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Neutrino reactions of two-nucleon system in core-collapse supernova

Shota Nasu

Graduate School of Science,
Department of Physics, Osaka University

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Abstract

Neutrino emission from electron/positron capture on the deuteron and the nucleon-nucleon fusion processes in the surface region of a supernova core are studied. These weak processes are evaluated in the standard nuclear physics approach, which consists of one-nucleon and two-nucleon-exchange currents and nuclear wave functions generated by a high precision nucleon-nucleon potential. In addition to the cross sections for these processes involving the deuteron, we present neutrino emissivities due to these processes calculated for typical profiles of core-collapsed supernovae. These novel neutrino emissivities are compared with the standard neutrino emission mechanisms. We find that the neutrino emissivity due to the electron capture on the deuteron is comparable to that on the proton in the deuteron abundant region. The implications of the new channels involving deuteron for the supernova mechanism are discussed.

The influences of the nuclear medium on the neutrino emissivity are studied for the electron capture on the deuteron. The medium effects are investigated based on the thermodynamic Green's function and T-matrix approach, using a simple model which consist of only the one-nucleon currents and the S-wave separable nucleon-nucleon potential. By considering the medium effects for two-nucleon system through the Pauli blocking and the self-energy, we find that the neutrino emissivity is moderately reduced. In the outer region $r > 30\text{km}$, the medium effect is negligible, however in the internal region $r < 12\text{km}$ reduction factor is about 40% within the current approximate treatment of medium effect.

Chapter 1

Introduction

The neutrinos play pivotal roles in core-collapse supernovae and the subsequent evolution to neutron stars. Neutrino reactions in dense matter of a supernova core are crucial for understanding the explosion mechanism, which is still elusive despite extensive studies over decades. It is therefore essential to identify all neutrino processes (both neutrino-emission and neutrino-absorption processes) that are important in the supernova environment. Failing to include all the relevant neutrino processes in the supernova modeling may have significant consequences for the theoretical understanding of supernova explosion and related observables. The emission of neutrinos and any additional neutrino emission mechanism so far not considered acts as a cooling mechanism of the proto-neutron star could increase the neutrino flux. A portion of the emitted neutrinos are subsequently absorbed by the material behind the shockwave and these act as a heating agent. Additional sources of neutrino flux could increase this neutrino heating mechanism and may help the revival of a stalled shockwave and lead to a successful supernova explosion [1, 2, 3]. These neutrinos which are emitted gradually (~ 20 s) from a nascent neutron star (proto-neutron star) in a supernova explosion, could be detected as supernova neutrinos at the terrestrial neutrino detector like in the case of SN1987A [4], and would be helpful in establishing the neutrino emission mechanisms.

Recent calculations have shown that deuterons, tritons and ^3He can appear copiously in the regions between the supernova core and the shockwave [5, 6, 7]. These light elements have so far not been included in the tables of equation of state (EOS) [8, 9, 10] that are routinely used in supernova simulations where the nuclear species are limited to the proton, neutron, ^4He and one “representative heavy nucleus” that is assumed to simulate the roles

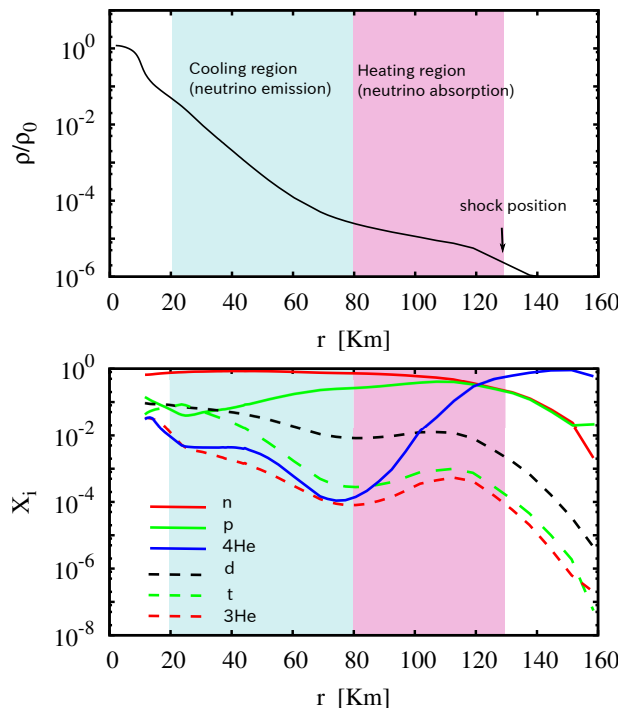


Figure 1.1: The density(top panel) and mass fraction X_i (bottom panel) at 150 ms after the core bounce taken from [5]. The horizontal axis, r , is the distance from the supernova center and ρ_0 is density of normal nuclear matter.

of all heavy nuclei. The light elements with mass number $A = 2$ and 3 can be abundant in hot and moderately dense matter ($< 10^{13} \text{ g/cm}^3$) under nuclear statistical equilibrium [5, 6, 7] and should be considered in studying the supernova mechanism. They appear in the heating region behind a shockwave, and also in the cooling region at the surface of a proto-neutron star; their appearance gives a new contribution to the neutrino opacity. As an example, the distributions of matter density(ρ) and the mass fraction X_i at 150 msec after the core bounce in Fig. 1.1 [5]. In this particular snapshot of the profile of supernova explosion, the shock wave is stalled around 130km from the center. The neutrino heating region due to neutrino absorption is from $r \sim 80\text{km}$ to behind the shock wave, while the neutrinos are emitted(cooling region) mainly $20 < r < 80\text{km}$. Neutrino processes in this cooling region are essential for determining the flux and spectra of emitted neutrinos, which in turn affect the efficiency of neutrino heating behind the shockwave. In the neutrino-emitting cooling region, the deuteron mass fraction amounts to about 10%, where the density is about $10^{-4} \sim 10^{-1}$ of nuclear matter density $\rho_0 \sim 2.8 \times 10^{14} \text{ [g/cm}^3\text{]}$.

The importance of the neutrino absorption reactions on light nuclei has first been pointed out by Haxton. He studied the neutrino- ^4He cross section [11] and its effect on spherical supernova simulations [12] however it has little impacts on the shock wave revival. Recently numerical simulations of supernova explosion has been performed [13] with neutrino- ^4He cross sections obtained with an advanced calculation [14, 15], however, they found small effect on the delayed explosion mechanism. Thermal averaged neutral-current cross sections and energy transfer cross sections of neutrino-triton and neutrino- ^3He are studied in [16]. They pointed out a larger influence of these processes on supernova mechanism, compared to the ^4He reactions. Arcones et al. [6] studied the cross sections for charged-current and neutral-current neutrino reactions on tritons and ^3He to evaluate their influences on the neutrino spectra at the outer layer of a proto-neutron star. The neutrino absorptions on deuterons was investigated by Nakamura et al. [17] as an additional heating mechanism on top of the neutrino reactions on nucleons and ^4He , and found that the energy transfer cross sections of the deuteron is larger than those of triton, ^3He and ^4He . It was shown recently by Furusawa et al. [18] that the neutrino absorption reactions on light nuclei indeed play important role on the shock wave expansion.

According to [5, 6], the deuteron fraction can be larger than the proton fraction in part of the neutrino-sphere region between the shock wave and the surface of the proto-neutron star. This indicates that weak-interaction deuteron breakup may play a significant role in neutrino emission processes, possibly altering the conventional understanding of the role of the protons in the neutrino-emission processes as well as the neutronization of matter. An additional deuteron formation in nucleon-nucleon scattering also leads to neutrino emission. Although both these neutrino emission processes certainly exist on top of the conventional neutrino-emission processes, they have so far not been considered in supernova simulations. The purpose of this thesis is to study neutrino emissions via deuteron breakup/formation (to be referred to as NEvDBF) in the surface region of a proto-neutron star, where the neutrino emissions act as a cooling mechanism. We examine the role of NEvDBF for profile of matter (density, temperature and mass fractions of light elements) using the results of [5] as one of the 'realistic' cases and provide the neutrino emission rates, which are basic quantities to be used in supernova simulation to account both neutrino emission and absorption reactions due to two nucleon system. In part I of the thesis, we present the first evaluation of the neutrino emissivities from electron/positron-capture

on the deuteron and from the nucleon-nucleon weak fusion processes; see (2.0.1)-(2.0.5) below. These neutrino emissivities arising from NEvDBF will be compared with those coming from the “conventional” processes. The neutrino emission processes that have been previously considered in the literature shall be referred to as *conventional processes*; They are listed in (2.0.7)-(2.0.11) below. We further will discuss the possible influences of NEvDBF on neutrino emission in the supernova environment. The reaction rates reported here will supplement the conventional ones and are expected to be useful for the numerical simulation of supernova explosion and proto-neutron star cooling. This part of thesis is based on our arXiv preprint [19]

Theoretical treatments of electroweak processes in two-nucleon systems have been well studied. For low-energy neutrino-deuteron reactions, serious efforts to minimize the theoretical uncertainty have been made in order to analyze data from the Sudbury Neutrino Observatory [20, 21]. One approach to this problem is the standard nuclear physics approach (SNPA) that involves nuclear wave functions that are derived from high-precision phenomenological nuclear potentials, and one-nucleon and two-nucleon electroweak currents. This method has been well tested by analyses of photo-reactions, electron scattering, and muon capture on the two-nucleon systems [22, 23, 24]. Another theoretical approach, effective field theory (EFT) for few-nucleon systems, has been developed and applied to low-energy electroweak processes [25]. Both methods essentially agree with each other for low-energy electroweak processes in the two-nucleon systems. The pp -fusion process, $pp \rightarrow de^- \nu_e$, is one of such processes relevant to this work. The pp -fusion reaction has been studied with both SNPA and EFT, and good agreement between the two methods has been found [26, 27]. Another nucleon-nucleon fusion process relevant to this work is neutron-neutron fusion, which was previously studied with EFT [28]. In this work, we need reaction rates for a relatively high energy region that is beyond the applicability of EFT consisting of the nucleon and pion only. We therefore adopt SNPA in the present work.

In part I of our thesis we analyze neutrino reactions in two-nucleon system without taking into account the modifications of reaction in nuclear medium. The matter density may vary over very wide-range from the very low density to the nuclear saturation density during the supernova explosion process. A natural question is at which matter density we can safely use the results obtained in part I and when we have to start to

worry about the effects of nuclear medium. The second purpose of the thesis in part II is that we formulate a method to study the neutrino reactions in nuclear medium and give an estimation of the neutrino reaction rates in the cooling and heating region where nuclear density is around $\rho < 0.1\rho_0$. Light cluster abundances in supernova explosion were investigated in the so-called the generalized Beth-Uhlenbeck approach in which total baryon density are decomposed into A-nucleon components [5]. In estimating the light cluster abundance, the energy shift for A-nucleon quasi-particle bound state and continuum state are considered. The energy shifts are due to the self-energy shift, the Pauli blocking term and the perturbative Coulomb correction. Besides this, the two-nucleon correlations, such as the deuteron binding energy and the scattering phase shifts were also studied with the generalized Beth-Uhlenbeck approach in a hot iso-symmetric nuclear matter [29, 30]. At a density near a critical value (Mott density), the two-nucleon clustering is strongly suppressed by the Pauli blocking and the quasi-particle energy shift. In [5], a finite temperature nuclear medium has studied using the thermodynamic imaginary-time Green's function formalism with the quasi-particle approximation. There have been many works on finite temperature nuclear medium based on the thermodynamic imaginary-time Green's function formalism with the ladder T-matrix approach [29, 31, 32]. Alternatively, the real-time Green's function formalism has been applied to compact star, which was reviewed in [33], recently. The thermodynamic Green's function formalism has also been applied to reactions in nuclear medium. For example, the nucleon-nucleon scattering with the quasi-particle picture has been studied in [34]. Also the neutrino emissivity for the modified URCA and neutron-neutron bremsstrahlung processes has been worked out with an approximation that the two-nucleon relative and center-of-mass momenta are set to the Fermi momentum for a certain profile of a neutron star matter [35]. They found the drastic reductions of emissivities in a high density because of low momentum exclusions by the Pauli blocking.

In part II of this thesis, we employ the thermodynamic imaginary-time Green's function and ladder T-matrix approach with the quasi-particle approximation. To simplify the analysis, we consider only S-wave of two-nucleon system, and adopt a single-term separable NN potential to describe the interaction. Although we investigate the medium effects within a simple setup, it can reveal to which extent our analysis on neutrino reactions without including in-medium effects works in the supernova environments.

This thesis is arranged as follows. In Part I we discuss neutrino emission via deuteron breakup or formation. In chapter 2, we discuss a possible role played by neutrino emissions via deuteron breakup/formation (NEvDBF) in the supernova core. The theoretical framework for calculating the cross sections for NEvDBF and the corresponding neutrino emissivities are outlined in chapter 3, and the numerical results are presented in chapters 4.1 and 4.2. The implications of these results for the supernova mechanism are discussed in section 4.2.2 and section 4.2.3, and section 4.2.4 is dedicated to a summary.

The part II are given for the investigations of the in-medium effects for the deuteron reaction. The thermodynamic Green's function method and T-matrix approximation approach are reviewed in chapter 5 and 6, respectively. The quasi-particle approximation, and the resulting nucleon self-energy, bound state and neutrino emissivity in this scheme are discussed in chapter 7. Summary and conclusion of our thesis are given in chapter 8.

Part I

Neutrino reactions, cross section and emissivity

Chapter 2

Neutrino Reaction via Deuteron

We consider the following NEvDBF reactions where the first four reactions occur via the charged-current (CC) whereas the the neutral-current (NC) acts in last one:

$$d + e^- \rightarrow n + n + \nu_e , \quad (2.0.1)$$

$$d + e^+ \rightarrow p + p + \bar{\nu}_e , \quad (2.0.2)$$

$$n + n \rightarrow d + e^- + \bar{\nu}_e , \quad (2.0.3)$$

$$p + p \rightarrow d + e^+ + \nu_e , \quad (2.0.4)$$

$$p + n \rightarrow d + \nu + \bar{\nu} . \quad (2.0.5)$$

It is to be noted that, while the first four reactions concern the emission of ν_e or $\bar{\nu}_e$ only, the last one provides the $\nu\bar{\nu}$ pair-emission of all three flavors. These processes will be compared with the neutrino reactions on the nucleon, which have been used routinely in studying supernovae and neutron stars. Those basic reactions for the neutrino emission are:

$$p + e^- \rightarrow n + \nu_e , \quad (2.0.6)$$

$$n + e^+ \rightarrow p + \bar{\nu}_e , \quad (2.0.7)$$

$$n + n \rightarrow p + n + e^- + \bar{\nu}_e , \quad (2.0.8)$$

$$p + p \rightarrow p + n + e^+ + \nu_e , \quad (2.0.9)$$

$$N + N' \rightarrow N + N' + \nu + \bar{\nu} , \quad (2.0.10)$$

$$e^- + e^+ \rightarrow \nu + \bar{\nu} . \quad (2.0.11)$$

As mentioned, these processes that are commonly considered in the literature are referred to as the *conventional* processes. The reactions (2.0.6) and (2.0.7) describe the e^-/e^+ -captures on the nucleon producing $\nu_e/\bar{\nu}_e$ and are called the direct Urca processes. The reactions (2.0.8) and (2.0.9) are the modified Urca processes where the nucleon-nucleon collisions emit $\bar{\nu}_e$ and ν_e . Note that in the two last reactions a pair of $\nu\bar{\nu}$ of all flavors is produced by what is called the nucleon-nucleon bremsstrahlung process (2.0.10) and by the e^+e^- annihilation (2.0.11) reaction, respectively.

The reactions listed above determine the neutrino emission and the detailed balance between neutrons and protons. The NC reactions produce pairs of $\nu\bar{\nu}$ and act as a cooling mechanism. When the CC reactions are frequent enough, the proton and neutron fractions are determined through the β -equilibrium, $\mu_e = \mu_n - \mu_p + \mu_\nu$. The chemical equilibrium is realized among electrons, positrons, nucleons and neutrinos in the proto-neutron star. At the surface of the proto-neutron star (densities $\sim 10^{11} - 10^{13} \text{g/cm}^3$) where neutrinos are emitted, one has to solve the neutrino transfer equation with detailed rates of the neutrino reactions to determine the neutrino distribution and its evolution associated with the change of composition of matter.

We remark that the reaction rates depend on the degeneracy of leptons and nucleons in the supernova environment. One has to consider their energy distributions in the initial and final states in evaluating these reaction rates. Especially when the leptons and/or nucleons in the matter are degenerate, the reaction rates are significantly suppressed.

2.1 Electron/Positron Capture on Deuteron

The e^- -capture on the deuteron, (2.0.1), instead of the proton, (2.0.6), acts as a source of ν_e *at the surface* of the proto-neutron star if deuterons are abundant. Similarly, the e^+ -capture on the deuteron (2.0.2) produces $\bar{\nu}_e$. In a supernova core, the total number of nucleons is the sum of free nucleons and those bound in nuclei. Thus a supernova simulation that takes into account the light element abundance, one has less number of free nucleons and more nucleons which are bound in light elements. The e^-/e^+ -capture on the free nucleons is a major source of the neutrino emission and the cooling process in a conventional treatment of supernova simulations that do not consider the light element abundance. Therefore, if the capture rates on the deuteron are different from those on the

free nucleon, the neutrino fluxes and the cooling rate will be modified from those based on the conventional supernova simulation. This could lead to a reduction of neutrino luminosities and if so may not only become a negative effect for the neutrino heating mechanism but could also slow down the neutronization speed of proto-neutron star cooling in addition.

Just like the first direct Urca process (2.0.6) the e^- -capture on the deuteron acts as a source of neutronization of proto-neutron star and drives the dense matter toward the neutron-rich side by changing protons into neutrons with neutrino emissions. Similar to the second direct Urca process (2.0.2) the e^+ -capture on the deuteron acts as a counter reaction. In addition, in the matter with trapped neutrinos (density $> 10^{12}\text{g/cm}^3$) the reversed reactions (neutrino absorptions) may take place as well. The balance between neutrons and protons are determined through the quasi-equilibrium and the speed of deleptonization by neutrino emissions.

2.2 Deuteron Formation from Nucleon-Nucleon Scattering

The deuteron formation from two nucleons, (2.0.3), (2.0.4) and (2.0.5), are new sources of neutrinos. They add neutrino flux to the one which is produced in the modified Urca process and the nucleon-nucleon bremsstrahlung. The only difference is that we allow the final two-nucleon states to form deuterons. These new reactions will take place regardless of the abundance of the deuteron in the region outside the proto-neutron star.

The CC processes, (2.0.3) and (2.0.4), occur in addition to the modified Urca processes, (2.0.8) and (2.0.9), and the conventional processes (2.0.6) and (2.0.7), for ν_e ($\bar{\nu}_e$)-emission. They will be part of the reactions which determine the composition under quasi-chemical equilibrium. As is well known, when the proton fraction is small enough in cold neutron stars, the direct Urca process (2.0.6) is hindered and the modified Urca process (2.0.8) is essential for the cooling. The processes with positrons in the initial state are hindered in the cold neutron stars, where electrons are degenerate and positrons are scarce.

In non-degenerate situation when the temperature is high enough, both the two processes, (2.0.3) and (2.0.4) can proceed in the supernova core environment.

The NC process, (2.0.5), is a new channel on top of the conventional nucleon-nucleon bremsstrahlung, (2.0.10). The importance of the nucleon-nucleon bremsstrahlung for the

$\nu\bar{\nu}$ pair emission is well known through studies on cold neutron stars and supernovae. The nucleon-nucleon bremsstrahlung is one of the main sources of cooling of cold neutron stars, and [36] pointed out this process' importance as a dominant source of the ν_μ and ν_τ pair creation in the proto-neutron star cooling. The NC processes are important in the supernova since ν_μ and ν_τ carry away the energy with almost no energy deposition in the heating region [1, 2, 3]. The nucleon-nucleon bremsstrahlung has been routinely implemented in the supernova simulations in addition to the $\nu\bar{\nu}$ production due to the e^-e^+ annihilation. As this paper suggests there is an additional $\nu\bar{\nu}$ producing channel, $NN \rightarrow d\nu\bar{\nu}$, which we will show in the next chapters can be of importance for the cooling of the proto-neutron star.

Chapter 3

The Formulation of the Neutrino Process

3.1 Weak interaction Hamiltonian

The standard low-energy interaction Hamiltonian for the semileptonic weak process is given by the product of the hadron current (J^λ) and the lepton current (L^λ) as

$$H_W^{CC} = \frac{G'_F V_{ud}}{\sqrt{2}} \int d\mathbf{x} [J_\lambda^{CC}(\mathbf{x}) L^\lambda(\mathbf{x}) + \text{h. c.}] , \quad (3.1.1)$$

$$H_W^{NC} = \frac{G'_F}{\sqrt{2}} \int d\mathbf{x} [J_\lambda^{NC}(\mathbf{x}) L^\lambda(\mathbf{x}) + \text{h. c.}] , \quad (3.1.2)$$

for the CC and the NC processes, respectively. The weak coupling constant $G'_F = 1.1803 \times 10^{-5} \text{ GeV}^{-2}$ is taken from [21], and the CKM matrix element $V_{ud} = 0.9740$ is given in [37]. The weak hadronic currents are a combination of a vector current V^λ and an axial vector current A^λ :

$$J_\lambda^{CC} = V_\lambda^\pm - A_\lambda^\pm , \quad (3.1.3)$$

$$J_\lambda^{NC} = (1 - 2 \sin^2 \theta_W) V_\lambda^3 - A_\lambda^3 - 2 \sin^2 \theta_W V_\lambda^s , \quad (3.1.4)$$

where θ_W is the Weinberg angle. The superscript $+$ ($-$) denotes the isospin raising (lowering) operator and the superscript '3' denotes the third component of the isovector current. V_λ^s is iso-scalar vector current. The lepton current L_λ is

$$L_\lambda = \bar{\psi}_l \gamma_\lambda (1 - \gamma_5) \psi_\nu . \quad (3.1.5)$$

The weak nuclear currents consist of one-nucleon [impulse-approximation (IA)] current and two-nucleon meson-exchange currents (MEC).

3.2 Impulse approximation current

The matrix elements of the vector and axial-vector IA currents are written as

$$\langle N(p') | V_\lambda^\pm(0) | N(p) \rangle = \bar{u}(p') \left[f_V \gamma_\lambda + i \frac{f_M}{2M_N} \sigma_{\lambda\rho} q^\rho \right] \tau^\pm u(p) \quad (3.2.1)$$

$$\langle N(p') | A_\lambda^\pm(0) | N(p) \rangle = \bar{u}(p') [f_A \gamma_\lambda \gamma_5 + f_P \gamma_5 q_\lambda] \tau^\pm u(p), \quad (3.2.2)$$

where M_N is the nucleon masses, $q^\mu = p'^\mu - p^\mu$ is the lepton momentum transfer and $\tau^\pm$ is the isospin raising/lowering operator. The third component of the isovector current is given by a replacement of $\tau^\pm$ with $\tau^3/2$. The isoscalar current is given as

$$\langle N(p') | V_\lambda^\pm(0) | N(p) \rangle = \bar{u}(p') \left[f_V \gamma_\lambda + i \frac{f_M^s}{2M_N} \sigma_{\lambda\rho} q^\rho \right] \frac{1}{2} u(p). \quad (3.2.3)$$

These general form of the matrix elements follow from the Lorentz invariance and the parity invariance of the strong interactions. Here the second class currents are dropped. The form factors f_V, f_M, f_A and f_P are the Lorentz invariant functions that depend on the squared momentum transfer q_μ^2 , which are so-called the vector, weak magnetism, axial-vector and pseudo-scalar type form factors, respectively. These are conventionally parametrized as [38, 39]

$$f_V(q_\mu^2) = G_D(q_\mu^2)(1 + \mu_p \eta)(1 + \eta)^{-1} \quad (3.2.4)$$

$$f_M(q_\mu^2) = G_D(q_\mu^2)(\mu_p - \mu_n - 1 - \mu_n \eta)(1 + \eta)^{-1} \quad (3.2.5)$$

$$f_A(q_\mu^2) = g_A G_A(q_\mu^2) \quad (3.2.6)$$

$$f_P(q_\mu^2) = -\frac{2m_N}{m_\pi - q_\mu^2} g_A(q_\mu^2) \quad (3.2.7)$$

$$f_M^s(q_\mu^2) = G_D(q_\mu^2)(\mu_p - \mu_n - 1 + \mu_n \eta)(1 + \eta)^{-1}, \quad (3.2.8)$$

with

$$G_D(q_\mu^2) = \left(1 - \frac{q_\mu^2}{0.71 \text{GeV}^2} \right)^{-2} \quad (3.2.9)$$

$$G_A(q_\mu^2) = \left(1 - \frac{q_\mu^2}{1.04 \text{GeV}^2} \right)^{-2}, \quad (3.2.10)$$

where $\mu_p = 2.793, \mu_n = -1.913$ [40], $g_A = 1.267$ [41], $\eta = -q_\mu^2/4m_N^2$, and m_π is the pion mass.

