\quad + \theta\theta\bar{\theta}\bar{\theta}(8\square d + 8i[v_m, \partial^m d] + i\text{Tr}[\bar{\sigma}^m\sigma^l\bar{\sigma}^n\sigma^k - \bar{\sigma}^m\sigma^k\bar{\sigma}^n\sigma^l]\mathcal{D}_n\mathcal{D}_l v_{mk} \\
&\quad + 4i\{\bar{\lambda}_{\dot{\alpha}}, (\bar{\sigma}^m\mathcal{D}_m\lambda)^{\dot{\alpha}}\} - 4i\{\lambda^\alpha, (\sigma^m\mathcal{D}_m\bar{\lambda})_\alpha\} - \frac{4}{3}[v_m, [v^m, d]]). \tag{10.17}
\end{aligned}$$

Finally, we obtain the flow equations for the each component of the vector multiplet as

$$\dot{c} = 0, \tag{10.18}$$

$$\dot{\chi} = 0, \tag{10.19}$$

$$\dot{\bar{\chi}} = 0, \tag{10.20}$$

$$\dot{m} = 0, \tag{10.21}$$

$$\dot{m}^* = 0, \tag{10.22}$$

$$\dot{v}_m = -16\mathcal{D}^k v_{mk} + 16\mathcal{D}_m\partial_kv^k - 8\{\bar{\lambda}_{\dot{\alpha}}, (\bar{\sigma}_m\lambda)^{\dot{\alpha}}\}, \tag{10.23}$$

$$\dot{\bar{\lambda}} = -16\bar{\sigma}^k\sigma^m\mathcal{D}_k\mathcal{D}_m\bar{\lambda} + 8[\bar{\lambda}, d + i\partial_m v^m], \quad (10.24)$$

$$\dot{\lambda} = -16\sigma^k\bar{\sigma}^m\mathcal{D}_k\mathcal{D}_m\lambda - 8[\lambda, d - i\partial_m v^m], \quad (10.25)$$

$$\begin{aligned} \dot{d} = & 16\Box d + 16i[v_m, \partial^m d] \\ & + 2i\text{Tr}[\bar{\sigma}^m\sigma^l\bar{\sigma}^n\sigma^k - \bar{\sigma}^m\sigma^k\bar{\sigma}^n\sigma^l]\mathcal{D}_n\mathcal{D}_l v_{mk} \\ & + 8i\{\bar{\lambda}_{\dot{\alpha}}, (\bar{\sigma}^m\mathcal{D}_m\lambda)^{\dot{\alpha}}\} - 8i\{\lambda^{\alpha}, (\sigma^m\mathcal{D}_m\bar{\lambda})_{\alpha}\} \\ & - \frac{8}{3}[v_m, [v^m, d]]. \end{aligned} \quad (10.26)$$

We find that the flow equations for each component are consistent with WZ gauge. Here we choose initial conditions to satisfy the WZ gauge at $t = 0$ as

$$c|_{t=0} = 0, \quad (10.27)$$

$$\chi|_{t=0} = 0, \quad (10.28)$$

$$\bar{\chi}|_{t=0} = 0, \quad (10.29)$$

$$m|_{t=0} = 0, \quad (10.30)$$

$$m^*|_{t=0} = 0, \quad (10.31)$$

$$v_m|_{t=0} = V_m, \quad (10.32)$$

$$\bar{\lambda}|_{t=0} = \bar{\Lambda}, \quad (10.33)$$

$$\lambda|_{t=0} = \Lambda, \quad (10.34)$$

$$d|_{t=0} = D. \quad (10.35)$$

Let us compare the flow equation for Yang-Mills theory proposed by Luscher with our results for super Yang-Mills theory in Eqs (10.24) and (10.25). In Ref. [7], Luscher claims that the gradient flow equations of the quark field are given as

$$\dot{\bar{\chi}} = \bar{\chi}\overleftarrow{\Delta} + \alpha_0\bar{\chi}\partial_{\nu}B_{\nu}, \quad (10.36)$$

$$\dot{\chi} = \Delta\chi - \alpha_0\partial_{\nu}B_{\nu}\chi. \quad (10.37)$$

On the other hand our results for the gradient flow equations of the gaugino field in Eqs. (10.24) and (10.25) are given as

$$\dot{\bar{\lambda}} = -16\bar{\sigma}^k\sigma^m\mathcal{D}_k\mathcal{D}_m\bar{\lambda} + 8[\bar{\lambda}, d + i\partial_m v^m], \quad (10.38)$$

$$\dot{\lambda} = -16\sigma^k\bar{\sigma}^m\mathcal{D}_k\mathcal{D}_m\lambda - 8[\lambda, d - i\partial_m v^m]. \quad (10.39)$$

We find that if we regard Δ as $\mathbb{D}^2$, Eqs. (10.36) and (10.37) are almost similar to our results Eqs. (10.38) and (10.39) respectively except for $[\bar{\lambda}, d]$ term and $[\lambda, d]$ term and the point that α_0 terms are described in terms of commutation relations.

11 Short Summary

We proposed a supersymmetric extension of the gradient flow equation using the superfield formalism. Since in the general gauge the equation involves infinite number of terms, it is difficult to solve it non-perturbatively. In order to obtain the gradient flow equation with finite number of terms, we need to take the WZ gauge. We find a special form of the α_0 term so that the gradient flow equation is consistent within the WZ gauge. Since the super Yang-Mills theory has a gaugino which is closely related to the gauge fields by SUSY, the gradient flow in the super Yang-Mills theory leads to the equation of the matter field. It is important to examine whether gauge invariant physical quantities require additional renormalization or not, which is under way.