The non-relativistic form of the IA currents that we will use are given by

$$V_{IA,0}^{\pm}(\mathbf{x}) = \sum_i f_V \tau_i^{\pm} \delta(\mathbf{x} - \mathbf{r}_i) \quad (3.2.11)$$

$$\mathbf{V}_{IA}^{\pm}(\mathbf{x}) = \sum_i \left[f_V \frac{\mathbf{p}' + \mathbf{p}}{2m_N} + \frac{f_V + f_M}{2m_N} \nabla \times \boldsymbol{\sigma}_i \right] \tau_i^{\pm} \delta(\mathbf{x} - \mathbf{r}_i) \quad (3.2.12)$$

$$A_{IA,0}^{\pm}(\mathbf{x}) = \sum_i \left[\frac{f_A}{2m_N} \boldsymbol{\sigma}_i \cdot (\mathbf{p}' + \mathbf{p}) - \frac{if_P q^0}{2m_N} \boldsymbol{\sigma}_i \cdot \nabla \right] \tau_i^{\pm} \delta(\mathbf{x} - \mathbf{r}_i) \quad (3.2.13)$$

$$\mathbf{A}_{IA}^{\pm}(\mathbf{x}) = \sum_i \left[f_A \boldsymbol{\sigma}_i + \frac{f_P}{2m_N} \nabla (\nabla \cdot \boldsymbol{\sigma}_i) \right] \tau_i^{\pm} \delta(\mathbf{x} - \mathbf{r}_i) \quad (3.2.14)$$

$$V_{IA,0}^s(\mathbf{x}) = \sum_i f_V \frac{1}{2} \delta(\mathbf{x} - \mathbf{r}_i) \quad (3.2.15)$$

$$\mathbf{V}_{IA}^s(\mathbf{x}) = \sum_i \left[f_V \frac{\mathbf{p}' + \mathbf{p}}{2m_N} + \frac{f_V + f_M^s}{2m_N} \nabla \times \boldsymbol{\sigma}_i \right] \frac{1}{2} \delta(\mathbf{x} - \mathbf{r}_i). \quad (3.2.16)$$

3.3 Exchange currents

We follow the NSGK formula for the exchange current [20, 21]. First we consider the axial-vector exchange currents, A_{MEC}^{μ} , consisting of a pion-pole term and non-pole term, $\bar{A}_{MEC}^{\mu}$. Using the PCAC hypothesis we can construct the A_{MEC}^{μ} from only non-pole term

$$A_{MEC}^{\mu} = \bar{A}_{MEC}^{\mu} - \frac{q^{\mu}}{m_{\pi} - q_{\mu}^2} (\mathbf{q} \cdot \bar{\mathbf{A}}_{MEC} - q^0 \bar{A}_{MEC,0}). \quad (3.3.1)$$

Following [26], for space components of the axial-vector exchange currents $\mathbf{A}_{MEC}$, we consider the π -pair current (denoted by πS), ρ -pair current (ρS), $\pi\rho$ -exchange current ($\pi\rho$), π -exchange Δ -excitation current ($\Delta\pi$) and ρ -exchange Δ -excitation current ($\Delta\rho$). These diagrammatic representations are shown in Fig. 3.1 The explicit form of the space

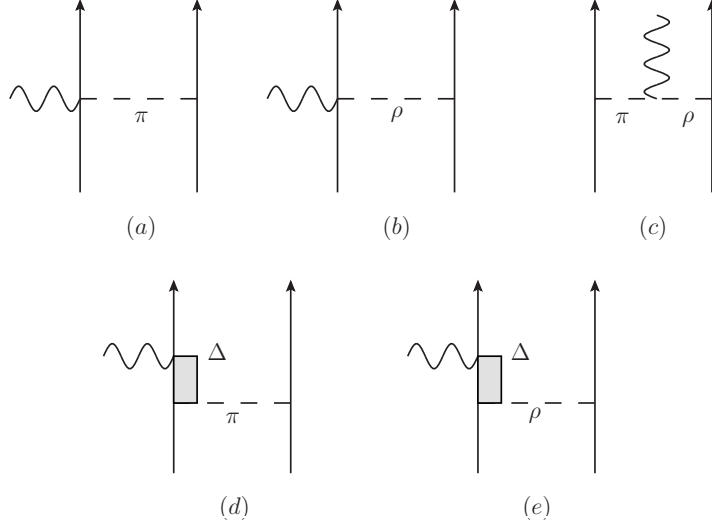


Figure 3.1: Diagrams contributing to the space components of axial-vector exchange current $\mathbf{A}_{MEC}$. Diagram (a) and (b) represent the π - (πS) and ρ - pair (ρS) current. (c) is the $\pi\rho$ -exchange current ($\pi\rho$). (d) and (e) are the π -exchange Δ -excitation ($\Delta\pi$) and ρ -exchange Δ -excitation ($\Delta\rho$) current.

components of the axial-vector exchange currents are as follows

$$\begin{aligned} \bar{A}_{ij}^{\pm}(\mathbf{q}; \pi S) &= -\frac{f_A f_{\pi NN}^2}{m_N m_{\pi}^2} \frac{\boldsymbol{\sigma}_i \cdot \mathbf{k}_j}{m_{\pi}^2 + \mathbf{k}_j^2} f_{\pi}^2(\mathbf{k}_j) \\ &\times \left\{ (\boldsymbol{\tau}_i \times \boldsymbol{\tau}_j)^{\pm} \boldsymbol{\sigma}_i \times \mathbf{k}_j - \tau_j^{\pm} [\mathbf{q} + i \boldsymbol{\sigma}_i \times (\mathbf{p}_i + \mathbf{p}'_i)] \right\} + (i \leftrightarrow j) \end{aligned} \quad (3.3.2)$$

$$\begin{aligned} \bar{A}_{ij}^{\pm}(\mathbf{q}; \rho S) &= f_A \frac{f_{\rho}^2 (1 + \kappa_{\rho})^2}{4m_N^3} \frac{f_{\rho}^2(\mathbf{k}_j)}{m_{\rho} + \mathbf{k}_j} \\ &\times \left(\tau_j^{\pm} \{ (\boldsymbol{\sigma}_j \times \mathbf{k}_j) \times \mathbf{k}_j - i [\boldsymbol{\sigma}_i \times (\boldsymbol{\sigma}_j \times \mathbf{k}_j) \times (\mathbf{p}_i + \mathbf{p}'_i)] \} \right. \\ &\quad + (\boldsymbol{\tau}_i \times \boldsymbol{\tau}_j)^{\pm} \{ \mathbf{q} \boldsymbol{\sigma}_i \cdot (\boldsymbol{\sigma}_j \times \mathbf{k}_j) + i (\boldsymbol{\sigma}_j \times \mathbf{k}_j) \times (\mathbf{p}_i + \mathbf{p}'_i) \\ &\quad \left. - [\boldsymbol{\sigma}_i \times (\boldsymbol{\sigma}_j \times \mathbf{k}_j)] \times \mathbf{k}_j \} \right) + (i \leftrightarrow j) \end{aligned} \quad (3.3.3)$$

$$\begin{aligned} \bar{A}_{ij}^{\pm}(\mathbf{q}; \Delta\pi) &= \kappa \frac{16}{25} f_A \frac{f_{\pi NN}}{m_{\pi}^2 (m_{\Delta} - m_N)} \frac{\boldsymbol{\sigma}_j \cdot \mathbf{k}_j}{m_{\pi} + \mathbf{k}_j^2} f_{\pi}^2 \left[4 \tau_j^{\pm} \mathbf{k}_j - (\boldsymbol{\tau}_i \times \boldsymbol{\tau}_j)^{\pm} \boldsymbol{\sigma}_i \times \mathbf{k}_j \right] \\ &+ (i \leftrightarrow j) \end{aligned} \quad (3.3.4)$$

$$\begin{aligned} \bar{A}_{ij}^{\pm}(\mathbf{q}; \Delta\rho) &= -\kappa \frac{4}{25} f_A \frac{f_{\rho}^2 (1 + \kappa_{\rho})^2}{m_N^2 (m_{\Delta} - m_N)} \frac{f_{\rho}^2(\mathbf{k}_j)}{m_{\rho}^2 + \mathbf{k}_j^2} \\ &\times \left\{ 4 \tau_j^{\pm} (\boldsymbol{\sigma}_j \times \mathbf{k}_j) \times \mathbf{k}_j - (\boldsymbol{\tau}_i \times \boldsymbol{\tau}_j)^{\pm} \boldsymbol{\sigma}_i \times [(\boldsymbol{\sigma}_j \times \mathbf{k}_j) \times \mathbf{k}_j] \right\} \\ &+ (i \leftrightarrow j) \end{aligned} \quad (3.3.5)$$

$$\begin{aligned} \bar{A}_{ij}^{\pm}(\mathbf{q}; \pi\rho) &= 2 f_A \frac{f_{\rho}}{m_N} \frac{\boldsymbol{\sigma}_J \cdot \mathbf{k}_j}{(m_{\rho}^2 + \mathbf{k}_i^2)(m_{\pi}^2 + \mathbf{k}_j^2)} f_{\rho}(\mathbf{k}_i) f_{\pi}(\mathbf{k}_j) (\boldsymbol{\tau}_i \times \boldsymbol{\tau}_j)^{\pm} \\ &\times [(1 + \kappa_{\rho}) \boldsymbol{\sigma}_i \times \mathbf{k}_i - i (\mathbf{p}_i + \mathbf{p}')] + (i \leftrightarrow j), \end{aligned} \quad (3.3.6)$$

where m_ρ and m_Δ are the masses of the ρ meson and Δ -particle respectively. $\mathbf{k}_{i(j)}$ denotes the momentum transfer of the i th (j th) nucleon and $\mathbf{q} \equiv \mathbf{k}_i + \mathbf{k}_j$ is the sum of these momentum transfer. The form factors, f_π, f_ρ , for the pion-nucleon and ρ -nucleon vertices are parametrized as

$$f_\pi(\mathbf{k}) = \frac{\Lambda_\pi^2 - m_\pi^2}{\Lambda_\pi^2 + \mathbf{k}^2}, \quad f_\rho(\mathbf{k}) = \frac{\Lambda_\rho^2 - m_\rho^2}{\Lambda_\rho^2 - \mathbf{k}^2} \quad (3.3.7)$$

with $\Lambda_\pi = 4.8\text{fm}^{-1}$ and $\Lambda_\rho = 6.8\text{fm}^{-1}$. The strength of the $\Delta\pi$ and $\Delta\rho$ currents is adjusted to reproduce the experimental triton beta decay rate [26]. The overall factor $\kappa \sim 0.8$ is multiplied to $\Delta\pi$ and $\Delta\rho$ currents.

For the time component we employ the pion-exchange current, so-called KDR-current [42] following from the low-energy theorem. The explicit form reads

$$\bar{A}_{0ij}^\pm(\mathbf{q}; KDR) = \frac{2}{if_A} \left(\frac{f_{\pi NN}}{m_\pi} \right) f_\pi^2(\mathbf{k}_j) \frac{\boldsymbol{\sigma}_j \cdot \mathbf{k}_j}{m_\pi^2 + \mathbf{k}_j^2} (\boldsymbol{\tau}_i \times \boldsymbol{\tau}_j)^\pm + (i \leftrightarrow j). \quad (3.3.8)$$

Regarding the vector currents we neglect the time components which vanish in the static limit. For the space components $\mathbf{V}$ we take account of the π -pair (*pair*), pionic (*pionic*), π -exchange Δ -excitation and ρ -exchange Δ -excitation (Δ) currents shown in Fig. 3.2. Their explicit form are as follows:

$$\mathbf{V}_{ij}^\pm(\mathbf{q}; \text{pair}) = -2if_V \left(\frac{f_{\pi NN}}{m_\pi} \right)^2 f_\pi^2(\mathbf{k}_j) \frac{\boldsymbol{\sigma}_i(\boldsymbol{\sigma}_j \cdot \mathbf{k}_j)}{m_\pi^2 + \mathbf{k}_j^2} (\boldsymbol{\tau}_i \times \boldsymbol{\tau}_j)^\pm + (i \leftrightarrow j) \quad (3.3.9)$$

$$\begin{aligned} \mathbf{V}_{ij}^\pm(\mathbf{q}; \text{pionic}) &= 2if_V \left(\frac{f_{\pi NN}}{m_\pi} \right)^2 f_\pi(\mathbf{k}_i) f_\pi(\mathbf{k}_j) \frac{(\boldsymbol{\sigma}_i \cdot \mathbf{k}_i)(\boldsymbol{\sigma}_j \cdot \mathbf{k}_j)(\mathbf{k}_i - \mathbf{k}_j)}{(m_\pi^2 + \mathbf{k}_i^2)(m_\pi^2 + \mathbf{k}_j^2)} \\ &\quad \times (\boldsymbol{\tau}_i \times \boldsymbol{\tau}_j)^\pm \end{aligned} \quad (3.3.10)$$

$$\begin{aligned} \mathbf{V}_{ij}^\pm(\mathbf{q}; \Delta) &= -4i\pi \frac{f_V + f_M}{2m} \left[\frac{f_\pi^2(\mathbf{k}_j)}{m_\pi^2 + \mathbf{k}_j^2} \mathbf{q} \times \{c_0 \mathbf{k}_j \boldsymbol{\tau}_j^\pm + d_1(\boldsymbol{\sigma}_i \times \mathbf{k}_j)(\boldsymbol{\tau}_i \times \boldsymbol{\tau}_j)^\pm\} (\boldsymbol{\sigma}_j \cdot \mathbf{k}_j) \right. \\ &\quad \left. + \frac{f_\rho^2(\mathbf{k}_j)}{m_\rho^2 + \mathbf{k}_j^2} \mathbf{q} \times \{c_\rho \mathbf{k}_j \times (\boldsymbol{\sigma}_j \times \mathbf{k}_j) \boldsymbol{\tau}_j^\pm + d_\rho \boldsymbol{\sigma}_i \times (\mathbf{k}_j \times (\boldsymbol{\sigma}_j \times \mathbf{k}_j))(\boldsymbol{\tau}_i \times \boldsymbol{\tau}_j)^\pm\} \right] \\ &\quad + (i \leftrightarrow j). \end{aligned} \quad (3.3.11)$$

The one-pion-exchange potential requires Eq. (3.3.9) and (3.3.10) to satisfy vector current conservations. The numerical values of the various parameters are

$$\frac{f^2}{4\pi} = 0.08, \quad c_0 m_\pi^3 = 0.188, \quad d_1 m_\pi^3 = -0.044, \quad c_\rho m_\rho^3 = 36.2, \quad d_\rho = -\frac{1}{4} c_\rho. \quad (3.3.12)$$

This model description of the vector current has been well tested by comparing the model predictions with, e.g., the measured $n + p \rightarrow d + \gamma$ reaction data [20].

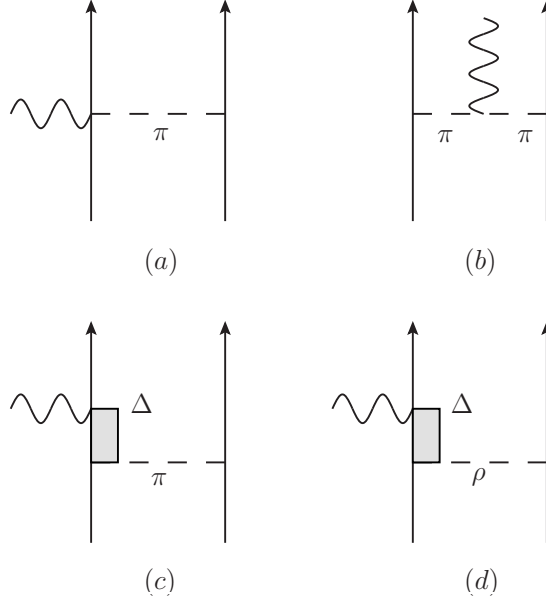


Figure 3.2: Diagrams contributing to the space components of vector exchange current V_{MEC} . Diagram (a) represents the π -pair (*pair*) current. (b) is the pionic (*pionic*) current. (c) and (d) are the π - and ρ -exchange Δ -excitation (Δ) currents.

3.4 Multipole operators

The matrix element of the current-current form of the semi-leptonic effective Hamiltonian is given by

$$\langle F, f | \int d\mathbf{x} J_\lambda(\mathbf{x}) L^\lambda(\mathbf{x}) | I, i \rangle = (2\pi)^3 \delta^3(\mathbf{P}_I + \mathbf{P}_i - \mathbf{P}_F - \mathbf{P}_f) j_\lambda l^\lambda \quad (3.4.1)$$

$$l^\lambda = \langle f | L^\lambda(0) | i \rangle, \quad j_\lambda = \langle F | J_\lambda(0) | I \rangle, \quad (3.4.2)$$

where $|I\rangle$ and $|F\rangle$ are the nuclear initial and final states, and $|i\rangle$ and $|f\rangle$ are the lepton initial and final states. Hereafter the dependence on the center-of-mass motion of nuclear system is eliminated from the nuclear current by using $J \rightarrow \mathcal{J}$. This matrix element can be expand by using the multipole operators T_x^{JM} as

$$\begin{aligned} l^\lambda j_\lambda &= \sum_{J_o M_o} 4\pi i^{J_o} (-1)^{M_o} \\ &\times \langle F | T_C^{J_o M_o} l_C^{J_o - M_o} + T_E^{J_o M_o} l_E^{J_o - M_o} + T_L^{J_o M_o} l_L^{J_o - M_o} + T_M^{J_o M_o} l_M^{J_o - M_o} | I \rangle. \end{aligned} \quad (3.4.3)$$

The nuclear multipole operators for the charge, electric, magnetic and longitudinal are defined by

$$T_E^{JM}(\mathcal{J}) = \frac{1}{q} \int d\mathbf{x} \nabla \times [j_J(qx) \mathbf{Y}_{JJM}(\hat{\mathbf{x}})] \cdot \mathcal{J}(\mathbf{x}), \quad (3.4.4)$$

$$T_M^{JM}(\mathcal{J}) = \int d\mathbf{x} j_J(qx) \mathbf{Y}_{JJM}(\hat{\mathbf{x}}) \cdot \mathcal{J}(\mathbf{x}) \quad (3.4.5)$$

$$T_L^{JM}(\mathcal{J}) = \frac{i}{q} \int d\mathbf{x} \nabla [j_J(qx) Y_{JM}(\hat{\mathbf{x}})] \cdot \mathcal{J}(\mathbf{x}) \quad (3.4.6)$$

$$T_C^{J_0}(\mathcal{J}) = \int d\mathbf{x} j_J(qx) Y_{JM}(\hat{\mathbf{x}}) \mathcal{J}_0(\mathbf{x}), \quad (3.4.7)$$

where $\mathbf{Y}_{JLM}(\hat{\mathbf{x}})$ are the vector spherical harmonics, q is the magnitude of momentum transfer from the lepton to the nuclei, $j_J(qx)$ is the spherical Bessel functions of order J , and $\hat{\mathbf{x}} \equiv \mathbf{x}/|\mathbf{x}|$. The longitudinal operator of the vector current is related to the charge operator through the CVC as

$$T_L^{J_0}(\mathcal{V}) = -\frac{\omega}{q} T_C^{J_0}(\mathcal{V}). \quad (3.4.8)$$

The expansion for the lepton matrix element are given as

$$\ell_C^{JM} = Y_{JM}(\hat{\mathbf{q}}) l^0, \quad (3.4.9)$$

$$\ell_E^{JM} = \left(\sqrt{\frac{J+1}{2J+1}} \mathbf{Y}_{J-1JM}(\hat{\mathbf{q}}) + \sqrt{\frac{J}{2J+1}} \mathbf{Y}_{J+1JM}(\hat{\mathbf{q}}) \right) \cdot \mathbf{l}, \quad (3.4.10)$$

$$\ell_M^{JM} = \mathbf{Y}_{JJM}(\hat{\mathbf{q}}) \cdot \mathbf{l}, \quad (3.4.11)$$

$$\ell_L^{JM} = \left(\sqrt{\frac{J}{2J+1}} \mathbf{Y}_{J-1JM}(\hat{\mathbf{q}}) - \sqrt{\frac{J+1}{2J+1}} \mathbf{Y}_{J+1JM}(\hat{\mathbf{q}}) \right) \cdot \mathbf{l}. \quad (3.4.12)$$

3.5 Nuclear matrix elements

The transition probability due to the electron(positron) capture $e^\mp(p) + i \rightarrow \nu(\bar{\nu})(p') + f$ of the initial two nucleon state $i(|LSJT, M\rangle)$ to the final state $f(|L'S'J'T', M'\rangle)$ can be written with $X_\alpha^\mp(f, i; p', p)$ as

$$\sum_{spin's} |\langle f(L'S'J'T', M'); \nu(\bar{\nu})(p') | H_W^\alpha | i(LSJT, M); e^\mp(p) \rangle|^2 = 2(4\pi) \sum_{spin's} X_\alpha^\mp(f, i; p', p). \quad (3.5.1)$$

Here L, S, J and T are orbital, spin, total angular momentum and isospin of two-nucleon state. We sum all spin states of leptons and two-nucleon states. For $\alpha = \text{CC}$ and NC reactions, $X_\alpha^\mp$ is given as

$$\begin{aligned}
X_{\alpha}^{\mp}(f, i; p', p) = & \frac{G_F^2}{2} F_Z(E) \begin{pmatrix} V_{ud}^2 \\ 1 \end{pmatrix} \sum_{J_o} [\\
& |\langle T_C^{J_o}(\mathcal{V}) \rangle|^2 (1 + \boldsymbol{\beta} \cdot \boldsymbol{\beta}' + \frac{q_0^2}{q^2} (1 - \boldsymbol{\beta} \cdot \boldsymbol{\beta}' + 2\hat{q} \cdot \boldsymbol{\beta} \hat{q} \cdot \boldsymbol{\beta}') - \frac{2q_0}{q} \hat{q} \cdot (\boldsymbol{\beta} + \boldsymbol{\beta}')) \\
& + |\langle T_C^{J_o}(\mathcal{A}) \rangle|^2 (1 + \boldsymbol{\beta} \cdot \boldsymbol{\beta}') + |\langle T_L^{J_o}(\mathcal{A}) \rangle|^2 (1 - \boldsymbol{\beta} \cdot \boldsymbol{\beta}' + 2\hat{q} \cdot \boldsymbol{\beta} \hat{q} \cdot \boldsymbol{\beta}') \\
& + 2\text{Re}[\langle T_C^{J_o}(\mathcal{A}) \rangle \langle T_L^{J_o}(\mathcal{A}) \rangle^*] \hat{q} \cdot (\boldsymbol{\beta} + \boldsymbol{\beta}') \\
& + [|\langle T_M^{J_o}(\mathcal{V}) \rangle|^2 + |\langle T_E^{J_o}(\mathcal{V}) \rangle|^2 + |\langle T_M^{J_o}(\mathcal{A}) \rangle|^2 + |\langle T_E^{J_o}(\mathcal{A}) \rangle|^2] \\
& \times (1 - \hat{q} \cdot \boldsymbol{\beta} \hat{q} \cdot \boldsymbol{\beta}') \\
& \mp 2\text{Re}[\langle T_M^{J_o}(\mathcal{V}) \rangle \langle T_E^{J_o}(\mathcal{A}) \rangle^* + \langle T_M^{J_o}(\mathcal{A}) \rangle \langle T_E^{J_o}(\mathcal{V}) \rangle^*] \hat{q} \cdot (\boldsymbol{\beta} - \boldsymbol{\beta}')] \\
& .
\end{aligned} \tag{3.5.2}$$

Here $\boldsymbol{\beta} = \mathbf{p}/e(p)$ is velocity of the lepton with $\mathbf{p}$ and $\mathbf{p}'$ being the momentum of electron or position and neutrino. $F_Z(E)$ is Fermi function to take account of Coulomb correction for the electron wave function. The nuclear reduced matrix element $\langle o \rangle = \langle f || O || i \rangle$ of the multipole operator O is defined in Eq. (60) of [20], which includes all information of nuclear current and nuclear wave functions. For e^- -capture on the deuteron, the reduced nuclear matrix elements should be understood as

$$\langle \mathcal{O} \rangle = \sum_{L=0,2} \langle L' S' J' T'; NN || \mathcal{O} || L, S = 1, J = 1, T = 0; d \rangle. \tag{3.5.3}$$

The deuteron bound state and the two nucleon scattering state wave functions are written as

$$|L, S = 1, J = 1, T = 0; d\rangle = \left[[Y_L(\hat{\mathbf{r}}) \otimes [\chi(1) \otimes \chi(2)]_{(1)}^{S_d}]_{(J)} \right] R_d(r) \tag{3.5.4}$$

$$|LSJT; NN\rangle = \frac{1 - (-1)^{L+S+T}}{\sqrt{2}} \sum_{L'} [Y_{L'}(\hat{\mathbf{r}}) \otimes \chi_S]_{(J)} \eta_T(\tau) R_{L',L;S}^J(r) \tag{3.5.5}$$

where χ_S and η_T are the two-nucleon spin and isospin wave functions with total spin S and isospin T . The radial wave function of scattering state is normalized, in the plane wave limit, so that

$$R_{L',L;S}^J(r) \rightarrow j_L(p'r) \delta_{L,L'}. \tag{3.5.6}$$

The deuteron formation probability from two nucleons $N + N \rightarrow l(p) + \bar{l}(p') + d$ is

$$\langle d \| T_{\kappa}^{J_o}(\mathcal{J}) \| NN \rangle = (phase) \langle NN \| T_{\kappa}^{J_o}(\mathcal{J}) \| d \rangle, \quad (3.5.10)$$

$\begin{array}{c} \mathcal{J} \\ \hline \kappa \end{array}$	$\mathcal{V}$	$\mathcal{A}$
C	$(-1)^{J_{NN}+1}$	$(-1)^{J_{NN}}$
L	$(-1)^{J_{NN}}$	$(-1)^{J_{NN}+1}$
E	$(-1)^{J_{NN}}$	$(-1)^{J_{NN}+1}$
M	$(-1)^{J_{NN}+1}$	$(-1)^{J_{NN}}$

Table 3.1: The relative phases between the reduced matrix elements of the electron capture and those of the deuteron formation.

written by

$$\begin{aligned} \sum_{spin's} |\langle d(L'S'J'T', M'); l(p), \bar{l}(p') | H_W^{\alpha} | i(LSJT, M) \rangle|^2 \\ = 2(4\pi) \sum_{LSJT, J_o} [\langle 1/2 \tau_1 \ 1/2 \tau_2 | TT_Z \rangle]^2 X_{\alpha}^+(f, i; p', p), \end{aligned} \quad (3.5.7)$$

where the lepton velocity and the momentum transfer are defined as follows

$$\boldsymbol{\beta} = \frac{\mathbf{p}_l}{E_l}, \quad \boldsymbol{\beta}' = \frac{\mathbf{p}_{\bar{l}}}{E_{\bar{l}}}, \quad \mathbf{q} = -\mathbf{p}_l - \mathbf{p}_{\bar{l}}. \quad (3.5.8)$$

For the deuteron formation reactions, the nuclear reduced matrix element is represented as

$$\langle \mathcal{O} \rangle = \sum_{L'=0,2} \langle L'S'J'T'; d \| \mathcal{O} \| LSJT; NN \rangle. \quad (3.5.9)$$

The difference between the electron capture and the deuteron formation in the nuclear reduced matrix element appears in the phase of the total angular momentum. For the vector($\mathcal{V}$) and axial-vector($\mathcal{A}$) currents, the relations of the reduced matrix elements in terms of the multipole operators(κ) are given as Eq. (3.5.10) and the Table 3.1.

3.6 Cross section and neutrino emissivity

In this work, we evaluate the cross sections and emissivities of the semileptonic processes, (2.0.1) - (2.0.5). The emissivities for the neutrino Q_{ν} , and the anti-neutrino $Q_{\bar{\nu}}$ are given

by integrating the transition probability over the momentum, $\mathbf{p}_{i,k}$ ($\mathbf{p}_{f,l}$), of the initial (final) particles labeled by k (l) with a weighting factor of the momentum distributions:

$$Q_{\nu(\bar{\nu})}^{\alpha} = \frac{(2\pi)^4}{s_i s_f} \int \prod_k \left[\frac{d\mathbf{p}_{i,k}}{(2\pi)^3} \right] \prod_l \left[\frac{d\mathbf{p}_{f,l}}{(2\pi)^3} \right] \delta^{(4)} \left(\sum_{l'} p_{f,l'} - \sum_{k'} p_{i,k'} \right) \times \omega_{\nu(\bar{\nu})} \sum_{i,f} |\langle f | H_W^{\alpha} | i \rangle|^2 \Xi, \quad (3.6.1)$$

where s_i (s_f) is a symmetry factor for the identical two nucleons in the initial (final) state, $\omega_{\nu(\bar{\nu})}$ is the energy of the emitted neutrino (anti-neutrino). The summation $\sum_{i,f}$ is over spin states of the initial and final particles. The symbol Ξ represents the occupation probability of incoming particles and the Pauli blocking for outgoing particles. Note that the Pauli blocking factor for the final state neutrino(anti-neutrino) is not included in Ξ for $Q_{\nu}(Q_{\bar{\nu}})$.

$$\Xi = \prod_{k=\text{initial particle}} f_k(p_k) \prod_{l=\text{final fermion} \neq \nu(\bar{\nu})} (1 - f_l(p_l)) \quad (3.6.2)$$

with

$$f_k = \frac{1}{\exp((e_k - \mu_k)/k_B T) \pm 1} \quad (3.6.3)$$

for a fermion (+) and a boson (-), with e_k (μ_k) being the energy (chemical potential) for particle k , and $k_B T$ is the temperature multiplied by the Boltzmann constant.