I would like to add comments as follows. There is not an explicit reason why we have to derive the gradient flow equation by the gradient of the action. If the theory has SUSY, there is an explicit relation between the gradient flow equation of the gauge field and the one of the matter field. In this work, we show that we can construct the each equation consistently under the WZ gauge. However it is valid to derive the gradient flow equation from the gradient of the action, when we study the correspondence between the ERG equation and the gradient flow equation. The scale which is the limit of the low energy corresponds to the flow time when the gradient of the action equals 0, that is, the equation of motion is valid.

When we extend the gradient flow equation to the supersymmetric one, we could derive the equation of the matter field naturally. In other words, owing to the SUSY transformation, which gives the relation between the gauge field and the gaugino field, we can obtain the consistent method to construct the flow equations. Using the method, there is no arbitrariness of the form of the flow equation of the matter fields.

We used the superfield formalism to extend the gradient flow equation

for the super Yang-Mills theory, but we can also use the supermultiplet formalism. Using the formalism, it is useful to extend for $N = 2$ SUSY. This is the futures work.

12 Summary and Disucussion

In this thesis, we study new directions in the ERG through the Lifshitz-type theory, the gradient flow equation and its supersymmetric extension.

In the Lifshitz-type theory, higher derivative terms in the kinetic terms suppress the UV divergence. However, there is a problem of broken Lorentz symmetry, therefore, we should examine whether the theory restores the Lorentz symmetry in the IR region.

In part I, we applied the Wegner-Houghton equation with the momentum cut-off in the cylindrical shape and analyzed the RG flow in the $z = 2$, $d = 3 + 1$ Lifshitz-type scalar model numerically. As a result, we find that the terms that break the Lorentz symmetry vanish at low energy for the Lifshitz-type scalar model. There are two significances in the results. First, Lifshitz-type theory has the advantage in terms of the renormalization. Since we found the restoration of Lorentz symmetry, the theory broadens the class of perturbatively renormalizable field theories which can be applied to particle physics. Secondly, it means that this is a concrete solution to the problem of triviality of ϕ^4 . The Lifshitz-type theory has nontrivial interaction terms $\hat{\lambda}_n \phi^n$ ($n = 4, 6, 8, 10$) even when the Lorentz symmetry is restored at low energy. We find a concrete solution to the long-standing problem of triviality of ϕ^4 theory in $d = 3 + 1$.

There is also room for discussion on the cut-off method. We used the cut-off to respect the spatial symmetry. More pertinent cut-off which keeps the symmetry explicitly may exist. The truncation method remains as a matter to be discussed further. Given some symmetries, for example SUSY, we may improve this analytic method. If the theory can include the gauge field, we could discuss the standard model, then we could extend the range of application of Lifshitz-type theory further. For this purpose, the ERG for supersymmetric theory and the gauge theory should be studied, that is to say, we should study the cut-off problem which means how to keep the symmetry explicitly of the cut-off in the ERG.

On the other hand, the gradient flow equation is also an attractive method in terms of the renormalization. In this method, the expectation value of any gauge invariant local operators constructed by the solution of a certain type

of diffusion equation called the gradient flow equation whose initial value is the bare gauge field, is finite without additional renormalization. It is interesting to find out in what physical system this method can be applied.

In part II, we give the general form of the supersymmetric extended gradient flow equation. However, since the equation has infinite number of terms, it is difficult to solve it non-perturbatively. In order to obtain the gradient flow equation with finite number of terms, we need to take the WZ gauge. We find that we can obtain a consistent gradient flow equation within the WZ gauge, if we choose a proper α_0 term, which is described in terms of Φ defined by Eq. (10.15). As a result, we give the gradient flow equation to the super Yang-Mills theory which is closed under the WZ gauge.

There are two significances in our result. First, we extend the method of the gradient flow equation to one of the super Yang-Mills theory. Since the super Yang-Mills theory is attractive for its own sake, the gradient flow equation will be useful for further studies. Secondly, we obtain the gradient flow equation of the matter field very naturally. Luscher proposed the gradient flow of the matter field. However the quark part of this equation is no longer defined from the gradient of the action, therefore there is an arbitrariness. Since the super Yang-Mills theory has a gaugino which is closely related to the gauge fields by SUSY, the gradient flow in the super Yang-Mills theory leads the equation of the matter field.

Of course it is also important to study whether the physical quantities do not require additional renormalization. This theory may be applied to the ERG equation, which keeps the gauge symmetry. Moreover if we formulate the equation, which is SUSY invariant at each flow time, it may lead to construct the ERG equation, which keeps the SUSY explicitly. In either case, the gradient flow equation is maybe a key to find a new ERG.

In conclusion, our results is not only attractive itself but also give the new direction in the ERG to keep the symmetry explicitly. We hope that our results in this thesis could give a first step towards the study on the new ERG.

Acknowledgments

Before anything else, we would like to thank PhD supervisor, Prof. Tetsuya Onogi for very stimulating discussion, meaningful advice, encouragements and many lecture throughout this thesis. We also would like to thank Prof. Kiyoshi Higashijima and Etsuko Itou. The work in Part I could not be accomplished without their great advising. Owing to Prof. Martin Luscher's great work, we could achieve our work. Prof. Martin Luscher is so kind to tell us detailed calculations. We would like to express our appreciation to him. We also thank Prof. Satoshi Yamaguchi for many useful comments. We acknowledge all our laboratory members for joyful discussions. This work was supported by Grant-in-Aid for JSPS Fellows Grant Number 25·1336.

Finally we would like to thank my parents, my brother, specially my fiancée for continuous encouragements.

A Notation in Part. I

We use the following notation. The definitions of integral symbols are

$$\int_x \equiv \int d^d x, \quad (\text{A.1})$$

$$\int_p \equiv \int \frac{d^d p}{(2\pi)^d}. \quad (\text{A.2})$$

The definitions of δ functions and its symbols are

$$\int_x \delta(x) g(x) = g(0), \quad (\text{A.3})$$

$$\int_p \hat{\delta}(p) f(p) = f(0), \quad (\text{A.4})$$

$$\delta(x) = \int_p e^{-ipx}, \quad (\text{A.5})$$

$$\hat{\delta}(p) \equiv (2\pi)^d \delta(p) = \int_x e^{ipx}. \quad (\text{A.6})$$

The definitions of Fourier transformation of fields are

$$\phi(x) = \int_p e^{ipx} \phi(p), \quad (\text{A.7})$$

$$\phi(p) = \int_x e^{-ipx} \phi(x). \quad (\text{A.8})$$

The definition of trace symbol is

$$\text{tr} \equiv \int_k \int_{k'} \hat{\delta}(k - k'). \quad (\text{A.9})$$

B Derivation of Wegner-Houghton Equation

This is a review of [28, 29] with the exception of a derivation of the extended Wegner-Houghton equation for Lifshitz-type theory. The general Wilsonian effective action is given as

$$\begin{aligned} S[\Omega; \Lambda] = \sum_n \frac{1}{n!} \int_{p_1} \cdots \int_{p_n} \hat{\delta}^{(D)}(p_1 + \cdots + p_n) g_{i_1, \dots, i_n}(p_1, \dots, p_n; \Lambda) \\ \times \Omega_{i_1}(p_1; \Lambda) \cdots \Omega_{i_n}(p_n; \Lambda), \end{aligned} \quad (\text{B.1})$$

where

$$\Omega(p_i; \Lambda) \equiv \Omega(p_i) \theta(\Lambda - p_i). \quad (\text{B.2})$$

$\Omega(p_i)$ denotes general fields, for example, in the case of scalar fields $\Omega(p_i) = \phi(p_i)$. We introduce the shell-mode wave functions $\Omega_s(p_i)$, which are nonzero only for $\Lambda(\delta t) = \Lambda e^{-\delta t} \leq p_i \leq \Lambda$, and $\Omega_{IR}(p_i; \Lambda(\delta t))$, which are nonzero only for $p_i \leq \Lambda(\delta t)$, where $\Lambda(t) \equiv \Lambda e^{-t}$, $\delta\Lambda \equiv \Lambda \delta t$. We then write

$$\Omega(p_i; \Lambda) = \Omega_{IR}(p_i; \Lambda(\delta t)) + \Omega_s(p_i). \quad (\text{B.3})$$

The partition function Z is given as

$$Z = \int [d\Omega] \exp\{-S[\Omega; \Lambda]\}. \quad (\text{B.4})$$

Using Eq. (B.3), we split $[d\Omega]$ into $[d\Omega_{IR}]$ and $[d\Omega_s]$ in the partition function Z to obtain

$$Z = \int [d\Omega_{IR}] \int [d\Omega_s] \exp\{-S[\Omega_{IR} + \Omega_s; \Lambda]\}. \quad (\text{B.5})$$

On the other hand, a partition function that is defined by the cut-off $\Lambda(\delta t)$ is given as

$$Z = \int [d\Omega_{IR}] \exp\{-S[\Omega_{IR}; \Lambda(\delta t)]\}, \quad (\text{B.6})$$

which gives the same value as (B.5). Therefore, we obtain the RG equation

$$\exp\{-S[\Omega_{IR}; \Lambda(\delta t)]\} = \int [d\Omega_s] \exp\{-S[\Omega_{IR} + \Omega_s; \Lambda]\}. \quad (\text{B.7})$$

On the L.H.S. of Eq. (B.7), expanding $S[\Omega_{IR} + \Omega_s; \Lambda]$ by Ω_s and integrating out Ω_s through the first order of $\delta\Lambda$, we obtain the equation

$$S[\Omega_{IR}; \Lambda(\delta t)] = S[\Omega_{IR}; \Lambda] + \frac{1}{2} \text{tr} \ln \left(\frac{\delta^2 S}{\delta \Omega^i \delta \Omega^j} \right) - \frac{1}{2} \frac{\delta S}{\delta \Omega^i} \left(\frac{\delta^2 S}{\delta \Omega^i \delta \Omega^j} \right)^{-1} \frac{\delta S}{\delta \Omega^j}. \quad (\text{B.8})$$

On the other hand, a general action at the scale $\Lambda(\delta t)$ or Λ is

$$\begin{aligned} S[\Omega_{IR}; \Lambda(\delta t)] &= \sum_n \frac{1}{n!} \int_{p_1}^{\Lambda(\delta t)} \cdots \int_{p_n}^{\Lambda(\delta t)} \hat{\delta}^{(D)}(p_1 + \cdots + p_n) g_n(p; \Lambda(\delta t)) \\ &\quad \times \Omega_{i_1}(p_1; \Lambda(\delta t)) \cdots \Omega_{i_n}(p_n; \Lambda(\delta t)), \end{aligned} \quad (\text{B.9})$$

$$\begin{aligned} S[\Omega_{IR}; \Lambda] &= \sum_n \frac{1}{n!} \int_{p_1}^{\Lambda} \cdots \int_{p_n}^{\Lambda} \hat{\delta}^{(D)}(p_1 + \cdots + p_n) g_n(p; \Lambda) \\ &\quad \times \Omega_{i_1}(p_1; \Lambda(\delta t)) \cdots \Omega_{i_n}(p_n; \Lambda(\delta t)), \end{aligned} \quad (\text{B.10})$$