The corresponding cross section formula is given in the standard way as

$$\sigma_{i \rightarrow f}^{\alpha} = \frac{(2\pi)^4}{s_f v_{rel}} \int \prod_{l=1} \frac{d\mathbf{p}_{f,l}}{(2\pi)^3} \delta^{(4)} \left(\sum_{l'} p_{f,l'} - \sum_{k'} p_{i,k'} \right) \prod_{k''} \frac{1}{2s_{i,k''} + 1} \sum_{i,f} |\langle f | H_W^{\alpha} | i \rangle|^2, \quad (3.6.4)$$

where v_{rel} is the relative velocity of the incoming particles. More detailed formula for the cross sections and emissivities for the different processes are given in the following sections.

3.6.1 Electron and positron capture on deuteron

The cross section formula for the electron/positron capture reaction $e^{-}(p_e) + d(P_d) \rightarrow \nu_e(p_{\nu}) + n(p'_1) + n(p'_2)$ / $e^{+}(p_e) + d(P_d) \rightarrow \bar{\nu}_e(p_{\nu}) + p(p'_1) + p(p'_2)$ is given as

$$\sigma_{e^{\mp}-cap} = \frac{m_N}{3\pi\beta} \int_0^{p_{\nu,max}} dp_{\nu} p_{\nu}^2 p'_{NN} \int_{-1}^1 d \cos \theta_{\bar{l}\bar{l}} \sum_{L',S',J',T'} X_{CC}^{\mp}(NN(L'S'J'T'=1), d; p_{\nu}, p_e). \quad (3.6.5)$$

Here N denotes neutron/proton for the electron/positron capture reaction. We introduced the relative momentum $\mathbf{p}'_{NN} = (\mathbf{p}'_1 - \mathbf{p}'_2)/2$ and the center of mass momentum $\mathbf{P}' = \mathbf{p}'_1 + \mathbf{p}'_2$ of final two nucleons. $\beta = p_e/e(p_e)$ denotes the velocity of the electron.

The neutrino emissivity $Q_{\nu_e/\bar{\nu}_e}$ for electron neutrino or anti-neutrino is given as

$$Q_{\nu_e/\bar{\nu}_e} = \frac{m_N}{8\pi^6} \int_0^\infty dp_e \int_0^{p_{\nu, max}} dp_\nu p_\nu^3 p_e^2 p'_{NN} \langle \Xi \rangle_{\nu_e/\bar{\nu}_e} \times \int_{-1}^1 d\cos\theta_{e\nu} \sum_{S' L' J' T'=1} X_{CC}^\mp(NN(L' S' J' T' = 1), d; p_e, p_\nu). \quad (3.6.6)$$

Here $\langle \Xi \rangle_{\nu_e/\bar{\nu}_e}$ is given as

$$\langle \Xi \rangle_{\nu_e/\bar{\nu}_e} = f_{e^\mp}(p_e) \int d\mathbf{P}_d f_d(\mathbf{P}_d) (1 - f_N(\mathbf{P}_d/2 + \mathbf{p}'_{NN})) (1 - f_N(\mathbf{P}_d/2 - \mathbf{p}'_{NN})). \quad (3.6.7)$$

Since the exact formula of emissivity includes 8 dimensional phase-space integration, we have introduced an approximation to make the numerical integration manageable. Namely, we factorize the angular dependence of the matrix element and that of Ξ , we obtain as approximated formula as

$$\int d\Omega_{p'_{NN}} |\langle f | H_W | i \rangle|^2 \Xi \sim \int d\Omega_{p'_{NN}} |\langle f | H_W | i \rangle|^2 \times \int \frac{d\Omega_{p'_{NN}}}{4\pi} \Xi. \quad (3.6.8)$$

Moreover, we have neglected the difference between the center of mass energy of two-nucleon and the deuteron, and we used $p'^2_{NN}/m_N = e_e(p_e) + m_d - p_\nu - 2m_n$.

3.6.2 Neutrino emission in deuteron formations

The cross section formula of the neutrino emission in nucleon-nucleon scattering $N(p_1) + N(p_2) \rightarrow d(P_d) + l(p_l) + \bar{l}(p_{\bar{l}})$ are given as

$$\sigma_{NN-fusion} = \frac{2\mu_{NN}}{\pi p_{NN}} f_\alpha \int_0^{p_{max}} dp_l p_l^2 p_{\bar{l}}^2 \times \int_{-1}^1 d\cos\theta_{l\bar{l}} \sum_{LSJT} X_\alpha^+(d, NN(LSJT); p_{\bar{l}}, p_l) \quad (3.6.9)$$

where $\alpha = CC$ for reactions (2.0.3) and (2.0.4) and $\alpha = NC$ for reaction (2.0.5). We denote the momentum of lepton as p_l and that of anti-lepton as $p_{\bar{l}}$ and the momentum transfer $\mathbf{q} = -\mathbf{p}_l - \mathbf{p}_{\bar{l}}$. $\cos\theta_{l\bar{l}} = \hat{p}_l \cdot \hat{p}_{\bar{l}}$ is the lepton angle.

The isospin factor f_α is $f_\alpha = 1$ and $1/2$ for the CC and NC reactions, respectively.

The emissivity is given as

$$\begin{aligned}
Q_{\nu/\bar{\nu}} &= \frac{1}{4\pi^6} \int_0^\infty dp_{NN} \int_0^{p_{l,max}} dp_l p_{NN}^2 p_l^2 p_{\bar{l}} e_{\bar{l}}(p_{\bar{l}}) p_\nu \langle \Xi \rangle_{\nu/\bar{\nu}} \\
&\times \int_{-1}^1 d\cos\theta_{e\nu} \sum_{SLJT} X_\alpha^+(d, NN(LSJT); p_{\bar{l}}, p_l)
\end{aligned} \tag{3.6.10}$$

Neutrino energy p_ν is either p_l or $p_{\bar{l}}$ for neutrino or anti-neutrino emissivity and

$$\langle \Xi \rangle_{\nu/\bar{\nu}} = F(p_l, p_{\bar{l}}) \int d\mathbf{P} f_N(\mathbf{P}/2 + \mathbf{p}_{NN}) f_N(\mathbf{P}/2 - \mathbf{p}_{NN}). \tag{3.6.11}$$

Here for CC reactions (2.0.3) and (2.0.4) F is given as

$$F(p_l, p_{\bar{l}}) = 1 - f_e(p_l) \text{ for (2.0.3)} \tag{3.6.12}$$

$$= 1 - f_e(p_{\bar{l}}) \text{ for (2.0.4)}, \tag{3.6.13}$$

while for NC reaction (2.0.5), F is given as

$$F(p_l, p_{\bar{l}}) = 1 - f_\nu(p_l) \text{ for } Q_{\bar{\nu}} \tag{3.6.14}$$

$$= 1 - f_{\bar{\nu}}(p_{\bar{l}}) \text{ for } Q_\nu. \tag{3.6.15}$$

We approximate the energy conservation relation as $e_{\bar{l}}(p_{\bar{l}}) = m_{N_1} + m_{N_2} + p_{NN}^2/(2\mu_{NN}) - e_l(p) - m_d$ with the relative momentum of two nucleon p_{NN} and reduced mass μ_{NN} .

Chapter 4

Results

In this chapter, we study the cross sections of the neutrino production reactions, (2.0.1)-(2.0.5) for kinetic energies of initial states up to around 100 MeV. Here we examine the total cross sections of electron capture on the deuteron and weak fusion processes in comparison with the electron capture on the nucleon. The role of new neutrino production reactions(2.0.1)-(2.0.5) in the supernova explosion depends on the temperature, density and fractions of electron, nucleon and deuteron. As a representation of typical environment, we examined the emissivities at particular time slice, which is 150ms after core bounce predicted in [5]. The emissivities of the new reactions are compared with those of routinely included reactions and possible effects of the new mechanisms are discussed.

4.1 Neutrino production cross sections

4.1.1 Total cross sections

The total cross sections of the electron-deuteron reaction are shown in Fig. 4.1 for $e^- + d \rightarrow n + n + \nu_e$, (left panel) and for the positron-deuteron reaction, $e^+ + d \rightarrow p + p + \bar{\nu}_e$, (right panel). The cross sections of the two reactions are almost the same magnitude except at the very low-energy region where the cross section of the e^+ -capture is larger than that of the e^- -capture because of a difference in the Q-value. In the low-energy region, $E_e < 50$ MeV, the cross section for the e^- (e^+)-capture on the deuteron is smaller than that of the proton (neutron) by a factor of more than 3, mainly due to the higher threshold energy. In other words, the neutrino production rate due to the e^- (e^+)-capture on the free proton (neutron) is reduced when the nucleon is bound in a deuteron. The

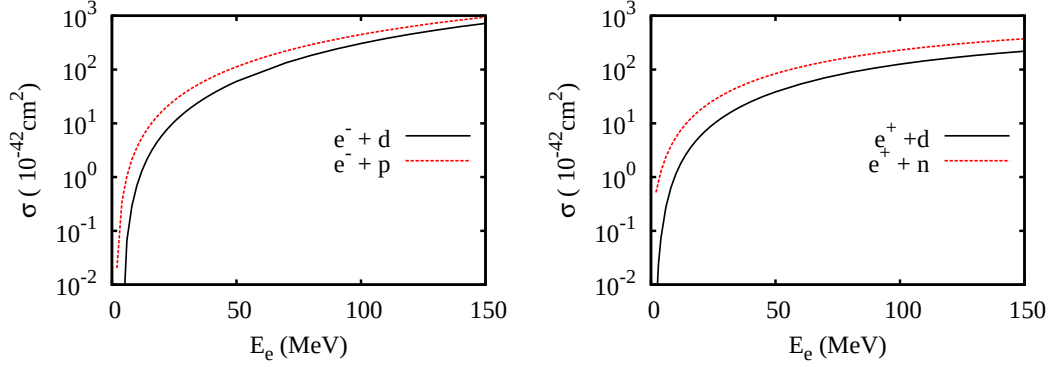


Figure 4.1: Total cross sections for the e^- -capture on the deuteron. The solid and dashed curves in the left (right) panel shows the total cross sections of the electron (positron) capture on the deuteron and the proton (neutron), respectively.

reduced rate of the e^- (e^+)-capture on the deuteron would play a detrimental role if one introduces light elements in a supernova simulation. We note that in the higher energy region ($50 \text{ MeV} < E_e < 150 \text{ MeV}$), the cross section for the e^- (e^+)-capture on a bound proton become comparable to that of a free proton (neutron) but still smaller, because of the phase space differences.

The neutrino productions due to the weak fusion processes in the nucleon-nucleon scattering, (2.0.3)-(2.0.5), are shown in Fig. 4.2. The cross sections for the CC processes are about four times larger than the NC process. This is partly due to the isospin- and the symmetry factor for the initial identical nucleons. Also note that the three processes (2.0.3), (2.0.4) and (2.0.5) are exothermic reactions. Thus cross sections follow the $1/v$ law in the low-energy region except for the pp -fusion reaction where the Coulomb repulsion between the protons reduces the transition probability. Our result of pp -fusion cross section at keV region agrees well with the previous work of [26].

The cross sections of the weak fusion processes are about $10^3 \sim 10^4$ times smaller than that of the e^- (e^+)-capture on the deuteron. This can be understood from the phase space of final state and incident flux at least for the low energy region. We assume that the nuclear matrix element of weak current is energy independent and the same for both reactions. We also neglect the electron mass and center of kinetic energy of the two-

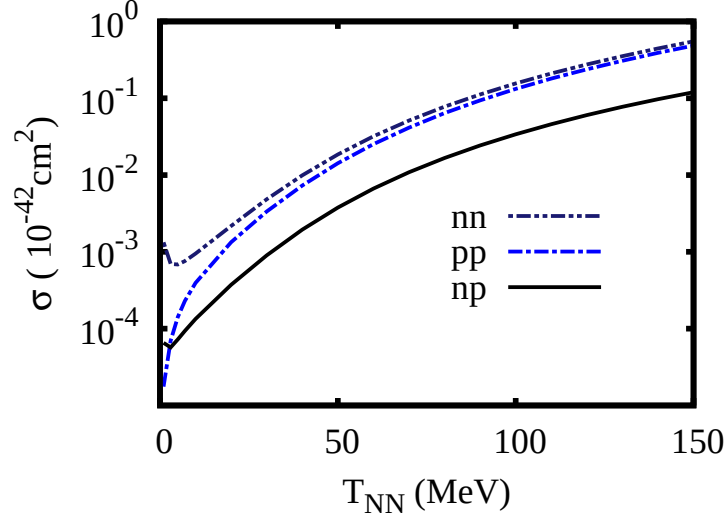


Figure 4.2: Total cross sections for $p+n \rightarrow d+\bar{\nu}_x+\nu_x$ (solid, black), $p+p \rightarrow d+e^++\nu_e$ (dash-dot, blue) and $n+n \rightarrow d+e^-+\bar{\nu}_e$ (dash-two-dot, dark-blue) reactions.

nucleon center of mass motion. Then the cross sections (3.6.5) and (3.6.9) can be written apart from the common energy independent factor as

$$I_{ed} = \frac{1}{3} \int_0^\omega dp_\nu p_\nu^2 p'_{NN} \quad (4.1.1)$$

$$I_{NN-fusion} = \int_0^\omega dp_\nu \frac{p_\nu^2 p_l^2}{p_{NN}}. \quad (4.1.2)$$

where $\omega = p_{\nu,max}$ denotes the maximum neutrino energy. The integral can be written as

$$I_{ed} = \frac{1}{3} \int_0^\omega dp_\nu p_\nu^2 \sqrt{m_N(\omega - p_\nu)} = \sqrt{m_N} \omega^{7/2} \cdot \frac{16}{315} \quad (4.1.3)$$

$$I_{NN-fusion} = \int_0^\omega dp_\nu \frac{p_\nu^2 (\omega - p_\nu)^2}{\sqrt{m_N} \omega} = \sqrt{m_N} \omega^{7/2} \cdot \frac{\omega}{30 m_N}. \quad (4.1.4)$$

Therefore, if maximum neutrino energies of those reactions are chosen to be the same, we obtain

$$I_{NN-fusion} \simeq \frac{\omega}{m_N} I_{ed}. \quad (4.1.5)$$

This relation explains $\sigma_{ed} \gg \sigma_{NN}$ for low energy region.

$\begin{matrix} E_{\text{lab}} \\ \backslash \\ {}^{2S+1}L_J \end{matrix}$	10MeV	50MeV	100MeV	150MeV
1S_0	0.995	0.808	0.556	0.400
3P_0	0.001	0.037	0.075	0.089
3P_1	0.002	0.044	0.080	0.087
1D_2	0.000	0.011	0.046	0.078
${}^3P_2 - {}^3F_2$	0.002	0.098	0.217	0.275
3F_3	0.000	0.001	0.009	0.020
1G_4	0.000	0.000	0.003	0.011
${}^3F_4 - {}^3H_4$	0.000	0.001	0.012	0.031
3H_5	0.000	0.000	0.001	0.003
1I_6	0.000	0.000	0.000	0.002
3H_6	0.000	0.000	0.001	0.005

Table 4.1: Contributions of the partial waves for the $e^- + d \rightarrow n + n + \nu_e$ cross sections. The first column shows the two nucleon partial wave ${}^{2S+1}L_J$. The ratios of $\sigma^{LSJ}/\sigma(\text{total})$ are shown for several representative values of the incident electron laboratory energy E_{lab} , where $\sigma(\text{total}) = \sum_{LSJ} \sigma^{LSJ}$.

4.1.2 Contributions of NN partial waves

It is informative to decompose the total cross section into partial wave components of the nucleon scattering states. The partial wave decomposition of total cross sections are shown in Table 4.1 for the $e^- + d \rightarrow n + n + \nu_e$ and Table 4.2 for the $e^+ + d \rightarrow p + p + \bar{\nu}_e$. Here the partial waves are summed up to $J_{NN} \leq 6$ (J_{NN} : total angular momentum). For each reactions, e^- - and e^+ -captures on the deuteron, it is enough to take the partial waves up to $J_{NN} \leq 4$ to obtain convergence of the cross sections. In the low-energy region, $E_e < 50\text{MeV}$, the dominant transition matrix element is the Gamow-Teller transition from the deuteron to the 1S_0 scattering state. P-wave contribution become important at higher energies.

The partial wave decompositions of the total cross sections for the weak fusion processes (2.0.3)-(2.0.5) are presented in Tables 4.3, 4.4 and 4.5. It is enough to take the two-nucleon partial waves up to $J_{NN} < 3$ in the energy region we studied. In contrast to the $e^-(e^+)$ -

$\begin{matrix} E_{\text{lab}} \\ 2S+1L_J \end{matrix}$	10MeV	50MeV	100MeV	150MeV
1S_0	0.992	0.790	0.529	0.378
3P_0	0.001	0.032	0.066	0.078
3P_1	0.002	0.043	0.094	0.125
1D_2	0.000	0.011	0.039	0.057
$^3P_2 - ^3F_2$	0.005	0.122	0.247	0.302
3F_3	0.000	0.001	0.009	0.019
1G_4	0.000	0.000	0.003	0.007
$^3F_4 - ^3H_4$	0.000	0.001	0.012	0.026
3H_5	0.000	0.000	0.001	0.002
1I_6	0.000	0.000	0.000	0.001
3H_6	0.000	0.000	0.001	0.004

Table 4.2: Contributions of the partial waves for the $e^+ + d \rightarrow p + p + \bar{\nu}_e$ cross sections.

$\begin{matrix} T_{\text{NN}} \\ 2S+1L_J \end{matrix}$	10MeV	50MeV	100MeV	150MeV
1S_0	0.913	0.185	0.030	0.006
3P_0	0.003	0.007	0.006	0.007
3P_1	0.020	0.085	0.072	0.063
1D_2	0.042	0.628	0.782	0.784
$^3P_2 - ^3F_2$	0.022	0.086	0.084	0.097
3F_3	0.000	0.009	0.026	0.042
1G_4	0.000	0.000	0.000	0.001
$^3F_4 - ^3H_4$	0.000	0.000	0.000	0.000
3H_5	0.000	0.000	0.000	0.000
1I_6	0.000	0.000	0.000	0.000
3H_6	0.000	0.000	0.000	0.000

Table 4.3: Contributions of the partial waves for the $n + n \rightarrow d + e^- + \bar{\nu}_e$ cross sections. The ratios of $\sigma^{LSJ}/\sigma(\text{total})$ are shown for several incident nucleon-nucleon kinetic energies T_{NN} .

$\begin{array}{c} T_{\text{NN}} \\ 2S+1L_J \end{array}$	10MeV	50MeV	100MeV	150MeV
1S_0	0.939	0.216	0.036	0.008
3P_0	0.002	0.006	0.006	0.007
3P_1	0.015	0.081	0.071	0.063
1D_2	0.032	0.608	0.779	0.785
$^3P_2 - ^3F_2$	0.012	0.080	0.083	0.095
3F_3	0.012	0.008	0.024	0.041
1G_4	0.000	0.000	0.000	0.001
$^3F_4 - ^3H_4$	0.000	0.000	0.000	0.000
3H_5	0.000	0.000	0.000	0.000
1I_6	0.000	0.000	0.000	0.000
3H_6	0.000	0.000	0.000	0.000

Table 4.4: Contributions of the partial waves for the $p + p \rightarrow d + e^+ + \nu_e$ cross sections.

$\begin{array}{c} T_{\text{NN}} \\ 2S+1L_J \end{array}$	10MeV	50MeV	100MeV	150MeV
1S_0	0.923	0.170	0.021	0.002
3P_0	0.002	0.006	0.006	0.007
3P_1	0.016	0.054	0.036	0.027
1D_2	0.047	0.701	0.861	0.869
$^3P_2 - ^3F_2$	0.013	0.059	0.048	0.047
3F_3	0.000	0.010	0.028	0.047
1G_4	0.000	0.000	0.000	0.001
$^3F_4 - ^3H_4$	0.000	0.000	0.000	0.000
3H_5	0.000	0.000	0.000	0.000
1I_6	0.000	0.000	0.000	0.000
3H_6	0.000	0.000	0.000	0.000

Table 4.5: Contributions of the partial waves for the $n + p \rightarrow d + \nu + \bar{\nu}$ cross sections.

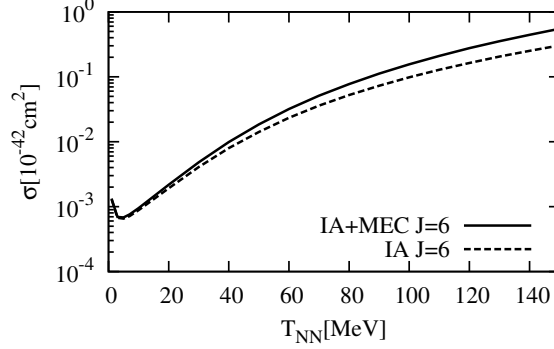


Figure 4.3: MEC contributions of the $n + n \rightarrow d + e^- + \bar{\nu}_e$ total cross section. The solid and dashed lines shows the results using IA + MEC for NN partial waves $J \leq 6$ (IA+MEC $J = 6$) and IA only for $J \leq 6$ (IA $J = 6$), respectively.

capture, we can see that the higher partial waves of initial two-nucleon state other than 1S_0 quickly increases its importance as the two-nucleon kinetic energy of the relative motion (T_{NN}) increases. For $T_{NN} > 20$ MeV, the Gamow-Teller transition between the 1D_2 -deuteron is the dominant transition amplitude.

4.1.3 Meson Exchange Current

Fig. 4.3 shows that the $n + n \rightarrow d + e^- + \bar{\nu}_e$ cross section using IA + MEC and IA only, for which all partial waves ($J \leq 6$) are included. Interestingly, it can be seen that the contribution of the MEC increases and becomes as important as the IA contribution as T_{NN} increases.

To see how much each component of the currents contributes to the cross sections for $nn \rightarrow d$ reaction, we decompose the lepton angle integrated matrix element

$$\int d\cos\theta_{l\bar{l}} X^\mp. \quad (4.1.6)$$

into IA and MEC contributions. $X^\mp$ is the square of the matrix elements from (3.5.2). Although interference terms of IA and MEC are important for evaluating cross section, this decomposition is useful as the indication of estimating current contributions. The results are displayed in Tables 4.6 and 4.7 for which the two-nucleon kinetic energy T_{NN} are 10 and 100 MeV, respectively. The chosen lepton energy corresponds to a peak position

	1S_0	3P_0	3P_1	1D_2	$^3P_2 - ^3F_2$	3F_3	total
IA	0.512×10^1	0.840×10^{-2}	0.834×10^{-1}	0.232×10^0	0.137×10^0	0.579×10^{-3}	0.558×10^1
pi-Delta(A)	0.148×10^{-1}	0.525×10^{-7}	0.140×10^{-5}	0.137×10^{-3}	0.903×10^{-6}	0.806×10^{-8}	0.150×10^{-1}
rho-Delta(A)	0.209×10^{-2}	0.187×10^{-9}	0.242×10^{-8}	0.139×10^{-5}	0.188×10^{-8}	0.856×10^{-12}	0.209×10^{-2}
pi-Delta(V)	0.106×10^{-4}	0.939×10^{-10}	0.115×10^{-9}	0.754×10^{-7}	0.492×10^{-9}	0.359×10^{-11}	0.107×10^{-4}
rho-Delta(V)	0.498×10^{-6}	0.168×10^{-12}	0.425×10^{-12}	0.304×10^{-9}	0.165×10^{-11}	0.145×10^{-15}	0.499×10^{-6}
pi-pair(V)	0.119×10^{-4}	0.426×10^{-10}	0.261×10^{-9}	0.578×10^{-7}	0.268×10^{-9}	0.206×10^{-11}	0.119×10^{-4}
pionic(V)	0.101×10^{-5}	0.152×10^{-9}	0.306×10^{-9}	0.251×10^{-7}	0.186×10^{-9}	0.214×10^{-13}	0.104×10^{-5}
KDR(A^0)	0.270×10^{-5}	0.000×10^0	0.439×10^{-2}	0.965×10^{-7}	0.113×10^{-8}	0.209×10^{-10}	0.439×10^{-2}
pi-pair(A)	0.386×10^{-4}	0.365×10^{-6}	0.150×10^{-6}	0.331×10^{-4}	0.195×10^{-6}	0.660×10^{-9}	0.724×10^{-4}
rho-pair(A)	0.290×10^{-3}	0.182×10^{-8}	0.377×10^{-9}	0.312×10^{-6}	0.127×10^{-8}	0.132×10^{-12}	0.291×10^{-3}
pi-rho(A)	0.385×10^{-3}	0.514×10^{-7}	0.272×10^{-6}	0.161×10^{-4}	0.318×10^{-6}	0.216×10^{-9}	0.401×10^{-3}

Table 4.6: Partial wave and nuclear current decompositions of $\int dx_e X$ for the reaction of $nn \rightarrow de^- \bar{\nu}_e$. The first column denotes the current contributions for $\int dx_e X$. The first row represents the two nucleon partial wave $^{2S+1}L_J$. 'total' is given as the summed $\int dx_e X$ over the partial waves. $T_{NN}=10\text{MeV}$, $E_\nu = 7.7\text{MeV}$ (peak position of the amplitude), $E_e = 5.8179\text{MeV}$

of the matrix element.

We see from Table 4.6 that the main contribution for a low kinetic energy ($T_{NN} \sim 10\text{MeV}$) are from IA, while MEC contributes less than 1%. In contrast, when the kinetic energy increases MEC contributions become more significant. Table 4.7 indicates that at the higher energies KDR and $\pi - \Delta$ components of the axial vector currents are essential. The KDR(A^0) current contains spin-momentum operator which induces $\Delta L = 1$ transition, then this produces the large matrix element between 3P_1 scattering state and the deuteron S-,D-wave. Correspondingly, the axial $\pi - \Delta$ current has the tensor character which produces the large matrix element between 1D_2 scattering state and the deuteron S-wave. For the impulse current, the dominant matrix element is $\langle d | \mathcal{O}_{IA} | 1D2 \rangle$ which is due to the Gamow-Teller operator, and $\langle s | \mathcal{O}_{Imp} | 3P1 \rangle$, $\langle s | \mathcal{O}_{Imp} | 1S0 \rangle$ give sub-leading contributions.

It is notable that even though the relevant temperature in a supernova is $T = 10 - 20 \text{ MeV}$, the emissivity of $NN \rightarrow d$ for $T \sim 15 \text{ MeV}$ receives the largest contribution from the energy region $T_{NN} \sim 100 \text{ MeV}$ as we will see later. To calculate the neutrino processes in a supernova environment it is necessary to sum the two-nucleon partial waves

	1S_0	3P_0	3P_1	1D_2	$^3P_2 - ^3F_2$	3F_3	total
IA	0.439×10^{-2}	0.411×10^{-3}	0.917×10^{-2}	0.520×10^{-1}	0.846×10^{-2}	0.232×10^{-2}	0.767×10^{-1}
pi-Delta(A)	0.130×10^{-2}	0.891×10^{-6}	0.759×10^{-4}	0.223×10^{-2}	0.345×10^{-4}	0.168×10^{-4}	0.366×10^{-2}
rho-Delta(A)	0.345×10^{-3}	0.175×10^{-7}	0.775×10^{-6}	0.909×10^{-4}	0.462×10^{-6}	0.267×10^{-7}	0.437×10^{-3}
pi-Delta(V)	0.564×10^{-4}	0.108×10^{-6}	0.340×10^{-6}	0.730×10^{-4}	0.138×10^{-5}	0.442×10^{-6}	0.132×10^{-3}
rho-Delta(V)	0.444×10^{-5}	0.984×10^{-9}	0.679×10^{-8}	0.109×10^{-5}	0.219×10^{-7}	0.255×10^{-9}	0.556×10^{-5}
pi-pair(V)	0.164×10^{-4}	0.266×10^{-7}	0.189×10^{-6}	0.146×10^{-4}	0.751×10^{-6}	0.292×10^{-7}	0.320×10^{-4}
pionic(V)	0.724×10^{-5}	0.194×10^{-7}	0.265×10^{-6}	0.230×10^{-5}	0.255×10^{-6}	0.239×10^{-8}	0.101×10^{-4}
KDR(A^0)	0.235×10^{-5}	0.000×10^0	0.667×10^{-2}	0.151×10^{-4}	0.757×10^{-6}	0.520×10^{-6}	0.668×10^{-2}
pi-pair(A)	0.109×10^{-3}	0.901×10^{-5}	0.766×10^{-5}	0.533×10^{-3}	0.151×10^{-4}	0.985×10^{-6}	0.675×10^{-3}
rho-pair(A)	0.843×10^{-4}	0.209×10^{-6}	0.117×10^{-6}	0.197×10^{-4}	0.320×10^{-6}	0.408×10^{-8}	0.105×10^{-3}
pi-rho(A)	0.204×10^{-4}	0.274×10^{-5}	0.194×10^{-4}	0.198×10^{-3}	0.251×10^{-4}	0.826×10^{-7}	0.266×10^{-3}

Table 4.7: Partial wave and nuclear current decompositions of $\int dx_e X$ for the reaction of $nn \rightarrow de^- \bar{\nu}_e$. $T_{NN} = 100\text{MeV}$, $E_\nu = 51.5\text{MeV}$ (peak of the amplitude), $E_e = 52.018\text{MeV}$

up to $J_{NN} < 3$, and to include the two-nucleon kinetic energy of the relative motion T_{NN} up to $T_{NN} \sim 100\text{MeV}$.

4.2 Neutrino emissivities

4.2.1 Supernova profiles

In order to study the consequences of neutrino emissions due to deuteron breakup and formation (DBF) for the supernova-explosion mechanism, we calculate neutrino emissivities for a given profile of a core-collapse supernovae, and compare the emissivities due to DBF with those arising from the conventional processes. To this end, we consider two representative profiles of a supernova core, Compositions I and II.

Composition I is the one obtained in [43] in simulating gravitational collapse and core bounce for a $15 M_\odot$ star ($M_\odot$: solar mass). This composition, which represents a typical situation of the post-bounce phase with a stalled shock wave, has been obtained from a numerical simulation adopting the Shen equation of state (EOS) [9, 10, 44]. Composition I includes only nucleons, ^4He and a single heavy nucleus in the Shen EOS. Fig. 4.5 shows the temperature (T) and the density (ρ) as functions of the distance r from the supernova center, pertaining to a snapshot at 150 ms after the core bounce. The neutrino emissivities due to these conventional processes [45] presented in this work are calculated

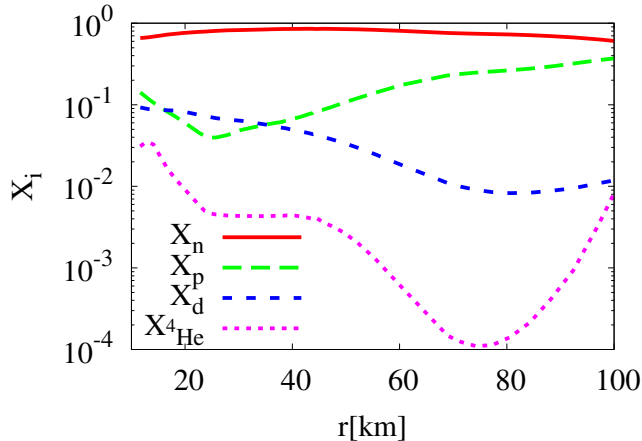


Figure 4.4: Mass fractions pertaining to a snapshot at 150 ms after the core bounce taken from [5]. The mass fraction for the neutron, proton, deuteron and ^4He are shown in solid(red), long-dashed(green), short-dashed(blue) and dotted(magenta) curves, respectively. The horizontal axis, r , is the distance from the supernova center.

with Composition I.

To assess the significance of the new additional emissivities due to DBF, we consider Composition II, which includes the mass fractions of the light elements obtained from the nuclear statistical equilibrium model [5], i.e., nucleons, deuterons, tritons, ^3He , ^4He and other nuclei are taken into account. Fig. 4.4 shows the mass fractions of neutron, proton, deuteron and ^4He . We remark that Compositions I and II share the same data for the profiles of T and ρ shown in Fig. 4.5. The nucleon chemical potentials needed to calculate the emissivities are also taken from the Shen EOS.

Two regions in the profile will be discussed separately: the surface region of a proto-neutron star ($r > 20$ km, $\rho < 10^{13}$ g/cm 3) and the inner region ($r < 20$ km, $\rho > 10^{13}$ g/cm 3). The former corresponds to the neutrino-sphere region between the surface of the nascent proto-neutron star and the shock wave, where neutrino cooling and heating are important. The latter corresponds to a high density region in the core of the proto-neutron star. One can legitimately question the existence of free-space deuterons in dense nuclear medium like the core region. Our aim here is to make a first study of possible influences of deuteron-like correlations that may persist even in the core region. Obviously our results for the core region obtained with the use of free-space deuterons are of exploratory nature

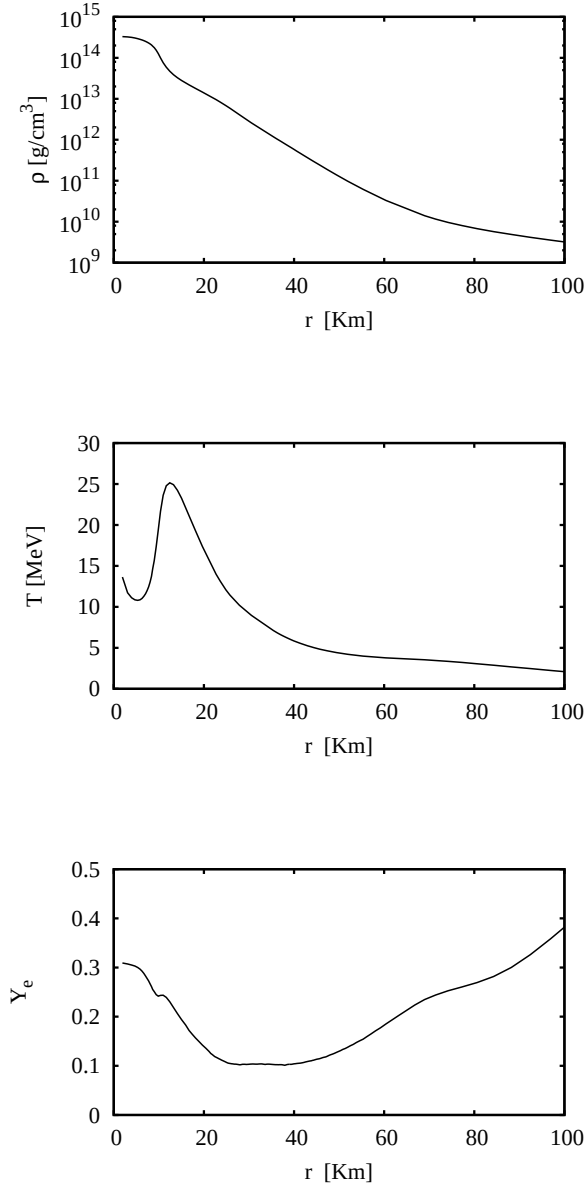


Figure 4.5: The density(top panel), temperature (middle panel) and electron fraction (bottom panel) distributions pertaining to a snapshot at 150 ms after the core bounce taken from [5]. The horizontal axis, r , is the distance from the supernova center.

and should be taken as such.

4.2.2 Emissivity from the surface region of a proto-neutron star

To set the stage for examining the possible influences of ν_e -emissivities due to DBF, we first present ν_e -emissivities due to the conventional reactions calculated with Composition I. The top panel in Fig. 4.6 shows the emissivities arising from $e^- - d$ (2.0.1) and e^-p capture (2.0.6), while the bottom panel gives the emissivities due to nucleon-nucleon bremsstrahlung (2.0.10). The figure indicates that e^-p capture gives a dominant contribution. The nucleon-nucleon bremsstrahlung (2.0.10) and the pair-production process (2.0.11) are about 10^{-3} smaller than that of e^-p capture and give only minor contributions to the emissivity for the present profile.

The top panel in Fig. 4.6 gives the neutrino emissivity due to e^- -capture on the deuteron (2.0.1) calculated for Composition II. The figure shows that the neutrino emissivity due to e^- -capture on the deuteron is smaller than that on the proton (2.0.6) by a factor of $2-10^2$ depending on the distance r_c . The relative importance between electron capture on proton and deuteron can be understood by the cross sections and the mass fractions. Since the non-degenerate approximation for the nucleon distribution is expected to be valid in the surface region, the emissivity can be expressed by the neutrino production cross sections and the numbers of the nucleon and deuteron:

$$\frac{\sigma_{e^-d}n_d}{\sigma_{e^-p}n_p} = \frac{\sigma_{e^-d}}{\sigma_{e^-p}} \cdot \frac{X_d/2m_N}{X_p/m_N} \sim \frac{1}{3} \cdot \frac{X_d}{2X_p}. \quad (4.2.1)$$

This results shows that the emissivity for the e^- -capture on deuteron could be as important as proton in a region where deuteron is abundant.

In Fig. 4.6 (bottom panel), it is shown that the neutrino emissivities from deuteron formation (2.0.4) and (2.0.5) are orders of magnitude smaller than those from e^- -captures, (2.0.1) and (2.0.6), and the pair-production process (2.0.11). However, the neutrino emissivities from deuteron formations become comparable to the $\nu\bar{\nu}$ emissivity from nucleon-nucleon bremsstrahlung for distances closer to 100 km in the cooling region.

As for the $\bar{\nu}_e$ -emissivity shown in Fig. 4.7, e^+ -capture on the neutron (2.0.7) is dominant over the other processes due to the very large neutron abundance as well as the relatively large cross sections. As seen, the emissivity due to e^+ -captures on the deuteron (2.0.2) is smaller than those on the neutron by a factor of 10^2-10^3 , but is compara-

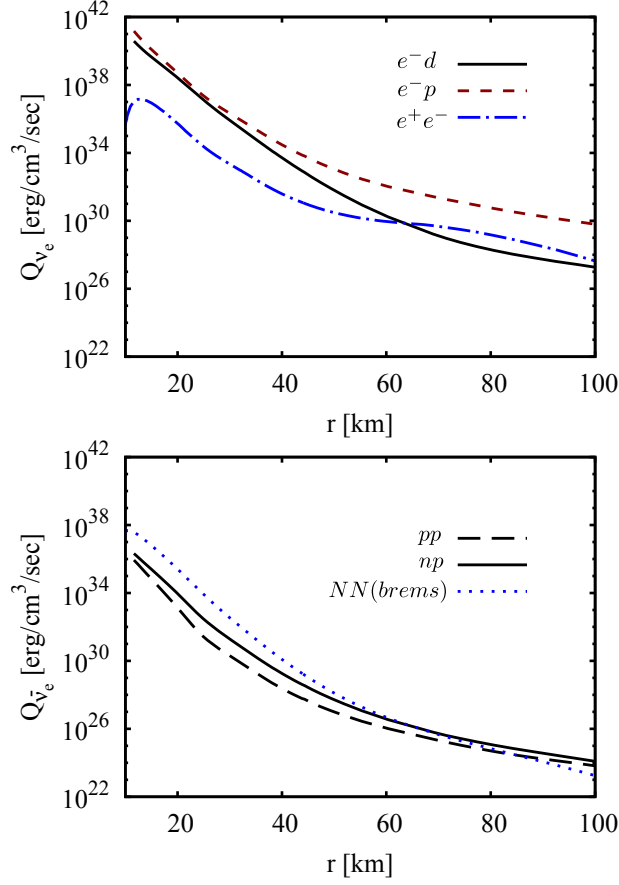


Figure 4.6: The ν_e -emissivities are shown as functions of the distance r from the center of the supernova evaluated with composition II except NN bremsstrahlung. In the top panel, the neutrino emissivities due to e^- captures on deuteron (2.0.1) and proton (2.0.6), and e^+e^- annihilation (2.0.11) are shown in solid, dashed and dash-dotted curves, respectively. In the bottom panel, the emissivities due to the pp and np fusion processes, (2.0.4) (2.0.5), and NN bremsstrahlung (2.0.10) are shown in long-dash, solid and dotted curves, respectively. The emissivities due to the reactions (2.0.10) and (2.0.11) are taken from [43]. The emissivity due to the reaction (2.0.6) is evaluated by using 'phase space constraints method' (see appendix A).

ble to the pair-production process for $r < 40$ km. The emissions from nucleon-nucleon bremsstrahlung and deuteron formation are also smaller than e^+ capture on the deuteron. We observe that the emissivities due to the deuteron formation from nn and np is comparable to that from NN bremsstrahlung and is more important in the outer region.

Let us compare the emissivity of the e -capture reactions for Composition I and Composition II. For the Composition II, abundances of light elements deuteron, triton and ^3He are taken into consideration, while they are not included for the Composition I. The left panel in Fig. 4.8 represents the mass fraction of proton and deuteron for Composition II, and that of proton for Composition I. The right panel in Fig. 4.8 shows the ratio of the neutrino emissivity from e^- -captures on the proton and deuteron with Composition II to that from e^- -captures on the proton with Composition I. As seen in Fig. 4.8 (left panel), the total ν_e -emissivity in some regions can be reduced by up to 40% due to the deuteron abundance. This is due to the smaller e^- -d capture cross section and also to the smaller abundance of deuterons for Composition II compared to those of the proton for Composition I. The free proton abundance in Composition II is smaller than that in Composition I by a factor of ~ 2 , and thus the total emissivity due to the e^- -capture on protons is effectively reduced. This indicates that the effective neutrino emissivity *per proton* via e^- -captures on the proton is reduced if a substantial amount of protons in a supernova are bound in deuterons or other light elements. This suggests that, in considering the neutrino emissivities due to e^- -captures, it is important to take account of the abundances of deuterons and other light elements. Meanwhile, as can be seen from Fig. 4.8 (right panel), the total anti-neutrino emissivity is hardly affected by the deuteron abundance, mainly due to the dominant abundance of the neutron in the matter.

Fig. 4.9 shows ν_μ -emissivities due to np fusion (deuteron formation), NN bremsstrahlung and e^+e^- annihilation. We note that the np fusion contribution is comparable to the NN bremsstrahlung contribution around $r = 60$ km, and the former becomes more important for $r \geq 80$ km. In other words, emission of neutrino pairs through deuteron formation may contribute to additional cooling when NN bremsstrahlung is an important process.

Finally, we have examined the role of partial waves and MEC for emissivities. Table 4.8 shows the contribution of two-nucleon each partial waves for the emissivity Q_{IA+MEC}^{LSJ} for $(n + n \rightarrow d + e^- + \bar{\nu}_e)$ and $(e^- + d \rightarrow n + n + \nu_e)$ at $r = 20\text{km}$. Table 4.8 also shows the enhancement factor due to MEC $Q_{IA+MEC}^{LSJ}/Q_{IA}^{LSJ}$ for each partial wave amplitudes.

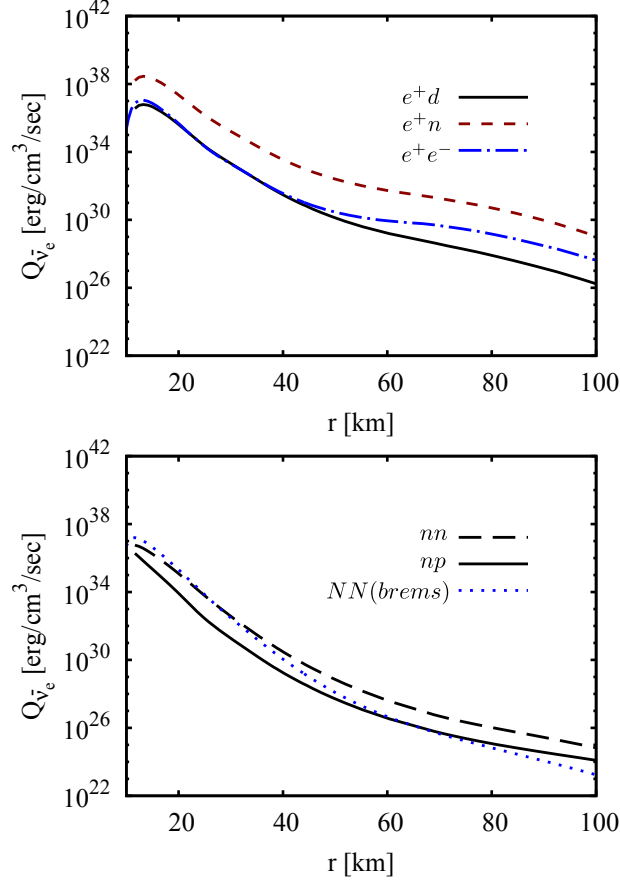


Figure 4.7: The $\bar{\nu}_e$ -emissivities evaluated with Composition II except NN bremsstrahlung. In the top panel, the emissivities due to e^+ captures on a deuteron (2.0.2) and a neutron (2.0.7), and e^+e^- annihilation (2.0.11) are shown in solid, dotted and long dashed curves, respectively. In the bottom panel, the emissivities due to the nn (2.0.3) and np (2.0.5) fusion processes and NN bremsstrahlung(2.0.10) are plotted in dash-two-dotted, solid and two-dotted curves, respectively. The emissivity due to (2.0.7) is evaluated by using 'phase space constraint method'.

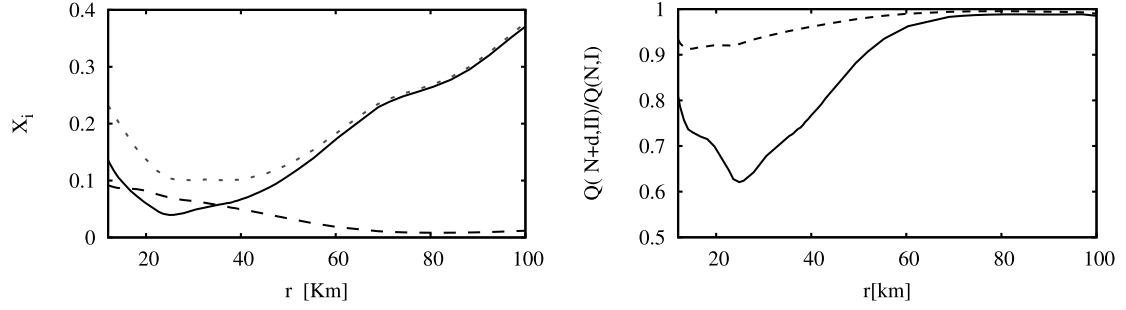


Figure 4.8: Left panel: Mass fraction of proton (solid curve) and deuteron (long dashed curve) for Composition II and that of proton (short dashed curve) for Composition I. Right panel: The ratio of the neutrino emissivity due to e^- -capture (solid curve) and e^+ -capture (short dashed curve) calculated for Composition II to that calculated for Composition I.

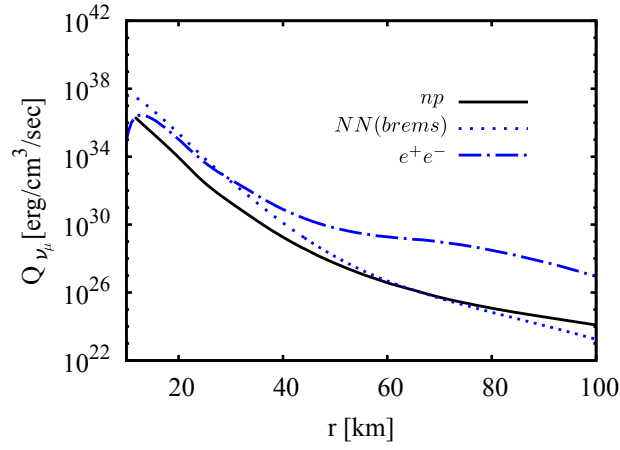


Figure 4.9: The ν_μ -emissivities evaluated with Composition II except NN bremsstrahlung. The contributions of np fusion, NN bremsstrahlung and e^+e^- annihilation are shown in solid, double-dotted and dashed lines, respectively.

$^{2S+1}L_J$	$nn \rightarrow de^- \bar{\nu}_e$			$e^- d \rightarrow nn\nu_e$		
	Q_{IA+MEC}^{LSJ}	$\frac{Q_{IA+MEC}^{LSJ}}{Q_{IA}^{LSJ}}$	$\frac{Q_{IA+MEC}^{LSJ}}{Q_{total}}$	Q_{IA+MEC}^{LSJ}	$\frac{Q_{IA+MEC}^{LSJ}}{Q_{IA}^{LSJ}}$	$\frac{Q_{IA+MEC}^{LSJ}}{Q_{total}}$
1S_0	0.51038×10^{34}	0.896	0.035	0.17036×10^{39}	1.089	0.519
3P_0	0.94236×10^{33}	1.243	0.007	0.24793×10^{38}	1.032	0.076
3P_1	0.10080×10^{35}	0.824	0.069	0.25130×10^{38}	0.993	0.077
1D_2	0.11116×10^{36}	2.023	0.762	0.18192×10^{38}	1.025	0.055
$^3P_2 - ^3F_2$	0.13421×10^{35}	1.175	0.092	0.74130×10^{38}	1.022	0.226
3F_3	0.50430×10^{34}	1.393	0.035	0.43573×10^{37}	1.016	0.013
1G_4	0.12697×10^{33}	1.325	0.001	0.23379×10^{37}	1.000	0.007
$^3F_4 - ^3H_4$	0.96435×10^{31}	1.123	0.000	0.68025×10^{37}	1.001	0.021
3H_5	0.63790×10^{31}	1.183	0.000	0.63264×10^{36}	1.001	0.002
1I_6	0.19232×10^{30}	1.155	0.000	0.43073×10^{36}	1.000	0.001
3H_6	0.11467×10^{29}	1.066	0.000	0.10691×10^{37}	1.000	0.003
total	0.14590×10^{36}	1.643	1.000	0.32823×10^{39}	1.053	1.000

Table 4.8: Contribution of NN partial wave to the neutrino emissivity Q_{IA+MEC}^{LSJ} for $nn \rightarrow de^- \bar{\nu}_e$ and $e^- d \rightarrow nn\nu_e$ reactions. The emissivities are calculated at $r = 20\text{km}$ in unit of $[erg/cm^3/sec]$. The first column shows the two nucleon partial wave $^{2S+1}L_J$. The emissivities of each partial wave are given in the 3rd and 6th column. In the 4th and 7th column, we show the enhancement factor due to MEC $Q_{IA+MEC}^{LSJ}/Q_{IA}^{LSJ}$ for each partial wave. The partial fraction of each partial wave $Q_{IA+MEC}^{LSJ}/Q_{total}$, where $Q_{total} = \sum_{LSJ} Q_{IA+MEC}^{LSJ}$ is shown at the 4th and 7th column.

The convergence of the partial wave expansion of each emissivity may be obtained up to $J = 3$. For the emissivity of $e^- d \rightarrow nn\nu_e$, the P-wave contribution is important next to the S-wave. The contribution of MEC is about 6%, which is similar to what we have observed in the usual electroweak process of deuteron. On the other hand, for $nn \rightarrow de^- \bar{\nu}_e$, the $J = 2$ contribution is very large and the contribution of MEC increases the emissivity about 64%. The different dependence of the emissivities between the two reactions can be partly understood by looking at the dependence on the two-nucleon kinetic of emissivity dQ/dT for $nn \rightarrow de^- \bar{\nu}_e$ and $e^- d \rightarrow nn\nu_e$ shown in Fig. 4.2.2. Since the emissivity is proportional to p_ν^3 , the higher neutrino energy region contribute more. For the $nn \rightarrow d$ reaction, neutrino energy increases for larger T_{NN} . Therefore, the emissivity has the peak at the $T \sim 110\text{MeV}$, which is relatively high energy region. On the other hand, for the

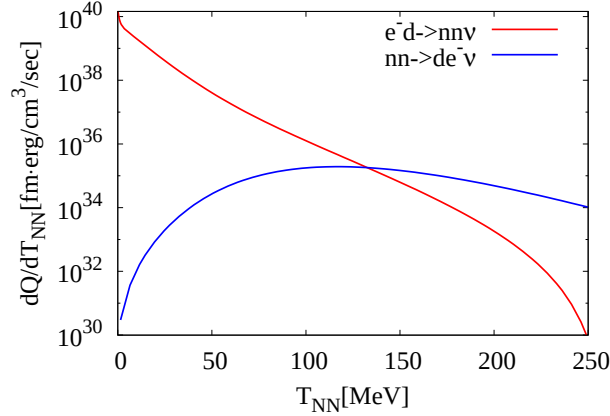


Figure 4.10: Two-nucleon kinetic energy dependence of emissivities for $nn \rightarrow de^- \bar{\nu}_e$ and $e^- d \rightarrow nn \nu_e$ at $r = 20\text{km}$. The emissivity is calculated including both the IA and MEC up to $J \leq 3$.

$e^- d \rightarrow nn \nu_e$ reactions, initial electron energy is shared between energy of the final states T_{NN} and E_ν . Therefore the emissivity decreases as T_{nn} increases. As a results, the higher partial waves and MEC currents becomes more important for $nn \rightarrow d$ than for $d \rightarrow nn$.

4.2.3 Inner region of a proto-neutron star

In this high density region the deuteron is strongly modified and is not bound. As mentioned earlier, the “deuteron” used in our calculation should be regarded as a simplistic devise to simulate possible two-nucleon tensor correlation in nuclear matter. It is hoped that the results in this section give us some hint on whether we need to go beyond the mean-field nuclear matter approach and include possible two-nucleon correlations. With this caveat in mind we present the neutrino emissivities via the “deuteron” formation processes in the central part of the supernova core. Figs. 4.11 and 4.12 show the results obtained with the use of Composition I.