where $p \equiv \{p_1, p_2, \dots, p_n\}$. The difference in effective actions is given by

$$\begin{aligned}
& S[\Omega_{IR}; \Lambda(\delta t)] - S[\Omega_{IR}; \Lambda] \\
&= \sum_n \frac{1}{n!} \int_{p_1}^{\Lambda(\delta t)} \cdots \int_{p_n}^{\Lambda(\delta t)} \hat{\delta}^{(D)}(p_1 + \cdots + p_n) [g_n(p; \Lambda(\delta t)) - g_n(p; \Lambda)] \\
&\quad \times \Omega_{i_1}(p_1; \Lambda(\delta t)) \cdots \Omega_{i_n}(p_n; \Lambda(\delta t)) \\
&= \sum_n \frac{1}{n!} \int_{p_1}^{\Lambda(\delta t)} \cdots \int_{p_n}^{\Lambda(\delta t)} \hat{\delta}^{(D)}(p_1 + \cdots + p_n) \left[-(\delta t) \Lambda \frac{\partial g_n(p; \Lambda)}{\partial \Lambda} \right] \\
&\quad \times \Omega_{i_1}(p_1; \Lambda(\delta t)) \cdots \Omega_{i_n}(p_n; \Lambda(\delta t)). \tag{B.11}
\end{aligned}$$

As mentioned above, we can write the difference in terms of the coupling constant differentiated with respect to Λ . We define $p = \Lambda \hat{p}$. When this coupling depends on the momentum explicitly, it is written as

$$\begin{aligned}
\delta g(\Lambda \hat{p}; \Lambda) &= \frac{\partial g(p; \Lambda)}{\partial \Lambda} \delta \Lambda + \sum_{i=1}^n \frac{\partial g(\Lambda \hat{p}; \Lambda)}{\partial \hat{p}_i} \delta \hat{p}_i \\
&= -\frac{\partial g(p; \Lambda)}{\partial \Lambda} \Lambda \delta t + \sum_{i=1}^n \frac{\partial g(\Lambda \hat{p}; \Lambda)}{\partial \hat{p}_i} \hat{p}_i \delta t. \tag{B.12}
\end{aligned}$$

Therefore, we obtain

$$\Lambda \frac{\partial}{\partial \Lambda} g(p; \Lambda) = \Lambda \frac{d}{d\Lambda} g(p; \Lambda) - \sum_{i=1}^n \hat{p}_i \frac{\partial}{\partial \hat{p}_i} g(p; \Lambda). \tag{B.13}$$

The dimensions of fields become $d_\Omega - \gamma$ as a consequence of the quantum effect. Using a dimensionless coupling $\hat{g}$, we write $g(\Lambda) = \Lambda^{[g]} \hat{g}(\Lambda)$, where $[g](= d - \sum_{\Omega_i} (d_{\Omega_i} - \gamma))$ is the dimensions of g ; then

$$\begin{aligned}
\Lambda \frac{\partial}{\partial \Lambda} g(\Lambda) &= \Lambda^{[g]} \Lambda \frac{d}{d\Lambda} \hat{g}(\Lambda) + [g] g(\Lambda) - \sum_{i=1}^n \hat{p}_i \frac{\partial}{\partial \hat{p}_i} g(\Lambda) \\
&= \Lambda^{[g]} \Lambda \frac{d}{d\Lambda} \hat{g}(\Lambda) + \left\{ d - \sum_{\Omega_i} (d_{\Omega_i} - \gamma) \right\} g(\Lambda) - \sum_{i=1}^n \hat{p}_i \frac{\partial}{\partial \hat{p}_i} g(\Lambda) \tag{B.14}
\end{aligned}$$

Substituting (B.14) into (B.11), and writing them in terms of the action S , we obtain

$$\begin{aligned}
& S[\Omega_{IR}; \Lambda(\delta t)] - S[\Omega_{IR}; \Lambda] \\
&= \delta t \left\{ -\Lambda \frac{d}{d\Lambda} S - dS + \int_p \Omega_p^i \left(d_\Omega - \gamma + \hat{p}^\mu \frac{\partial'}{\partial \hat{p}^\mu} \right) \frac{\delta}{\delta \Omega_p^i} S \right\}. \tag{B.15}
\end{aligned}$$

The term $\Lambda \frac{d}{d\Lambda} S$ denotes $\sum_n \frac{1}{n!} \int_{p_1} \cdots \int_{p_n} \hat{\delta}^{(D)}(p_1 + \cdots + p_n) \Lambda \frac{d}{d\Lambda} \hat{g}_{i_1, \dots, i_n} \hat{\Omega}_{i_1} \cdots \hat{\Omega}_{i_n}$, where $\hat{\Omega}$ is a dimensionless field. The operator $\int_p \Omega_p^i (d_\Omega - \gamma) \frac{\delta}{\delta \Omega_p^i}$ of the third term on the R.H.S. counts the degrees of powers of the fields multiplied by $(d_\Omega - \gamma)$ for each term. The operator $\int_p \Omega_p^i \hat{p}^\mu \frac{\partial'}{\partial \hat{p}^\mu} \frac{\delta}{\delta \Omega_p^i}$ of the third term on the R.H.S. counts the number of derivative operators for each term. The prime mark of ∂' denotes that the operator does not operate on momentum arguments of delta functions or step functions but only on the coupling constants. Comparing Eq. (B.8) with Eq. (B.15), we finally obtain the Wegner-Houghton equation,

$$\begin{aligned} \Lambda \frac{d}{d\Lambda} S &= -\frac{1}{2\delta t} \left\{ \text{tr} \ln \left(\frac{\delta^2 S}{\delta \Omega \delta \Omega} \right) - \frac{\delta S}{\delta \Omega} \left(\frac{\delta^2 S}{\delta \Omega \delta \Omega} \right)^{-1} \frac{\delta S}{\delta \Omega} \right\} \\ &\quad - dS + \int_p \Omega_p^i \left(d_\Omega - \gamma + \hat{p}^\mu \frac{\partial'}{\partial \hat{p}^\mu} \right) \frac{\delta}{\delta \Omega_p^i} S. \end{aligned} \quad (\text{B.16})$$