The graphs in Figs. 4.11 and 4.12 indicate that neutrino emissions via “deuteron” formation become dominant for $\bar{\nu}_e$ and ν_μ . Since the electrons are highly degenerate in the core ($r < 10\text{ km}$), the pair process is strongly suppressed and nucleon-nucleon bremsstrahlung is a dominant channel in the conventional models. However, the figures show that neutrino pair emission from the neutron-proton weak fusion process (2.0.5) is much larger than those from the conventional processes, in particular for $\bar{\nu}_e$. The reaction

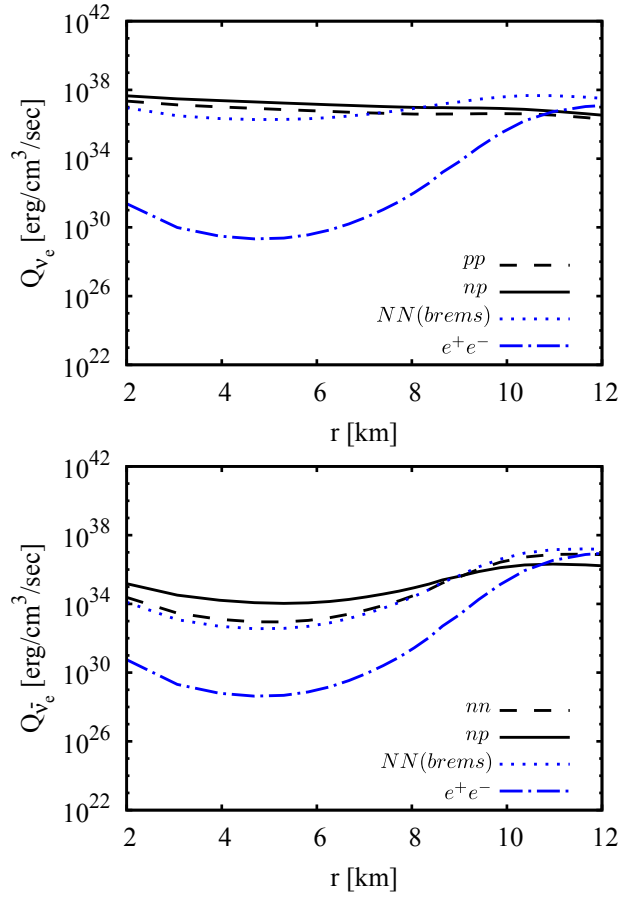


Figure 4.11: The ν_e -emissivities (top panel) and $\bar{\nu}_e$ -emissivities (bottom panel) in the inner core region evaluated with Composition I. See captions for Figs. 4.6 and 4.7 for details.

(2.0.5) is favored by its positive Q-value due to the “deuteron” binding energy, and by the absence of Pauli-blocking in the final state. Hence, neutrino emission via “deuteron” formation from nucleon-nucleon scattering may play an important role in the neutrino pair production process and for the transport of heat and leptons inside proto-neutron stars. It is desirable to examine further the abundance of “deuteron” (n-p tensor correlations) in dense nuclear matter.

4.2.4 Discussion

It was pointed out in [5, 6] that, in addition to deuterons, tritons can also have large abundances in high density regions (10^{11} – 10^{14} g/cm³), where the electron fraction Y_e is low. This suggests the possible importance of neutrino emissivities involving the triton

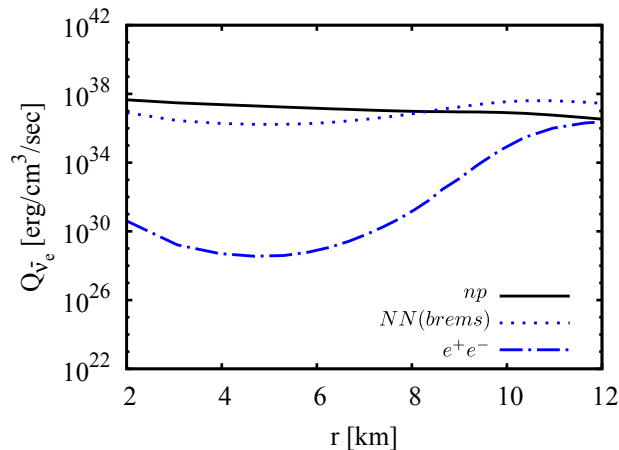


Figure 4.12: The ν_μ -emissivities in the inner core region calculated with Composition I. See the caption for Fig. 4.9 for details.

or “triton” (triton-like three-nucleon correlation in dense matter). In the present work, however, we have not considered these effects.

Our study here is based on the spherical (1D) configurations of supernovae. It would be interesting to study neutrino emission with the abundance of light elements in 2D/3D profiles. Hydrodynamical instabilities can generate a cooling region around a proto-neutron star and a heating region behind a stalled shock wave in a non-spherical manner. Since the density, temperature and electron fractions can have wider ranges in 2D/3D profiles, there may be regions of high deuteron abundance that cannot be found in the 1D profile. The existence of deuterons in the heating regions can contribute to the additional source of heating as studied by [17]. It thus seems interesting to study the possible effects of the neutrino emission and absorption channels involving the deuteron in multi-D supernova explosion simulations. In principle, one must study effects of all neutrino processes by solving the neutrino transfer and hydrodynamics, with detailed information on composition from the equation of state of supernova matter. Such a study is in progress [18, 46], taking into account the neutrino processes involving deuteron breakup/formation and the light element abundance.

We now summarize. Neutrino emissions from $e^\mp$ -capture on the deuteron and from deuteron formation in nucleon-nucleon weak-fusion processes have been studied as new neutrino emission mechanisms in supernovae. These weak processes are evaluated with the standard nuclear physics approach, which consists of the one-nucleon impulse current

and two-nucleon exchange current and nuclear wave functions derived from high-precision phenomenological NN potentials. It is found that the contribution of the two-nucleon meson-exchange current is only a few % for the $e^\mp$ -capture reactions, while it can be as large as the one-nucleon current contribution for the NN fusion reaction at higher energies. The consequences of these new neutrino-emission channels have been examined for representative profiles of core-collapse supernovae at 150 ms after core bounce. The emissivity due to the $e^\mp$ capture reaction on the deuteron is found to be smaller than that on the free nucleon. Therefore, as Fig. 4.8 indicates, the total neutrino emissivity due to electron capture on protons and deuterons is suppressed when an appreciable amount of protons in a supernova are bound inside deuterons. This results in a smaller neutrino luminosity and the lower efficiency of neutrino heating behind a stalled shock wave. Therefore, this new process contributes unfavorably towards a successful supernova explosions. It also leads to a slower speed of the deleptonization and, hence, a slower evolution of nascent proto-neutron stars. On the other hand, as seen in Figs. 4.6 and 4.7, neutrino emission via deuteron formation can be comparable to nucleon-nucleon bremsstrahlung in the outer region. This implies that there might exist situations in which the weak-interaction deuteron-formation processes are the main channels for neutrino emission.

In the inner core region, where the electrons are highly degenerate (high densities at low temperatures), pair-production via e^-e^+ annihilation is suppressed, making nucleon-nucleon bremsstrahlung a main channel to produce $\nu\bar{\nu}$ pairs among the conventional processes [36, 47]. Meanwhile, “deuteron” formation processes in NN scattering can have large rates for ν_μ and ν_τ emissions, a feature that may have significant consequences for the cooling of compact stars. Furthermore, the possible modification of the energy spectra of ν_e , ν_μ and ν_τ due to “deuteron” formation may influence supernova nucleosynthesis and the terrestrial observation of supernova neutrinos [48, 49]. On the other hand, the possible increase of the ν_μ and ν_τ fluxes due to “deuteron” formation hardly affects the heating process behind a shockwave, because these low-energy ν_μ and ν_τ interact with stellar matter only through the NC. As explained earlier, the “deuteron” here stands for a tensor-correlated NN pair that may persist even in dense nuclear matter. A detailed study of deuteron-like two-nucleon tensor correlation in dense matter seems well warranted, but it is beyond the scope of our present exploratory work.

Part II

Neutrino emission via deuteron processes in nuclear medium

The understanding of the explosion mechanism requires a knowledge of the nuclear matter properties and their weak reactions over the wide range of nuclear density and temperature. For example, at 150ms after the core bounce in [5], we are interested in the range of nuclear density ρ from the saturation density of nuclear matter ρ_0 to very low density $\sim 10^{-5}\rho_0$ and the temperature $T \sim 20\text{MeV}$ to a few MeV region. In those region, the light clusters such as deuteron, triton, ^3He can be produced. The formation of light clusters were investigated with the ladder approximation for the thermodynamic T-matrix based on the generalized Beth-Uhlenbeck approach [29].

In this part, we investigate modification of deuteron properties, nucleon-nucleon scattering and the effects of those modification for neutrino emissivity. At first, we briefly summarize the thermodynamic Green's function approach within the T-matrix approximation. In this thesis, as a first step of our investigation, we employed further quasi-particle approximation, where we neglect the imaginary part of the self-energy, which can be justified only in the low density region. Within the t-matrix and quasi-particle approximation, we have studied the neutrino emissivity in nuclear environment.

Chapter 5

Green's function method

In this chapter, we briefly review the method of thermal Green's function. Our description follows closely in [50, 51, 52, 53, 54]. We start from the the Hamiltonian of a system of Fermi particles of mass m that are interacting via instantaneous two-body potential V given as,

$$\begin{aligned} H &= H_0 + H_I \\ &= \int d^3x \psi^\dagger(\mathbf{x}t) \left[-\frac{\nabla^2}{2m} \right] \psi(\mathbf{x}t) + \frac{1}{2} \int d^3x \int d^3x' \psi^\dagger(\mathbf{x}t) \psi^\dagger(\mathbf{x}'t) V(\mathbf{x}, \mathbf{x}') \psi(\mathbf{x}'t) \psi(\mathbf{x}t). \end{aligned} \quad (5.0.1)$$

Here $\psi^\dagger(\mathbf{x}t)$ and $\psi(\mathbf{x}t)$ are quantum field operators in the Heisenberg picture, where we suppress the other quantum numbers such as spin or isospin for simplicity. The field operators satisfy the equal-time anticommutation relations.

$$\{\psi^\dagger(\mathbf{x}t), \psi(\mathbf{x}'t)\} = \delta(\mathbf{x} - \mathbf{x}') \quad (5.0.2)$$

$$\{\psi(\mathbf{x}t), \psi(\mathbf{x}'t)\} = 0 \quad (5.0.3)$$

$$\{\psi^\dagger(\mathbf{x}t), \psi^\dagger(\mathbf{x}'t)\} = 0. \quad (5.0.4)$$

For a system in thermodynamic equilibrium, the expectation value of any operator $\mathcal{O}$ is computed as follow for the grand-canonical ensemble

$$\langle \mathcal{O} \rangle = \frac{\text{Tr} [e^{-\beta(H-\mu N)} \mathcal{O}]}{\text{Tr} [e^{-\beta(H-\mu N)}]}, \quad (5.0.5)$$

where μ is the chemical potential and $\beta = 1/T$ is the inverse temperature. The Boltzmann constant is given to unity, $k_B = 1$ throughout this thesis. N is Fermion number operator

given as

$$N = \int d^3x \psi^\dagger(\mathbf{x}t) \psi(\mathbf{x}t). \quad (5.0.6)$$

We use a simplified notation for the expectation value expression (5.0.5) by introducing the finite temperature density matrix operator ρ and the partition function Z that

$$\rho = \frac{1}{Z} e^{-\beta(H - \mu N)} \quad (5.0.7)$$

$$Z = \text{Tr} e^{-\beta(H - \mu N)}, \quad (5.0.8)$$

and one find

$$\langle \mathcal{O} \rangle = \text{Tr}[\rho \mathcal{O}]. \quad (5.0.9)$$

5.1 Real time Green's function

The two-time one-particle causal Green's function is defined as

$$G_1(\mathbf{x}t, \mathbf{x}'t') = \frac{1}{i} \langle T[\psi(\mathbf{x}t) \psi^\dagger(\mathbf{x}'t')] \rangle, \quad (5.1.1)$$

where T is the time-ordering operator that

$$T[\psi(\mathbf{x}t) \psi^\dagger(\mathbf{x}'t')] = \begin{cases} \psi(\mathbf{x}t) \psi^\dagger(\mathbf{x}'t') & \text{for } t > t' \\ -\psi^\dagger(\mathbf{x}'t') \psi(\mathbf{x}t) & \text{for } t' > t. \end{cases} \quad (5.1.2)$$

The two-body Green's function has the form

$$G_2(\mathbf{x}_1 t_1, \mathbf{x}_2 t_2; \mathbf{x}'_1 t'_1, \mathbf{x}'_2 t'_2) = \left(\frac{1}{i} \right)^2 \langle T[\psi(\mathbf{x}_1 t_1) \psi(\mathbf{x}_2 t_2) \psi^\dagger(\mathbf{x}'_1 t'_1) \psi^\dagger(\mathbf{x}'_2 t'_2)] \rangle. \quad (5.1.3)$$

In general, N-particle Green's functions are defined analogously.

In addition to the causal Green's function, we define the correlation functions

$$G_1^>(\mathbf{x}t, \mathbf{x}'t') = \frac{1}{i} \langle \psi(\mathbf{x}t) \psi^\dagger(\mathbf{x}'t') \rangle \quad (5.1.4)$$

$$G_1^<(\mathbf{x}t, \mathbf{x}'t') = -\frac{1}{i} \langle \psi^\dagger(\mathbf{x}'t') \psi(\mathbf{x}t) \rangle. \quad (5.1.5)$$

Using those correlation functions, the causal Green's function can be written as

$$G_1(\mathbf{x}t, \mathbf{x}'t') = \theta(t - t') G_1^>(\mathbf{x}t, \mathbf{x}'t') + \theta(t' - t) G_1^<(\mathbf{x}t, \mathbf{x}'t'). \quad (5.1.6)$$

We also define the retarded and the advanced propagator

$$G_1^R(\mathbf{x}t, \mathbf{x}'t') = \frac{1}{i}\theta(t-t')\langle\{\psi(\mathbf{x}t), \psi^\dagger(\mathbf{x}'t')\}\rangle \quad (5.1.7)$$

$$G_1^A(\mathbf{x}t, \mathbf{x}'t') = -\frac{1}{i}\theta(t'-t)\langle\{\psi(\mathbf{x}t), \psi^\dagger(\mathbf{x}'t')\}\rangle. \quad (5.1.8)$$

In the infinite system in the equilibrium, the system is supposed to be invariant under the translations in space and time. Then these Green's functions depend only on $\mathbf{r} = \mathbf{x} - \mathbf{x}'$ and $\tau = t - t'$. The Fourier transformations of correlation functions are defined as

$$G_1^>(\mathbf{p}, \omega) = i \int d^3r \int_{-\infty}^{\infty} d\tau e^{-i\mathbf{p}\cdot\mathbf{r} + i\omega\tau} G_1^>(\mathbf{r}, \tau) \quad (5.1.9)$$

$$G_1^<(\mathbf{p}, \omega) = -i \int d^3r \int_{-\infty}^{\infty} d\tau e^{-i\mathbf{p}\cdot\mathbf{r} + i\omega\tau} G_1^<(\mathbf{r}, \tau). \quad (5.1.10)$$

The i and $-i$ factors are introduced so that both $G_1^>$ and $G_1^<$ become real and non-negative quantities. The inverse transformations are defined alternatively

$$iG_1^>(\mathbf{r}, \tau) = \int \frac{d^3p}{(2\pi)^3} \int_{-\infty}^{\infty} d\omega e^{i\mathbf{p}\cdot\mathbf{r} - i\omega\tau} G_1^>(\mathbf{p}, \omega) \quad (5.1.11)$$

$$-iG_1^<(\mathbf{r}, \tau) = \int \frac{d^3p}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\mathbf{p}\cdot\mathbf{r} - i\omega\tau} G_1^<(\mathbf{p}, \omega). \quad (5.1.12)$$

The Fourier transformations of G_1 , G_1^R and G_1^A are defined in a similar way.

Inserting the complete set of eigenstates of Hamiltonian H and number operator N in Eqs. (5.1.4, 5.1.5) and (5.1.9, 5.1.10), we obtain spectral representation for the correlation functions as

$$G_1^>(\mathbf{p}, \omega) = 2\pi \sum_{m,n} \frac{e^{-\beta(E_n - \mu N_n)}}{Z} |\langle m | \psi_{\mathbf{p}}^\dagger | n \rangle|^2 \delta(\omega - (E_n - E_m)) \quad (5.1.13)$$

$$G_1^<(\mathbf{p}, \omega) = 2\pi \sum_{m,n} \frac{e^{-\beta(E_n - \mu N_n)}}{Z} |\langle n | \psi_{\mathbf{p}} | m \rangle|^2 \delta(\omega - (E_m - E_n)). \quad (5.1.14)$$

$$(5.1.15)$$

Here $\psi_{\mathbf{p}} = \int d\mathbf{x} \psi(\mathbf{x}, 0) e^{i\mathbf{p}\cdot\mathbf{x}}$. $G_1^>(\mathbf{p}, \omega)$ can be understood as the density of state available for the addition of an extra particle with momentum $\mathbf{p}$ and energy ω , while $G_1^<(\mathbf{p}, \omega)$ is the density of particles with momentum $\mathbf{p}$ and energy ω .

Using the above spectral representation of the correlation functions, we can derive the important Kubo-Martin-Schwinger relation (KMS) :

$$G_1^>(\mathbf{p}, \omega) = G_1^<(\mathbf{p}, \omega) e^{\beta(\omega - \mu)}. \quad (5.1.16)$$

From the KMS relation, the following spectral representations of the correlation functions are obtained

$$G_1^>(\mathbf{p}, \omega) = A(\mathbf{p}, \omega)[1 - f(\omega)] \quad (5.1.17)$$

$$G_1^<(\mathbf{p}, \omega) = A(\mathbf{p}, \omega)f(\omega), \quad (5.1.18)$$

where $f(\omega)$ is the Fermi distribution function:

$$f(\omega) = \frac{1}{e^{\beta(\omega - \mu)} + 1}. \quad (5.1.19)$$

The spectral function $A(\mathbf{p}, \omega)$ is defined as

$$\begin{aligned} A(\mathbf{p}, \omega) &= G_1^>(\mathbf{p}, \omega) + G_1^<(\mathbf{p}, \omega) \\ &= 2\pi \sum_{m,n} \frac{e^{-\beta(E_m - \mu N_m)} + e^{-\beta(E_n - \mu N_n)}}{Z} |\langle n | \psi_{\mathbf{p}} | m \rangle|^2 \delta(\omega - (E_m - E_n)). \end{aligned} \quad (5.1.20)$$

An important property of the spectral function is that the following sum rule must be fulfilled:

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A(\mathbf{p}, \omega) = 1. \quad (5.1.21)$$

The spectral function $A(\mathbf{p}, \omega)$ can be interpreted as a probability that a particle with momentum $\mathbf{p}$ has energy ω in the medium. In the zero temperature limit, $G_1^<$ and $G_1^>$ are the spectral function of the particle and the hole, respectively.

The Fourier transformation of the the causal and retarded/advanced Green's functions is obtained by using the following representation of theta function,

$$\theta(t - t') = \frac{i}{2\pi} \int_{-\infty}^{\infty} dx \frac{e^{-ix(t-t')}}{x + i\epsilon}. \quad (5.1.22)$$

The Lehmann representation of $G_1^{R/A}$ are

$$G_1^R(\mathbf{p}, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega' \frac{A(\mathbf{p}, \omega')}{\omega - \omega' + i\eta} \quad (5.1.23)$$

$$G_1^A(\mathbf{p}, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega' \frac{A(\mathbf{p}, \omega')}{\omega - \omega' - i\eta}. \quad (5.1.24)$$

If $G^{R/A}$ are analytically continued for complex ω , $G_1^{R/A}(\mathbf{p}, \omega)$ is analytic in the up-(lower) half plane of complex ω . The Lehmann representation of the causal Green's

function is also derived as

$$G_1(\mathbf{p}, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega' \frac{A(\mathbf{p}, \omega')}{1 + e^{-\beta\omega'}} \left[\frac{1}{\omega - \omega' + i\eta} + \frac{e^{-\beta\omega'}}{\omega - \omega' - i\eta} \right] \quad (5.1.25)$$

$$= \frac{1}{2\pi} P \int_{-\infty}^{\infty} d\omega' \frac{A(\mathbf{p}, \omega')}{\omega - \omega'} - \frac{i}{2} \tanh\left(\frac{\beta\omega}{2}\right) A(\mathbf{p}, \omega), \quad (5.1.26)$$

which it can be written as

$$G_1(\mathbf{p}, \omega) = \frac{G_1^R(\mathbf{p}, \omega)}{1 + e^{-\beta\omega}} + \frac{G_1^A(\mathbf{p}, \omega)}{1 + e^{\beta\omega}}. \quad (5.1.27)$$

Other useful relations are shown below:

$$\text{Re } G_1(\mathbf{p}, \omega) = \text{Re } G_1^R(\mathbf{p}, \omega) = \text{Re } G_1^A(\mathbf{p}, \omega) \quad (5.1.28)$$

$$A(\mathbf{p}, \omega) = -2\text{Im } G_1^R(\mathbf{p}, \omega) \quad (5.1.29)$$

$$= 2\text{Im } G_1^A(\mathbf{p}, \omega) \quad (5.1.30)$$

$$= i[G_1^R(\mathbf{p}, \omega) - G_1^A(\mathbf{p}, \omega)] \quad (5.1.31)$$

$$= G_1^>(\mathbf{p}, \omega) + G_1^<(\mathbf{p}, \omega). \quad (5.1.32)$$

As a consequence, the Green's functions, $G_1, G_1^{R/A}, G_1^{> / <}$ is determined through the spectral function $A(\mathbf{p}, \omega)$.

5.2 Thermal Green's function

In this section, we define the “thermal” or “Matsubara” Green's functions $\tilde{G}$. The real time Green's functions introduced in the previous section describes the time evolution of the system using Heisenberg operator. From the similarity between the time evolution operator e^{-iHt} and the statistical weight $e^{-\beta H}$, the thermal Green's function $\tilde{G}$ can be obtained by extending the G for real time variables to the complex number.

In order to ensure the convergence of the correlation functions, one restrict the imaginary time region such that

$$-\beta < \text{Im}\tau = \text{Im}(t - t') < \beta, \quad (5.2.1)$$

or it can be rewritten for the two imaginary times t and t' as

$$-\beta < \text{Im}t < 0. \quad (5.2.2)$$

The Green's function $\tilde{G}_1$ defined in the above region of complex time, can be expressed by using the correlation functions with the following definition of time ordering.

$$\tilde{G}_1(\mathbf{x}t, \mathbf{x}'t') = \begin{cases} \tilde{G}_1^>(\mathbf{x}t, \mathbf{x}'t') & \text{for } -\beta < \text{Im}\tau < 0 \\ & \text{Re}\tau > 0, \text{Im}\tau = 0 \\ \tilde{G}_1^<(\mathbf{x}t, \mathbf{x}'t') & \text{for } 0 < \text{Im}\tau < \beta \\ & \text{Re}\tau < 0, \text{Im}\tau = 0 \end{cases}. \quad (5.2.3)$$

One can derive the KMS relation for the thermal correlation functions $G_1^<$ at $t = 0$ and $G_1^>$ at the boundary $t = -i\beta$ as

$$\tilde{G}_1^<(\mathbf{x}, t = 0; \mathbf{x}'t') = -e^{\beta\mu} \tilde{G}_1^>(\mathbf{x}, t = -i\beta; \mathbf{x}'t'), \quad (5.2.4)$$

which gives the KMS relation for the thermal Green's function:

$$\tilde{G}_1(\mathbf{x}, 0; \mathbf{x}', t') = -e^{\beta\mu} \tilde{G}_1(\mathbf{x}, -i\beta; \mathbf{x}', t'). \quad (5.2.5)$$

Eq. (5.2.5) can be rewritten using the Fourier transformation of $\mathbf{r}$ and relative time variable τ ,

$$\tilde{G}_1(\mathbf{p}, \tau) = -e^{\beta\mu} \tilde{G}_1(\mathbf{p}, \tau - i\beta). \quad (5.2.6)$$

With the periodic condition for the imaginary time in Eq. (5.2.6), we can make Fourier expansion of thermal Green's function as

$$\tilde{G}_1(\mathbf{p}, \tau) = \frac{1}{-i\beta} \sum_{\nu=\text{odd}} e^{-iz_\nu\tau} \tilde{G}_1(\mathbf{p}, z_\nu) \quad (5.2.7)$$

$$z_\nu = \frac{\nu\pi}{-i\beta} + \mu, \quad (5.2.8)$$

where z_ν is the so-called Matsubara frequency and ν is odd integer. Those Fourier coefficients are obtained as

$$\tilde{G}_1(\mathbf{p}, z_\nu) = \int_0^{-i\beta} d\tau e^{iz_\nu\tau} \tilde{G}_1(\mathbf{p}, \tau). \quad (5.2.9)$$

Using the spectral representation of correlation functions, the Lehmann representation of the thermal Green's function is obtained:

$$\tilde{G}_1(\mathbf{p}, z_\nu) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{A(\mathbf{p}, \omega)}{z_\nu - \omega}. \quad (5.2.10)$$

The spectral function $A(\mathbf{p}, \omega)$ determines both the real-time Green's functions and the thermal Green's functions. We now analytically continue the thermal Green's function $\tilde{G}_1(\mathbf{p}, z_\nu)$ at a given set of Matsubara frequencies, to entire complex plane z . We obtain

$$\tilde{G}_1(\mathbf{p}, z) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{A(\mathbf{p}, \omega)}{z - \omega}, \quad (5.2.11)$$

and

$$\tilde{G}_1(\mathbf{p}, z) = \begin{cases} G_1^R(\mathbf{p}, z) & \text{for } \text{Im}z > 0 \\ G_1^A(\mathbf{p}, z) & \text{for } \text{Im}z < 0. \end{cases} \quad (5.2.12)$$

5.3 Equation of motion and self-energy

In the next step, we derive set of equations to determine Green's function. The equation of motion for one-particle Green's function involves the two-body potential and the two-body Green's function such as [50]

$$\left[i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} \right] G_1(1, 1') = \delta(1 - 1') - i \int d\bar{1} V(1 - \bar{1}) G_2(1\bar{1}; 1'\bar{1}^+), \quad (5.3.1)$$

where we denote $(\mathbf{x}_1, t_1)$ as simply 1, $d1 = dt_1 d\mathbf{x}_1$ and $1^+ = (\mathbf{x}_1, t_1 + \epsilon)$, where t_1^+ is infinitesimally larger than t_1 . We also assume instantaneous two-Fermion interaction $V(1 - 2) = V(|\mathbf{x}_1 - \mathbf{x}_2|) \delta(t_1 - t_2)$. The above equation is derived by using the Heisenberg equation of motion:

$$i \frac{\partial \mathcal{O}(t)}{\partial t} = [\mathcal{O}(t), H(t)]. \quad (5.3.2)$$

To solve the equation of motion Eq. (5.3.1), one needs to know G_2 . The equation of motion for G_2 involves G_3 . Then we obtain a set of infinite coupled equations, which is hard to solve in general. Formally, the equation for one-body Green's function can be rewritten in a closed form by introducing self-energy $\Sigma(1, 1')$ as

$$\int d\bar{1} \Sigma(1, \bar{1}) G_1(\bar{1}, 1') = -i \int d\bar{1} V(1 - \bar{1}) G_2(1\bar{1}; 1'\bar{1}^+). \quad (5.3.3)$$

Eq. (5.3.1) is rewritten by using self energy as

$$\left[i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} \right] G_1(1, 1') = \delta(1 - 1') - \int d\bar{1} \Sigma(1, \bar{1}) G_1(\bar{1}, 1'). \quad (5.3.4)$$

The above equation can be written in the integral equation with appropriate boundary condition as

$$G_1(1, 1') = G_1^0(1, 1') + \int d2 \int d\bar{2} G_1^0(1, 2) \Sigma(2, \bar{2}) G_1(\bar{2}, 1'). \quad (5.3.5)$$

For convenience, let us decompose the self-energy of the Hartree-Fock part Σ_{HF} and the part Σ_c due to the correlation of particles in the medium,

$$\Sigma(1, 1') = \Sigma_{HF}(1, 1') + \Sigma_c(1, 1'). \quad (5.3.6)$$

Σ_{HF} is obtained by using the uncorrelated one-body Green's functions as

$$G_2(12; 1'2') \sim G_1(1, 1')G_1(2, 2') - G_1(1, 2')G_1(2, 1'). \quad (5.3.7)$$

Similar to the Green's function we define

$$\Sigma_c(1, 1') = \begin{cases} \Sigma^>(1, 1') & \text{for } it_1 > it'_1 \\ \Sigma^<(1, 1') & \text{for } it_1 < it'_1 \end{cases} \quad (5.3.8)$$

$$\Gamma(1, 1') = \Sigma^<(1, 1') + \Sigma^>(1, 1'). \quad (5.3.9)$$

An important property of the Σ_c is the periodic condition for the imaginary time like Green's functions,

$$\Sigma_c(1, 1')|_{t_1=0} = -e^{\beta\mu}\Sigma_c(1, 1')|_{t_1=-i\beta}. \quad (5.3.10)$$

Using a similar procedure to derive the spectral representations of the Green's functions, one can obtain the spectral representation of Σ_c as

$$\Sigma_c(\mathbf{p}, z) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\Gamma(\mathbf{p}, \omega)}{z - \omega}. \quad (5.3.11)$$

The function Γ is obtained from Σ_c using the above spectral representation,

$$\Gamma(\mathbf{p}, \omega) = i[\Sigma_c(\mathbf{p}, \omega + i\eta) - \Sigma_c(\mathbf{p}, \omega - i\eta)] \quad (5.3.12)$$

$$= \mp 2\text{Im}\Sigma(\mathbf{p}, \omega \pm i\eta). \quad (5.3.13)$$

The real-part of the self-energy is given by using the dispersion relation of the self-energy,

$$\text{Re}\Sigma(\mathbf{p}, \omega) = \Sigma_{HF} + P \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\Gamma(\mathbf{p}, \omega')}{\omega - \omega'}. \quad (5.3.14)$$

An example, for the simplest case is the non-interacting system, one obtains the free thermal Green's function

$$\tilde{G}_1^0(\mathbf{p}, z_\nu) = \frac{1}{z_\nu - p^2/2m}. \quad (5.3.15)$$

From the analytic continuation, one simply get

$$\tilde{G}_1^0(\mathbf{p}, z) = \frac{1}{z - p^2/2m}. \quad (5.3.16)$$

Then the spectral function for the non-interacting particle can be obtained as

$$A(\mathbf{p}, \omega) = i[\tilde{G}_1^0(\mathbf{p}, \omega + i\eta) - \tilde{G}_1^0(\mathbf{p}, \omega - i\eta)] \quad (5.3.17)$$

$$= 2\pi\delta\left(\omega - \frac{p^2}{2m}\right). \quad (5.3.18)$$

For the interacting case, taking the Fourier transformation of Eq. (5.3.5), we obtain,

$$[z_\nu - E_0(p) - \Sigma(\mathbf{p}, z_\nu)]\tilde{G}_1(\mathbf{p}, z_\nu) = 1, \quad (5.3.19)$$

where $E_0(p) = p^2/2m$ is the kinetic energy of a quasi-particle. After the analytic continuation to complex plane, the thermal Green's function is expressed in terms of self-energy as

$$\tilde{G}_1(\mathbf{p}, z) = \frac{1}{[z - E_0(p) - \Sigma(\mathbf{p}, z)]}. \quad (5.3.20)$$

Since the spectral function A is given in terms of the discontinuity of G_1 across the real axis, it can be expressed in terms of the real and imaginary part of the self-energy:

$$A(\mathbf{p}, \omega) = \frac{\Gamma(\mathbf{p}, \omega)}{[\omega - E_0(\mathbf{p}) - \text{Re}\Sigma(\mathbf{p}, \omega)]^2 + \left[\frac{\Gamma(\mathbf{p}, \omega)}{2}\right]^2}. \quad (5.3.21)$$

In the Hartree-Fock approximation, we find $\Gamma = 0$ and the spectral function is given as

$$A(\mathbf{p}, \omega) = 2\pi\delta(\omega - E_0(\mathbf{p}) - \Sigma_{HF}), \quad (5.3.22)$$

which describes the particles as moving independently through an average potential Σ_{HF} . Generally Γ is a finite value and $A(\mathbf{p}, \omega)$ has not the simple delta-shape but a Lorentz form with a finite width Γ . Thus for an relatively small width, we can think of Γ as a lifetime of the single-particle excited state(quasi-particle) with momentum $\mathbf{p}$.

Chapter 6

T-matrix approximation of two-particle Green's function

6.1 Ladder approximation

The equation of motion for the Green's function is not yet in a closed form, since we need to know the Green's function G_2 to evaluate self-energy. A perturbative treatment of the two-particle Green's function is not sufficient to investigate the bound state. Here we adopt the ladder approximation to sum up all order of the two-particle interaction as illustrated in Fig. 6.1.

This infinite series of interaction can be expressed by the integral equation for the

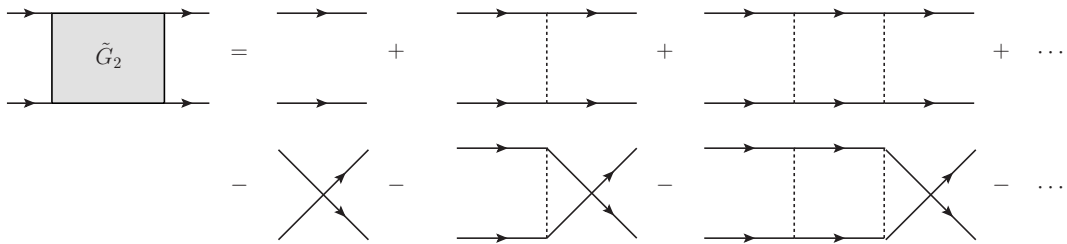


Figure 6.1: The diagrammatic representation for ladder approximation of $\tilde{G}_2$. The dashed line denotes the two-body interaction.

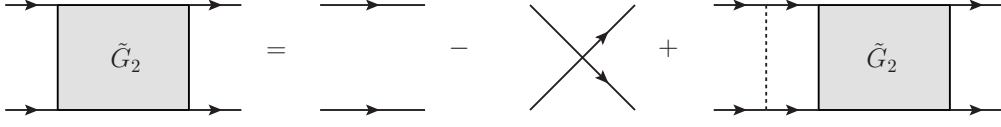


Figure 6.2: The diagrammatic representation for integral equation of $\tilde{G}_2$.

Green's function in Fig. 6.2 as

$$\begin{aligned} \tilde{G}_2(12; 1'2') &= \tilde{G}_1(1, 1')\tilde{G}_1(2, 2') - \tilde{G}_1(1, 2')\tilde{G}_1(2, 1') \\ &+ i \int_0^{-i\beta} d\bar{1} \int_0^{-i\beta} d\bar{2} \tilde{G}_1(1, \bar{1})\tilde{G}_1(2, \bar{2})V(\bar{1} - \bar{2})\tilde{G}_2(\bar{1}\bar{2}; 1'2'). \end{aligned} \quad (6.1.1)$$

It is noticed that the range of the imaginary-time t_i integration is from 0 to $-i\beta$.

It is convenient to introduce T-matrix to obtain the two-particle Green's function in a ladder approximation. The T-matrix satisfies the following integral equation,

$$\begin{aligned} T(12; 1'2') &= \delta(1 - 1')\delta(2 - 2')V(1 - 2) \\ &+ i \int_0^{-i\beta} d\bar{1} \int_0^{-i\beta} d\bar{2} V(1 - 2)\tilde{G}_1(1, \bar{1})\tilde{G}_1(2, \bar{2})T(\bar{1}\bar{2}; 1'2') \\ &= \delta(1 - 1')\delta(2 - 2')V(1 - 2) \\ &+ iV(1 - 2)\tilde{G}_1(1, 1')\tilde{G}_1(2, 2')V(1' - 2') \\ &+ i^2 \int_0^{-i\beta} d\bar{1} \int_0^{-i\beta} d\bar{2} V(1 - 2)\tilde{G}_1(1, \bar{1})\tilde{G}_1(2, \bar{2})V(\bar{1} - \bar{2})\tilde{G}_1(\bar{1}, 1')\tilde{G}_1(\bar{2}, 2')V(1' - 2') \\ &+ \dots, \end{aligned} \quad (6.1.2)$$

$$(6.1.3)$$

which are shown in Fig. 6.3. With the use of the T-matrix, the two-body Green's function $\tilde{G}_2$ is given as

$$V(1 - 2)\tilde{G}_2(12; 1'2') = \int_0^{-i\beta} d\bar{1} \int_0^{-i\beta} d\bar{2} T(12; \bar{1}\bar{2}) \left[\tilde{G}_1(\bar{1}, 1')\tilde{G}_1(\bar{2}, 2') - \tilde{G}_1(\bar{1}, 2')\tilde{G}_1(\bar{2}, 1') \right]. \quad (6.1.4)$$

Furthermore, from Eq. (5.3.3), the self-energy Σ is expressed using T-matrix instead of the two-particle Green's function as

$$\Sigma(1, 1') = -i \int_0^{-i\beta} d2 \int_0^{-i\beta} d\bar{2} [T(12; 1'\bar{2}) - T(12; \bar{2}1')] \tilde{G}_1(\bar{2}, 2^+). \quad (6.1.5)$$

Now we examine the analytic property of the T-matrix. Here we use instantaneous two-particle interaction. The T-matrix depends only on the time difference $\tau = t_1 - t_{1'}$

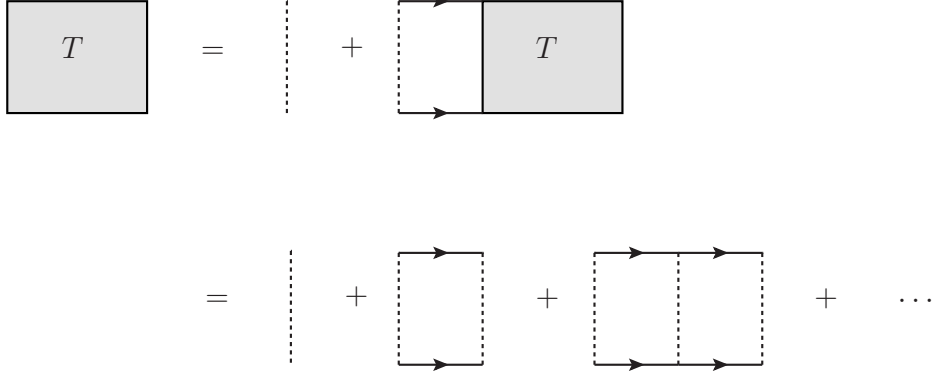


Figure 6.3: The diagrammatic representation of ladder T-matrix.

such as

$$T(12; 1'2') = \delta(t_1 - t_2)\delta(t_{1'} - t_{2'})\langle \mathbf{x}_1 \mathbf{x}_2 | T(t_1 - t_{1'}) | \mathbf{x}'_1 \mathbf{x}'_2 \rangle. \quad (6.1.6)$$

For imaginary time, we define $T^>, T^<, T^0$ according to $\text{Im}(\tau)$.

$$\langle \mathbf{x}_1 \mathbf{x}_2 | T(\tau) | \mathbf{x}'_1 \mathbf{x}'_2 \rangle = \begin{cases} \langle \mathbf{x}_1 \mathbf{x}_2 | T^>(\tau) | \mathbf{x}'_1 \mathbf{x}'_2 \rangle, & \text{Im}\tau < 0 \\ \langle \mathbf{x}_1 \mathbf{x}_2 | T^0(\tau) | \mathbf{x}'_1 \mathbf{x}'_2 \rangle, & \text{Im}\tau = 0 \\ \langle \mathbf{x}_1 \mathbf{x}_2 | T^<(\tau) | \mathbf{x}'_1 \mathbf{x}'_2 \rangle, & \text{Im}\tau > 0, \end{cases} \quad (6.1.7)$$

or equivalently we can write the T-matrix as

$$\begin{aligned} \langle \mathbf{x}_1 \mathbf{x}_2 | T(\tau) | \mathbf{x}'_1 \mathbf{x}'_2 \rangle &= \theta(-\text{Im}\tau) \langle \mathbf{x}_1 \mathbf{x}_2 | T^>(\tau) | \mathbf{x}'_1 \mathbf{x}'_2 \rangle \\ &+ \delta(\tau) \langle \mathbf{x}_1 \mathbf{x}_2 | T^0(\tau) | \mathbf{x}'_1 \mathbf{x}'_2 \rangle \\ &+ \theta(\text{Im}\tau) \langle \mathbf{x}_1 \mathbf{x}_2 | T^<(\tau) | \mathbf{x}'_1 \mathbf{x}'_2 \rangle. \end{aligned} \quad (6.1.8)$$

As one can see from Eq. (6.1.2), T^0 is just two-particle interaction V . The T-matrix satisfies the KMS boundary conditions similar to $\tilde{G}_1 \tilde{G}_1$,

$$\langle \mathbf{x}_1 \mathbf{x}_2 | T(\tau) | \mathbf{x}'_1 \mathbf{x}'_2 \rangle = e^{\beta(\mu_1 + \mu_2)} \langle \mathbf{x}_1 \mathbf{x}_2 | T(\tau - i\beta) | \mathbf{x}'_1 \mathbf{x}'_2 \rangle, \quad (6.1.9)$$

where μ_1, μ_2 are chemical potentials of particle 1 and 2. In the rest of this section, we use simplified notation $T(\tau)$ without explicit position dependence $\langle \mathbf{x}_1 \mathbf{x}_2 | T(\tau) | \mathbf{x}'_1 \mathbf{x}'_2 \rangle$. Using the Fourier transformations of the $T^{> / <}$ along to the real time axis defined as

$$T^{> / <}(\omega) = i \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} T^{> / <}(\tau), \quad (6.1.10)$$

the KMS relation can be written as

$$T^>(\omega) = e^{\beta(\omega - \mu_1 - \mu_2)} T^<(\omega). \quad (6.1.11)$$

Similar to the spectral representations of Green's function, we obtain

$$T^>(\omega) = [1 + g(\omega)] \mathcal{T}(\omega) \quad (6.1.12)$$

$$T^<(\omega) = g(\omega) \mathcal{T}(\omega), \quad (6.1.13)$$

where

$$g(\omega) = \frac{1}{e^{\beta(\omega - \mu_1 - \mu_2)} - 1}, \quad (6.1.14)$$

and

$$\mathcal{T}(\omega) = T^>(\omega) - T^<(\omega). \quad (6.1.15)$$

From Eq. (6.1.11) and (6.1.15), one see that the function $T^>(\omega)$ can be written by $T^<(\omega)$ using the the Bose distribution function $g(\omega)$. The appearance of the Bose distribution function is due to the periodic KMS condition for the T-matrix.

From the KMS condition for the T-matrix, the Fourier series expansion for T-matrix is given as

$$T(\tau) = \frac{1}{-i\beta} \sum_{\nu=\text{even}} e^{-iz_{11'}\tau} T(z_{11'}), \quad (6.1.16)$$

where $z_{11'} = \frac{\nu\pi}{-i\beta} + \mu_1 + \mu_2$ and ν is even integer. The Fourier coefficients in Eq. (6.1.16) is obtained by the integration along the imaginary time axis $+i\beta < \tau < -i\beta$,

$$T(z_{11'}) = \frac{1}{2} \int_{i\beta}^{-i\beta} d\tau e^{iz_{11'}\tau} T(\tau) \quad (6.1.17)$$

$$= \int_0^{-i\beta} d\tau e^{iz_{11'}\tau} T(\tau), \quad (6.1.18)$$

where the KMS relation is used in the second line. Using Eq. (6.1.8),

$$\begin{aligned} T(z_{11'}) &= \int_0^{-i\beta} d\tau e^{iz_{11'}\tau} [\theta(\text{Im}\tau) T^<(\tau) + \delta(\tau) V + \theta(-\text{Im}\tau) T^>(\tau)] \\ &= \int_0^{-i\beta} d\tau e^{iz_{11'}\tau} [\delta(\tau) V + \theta(-\text{Im}\tau) T^>(\tau)] \\ &= V + \int_0^{-i\beta} d\tau e^{iz_{11'}\tau} T^>(\tau). \end{aligned} \quad (6.1.19)$$

Because of the integration region $0 < \tau < -i\beta$, $T^<(\tau)$ does not contribute to the above integration. Now we express $T^>(\tau)$ using the Fourier transformation for the real time axis, one obtain,

$$\begin{aligned} T(z_{11'}) &= V + \int_0^{-i\beta} d\tau \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi i} e^{i(z_{11'} - \omega')\tau} T^>(\omega') \\ &= V + \int_0^{-i\beta} d\tau \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi i} e^{i(z_{11'} - \omega')\tau} [1 + g(\omega')] \mathcal{T}(\omega'), \end{aligned} \quad (6.1.20)$$

which leads to the following formula.

$$T(z_{11'}) = V + \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\mathcal{T}(\omega')}{z_{11'} - \omega'}. \quad (6.1.21)$$

We now analytic continue the above function for complex z plane, one obtain

$$T(z) = V + \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\mathcal{T}(\omega')}{z - \omega'}. \quad (6.1.22)$$

From this equation, we can derive

$$\mathcal{T}(\omega) = -2\text{Im}T(\omega + i\eta), \quad (6.1.23)$$

and the dispersion relation

$$\text{Re}T(\omega + i\eta) = V - P \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\text{Im}T(\omega' + i\eta)}{\omega - \omega'}. \quad (6.1.24)$$

6.2 Lippmann-Schwinger equation

We rewrite the Lippmann-Schwinger equation (Eq. (6.1.2)) into a momentum space integral equation. For instantaneous interaction V , the Lippmann-Schwinger equation can be written as

$$\begin{aligned} &\langle \mathbf{x}_1 \mathbf{x}_2 | T(t_1 - t_{1'}) | \mathbf{x}'_1 \mathbf{x}'_2 \rangle \\ &= \delta(t_1 - t_{1'}) \delta(\mathbf{x}_1 - \mathbf{x}'_1) \delta(\mathbf{x}_2 - \mathbf{x}'_2) V(\mathbf{x}_1, \mathbf{x}_2) \\ &+ i \int_0^{-i\beta} dt_{\bar{1}} \int d\mathbf{x}_{\bar{1}} d\mathbf{x}_{\bar{2}} V(\mathbf{x}_1, \mathbf{x}_2) \langle \mathbf{x}_1 \mathbf{x}_2 | \mathcal{G}_2^0(t_1 - t_{\bar{1}}) | \mathbf{x}'_1 \mathbf{x}'_2 \rangle \langle \mathbf{x}_{\bar{1}} \mathbf{x}_{\bar{2}} | T(t_{\bar{1}} - t_{1'}) | \mathbf{x}'_1 \mathbf{x}'_2 \rangle, \end{aligned} \quad (6.2.1)$$

where we have introduced two-particle Green's function

$$\langle \mathbf{x}_1 \mathbf{x}_2 | \mathcal{G}_2^0(t_1 - t'_1) | \mathbf{x}'_1 \mathbf{x}'_2 \rangle \equiv i \tilde{G}_1(\mathbf{x}_1 t_1, \mathbf{x}'_1 t'_1) \tilde{G}_1(\mathbf{x}_2 t_1, \mathbf{x}'_2 t'_1). \quad (6.2.2)$$

By using the similar procedure to derive the spectral representation of T-matrix, we will derive the spectral representation of $\mathcal{G}_2^0$. In the rest of this section we use simple notation $\mathcal{G}_2^0(\tau)$ for $\langle \mathbf{x}_1 \mathbf{x}_2 | \mathcal{G}_2^0(\tau) | \mathbf{x}'_1 \mathbf{x}'_2 \rangle$. The correlation functions for $\mathcal{G}_2^0$ for an imaginary time ordering can be defined

$$\mathcal{G}_2^0(\tau) = \begin{cases} \mathcal{G}_2^{0>}(\tau) = i\tilde{G}_1^>(\tau)\tilde{G}_1^>(\tau) & \text{for } \text{Im}\tau < 0 \\ \mathcal{G}_2^{0<}(\tau) = i\tilde{G}_1^<(\tau)\tilde{G}_1^<(\tau) & \text{for } \text{Im}\tau > 0 \end{cases}. \quad (6.2.3)$$

The KMS relation of $\mathcal{G}_2^0$ can be obtained from that of one-particle Green's function as

$$\mathcal{G}_2^0(t_1 = 0, t_{1'}) = e^{\beta(\mu_1 + \mu_2)} \mathcal{G}_2^0(t_1 = -i\beta, t_{1'}). \quad (6.2.4)$$

From this relation, the spectral decomposition of $\mathcal{G}_2^0$ can be obtained following as,

$$\mathcal{G}_2^0(z) = \int \frac{d\Omega}{2\pi} \frac{\mathcal{G}_2^{0>}(\Omega) - \mathcal{G}_2^{0<}(\Omega)}{z - \Omega}. \quad (6.2.5)$$

The Fourier transformations for correlation function $\mathcal{G}_2^{0> / <}(\Omega)$ is given as

$$\mathcal{G}_2^{0> / <}(\Omega) = i \int_{-\infty}^{\infty} d\tau e^{i\Omega\tau} \mathcal{G}_2^{0> / <}(\tau) \quad (6.2.6)$$

$$= \int \frac{d\omega'}{2\pi} \tilde{G}_1^{> / <}(\Omega - \omega') \tilde{G}_1^{> / <}(\Omega - \omega'). \quad (6.2.7)$$

Then the spectral representation for $\mathcal{G}_2^0$ reads

$$\begin{aligned} \mathcal{G}_2^0(z) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\tilde{G}_1^>(\omega)\tilde{G}_1^>(\omega') - \tilde{G}_1^<(\omega)\tilde{G}_1^<(\omega')}{z - \omega - \omega'} \\ &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{1 - f(\omega) - f(\omega')}{z - \omega - \omega'} A(\omega) A(\omega'). \end{aligned} \quad (6.2.8)$$

It is noticed that the exact form of $\mathcal{G}_2^0$ includes the effect of Pauli blocking

$$1 - f(\omega) - f(\omega') = [1 - f(\omega)][1 - f(\omega')] - f(\omega)f(\omega'), \quad (6.2.9)$$

where the first term represents the standard Pauli blocking of particle and the second term is that of hole.

Using this explicit form of $\mathcal{G}_2^0(z)$, one can derive the Lippmann-Schwinger equation as

$$\begin{aligned} &\langle \mathbf{x}_1 \mathbf{x}_2 | T(z_{11'}) | \mathbf{x}'_1 \mathbf{x}'_2 \rangle \\ &= \delta(\mathbf{x}_1 - \mathbf{x}'_1) \delta(\mathbf{x}_2 - \mathbf{x}'_2) V(\mathbf{x}_1, \mathbf{x}_2) \\ &+ \int d\mathbf{x}_{\bar{1}} d\mathbf{x}_{\bar{2}} V(\mathbf{x}_1, \mathbf{x}_2) \langle \mathbf{x}_1 \mathbf{x}_2 | \mathcal{G}_2^0(z_{11'}) | \mathbf{x}_{\bar{1}} \mathbf{x}_{\bar{2}} \rangle \langle \mathbf{x}_{\bar{1}} \mathbf{x}_{\bar{2}} | T(z_{11'}) | \mathbf{x}'_1 \mathbf{x}'_2 \rangle. \end{aligned} \quad (6.2.10)$$

We then use momentum space expression as

$$T(\mathbf{p}, \mathbf{p}', \mathbf{P}, z_{11'}) = V(\mathbf{p}, \mathbf{p}') + \int d^3q V(\mathbf{p}, \mathbf{q}) \mathcal{G}_2^0(\mathbf{q}, \mathbf{P}, z_{11'}) T(\mathbf{q}, \mathbf{p}', \mathbf{P}, z_{11'}), \quad (6.2.11)$$

where the relative and the center-of-mass momenta are defined as follows

$$\mathbf{p} = \frac{\mathbf{p}_1 - \mathbf{p}_2}{2}, \quad \mathbf{p}' = \frac{\mathbf{p}'_1 - \mathbf{p}'_2}{2} \quad (6.2.12)$$

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}'_1 + \mathbf{p}'_2. \quad (6.2.13)$$

Here we assume that the masses of the proton and the neutron mass are the same. After analytic continuation above the real energy axis, we obtain,

$$T(\mathbf{p}, \mathbf{p}', \mathbf{P}, \omega + i\eta) = V(\mathbf{p}, \mathbf{p}') + \int d^3q V(\mathbf{p}, \mathbf{q}) \mathcal{G}_2^0(\mathbf{q}, \mathbf{P}, \omega + i\eta) T(\mathbf{q}, \mathbf{p}', \mathbf{P}, \omega + i\eta). \quad (6.2.14)$$

Usual decomposition on the dependence of relative and center of mass variables of the two particles and the angular momentum decomposition does not work. Therefore, the above equation should be solved in three dimensional integral equation, which is numerically a heavy task. Here we introduce average of the angle between relative and center of mass momentum so that the partial wave decomposition as usual two-particle scattering system can be done. We introduce the angle averaged two-particle Green's function as

$$\mathcal{G}_2^0(\mathbf{p}, \mathbf{P}, \omega + i\eta) \sim \bar{\mathcal{G}}_2^0(p, P, \omega + i\eta) \quad (6.2.15)$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{1 - f_1(\omega) - f_2(\omega')}{\Omega - \omega - \omega' + i\eta} \langle A_1(p, P, \omega) A_2(p, P, \omega') \rangle, \quad (6.2.16)$$

where the angle averaged spectral functions is given as

$$\langle A_1(p, P, \omega) A_2(p, P, \omega') \rangle \equiv \frac{1}{2} \int_{-1}^1 d(\cos \theta_{pP}) A_1(\mathbf{P}/2 + \mathbf{p}, \omega) A_2(\mathbf{P}/2 - \mathbf{p}, \omega'). \quad (6.2.17)$$

With this simplification, the Lippmann-Schwinger equation can be written as a one-dimensional integral equation for total(J), spin(S) and orbital(L) angular momentum state

$$\begin{aligned} & T^J(p, p', P, \omega + i\eta; L'S', LS) \\ &= V^J(p, p'; L'S', LS) \\ &+ \sum_{\bar{L}\bar{S}} \int_0^\infty dq q^2 V^J(p, q; LS, \bar{L}\bar{S}) \bar{\mathcal{G}}_2^0(q, P, \omega + i\eta) T^J(q, p', P, \omega + i\eta; \bar{L}\bar{S}, LS). \end{aligned} \quad (6.2.18)$$

6.3 Self-energy in ladder approximation of two-particle Green's function

We will derive formula of the the self-energy in terms of T-matrix for the practical evaluation. From KMS relation, the Fourier expansion of Eq. (6.1.5) gives

$$\begin{aligned}
\Sigma(\mathbf{p}_1, z_\nu) &= \frac{-i}{-i\beta} \sum_{\nu'} \int d^3 p_2 [T(\mathbf{p}, \mathbf{p}, \mathbf{P}, z_\nu + z_{\nu'}) - T(\mathbf{p}, -\mathbf{p}, \mathbf{P}, z_\nu + z_{\nu'})] \tilde{G}_1(\mathbf{p}_2, z_{\nu'}) \\
&\equiv \frac{-i}{-i\beta} \sum_{\nu'} \int d^3 p_2 T_{ex}(\mathbf{p}, \mathbf{p}, \mathbf{P}, z_\nu + z_{\nu'}) \tilde{G}_1(\mathbf{p}_2, z_{\nu'}).
\end{aligned} \tag{6.3.1}$$

The self-energy can be decomposed into the Hartree-Fock term Σ_{HF} and the collision term Σ_c ,

$$\Sigma(\mathbf{p}_1, z_\nu) = \Sigma_{HF}(\mathbf{p}_1) + \Sigma_c(\mathbf{p}_1, z_\nu). \tag{6.3.2}$$

The Hartree-Fock term is due to the first potential term of Eq. (6.1.22) and is independent on z_ν given as

$$\begin{aligned}
\Sigma_{HF}(\mathbf{p}_1) &= \frac{-i}{-i\beta} \int d^3 p_2 V_{ex}(\mathbf{p}, \mathbf{p}) \sum_{\nu'} \tilde{G}_1(\mathbf{p}_2, z_{\nu'}) \\
&\equiv \int d^3 p_2 V_{ex}(\mathbf{p}, \mathbf{p}) n(\mathbf{p}_2),
\end{aligned} \tag{6.3.3}$$

where $n(\mathbf{p}_2) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A(\mathbf{p}_2, \omega) f(\omega)$ and $V_{ex}(\mathbf{p}, \mathbf{p})$ includes exchange term given by exchanging momentum and spin of the two particles.