In addition, we extend the equation to the one for the Lifshitz-type theory by replacing the $-dS$ and $\hat{p}^\mu \frac{\partial'}{\partial \hat{p}^\mu}$ with $-(D+z)$ and $z\hat{p}^0 \frac{\partial'}{\partial \hat{p}^0} + \hat{p}^i \frac{\partial'}{\partial \hat{p}^i}$ respectively. Thus we obtain the extended Wegner-Houghton equation for the Lifshitz-type theory.

$$\begin{aligned} \Lambda \frac{d}{d\Lambda} S &= -\frac{1}{2\delta t} \left\{ \text{tr} \ln \left(\frac{\delta^2 S}{\delta \Omega \delta \Omega} \right) - \frac{\delta S}{\delta \Omega} \left(\frac{\delta^2 S}{\delta \Omega \delta \Omega} \right)^{-1} \frac{\delta S}{\delta \Omega} \right\} \\ &\quad - (D+z)S + \int_p \Omega_p^i \left(d_\Omega - \gamma + z\hat{p}^0 \frac{\partial'}{\partial \hat{p}^0} + \hat{p}^i \frac{\partial'}{\partial \hat{p}^i} \right) \frac{\delta}{\delta \Omega_p^i} S. \end{aligned} \quad (\text{B.17})$$

There is room, however, how to choose the cut-off function in the Lifshitz-type theory.

C Notation in Part. II

We use the following notation. The definition of the covariant derivative and the gauge field strength are

$$\mathcal{D}_m \cdot \equiv \partial_m \cdot + \frac{i}{2} [v_m, \cdot], \quad (\text{C.1})$$

$$v_{mn} \equiv \partial_m v_n - \partial_n v_m + \frac{i}{2} [v_m, v_n]. \quad (\text{C.2})$$

respectively. The differential operators D and $\bar{D}$ are

$$D_\alpha(x) = \frac{\partial}{\partial \theta^\alpha} + i(\sigma^m \bar{\theta})_\alpha \partial_m, \quad (\text{C.3})$$

$$\bar{D}_{\dot{\alpha}}(x) = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i(\theta \bar{\sigma}^m)_{\dot{\alpha}} \partial_m, \quad (\text{C.4})$$

respectively. We introduce y and $y^\dagger$ as

$$y^m = x^m + i\theta \sigma^m \bar{\theta}, \quad (\text{C.5})$$

$$y^{\dagger m} = x^m - i\theta \sigma^m \bar{\theta}. \quad (\text{C.6})$$

respectively. For the sake of ease, we give D and $\bar{D}$ in terms of $(y, \theta, \bar{\theta})$ or $(y^\dagger, \theta, \bar{\theta})$ coordinates as

$$D_\alpha(y, \theta, \bar{\theta}) = \frac{\partial}{\partial \theta^\alpha} + 2i(\sigma^m \bar{\theta})_\alpha \frac{\partial}{\partial y^m}, \quad (\text{C.7})$$

$$\bar{D}_{\dot{\alpha}}(y, \theta, \bar{\theta}) = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}}, \quad (\text{C.8})$$

$$D_\alpha(y^\dagger, \theta, \bar{\theta}) = \frac{\partial}{\partial \theta^\alpha}, \quad (\text{C.9})$$

$$\bar{D}_{\dot{\alpha}}(y^\dagger, \theta, \bar{\theta}) = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - 2i(\theta \bar{\sigma}^m)_{\dot{\alpha}} \frac{\partial}{\partial y^{\dagger m}}. \quad (\text{C.10})$$

D Solution of Gradient Flow Equation

This appendix is review of Ref. [33]. We solve the gradient flow iteratively. R.H.S. of Eq. (7.1) is split into the linear part of the B field and the non-linear part R . We obtain

$$\partial_t B_\mu = \partial_\nu \partial_\nu B_\mu + (\alpha_0 - 1) \partial_\mu \partial_\nu B_\nu + R_\mu, \quad (\text{D.1})$$

$$\begin{aligned} R_\mu &= 2[B_\nu, \partial_\nu B_\mu] - [B_\nu, \partial_\mu B_\nu] \\ &\quad + (\alpha_0 - 1)[B_\mu, \partial_\nu B_\mu] + [B_\nu, [B_\nu, B_\mu]]. \end{aligned} \quad (\text{D.2})$$

Then the general solution of (7.1) is

$$B_\mu(t, x) = \int d^D y \left\{ K_t(x - y)_{\mu\nu} A_\nu(y) + \int_0^t ds K_{t-s}(x - y)_{\mu\nu} R_\nu(s, y) \right\}, \quad (\text{D.3})$$

where K is a heat kernel,

$$K_t(z)_{\mu\nu} = \int_p \frac{e^{ipz}}{p^2} \{(\delta_{\mu\nu}p^2 - p_\mu p_\nu)e^{-tp^2} + p_\mu p_\nu e^{-\alpha_0 tp^2}\}. \quad (\text{D.4})$$