The second term of Eq. (6.1.22) gives Σ_c given as

$$\begin{aligned}
\Sigma_c(\mathbf{p}_1, z_\nu) &= -\frac{-i}{-i\beta} \sum_{\nu_2} \int d^3 p_2 \int_{-\infty}^{\infty} \frac{d\Omega}{\pi} \frac{\text{Im} T_{ex}(\Omega + i\eta)}{z_\nu - z_{\nu'} - \Omega} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{A(\mathbf{p}_2, \omega)}{z_{\nu'} - \omega} \\
&= -\frac{1}{\beta} \int d^3 p_2 \int_{-\infty}^{\infty} \frac{d\Omega}{\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A(\mathbf{p}_2, \omega) \text{Im} T_{ex}(\Omega + i\eta) F(z_\nu, \Omega, \omega),
\end{aligned} \tag{6.3.4}$$

where the momentum variables for the T-matrix are not written for simplicity. Here $F(z_\nu, \Omega, \omega)$ is given by the sum of Matsubara frequency as(see Appendix B)

$$F(z_\nu, \Omega, \omega) = \sum_{\nu'} \frac{1}{z_{\nu'} + z_\nu - \Omega} \frac{1}{z_{\nu'} - \omega} \tag{6.3.5}$$

$$= \beta \frac{f(\omega) + g(\Omega)}{z_\nu + \omega - \Omega}. \tag{6.3.6}$$

Here the Bose distribution function $g(\Omega)$ is defined in Eq. (6.1.14). The sum of two terms is given as

$$\begin{aligned}\Sigma(\mathbf{p}_1, z_\nu) &= \int d^3 p_2 V_{ex}(\mathbf{p}, \mathbf{p}) n(\mathbf{p}_2) \\ &\quad - \int d^3 p_2 \int_{-\infty}^{\infty} \frac{d\Omega}{\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A(\mathbf{p}_2, \omega) \text{Im} T_{ex}(\Omega + i\eta) \frac{f(\omega) + g(\Omega)}{z_\nu + \omega - \Omega}.\end{aligned}\quad (6.3.7)$$

We analytic continue z_ν to the real axis $\omega + i\eta$ and then the imaginary part of the self-energy is given as

$$\text{Im}\Sigma(\mathbf{p}_1, \omega + i\eta) = \int d^3 p_2 \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} A(\mathbf{p}_2, \omega') \text{Im} T_{ex}(\omega + \omega' + i\eta) [f(\omega') + g(\omega + \omega')]. \quad (6.3.8)$$

Finally, the real-part of the self-energy can be obtained by using the dispersion relation, Eq. (5.3.14) as

$$\text{Re}\Sigma(\mathbf{p}_1, \omega) = \Sigma_{HF}(\mathbf{p}_1) - P \int_{-\infty}^{\infty} \frac{d\lambda}{\pi} \frac{\text{Im}\Sigma(\mathbf{p}_1, \lambda + i\eta)}{\omega - \lambda}. \quad (6.3.9)$$

Alternatively, one can write down by the spectral function A and T-matrix. Substituting Eq. (6.3.8) into Eq. (6.3.9) and writing an explicit form of Σ_{HF} , Eq. (6.3.3), the real-part of the self-energy follows

$$\begin{aligned}\text{Re}\Sigma(\mathbf{p}_1, \omega) &= \int d^3 p_2 \left[V_{ex}(\mathbf{p}, \mathbf{p}) \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} A(\mathbf{p}_2, \omega') f(\omega') \right. \\ &\quad \left. - \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} A(\mathbf{p}_2, \omega') \left\{ f(\omega') P \int_{-\infty}^{\infty} \frac{d\lambda'}{\pi} \frac{\text{Im} T_{ex}(\lambda' + i\eta)}{\omega + \omega' - \lambda'} + P \int_{-\infty}^{\infty} \frac{d\lambda'}{\pi} \frac{g(\lambda') \text{Im} T_{ex}(\lambda' + i\eta)}{\omega + \omega' - \lambda'} \right\} \right],\end{aligned}\quad (6.3.10)$$

where a variable transformation $\lambda' = \lambda + \omega$ has introduced. The second term, involving the Fermi distribution function times principal integration of T-matrix, can decompose using the dispersion relation of T-matrix (6.1.24) such as

$$-P \int_{-\infty}^{\infty} \frac{d\lambda}{\pi} \frac{\text{Im} T_{ex}(\lambda + i\eta)}{\omega - \lambda} = \text{Re} T_{ex}(\omega) - V, \quad (6.3.11)$$

then the potential dependent Σ_{HF} term is canceled and the following expression can be obtained

$$\begin{aligned}\text{Re}\Sigma(\mathbf{p}_1, \omega) &= \int d^3 p_2 \left[\int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} A(\mathbf{p}_2, \omega') \left\{ f(\omega') \text{Re} T_{ex}(\omega + \omega') - P \int_{-\infty}^{\infty} \frac{d\lambda'}{\pi} \frac{g(\lambda') \text{Im} T_{ex}(\lambda' + i\eta)}{\omega + \omega' - \lambda'} \right\} \right].\end{aligned}\quad (6.3.12)$$

6.4 Self-consistent iterative scheme

Now all necessary formula for the analysis of the two-nucleon system in nuclear medium are ready. We describe the procedure to use those formulae. Here we explain 'self-consistent iterative scheme' to obtain the spectral function and self-energy.

1. Initial spectral function

At first, we start from a free particle spectral function:

$$A_i(\mathbf{p}, \omega) = 2\pi\delta(\omega - E_0(p)), \quad (6.4.1)$$

where $E_0(p) = p^2/2m$ is the free particle kinetic energy.

2. Chemical potentials

The chemical potential can be obtained from

$$n_i = \frac{X_i \rho}{m} = (2s + 1) \int \frac{d^3p}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A_i(\mathbf{p}, \omega) f_i(\omega), \quad (6.4.2)$$

where s is the spin of the particle. Here the baryon mass density ρ , the temperature T and the mass fraction X_i of particle i are input for our analysis depending on the region of our interest of supernova explosion.

3. $\mathcal{G}_2^0$

The un-correlated two-body Green's function $\mathcal{G}_2^0$ can be computed using given spectral function and chemical potential

$$\mathcal{G}_2^0(\mathbf{p}, \mathbf{P}, \Omega + i\eta) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{1 - f_1(\omega) - f_2(\omega')}{\Omega - \omega - \omega' + i\eta} A_1(\mathbf{p}_1, \omega) A_2(\mathbf{p}_2, \omega'), \quad (6.4.3)$$

where momenta are related by Eq. (6.2.12) and (6.2.13).

4. T-matrix

The T-matrix can be obtained by solving the Lippmann-Schwinger equation involving angle-averaged non-correlated two-body Green's function defined by Eq. (6.2.16):

$$T(\mathbf{p}, \mathbf{p}', \mathbf{P}, \omega + i\eta) = V(\mathbf{p}, \mathbf{p}') + \int d^3q V(\mathbf{p}, \mathbf{q}) \bar{\mathcal{G}}_2^0(q, P, \omega + i\eta) T(\mathbf{q}, \mathbf{p}', \mathbf{P}, \omega + i\eta). \quad (6.4.4)$$

5. Self-energy

The real- and imaginary- parts of the self-energy can be computed by the T-matrix obtained at the previous step,

$$\text{Im}\Sigma(\mathbf{p}_1, \omega + i\eta) = \int d^3p_2 \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} A(\mathbf{p}_2, \omega') \text{Im}T_{ex}(\omega + \omega' + i\eta) [f(\omega') + g(\omega + \omega')] , \quad (6.4.5)$$

$$\begin{aligned} & \text{Re}\Sigma(\mathbf{p}_1, \omega) \\ &= \int d^3p_2 \left[\int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} A(\mathbf{p}_2, \omega') \left\{ f(\omega') \text{Re}T_{ex}(\omega + \omega') - P \int_{-\infty}^{\infty} \frac{d\lambda'}{\pi} \frac{g(\lambda') \text{Im}T_{ex}(\lambda' + i\eta)}{\omega + \omega' - \lambda'} \right\} \right] . \end{aligned} \quad (6.4.6)$$

6. Spectral function

The resultant spectral function involving self-energy can be obtained as

$$A(\mathbf{p}_1, \omega) = \frac{-2\text{Im}\Sigma(\mathbf{p}_1, \omega + i\eta)}{\left[\omega - \frac{p_1^2}{2m} - \text{Re}\Sigma(\mathbf{p}_1, \omega) \right]^2 + [\text{Im}\Sigma(\mathbf{p}_1, \omega + i\eta)]^2} . \quad (6.4.7)$$

7. Iterations

Note the obtained spectral function is the new input to be used in the step 2. The step 2 to 6 will be repeated until we get convergent results.

Chapter 7

Quasi-particle approximation and neutrino emissivity

The purpose of this chapter is to examine the role of nuclear medium for the neutrino reactions of a few nucleon system to find the limitation or valid density region of the application the reaction rates studied in part I. The medium effects summarized in terms of self-energy Σ are modification of the energy spectra from the real part of the self-energy and the finite life time or width Γ from the imaginary part of the self-energy appearing in the spectral function (6.4.7).

As the first approximation, we employ the small width ($\Gamma \ll |Re(\Sigma)|$) approximation called 'quasi-particle approximation' where quasi-particle has long life time. Here we will make further approximation $\omega = p^2/(2m_N)$ to evaluate the self-energy, where only the first step of the iterative procedure is taken. Within this approximation, we examine the deuteron states by searching the pole of the T-matrix in the medium. Here we take into account Pauli exclusion and shift of the single particle energy due to the interaction of the nucleon with the other nucleon in the medium. We also examined modification of the emissivity of the electron capture.

The small width approximation can be relaxed while keeping $\omega = p^2/(2m_N)$ approximation. We estimate the effect of the imaginary part of the self-energy and study the pole of the T-matrix on the physical plane energy plane. Here by taking into account for the finite life time of the single particle excitation, we examine the life time of deuteron in medium compared with its binding energy.

7.1 Quasi-particle approximation

In the quasi-particle approximation, we use a limit of small imaginary part of the self-energy. This approximation might be valid only in the low density region. We approximate the spectral function as

$$A(\mathbf{p}, \omega) = 2\pi\delta(\omega - E(\mathbf{p})), \quad (7.1.1)$$

where the one-particle energy is evaluated from the self-energy:

$$E(\mathbf{p}) = \frac{\mathbf{p}^2}{2m} + \text{Re}\Sigma(\mathbf{p}, E(\mathbf{p})). \quad (7.1.2)$$

Within this approximation, the self-energy is obtained from

$$\begin{aligned} \text{Im}\Sigma(\mathbf{p}_1, \omega + i\eta)|_{\omega=E(p_1)} = \\ \int d^3p_2 \text{Im}T_{ex}(\omega + E(p_2) + i\eta) [f(E(p_2)) + g(\omega + E(p_2))] |_{\omega=E(p_1)}. \end{aligned} \quad (7.1.3)$$

$$\begin{aligned} \text{Re}\Sigma(\mathbf{p}_1, \omega)|_{\omega=E(p_1)} \\ = \int d^3p_2 \left[\left\{ f(E(p_2)) \text{Re}T_{ex}(\omega + E(p_2)) - P \int_{-\infty}^{\infty} \frac{d\lambda'}{\pi} \frac{g(\lambda') \text{Im}T_{ex}(\lambda' + i\eta)}{\omega + E(p_2) - \lambda'} \right\} \right] |_{\omega=E(p_1)}. \end{aligned} \quad (7.1.4)$$

We start from the spectrum of a free particle $E(p) = p^2/(2m)$ and go through the step 1 to 6 within the quasi-particle approximation. The obtained single particle energy $E(p)$ is used for the steps 1 to 4 of the second iteration procedure. We investigate the properties of the deuteron, scattering state and neutrino emissivity by using the Lippmann-Schwinger equation of the second iteration procedure. In this approximation, the two-particle bound state appear as a pole of the T-matrix at the energy below the scattering state.

In each step of the above procedure, we need to transform variables into relative and center of mass variables as one faces always in the many body problem. In our two-particle system, we use the following average of the angle between relative and center of mass momentum. The two particle Green's function $\bar{\mathcal{G}}_2^0$ is given as

$$\bar{\mathcal{G}}_2^0(p, P, \omega + i\eta) = \frac{\langle Q(p, P) \rangle}{\omega - \langle E(p, P) \rangle + i\eta}, \quad (7.1.5)$$

where we perform angle average for each Pauli blocking and two-body quasi-particle energy

independently:

$$\langle Q(p, P) \rangle = \int \frac{d\Omega}{4\pi} [1 - f_1(\mathbf{p}, \mathbf{P}) - f_2(\mathbf{p}, \mathbf{P})] \quad (7.1.6)$$

$$\langle E(p, P) \rangle = \int \frac{d\Omega}{4\pi} [E_1(\mathbf{p}, \mathbf{P}) + E_2(\mathbf{p}, \mathbf{P})] \quad (7.1.7)$$

$$= \frac{p^2}{2\mu} + \frac{P^2}{2M} + \langle U_{12}(p, P) \rangle, \quad (7.1.8)$$

where we defined

$$\langle U_{12}(p, P) \rangle = \int \frac{d\Omega}{4\pi} [\text{Re}\Sigma_1(\mathbf{p}, \mathbf{P}) + \text{Re}\Sigma_2(\mathbf{p}, \mathbf{P})]. \quad (7.1.9)$$

Ω represents the angle between the relative and center-of-mass momenta:

$$d\Omega = 2\pi d\cos\theta_{pP}, \quad \cos\theta_{pP} = \hat{\mathbf{p}} \cdot \hat{\mathbf{P}}. \quad (7.1.10)$$

7.2 Effective Schrödinger equation

To evaluate neutrino emissivity, it is useful to introduce wave function φ for bound state and scattering state in-medium effective Schrödinger equation.

$$(E(p, P) - E)\varphi(\mathbf{p}, \mathbf{P}) = -Q(\mathbf{p}, \mathbf{P}) \int V_\alpha(\mathbf{p}, \mathbf{p}')\varphi(\mathbf{p}', \mathbf{P})d\mathbf{p}'. \quad (7.2.1)$$

The T-matrix and the scattering wave function $\varphi(\mathbf{p}, \mathbf{P})$ is related as

$$\varphi(\mathbf{p}, \mathbf{P}) = \delta(\mathbf{p} - \mathbf{p}_0) + \mathcal{G}_2(\mathbf{P}, \mathbf{p}, E(p_0, P))T(\mathbf{p}, \mathbf{p}_0, E(p_0, P)) \quad (7.2.2)$$

while T-matrix is related to the bound state pole using the wave function $\varphi_B(\mathbf{p}, \mathbf{P})$ as

$$T(\mathbf{p}, \mathbf{p}', E) = \frac{\varphi_B(\mathbf{p}, \mathbf{P})^\dagger \varphi_B(\mathbf{p}', \mathbf{P})}{E - E_B(P)}. \quad (7.2.3)$$

From now we consider 3S_1 and 1S_0 partial waves which are the most important nucleon-nucleon partial wave states in low-energy. The deuteron is in the 3S_1 channel. We use a separable potential for V that reproduces well the low energy nucleon-nucleon scattering phase shift and the binding energy of the deuteron [55]. The potential is given as

$$V_\alpha(p, p') = -4\pi \frac{\lambda_\alpha}{2\mu} g_\alpha(p) g_\alpha(p'), \quad (7.2.4)$$

$$g_\alpha(p) = \frac{1}{p^2 + b^2} \quad (7.2.5)$$

$$\lambda_\alpha = \frac{b(b+a)^2}{\pi}, \quad (7.2.6)$$

where μ is the reduced mass of the two nucleons. With this simple potential, the T-matrix is given in a compact form as

$$T_\alpha(p, p', P, E) = g_\alpha(p) \left[\frac{\lambda_\alpha}{1 - \lambda_\alpha J_\alpha(P, E)} \right] g_\alpha(p'), \quad (7.2.7)$$

$$J_\alpha(P, E) = \int_0^\infty dq \frac{q^2 g_\alpha^2(q) \langle Q(q, P) \rangle}{E - E(q, P) + i\eta}. \quad (7.2.8)$$

The bound state can be found by searching for a pole of the T-matrix. The bound state condition is given as

$$1 - \lambda_\alpha J_\alpha(P, E_d) = 0. \quad (7.2.9)$$

Here the deuteron energy is $E_d(P)$ and the binding energy is $-B(P) > 0$ which are evaluated from the continuum edge ($E(0, P)$) as

$$E_d(P) = \frac{P^2}{2M} + \Delta(P) \quad (7.2.10)$$

$$-B(P) = \Delta(P) - \langle U_{pn}(0, P) \rangle. \quad (7.2.11)$$

The deuteron effective wave function can be written in the following form

$$\varphi_{t,d}(p, P) = N_{t,d}(P) \frac{g_t(p) \langle Q_{pn}(p, P) \rangle}{\Delta(P) - \frac{p^2}{2\mu} - \langle U_{pn}(p, P) \rangle} \quad (7.2.12)$$

$$\equiv N_{t,d}(P) F_{t,d}(p, P), \quad (7.2.13)$$

where $N_{t,d}$ is the normalization constant

$$N_{t,d}(P) = \left[\int_0^\infty dp p^2 [F_{t,d}(p, P)]^2 \right]^{-1/2} \quad (7.2.14)$$

The scattering wave function of the 1S_0 channel can be written with the Green's function and T-matrix as

$$\varphi_s^{p_o}(p, P) = \sqrt{\frac{\pi}{2}} \left[\frac{\delta(p - p_o)}{p^2} + \mathcal{G}_2^0(p, P) T_{s,sc}(p, p_o, P, E(p_o, P)) \right], \quad (7.2.15)$$

where p_o denotes the on-energy shell relative momentum. Using the explicit form of the separable T-matrix (7.2.7), one finds

$$\varphi_s^{p_o}(p, P) = \sqrt{\frac{\pi}{2}} \left[\frac{\delta(p - p_o)}{p^2} + F_{s,sc}(p, p_o, P) D(p_o, P) g_s(p_o) \right] \quad (7.2.16)$$

$$F_{s,sc}(p, p_o, P) \equiv \frac{g_s(p) \langle Q_{NN}(p, P) \rangle}{\frac{p_o^2}{2\mu} + \langle U_{NN}(p_o, P) \rangle - \frac{p^2}{2\mu} - \langle U_{NN}(p, P) \rangle + i\eta} \quad (7.2.17)$$

$$D^{-1}(p_o, P) \equiv \frac{1}{\lambda_s} - \int_0^\infty d\bar{p} \bar{p}^2 F_{s,sc}(\bar{p}, p_o, P) g_s(\bar{p}). \quad (7.2.18)$$

The phase shift δ for the two-nucleon scattering state is defined as

$$\tan \delta = \frac{\text{Im}T(p_o, p_o, P, E(p_o, P))}{\text{Re}T(p_o, p_o, P, E(p_o, P))}. \quad (7.2.19)$$

7.3 Neutrino emissivity

Now we estimate these in-medium effect on the neutrino emissivity for the deuteron reactions. We focus on charged current electron capture on the deuteron which is one of the major reactions for the neutrino cooling as we have discussed in Part I.

$$d + e^- \rightarrow n + n + \nu_e. \quad (7.3.1)$$

Here we use a simple setup of impulse nuclear current, s-wave scattering state and s-wave deuteron wave function to examine the effects of nuclear medium. For this case the deuteron break-up process is given by the Gamow-Teller transition of 3S_1 deuteron to the 1S_0 scattering state. The emissivity is proportional to the transition amplitude

$$\begin{aligned} & \int \frac{d\Omega_{nn}}{4\pi} \sum_{spin's} |\langle nn({}^1S_0), \nu_e(\mathbf{p}_\nu) | H_W^{CC} | d({}^3S_1), e^-(\mathbf{p}_e) \rangle|^2 \\ &= \left[\frac{G_F V_{ud}}{\sqrt{2}} \right]^2 2(4\pi)^2 \frac{g_A^2}{\pi} \left(3 - \frac{\mathbf{p}_e}{E_e} \cdot \hat{\mathbf{p}}_\nu \right) \mathcal{I}^2, \end{aligned} \quad (7.3.2)$$

where $\mathcal{I}$ is the overlap integral of the initial and final two-nucleon states given as

$$\mathcal{I} = \int_0^\infty dr r^2 \varphi_{s,sc}(r) j_0(qr/2) \varphi_{t,d}(r). \quad (7.3.3)$$

Here $q = |\mathbf{q}| = |\mathbf{p}_e - \mathbf{p}_\nu|$ is the momentum transfer to the nucleon system from the lepton.

The integral $\mathcal{I}$ can be written with the momentum-space wave functions as

$$\begin{aligned} \mathcal{I}(p_o, q, P) &= \sqrt{\frac{\pi}{2}} \left[\langle \varphi_{t,d}(\mathbf{p}_o - \mathbf{q}/2, P) \rangle \right. \\ &\quad \left. + D(p_o, P) g_s(p_o) \int dp p^2 F_{s,sc}(p_o, p, P) \langle \varphi_{t,d}(\mathbf{p} - \mathbf{q}/2, P) \rangle \right], \end{aligned} \quad (7.3.4)$$

where

$$\langle \varphi_{t,d}(\mathbf{p} - \mathbf{q}/2, P) \rangle = \frac{1}{2} \int_{-1}^1 d(\cos \theta_{pq}) \varphi_{t,d}(\mathbf{p} - \mathbf{q}/2, P). \quad (7.3.5)$$

It is noticed that the same overlap integral can be used to evaluate the neutrino absorption reaction $\nu + d \rightarrow N + N + l$ in the neutrino heating region. Therefore the results of our analysis also gives suggestion to the modification of neutrino heating rate due to the medium effects.

	$a[\text{fm}^{-1}]$	$b[\text{fm}^{-1}]$
3S_1	0.231713	1.40552
1S_0	-0.039925	1.17712

Table 7.1: Parameters of nucleon-nucleon interaction in the separable approximation [56].

With the same procedure of calculating neutrino emissivity, presented in Part I, we obtain the neutrino emissivity $Q_{d,nn}$ with medium modification as

$$\begin{aligned}
Q_{d,nn} = & \frac{G_F^2 V_{ud}^2 g_A^2}{2\pi^6} \int dP f_d(P) \int dp_e f_e(p_e) \int dp \int_{-1}^1 dx_{e\nu} P^2 p_e^2 p_\nu^3 \\
& \times \left(3 - \frac{p_e}{E_e} x_{e\nu} \right) \langle (1 - f_n(\mathbf{P}/2 - \mathbf{p}))(1 - f_n(\mathbf{P}/2 + \mathbf{p})) \rangle \mathcal{I}^2(p, q, P),
\end{aligned} \tag{7.3.6}$$

with the energy conservation

$$\Delta(P) + E_e(p_e) = \frac{p^2}{2\mu} + \langle U_{nn}(p, P) \rangle + p_\nu. \tag{7.3.7}$$

In the following, we do not use the iso-spin formalism, since proton and neutron densities are very different in the environment of supernova explosion. The parameters of the separable potential are given in Table 7.1 for the triplet 3S_1 and the singlet 1S_0 two nucleon states.

7.3.1 Nucleon self-energy

The real part of the self-energy is decomposed into the contribution of various angular momentum states. For the identical particles such as $p - p$ and $n - n$, the exchange term of the T-matrix contributes. The self-energy of proton(neutron) due to the $p - p(n - n)$ interaction is given as

$$U_N(NN, p_1) = \sum_{LSJ} U_N(NN, LSJ, p_1) \tag{7.3.8}$$

where

$$U_N(NN, LSJ, p_1) \equiv \text{Re}\Sigma(p_1, E(p_1)) \quad (7.3.9)$$

$$\begin{aligned} &= (2J+1) \frac{1 - (-1)^{1+l+S}}{4} \int dp_2 p_2^2 \int d(\cos\theta) \\ &\quad \left\{ f_2[E(p_2)] \text{Re}T^J(p, P, E(p_1) + E(p_2); LS, LS) \right. \\ &\quad \left. + \int_{-\infty}^{\infty} \frac{dE}{\pi} g(E) \frac{\mathcal{P}}{E - E(p_1) - E(p_2)} \text{Im}T^J(p, P, E + i\eta; LS, LS) \right\}. \end{aligned} \quad (7.3.10)$$

The exchange term of the T-matrix appears as the factor $(-1)^{1+l+S}$. Here we take into account only 1S_0 channel. The self-energy due to $p-n$ interaction includes 1S_0 and 3S_1 channel given as

$$\begin{aligned} U_N(NN', LSJ, p_1) &= (2J+1) \frac{1}{4} \int dp_2 p_2^2 \int d(\cos\theta) \\ &\quad \left\{ f_2[E(p_2)] \text{Re}T^J(p, P, E(p_1) + E(p_2); LS, LS) \right. \\ &\quad \left. + \int_{-\infty}^{\infty} \frac{dE}{\pi} g(E) \frac{\mathcal{P}}{E - E(p_1) - E(p_2)} \text{Im}T^J(p, P, E + i\eta; LS, LS) \right\}. \end{aligned} \quad (7.3.11)$$

Adding the above contributions, we obtain the self-energy for the proton and neutron, respectively

$$U_p(p) = U_p(pn, ^3S_1, p) + U_p(pn, ^1S_0, p) + U_p(pp, ^1S_0, p) \quad (7.3.12)$$

$$U_n(p) = U_n(np, ^3S_1, p) + U_n(np, ^1S_0, p) + U_n(nn, ^1S_0, p). \quad (7.3.13)$$

Fig. 7.1 shows the momentum dependence of the self-energy at $r = 12.4, 15.0$ and 20km of the Composition II for proton and neutron. The self-energy of neutron becomes slightly larger than that of proton for outer r region. The matter density is about $0.17, 0.1$ and $0.05\rho_0$ at $r = 12.4, 15.0$ and 20km , respectively. Since the density increase with decreasing radius, the self-energy increases and $U_p(0) \sim -6\text{MeV}$ at $r = 12.4\text{km}$. As p increases, the self-energy decreases due to the dependence on relative momentum of the nucleon-nucleon interaction and higher momentum component of nucleons.