Here we define Fourier transformation of the B fields as

$$B_\mu(t, x) = \int_t e^{ipx} \tilde{B}_\mu(t, p). \quad (\text{D.5})$$

Then (D.3) is described by

$$\tilde{B}_\mu(t, p) = \tilde{K}_t(p)_{\mu\nu} \tilde{A}_\nu(p) + \int_0^t ds \tilde{K}_{t-s}(p)_{\mu\nu} \tilde{R}_\nu(s, p). \quad (\text{D.6})$$

Noting that the R is constructed of at most third order in B , the R is described in momentum space by

$$\begin{aligned} \tilde{R}_\mu^a(t, p) &= \sum_{n=2}^3 \frac{1}{n!} \int_{q_1} \cdots \int_{q_n} (2\pi)^D \delta(p + q_1 + \cdots + q_n) \\ &\quad \times X^{n,0}(p, q_1, \cdots, q_n)_{\mu\nu_1 \cdots \nu_n}^{ab_1 \cdots b_n} \tilde{B}_{\nu_1}^{b_1}(t, -q_1) \cdots \tilde{B}_{\nu_n}^{b_n}(t, -q_n) \end{aligned} \quad (\text{D.7})$$

where $X^{2,0}, X^{3,0}$ are vertex operators. They are coefficients which come from the Fourier expansion. Substituting Eq. (D.7) into Eq. (D.6) iteratively, we obtain the B field which is the solution of the gradient flow as

$$\begin{aligned} \tilde{B}_\mu^a(t, p) &= \tilde{K}_t(p)_{\mu\nu} + \frac{1}{2} \int_0^t ds \tilde{K}_{t-s}(p)_{\mu\nu} \int_q \int_r (2\pi)^D \delta(p - q - r) \\ &\quad \times X^{2,0}(p, -q, -r)_{\mu\nu\rho}^{abc} \tilde{K}_s(q)_{\rho\sigma} \tilde{K}_s(r)_{\sigma\tau} \tilde{A}_\sigma^b(q) \tilde{A}_\tau^c(r) + \cdots \end{aligned} \quad (\text{D.8})$$

Here $X^{2,0}$ and $X^{3,0}$ defined through Eq. (D.7) are described concretely by

$$\begin{aligned} X^{2,0}(p, q, r)_{\mu\nu\rho}^{abc} &= if^{abc} \{ (r - q)_\mu \delta_{\mu\rho} + 2q_\rho \delta_{\mu\nu} - 2r_\nu \delta_{\mu\rho} \\ &\quad + (\alpha_0 - 1)(q_\nu \delta_{\mu\rho} - r_\rho \delta_{\mu\nu}) \} \end{aligned} \quad (\text{D.9})$$

$$\begin{aligned} X^{3,0}(p, q, r, s)_{\mu\nu\rho\sigma}^{abcd} &= f^{abe} f^{cde} (\delta_{\mu\sigma} \delta_{\nu\rho} - \delta_{\mu\rho} \delta_{\sigma\nu}) + f^{ade} f^{bce} (\delta_{\mu\rho} \delta_{\sigma\nu} - \delta_{\mu\nu} \delta_{\rho\sigma}) \\ &\quad + f^{ace} f^{dbe} (\delta_{\mu\nu} \delta_{\rho\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho}), \end{aligned} \quad (\text{D.10})$$

respectively. Thus the B field is described by the expansion of the A field.

E Short Summary of Supersymmetry

E.1 Definition

This is review of [34]. The chiral superfield is defined by

$$\bar{D}_\alpha \Psi = 0. \quad (\text{E.1})$$

We described chiral multiplet $\Psi = \{A, \psi, F\}$ in terms of $(x, \theta, \bar{\theta})$ coordinates as

$$\begin{aligned} \Psi(x, \theta, \bar{\theta}) = & A + i\theta\sigma^m\bar{\theta}\partial_m A + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\square A \\ & \sqrt{2}\theta\psi - \frac{i}{\sqrt{2}}\theta\theta\partial_m\psi\sigma^m\bar{\theta} + \theta\theta F \end{aligned} \quad (\text{E.2})$$

The vector superfield is defined by

$$V = V^\dagger. \quad (\text{E.3})$$

We described vector multiplet $V = \{C, X, \bar{X}, M, M^*, V_m, \Lambda, \bar{\Lambda}, D\}$ in terms of $(x, \theta, \bar{\theta})$ coordinates as

$$\begin{aligned} V(x, \theta, \bar{\theta}) = & C + i\theta X - i\bar{\theta}\bar{X} + \frac{i}{2}\theta\theta M - \frac{i}{2}\bar{\theta}\bar{\theta}M^* \\ & -\theta\sigma^m\bar{\theta}V_m + i\theta\theta\bar{\theta}[\bar{\Lambda} + \frac{i}{2}\bar{\sigma}^m\partial_m X] \\ & -i\bar{\theta}\bar{\theta}\theta[\Lambda + \frac{i}{2}\sigma^m\partial_m\bar{X}] + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}[D + \frac{1}{2}\square C]. \end{aligned} \quad (\text{E.4})$$

E.2 Wess-Zumino Gauge

The Super gauge transformation is defined by

$$V' = V + \Psi + \Psi^\dagger. \quad (\text{E.5})$$

Under this transformation, the each component of the vector multiplet transforms as follows:

$$C' = C + A + A^* \quad (\text{E.6})$$

$$X' = X - i\sqrt{2}\psi \quad (\text{E.7})$$

$$M' = M - 2iF \quad (\text{E.8})$$

$$V'_m = V_m - i\partial_m(A - A^*) \quad (\text{E.9})$$

$$\Lambda' = \Lambda \quad (\text{E.10})$$

$$D' = D \quad (\text{E.11})$$

Using this gauge transformation, we fixed the WZ gauge, which is $C, X, M = 0$. Under this gauge, V is described in terms $(x, \theta, \bar{\theta})$ coordinates as

$$V(x, \theta, \bar{\theta}) = -\theta\sigma^m\bar{\theta}V_m + i\theta\theta\bar{\theta}\bar{\Lambda} - i\bar{\theta}\bar{\theta}\Lambda + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D, \quad (\text{E.12})$$

$$V^2(x, \theta, \bar{\theta}) = -\frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}V_mV^m, \quad (\text{E.13})$$

$$V^3(x, \theta, \bar{\theta}) = 0. \quad (\text{E.14})$$

And V is also described in terms $(y, \theta, \bar{\theta})$ coordinate as

$$\begin{aligned} V(y, \theta, \bar{\theta}) &= -\theta\sigma^m\bar{\theta}V_m + i\theta\theta\bar{\theta}\bar{\Lambda} - i\bar{\theta}\bar{\theta}\Lambda \\ &\quad + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}[D - i\partial_mV^m], \end{aligned} \quad (\text{E.15})$$

$$V^2(y, \theta, \bar{\theta}) = -\frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}V_mV^m, \quad (\text{E.16})$$

$$V^3(y, \theta, \bar{\theta}) = 0. \quad (\text{E.17})$$

F Derivation of Gauge Covariant Term

The super Yang-Mills action is given as

$$S = \int d^4x \int d^2\theta \text{Tr}[W^\alpha W_\alpha] + h.c. \quad (\text{F.1})$$

$$= - \int d^8z \text{Tr}[e^{-V}(D^\alpha e^V)W_\alpha] + h.c., \quad (\text{F.2})$$

When we make a variation over the V field, we obtain

$$\frac{\delta S}{\delta V} = \int d^8 z \text{Tr} \left[\frac{\delta}{\delta V} \{ e^{-V} (D^\alpha e^V) W_\alpha \} \right] + h.c. \quad (\text{F.3})$$

$$= 2 \int d^8 z \text{Tr} \left[\frac{\delta e^V}{\delta V} \{ (D^\alpha W_\alpha) e^{-V} + W^\alpha (D_\alpha e^{-V}) \} \right. \\ \left. - \frac{\delta e^{-V}}{\delta V} (D^\alpha e^V) W_\alpha \right] + h.c. \quad (\text{F.4})$$

$$= \frac{e^{L_V} - 1}{L_V} \cdot (D^\alpha W_\alpha + \{ e^{-V} D^\alpha e^V, W_\alpha \}) + h.c., \quad (\text{F.5})$$

where the normalization condition,

$$\text{Tr}[T^a T^b] = \frac{1}{2} \delta^{ab} \quad (\text{F.6})$$

is imposed. Here we used the useful formulae as

$$\delta(e^V) = e^V \left[\frac{1 - e^{-L_V}}{L_V} \cdot \delta V \right] \quad (\text{F.7})$$

$$= \left[\frac{e^{L_V} - 1}{L_V} \cdot \delta V \right] e^V, \quad (\text{F.8})$$

$$\delta(e^{-V}) = e^{-V} \left[\frac{1 - e^{L_V}}{L_V} \cdot \delta V \right] \quad (\text{F.9})$$

$$= \left[\frac{e^{-L_V} - 1}{L_V} \cdot \delta V \right] e^{-V}. \quad (\text{F.10})$$

For example, we give the proof of Eq. (F.7). We define $\delta\chi(t)$ as

$$\delta\chi(t) \equiv e^{-tV} e^{t(V+\delta V)} - 1 \quad (\text{F.11})$$

$$= e^{-tV} \delta(e^{tV}), \quad (\text{F.12})$$

and we differentiate Eq. (F.11) in respect to t . We obtain

$$\delta\dot{\chi}(t) = -[V, \delta\chi(t)] + \delta V. \quad (\text{F.13})$$

The solution of Eq. (F.13) is given as

$$\delta\chi(t) = \left(\frac{1 - e^{-tL_V}}{L_V} \right) \cdot \delta V. \quad (\text{F.14})$$

When we set $t = 1$ in Eqs (F.12) and (F.13), we obtain Eq. (F.7). Other equations also can be proved in a similar way or using the formula

$$e^V X e^{-V} = e^{L_V} X \quad (\text{F.15})$$

where X is any function.

G Expansion of Equation (10.8) with Component Fields

For the convenience of the expansion of (10.8) with the component fields, we give useful methods and formulae.

G.1 Coordinate Transformation

To clarify the covariance under the residual super gauge transformation which means the transformation restricted under WZ gauge, it is useful to calculate w_α in terms of $(y, \theta, \bar{\theta})$ coordinates. We obtain w_α as

$$w_\alpha(y, \theta, \bar{\theta}) = -\bar{D}^2(e^{-v}D_\alpha e^v) \quad (\text{G.1})$$

$$\begin{aligned} &= -4i\lambda_\alpha + 4\theta_\alpha d - 2i(\sigma^m \bar{\sigma}^k \theta)_\alpha v_{mk} \\ &\quad + 4\theta\theta\{\sigma^m \mathcal{D}_m \bar{\lambda}\}_\alpha. \end{aligned} \quad (\text{G.2})$$

Using the expansion formula,

$$f(y, \theta, \bar{\theta}) = f(x) + i\theta\sigma^m \bar{\theta} \partial_m f(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\square f(x), \quad (\text{G.3})$$

and

$$f(x, \theta, \bar{\theta}) = f(y) - i\theta\sigma^m \bar{\theta} \partial_m f(y) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\square f(y), \quad (\text{G.4})$$

we always rewrite the results in the $(y, \theta, \bar{\theta})$ coordinate or $(x, \theta, \bar{\theta})$ either. For example,

$$\begin{aligned} w_\alpha(x, \theta, \bar{\theta}) &= -4i\lambda_\alpha + 4\theta_\alpha d - 2i(\sigma^m \bar{\sigma}^k \theta)_\alpha v_{mk} \\ &\quad + 4\theta\theta\{\sigma^m \mathcal{D}_m \bar{\lambda}\}_\alpha + 4(\theta\sigma^m \bar{\theta})\partial_m \lambda_\alpha \\ &\quad + 2\theta\theta(\sigma^m \bar{\theta})_\alpha \{-i\partial_m d + \partial_m \partial^k v_k - \square v_m\} \\ &\quad + \frac{i}{2}\theta\theta(\sigma^m \bar{\sigma}^k \sigma^l \bar{\theta})_\alpha \partial_l [v_k, v_m] - i\theta\theta\bar{\theta}\bar{\theta}\square \lambda_\alpha. \end{aligned} \quad (\text{G.5})$$

Note that they are not covariant under the super gauge transformation, because we take the WZ gauge fixing. Using (C.7), we obtain the result of calculation of $D^\alpha w_\alpha$ which is first term of the R.H.S of (10.8) as

$$\begin{aligned} D^\alpha w_\alpha(y, \theta, \bar{\theta}) &= -8d + 8\theta\sigma^m \mathcal{D}_m \bar{\lambda} - 8\bar{\theta}\bar{\sigma}^m \partial_m \lambda \\ &\quad - 8i(\bar{\theta}\bar{\sigma}^m \theta)\partial_m d - 4(\bar{\theta}\bar{\sigma}^l \sigma^m \bar{\sigma}^k \theta)\partial_l v_{mk} \\ &\quad - 8i\theta\theta\{\bar{\theta}\bar{\sigma}^l \sigma^m \partial_l \mathcal{D}_m \bar{\lambda}\}. \end{aligned} \quad (\text{G.6})$$

G.2 Useful Formulae

We also give useful formulae to obtain the second term with component field of R.H.S. of (10.8) in terms of $(y, \theta, \bar{\theta})$ coordinates as

$$\begin{aligned}
e^{-v} D^\alpha e^v(y, \theta, \bar{\theta}) &= (\bar{\theta} \bar{\sigma}^m)^\alpha v_m + 2i\theta^\alpha \bar{\theta} \bar{\lambda} - i\bar{\theta} \bar{\theta} \lambda^\alpha \\
&\quad + \bar{\theta} \bar{\theta} (\theta^\alpha d - \frac{i}{2} (\theta \sigma^m \bar{\sigma}^k)^\alpha v_{km}) \\
&\quad - \theta \bar{\theta} \bar{\theta} \bar{\theta} \mathcal{D}_m (\bar{\lambda} \bar{\sigma}^m)^\alpha.
\end{aligned} \tag{G.7}$$

Finally we obtain the A in terms of $(y, \theta, \bar{\theta})$ coordinates as

$$\begin{aligned}
&(D^\alpha w_\alpha + \{e^{-v} D^\alpha e^v, w_\alpha\})(y, \theta, \bar{\theta}) \\
&= -8d + 8\theta \sigma^m \mathcal{D}_m \bar{\lambda} - 8\bar{\theta} \bar{\sigma}^m \mathcal{D}_m \lambda \\
&\quad + 8[\bar{\theta} \bar{\lambda}, \theta \lambda] - 8i(\bar{\theta} \bar{\sigma}^m \theta) \mathcal{D}_m d \\
&\quad + 4(\theta \sigma^k \bar{\sigma}^m \sigma^l \bar{\theta}) \mathcal{D}_l v_{mk} \\
&\quad - 8i\theta \bar{\theta} (\bar{\theta} \bar{\sigma}^l \sigma^m \mathcal{D}_l \mathcal{D}_m \bar{\lambda}) \\
&\quad + 8i\theta \bar{\theta} [\bar{\theta} \bar{\lambda}, d].
\end{aligned} \tag{G.8}$$

The $A^\dagger$ in terms of $(y^\dagger, \theta, \bar{\theta})$ coordinates is

$$\begin{aligned}
&(D^\alpha w_\alpha + \{e^{-v} D^\alpha e^v, w_\alpha\})^\dagger(y^\dagger, \theta, \bar{\theta}) \\
&= -8d - 8\bar{\theta} \bar{\sigma}^m \mathcal{D}_m \lambda + 8\theta \sigma^m \mathcal{D}_m \bar{\lambda} \\
&\quad + 8[\bar{\theta} \bar{\lambda}, \theta \lambda] + 8i(\bar{\theta} \bar{\sigma}^m \theta) \mathcal{D}_m d \\
&\quad + 4(\theta \sigma^l \bar{\sigma}^m \sigma^k \bar{\theta}) \mathcal{D}_l v_{mk} \\
&\quad + 8i\bar{\theta} \bar{\theta} (\mathcal{D}_l \mathcal{D}_m \lambda \sigma^m \bar{\sigma}^l \theta) \\
&\quad + 8i\bar{\theta} \bar{\theta} [\lambda(y^\dagger) \theta, d].
\end{aligned} \tag{G.9}$$

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