We examine the role of each mechanisms to the self-energy. The contribution of the 3S_1 and 1S_0 waves are almost the same magnitude for the proton self-energy at $r = 12.4\text{km}$ as shown in the left panel of Fig. 7.2. The first term of Eqs. (7.3.10) and (7.3.11), is the

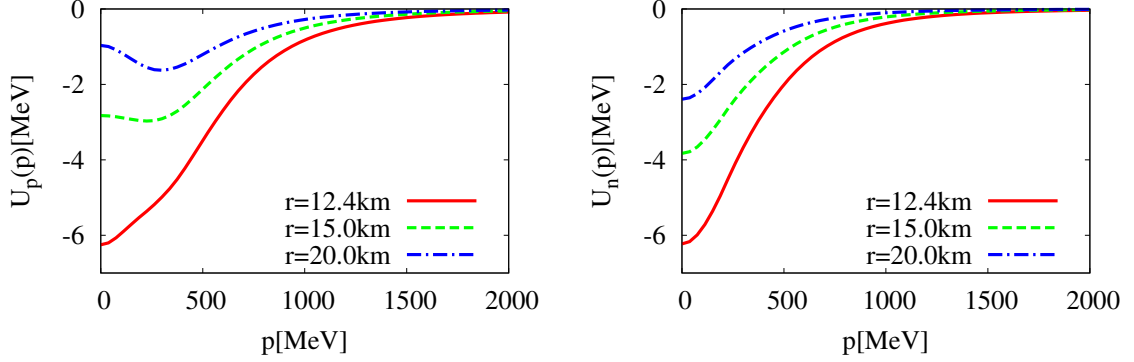


Figure 7.1: Momentum dependence of the real-part of the self-energy. Self-energy for the proton(left) and neutron(right) at $r = 12.4$ km, 15.0km and 20.0km are shown in solid(red), dashed(green) and dash-dotted(blue) curves.

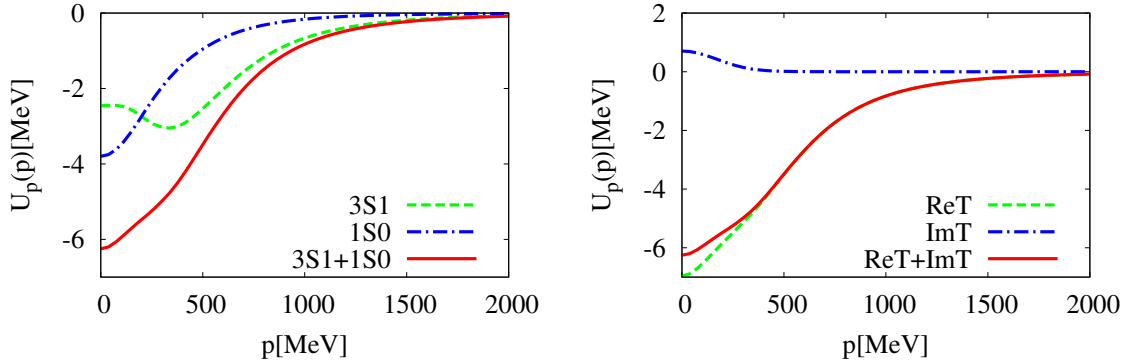


Figure 7.2: Self-energy of proton at $r = 12.5$ km. The left panel shows the contribution of 3S_1 (green dashed curve) and 1S_0 (blue dash-dotted curve) nucleon-nucleon interaction and the sum of the two partial waves(red solid curve). The right panel shows contribution of the first(green dash curve) and the second term(blue dash-dotted curve) in the bracket of Eqs. (7.3.10) and (7.3.11) and the sum of the two-terms(red solid curve).

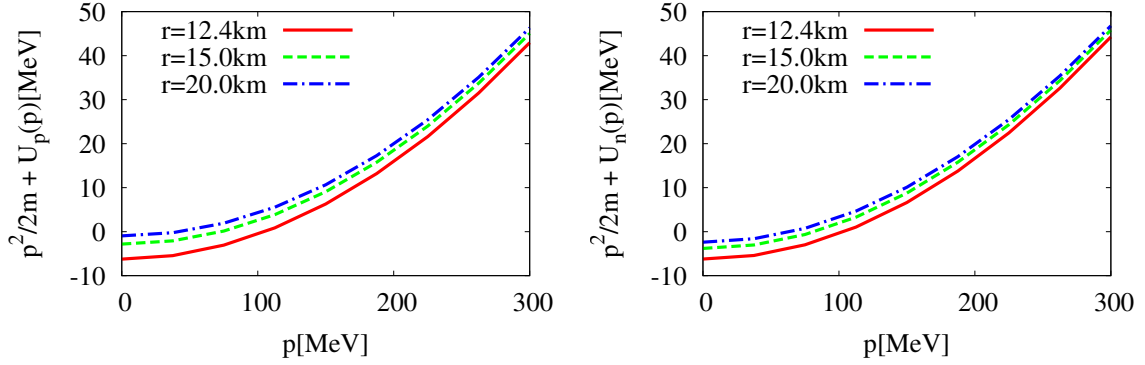


Figure 7.3: Momentum dependence of single particle energy at $r = 12.4\text{km}$ (red solid curve), 15.0km (green dashed curve) and 20.0km (blue dash-dotted curve) for the proton(left) and neutron(right).

main contribution to the self-energy for this case as shown in the right panel of Fig. 7.2. For neutron self-energy, situation is the same.

The single particle energy $E_i(p) = p^2/(2m) + U_i(p)$ is sum of kinetic energy and the calculated self-energy, which are shown in Fig. 7.3. The single particle energy increases monotonically as the momentum increases. It might be possible to parametrize the single-particle energy using the effective mass and the constant energy shift. Here we used the full momentum dependence of the calculated self-energy for the calculation of bound state and scattering state. The energy of the two-nucleon system for relative momentum p and center of mass momentum P is given as $E(p, P) = \frac{P^2}{2M} + \langle U_{12}(p, P) \rangle$. For given center of mass momentum P , the continuum state start at the minimum of this energy

$$E(p, P)|_{min} = E(0, P) = \frac{P^2}{2M} + \langle U_{12}(0, P) \rangle. \quad (7.3.14)$$

$E(0, P)$ is the energy of continuum edge and the binding energy is measured from the continuum edge.

7.3.2 Bound state

By using the self-energy calculated in the previous section, we examined the deuteron bound state in 3S_1 channel.

The calculated deuteron binding energy is shown in Fig. 7.4. The deuteron binding energy decreases as the matter density increases. At $r = 11.7\text{km}$, deuteron at rest does not

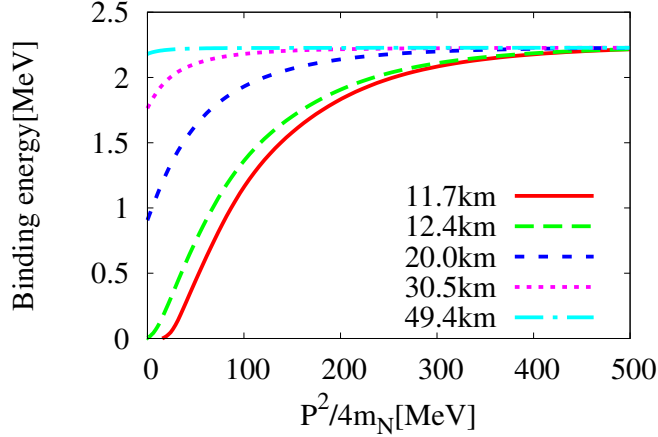


Figure 7.4: Deuteron binding energy as a function of the two-nucleon center-of-mass kinetic energy $P^2/4m_N$ at $r=11.7\text{km}$ (red solid curve), $r=12.4\text{km}$ (green long-dashed curve), $r=20.0\text{km}$ (blue short-dashed curve), $r=30.5\text{km}$ (magenta dotted curve) and $r=49.4\text{km}$ (turquoise dash-dotted curve).

exist and the binding energy increases $P^2/4m_N \sim 12\text{MeV}$ as the center of mass kinetic energy increases. The main effect of the weak binding of deuteron is due to the Pauli blocking factor $1 - f_p(p_p) - f_n(p_n)$ in the effective Schrödinger equation, which forbids the low momentum component of deuteron. However as the deuteron moves with larger momentum in the medium, the Pauli blocking becomes less effective and the binding energy approaches to that of vacuum. At $r = 30\text{km}$, where $\rho \sim 0.01\rho_0$, medium effect is still seen in the bound state energy. At $r = 50\text{km}$, the modification of binding energy in the medium is small. This suggests that deuteron heating mechanism calculated with the free space deuteron may be safe in the heating region of supernova explosion.

7.3.3 Neutrino emissivity

At first we compare the emissivity evaluated from a simple separable potential and realistic ANLV18 potential in the 5th and 6th column of Table 7.2, where the radius, matter density and temperature for the Composition II are given in the 1st, 2nd and 3rd column. We used impulse approximation and kept the s-wave component of the deuteron and the scattering state. The emissivities due to two models of nuclear potential differ about 20%. The separable potential will be good enough for the first study of the medium effects

r[km]	$\rho[\text{g}/\text{cm}^3]$	T[MeV]	med,sep.	free,sep.	free,AV18
12.4	0.47220×10^{14}	25.15	0.36620×10^{40}	0.60557×10^{40}	0.75702×10^{40}
15.0	0.27993×10^{14}	23.27	0.11029×10^{40}	0.13379×10^{40}	0.16531×10^{40}
19.7	0.14610×10^{14}	17.37	0.11704×10^{39}	0.12577×10^{39}	0.15228×10^{39}
30.5	0.25788×10^{13}	8.97	0.39009×10^{36}	0.38131×10^{36}	0.43504×10^{36}
49.4	0.13967×10^{12}	4.42	0.66458×10^{32}	0.65975×10^{32}	0.69643×10^{32}

Table 7.2: Neutrino emissivity for $d + e^- \rightarrow n + n + \nu_e$ for the Composition II in unit of $[\text{erg}/\text{cm}^3/\text{sec}]$. The second and the third column show the matter density and temperature. The 4th(med,sep.), 5th(free,sep.) and 6th(free,AV18) column shows the emissivity calculated with separable potential, separable potential without medium effect and ANLV18 [57] potential without medium effect, respectively.

on emissivity. Comparing the emissivities on the 4th and 5th column of Table 7.2, the medium effect reduces the emissivity about 40% and 20% from free case at $r = 12.4\text{km}$ and 15km . The reduction factor looks small in spite of rather large modification of the binding energy for deuteron at rest. However, it might be understood that rather large part of the contribution is from the moving deuteron, where the modification of medium is less effective. At $r = 50\text{km}$, we found medium effect is almost negligible. Therefore one expects that the deuteron reaction rates evaluated in [17] for the neutrino heating region might not be significantly affected by the medium effects.

7.3.4 Beyond the quasi-particle approximation

We have studied the two-nucleon bound state in the medium within the quasi-particle approximation. Here we estimate the imaginary part of the self-energy and its effects on the bound state energy as a first step to improve our previous study. In order to do this, we start from the self-energy in the approximation $\omega = p^2/(2m)$. Within this approximation, the spectral function (6.4.7) is given as

$$A_i(p, \omega) = \frac{\Gamma_i(p)}{(\omega - E_i(p))^2 + \Gamma_i^2/4}, \quad (7.3.15)$$

where $E_i(p)$ and $\Gamma_i(p)$ are calculated from the self-energy as

$$E_i(p) = \frac{p^2}{2m} + \text{Re}\Sigma_i(p, p^2/(2m) + i\epsilon), \quad (7.3.16)$$

$$\Gamma_i(p) = -2\text{Im}\Sigma_i(p, p^2/(2m) + i\epsilon) \quad (7.3.17)$$

The two-nucleon Green's function (6.4.3) can be evaluated using the above formula. Here we use approximation which will be valid for the non-degenerate situation, where we approximate $1 - f_1(\omega) - f_2(\omega') \sim 1$. Then the Green's function is given as

$$\mathcal{G}_2^0(p, P, \Omega + i\epsilon) \sim \frac{1}{\Omega - E_1(p_1) - E_2(p_2) - \text{Im}(\Sigma_1(p_1, p_1^2/(2m) + i\epsilon) + \Sigma_2(p_2, p_2^2/(2m) + i\epsilon))} \quad (7.3.18)$$

With this Green's function we solve the Lippmann-Schwinger equation (6.4.4) and find the pole as Eq. (7.2.9) where the two-body quasi-particle energy is replaced by complex value $E_1(p_1) + E_2(p_2) + \text{Im}(\Sigma_1(p_1) + \Sigma_2(p_2))$. Now the poles of the T-matrix below the real energy axis can be interpreted as resonance energy.

Fig. 7.5 shows that the momentum dependence of the $\text{Im}\Sigma$ at $r = 12.4, 15.0, 20.0$ and 30.0km for the proton and the neutron. One can see from Eq. (7.1.3) that $\text{Im}\Sigma$ is generated from the imaginary part of the T-matrix. Although the imaginary part of the T-matrix vanishes at threshold, the imaginary part of the self-energy is non-zero at $p = 0$. This is because we sum the momentum distribution of the second colliding nucleon. In the $r < 20\text{km}$ region, $\text{Im}\Sigma$ becomes relatively large compared to the $\text{Re}\Sigma$ or free deuteron binding energy.

Fig. 7.6 shows the pole of the T-matrix evaluated using Eq. (6.4.4) for the zero center of mass momentum $P = 0\text{MeV}$ at $r = 20.0, 30.0$ and 50.0km . At $r = 20\text{km}$, the deuteron width is about 10MeV which is large value compared to the real part of the pole energy. It suggests again that we need to beyond the quasi-particle approximation in $r < 30\text{km}$ region. However, the approximation will be reliable in the outer region, $r > 30\text{km}$, which covers a large part of the cooling region.

7.4 Discussion

In this part we have studied effects of the nuclear medium for the neutrino emissivity for the explosion process of supernova and studied how the previous calculations of neutrino

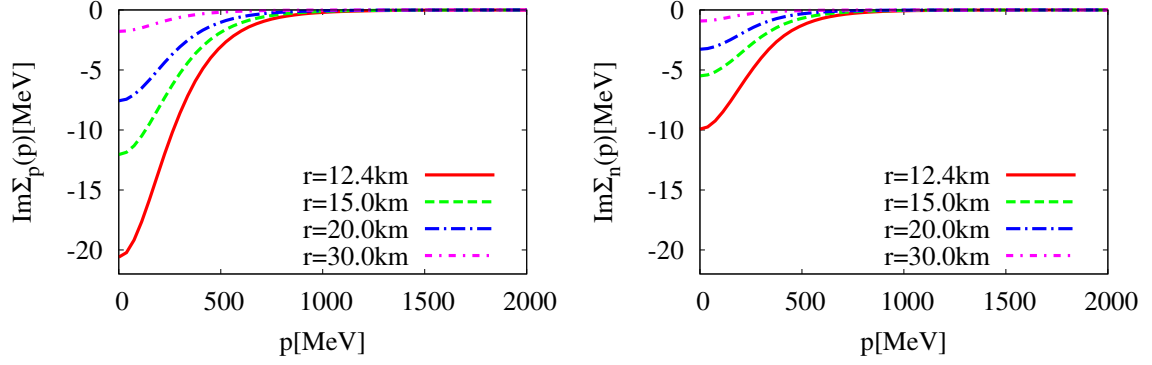


Figure 7.5: Momentum dependence of the imaginary part of the self-energy. The left (right) panel shows the imaginary part of the proton(neutron) self-energy at $r = 12.4\text{km}$ (red solid curve), 15.0km (green dashed curve), 20.0km (blue dash-dotted curve) and 30.0km (purple short dash-dotted curve).

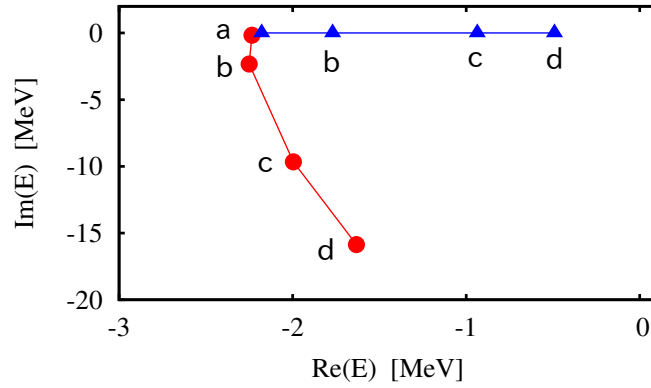


Figure 7.6: Pole position of T-matrix. The red circles(blue triangles) a,b,c,d show the pole positions at $r = 50, 30, 20, 15\text{km}$ with(without) imaginary part of the self-energy. Two-nucleon center-of mass momentum is set to be zero.

reaction rate for free space can be modified in the medium. We adopt the thermodynamic Green's function approach and the T-matrix approximation. In the T-matrix approximation, two-nucleon interact successively with each other and two-nucleon propagator includes Pauli blocking. We have made two major approximation in this work. In a self-consistent Green's function scheme, we used the first order iteration to obtain the self-energy of nucleon. Furthermore, we have used quasi-particle approximation that is the small width limit of the spectral function. Under this approximation, we have studied the properties of the two nucleon system using the typical profile of nuclear medium at the 150ms after core collapse.

We have studied the deuteron state in medium and found that the deuteron can exist as an proton-neutron bound in the $r \geq 12.4\text{km}$ region. The binding energy of deuteron is significantly reduced in the inner region. When a deuteron starts moving, the binding energy increases and approaches to the value of free deuteron. Further inner radius at $r = 11.7\text{km}$, the proton and neutron can not form deuteron for the CM kinetic energy $P^2/4m_N \lesssim 12\text{MeV}$. These results indicate that one can treat the deuteron in $r \geq 50\text{km}$ as the one in the free-space.

The electron capture on deuteron was studied as an example to examine the medium effects on emissivity. This reaction is known to be dominated by the Gamow-Teller transition. We used impulse nuclear current and the single-term separable potential for the two-nucleon interaction for the S-wave. The neutrino emissivity is modified in the medium through the modification of the single particle energy and the Pauli blocking. At $r = 12.4\text{km}$ the emissivity with the medium effects is reduced by 40% from the one evaluated in free space. On the other hand, at $r = 19.7\text{km}$ the difference between the emissivity with medium effects and without ones is already within 7%. This estimation implies that our evaluations of neutrino emissivity for the electron capture without the medium effects seems to be reasonable for the outer layer of supernova explosion.

We have estimated the effect of the imaginary part of the self-energy on the deuteron binding energy to examine the validity of the quasi-particle approximation. Within the $\omega = p^2/(2m)$ approximation of the self-energy which corresponds to take the first iteration, the pole of the T-matrix was studied for the non-degenerate nucleon region. We found the resonance pole corresponding to deuteron has a small imaginary part for $r > 30\text{km}$, which is the cooling region for this snap shot of supernova and it will be safe to use the

emissivity calculated from the free-particle for $r > 50\text{km}$.

In the next step, we need to perform a self-consistent calculation and further we need to use realistic nuclear potential instead of separable potential including higher partial waves, which however one expects to involve quite large numerical task.

Chapter 8

Conclusion

We have studied the significance of the neutrino production reactions with deuteron in core-collapse supernova.

In the first part of the thesis, the cross sections and the neutrino emissivity for these reaction has been investigated with the standard nuclear physics approach for which nuclear weak currents are constructed from one-body impulse currents and two-body exchange currents combined with high-precision realistic two-nucleon potential.

- We have completed a code for neutrino emissivity to provide the neutrino emissivities of the new deuteron related processes for the supernova simulation. The newly evaluated mechanisms are the $e^\pm$ capture on deuteron and $NN \rightarrow d+l+l'$ processes:

$$d + e^- \rightarrow n + n + \nu_e , \quad (8.0.1)$$

$$d + e^+ \rightarrow p + p + \bar{\nu}_e , \quad (8.0.2)$$

$$n + n \rightarrow d + e^- + \bar{\nu}_e , \quad (8.0.3)$$

$$p + p \rightarrow d + e^+ + \nu_e , \quad (8.0.4)$$

$$p + n \rightarrow d + \nu + \bar{\nu} . \quad (8.0.5)$$

- The role of the new processes has been studied in a typical supernova profile, which is the snapshot at 150 ms after core bounce for a $15 M_\odot$ star for the cooling region. In this region neutrino emission due to the electron capture on proton is the main mechanism. Anti-neutrino emission is emitted mainly by positron capture on neutron and NN bremsstrahlung, whose emissivity is about three orders of magnitude smaller than that of electron neutrino.

The emissivity of neutrino due to the electron capture on deuteron is comparable to that of the main process on proton especially in the $r < 60\text{km}$ region for the current profile. The emissivity of the anti-neutrino due to deuteron process is smaller than that of neutron, however it can be comparable to that of NN bremsstrahlung.

We found that the emissivity due to $e^\pm$ -capture on the deuteron is smaller than that of the free nucleon if the densities of the deuteron and nucleon are the same. Therefore the effective neutrino emissivity per nucleon via $e^\pm$ -capture is reduced by the existence of the nucleon as the element of deuteron, which leads unfavorably contributions for the neutrino heating mechanism. On the other hand, the emissivity for the neutrino production via deuteron formation may be comparable to the nucleon-nucleon bremsstrahlung.

- The analysis suggests, neutrino and anti-neutrino emission due to the deuteron breakup/formation play an important role in the supernova explosion. Since the present analysis depends on heavily on the temperature, density and mass fractions of the particular supernova environment and our current evaluation is numerically to demanded to be used in the simulation of supernova, it is useful to provide a compact table of the emissivities. For this purpose we plan to evaluate the emissivity in the non-degenerate approximation, where the emissivity can be expressed as

$$Q(e^-d \rightarrow \nu_e + n + n) = n_e n_d Q_{eff}(T) \quad (8.0.6)$$

where n_e, n_d are number density of electron and deuteron and $Q_{eff}(T)$ is effective emissivity as a function of one parameter temperature T .

In the second part of the thesis, the deuteron binding energy and emissivity of electron capture reaction in the medium is studied. The emissivities in part I is evaluated assuming the deuteron remains the same as free space in the medium. In the environment of supernova explosion, the profile of nuclear medium varies a lot from the inner to the outer region of supernova and the medium modification might be important even in some part of the cooling region of supernova. The medium effect is investigated based on the thermodynamic Green's function and T-matrix approach. For this analysis we used a simple model of nuclear interaction and current and took into account only s-wave NN continuum state.

- We have evaluated the nucleon self-energy in the quasi-particle approximation and the non-iterative calculation of the self-energy ($\omega = p^2/(2m)$ approximation). Within this two approximation, Two-nucleon scattering T-matrix is solved for the matter profile supernova used in part I.

The real part of the self-energy is negative due to the attractive s-wave NN interaction. This attraction shifts the threshold energy of continuum state. The deuteron bound state was studied by searching the pole of T-matrix. As the density increases, deuteron binding energy decreases. For $r > 12\text{km}$, $\rho > 0.17\rho_0$, the deuteron bound state at rest does not exist. However when total center of mass momentum increases, binding energy also increases because of less effective Pauli blocking. We found that the medium effect reduces the emissivity especially around $10 < r < 30\text{km}$. For the further outer region the medium modification is small within the current formalism.

- The effect of the imaginary part of the self-energy, which makes finite life time of the quasi particle, is taken into account within the ($\omega = p^2/(2m)$ approximation). The imaginary part of the self-energy is large compared with its real part in particular for $r < 20\text{km}$. The consequence of this imaginary part of self-energy on the deuteron energy is studied for the non-degenerate region. The pole of the T-matrix in complex energy plane is identified as 'deuteron'. In this analysis, we have neglected Pauli effects in the intermediate Green function, therefore the real part of the pole energy is moderately affected. On the other hand, the imaginary part of the pole energy, which corresponds to the width of the resonance, becomes larger as the r decreases. We found at $r > 30\text{km}$, the description of part I without medium effect can be applied. On the other hand, in the region of higher density around $10 < r < 30\text{km}$, the careful analysis of the medium dynamics must be further examined even by going back to the estimation of the deuteron fraction.

Acknowledgments

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Appendix A

Emissivity for the electron capture

The reactions of the electron and positron captures, which are so-called the direct URCA processes, are

$$e^- + p \rightarrow \nu_e + n \quad (\text{A.0.1})$$

$$e^+ + n \rightarrow \bar{\nu}_e + p. \quad (\text{A.0.2})$$

Neutrino emissivity of the electron capture is defined as

$$Q = \int \frac{d\mathbf{p}_1}{(2\pi)^3} \frac{d\mathbf{p}_2}{(2\pi)^3} \frac{d\mathbf{p}_3}{(2\pi)^3} \frac{d\mathbf{p}_4}{(2\pi)^3} E_4 (2\pi)^4 \delta(\sum_i p_i - \sum_f p_f) \sum_{\text{spin}} |M|^2 \Xi \quad (\text{A.0.3})$$

$$\Xi \equiv \frac{1}{1 + e^{(E_1 + \mu_1)/T}} \frac{1}{1 + e^{(E_2 + \mu_2)/T}} \left(1 - \frac{1}{1 + e^{(E_3 + \mu_3)/T}} \right). \quad (\text{A.0.4})$$

Ξ is the Fermi distribution functions of initial and final states. For $e^- + p \rightarrow \nu_e + n$ we label 1=p, 2= e^- , 3=n, 4= ν_e , and for $e^+ + n \rightarrow \bar{\nu}_e + p$ we label 1=n, 2= e^+ , 3=p, 4= $\bar{\nu}_e$. The matrix elements for the electron/positron captures are

$$\sum_{\text{spin}} |M|^2 = 4 \left(\frac{G_F V_{ud}}{\sqrt{2}} \right)^2 \frac{X(p_1, p_2, p_3, p_4)}{E_1 E_2 E_3 E_4} \quad (\text{A.0.5})$$

$$\begin{aligned} X(p_1, p_2, p_3, p_4) &\equiv m_1 m_3 (g_A^2 - g_V^2) (p_2 \cdot p_4) \\ &\quad + |g_A \pm g_V|^2 (p_1 \cdot p_2) (p_3 \cdot p_4) \\ &\quad + |g_A \mp g_V|^2 (p_1 \cdot p_4) (p_2 \cdot p_3), \end{aligned} \quad (\text{A.0.6})$$

where $G_F = 1.1803 \times 10^{-5} [\text{GeV}^{-2}]$ is the Fermi constant, $V_{ud} = 0.974$ is the CKM matrix element, and $g_A = 1.267$ and $g_V = 1$ are the axial and vector coupling constants. The upper and lower signs denote the electron capture (A.0.1) and positron capture (A.0.2)

respectively. X can be rewritten with the Mandelstam variables as

$$\begin{aligned}
X(s, t, u) = & \frac{m_1 m_3 (g_V^2 - g_A^2)}{2} (t - m_2^2 - m_4^2) \\
& + \frac{|g_A \pm g_V|^2}{4} (s - m_1^2 - m_2^2)(s - m_3^2 - m_4^2) \\
& + \frac{|g_A \mp g_V|^2}{4} (s + t - m_2^2 - m_3^2)(s + t - m_1^2 - m_4^2), \quad (\text{A.0.7})
\end{aligned}$$

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \quad (\text{A.0.8})$$

$$t = (p_4 - p_2)^2 = (p_3 - p_1)^2 \quad (\text{A.0.9})$$

$$u = (p_4 - p_1)^2 = (p_3 - p_2)^2 \quad (\text{A.0.10})$$

$$s + t + u = \sum_{i=1,2,3,4} m_i^2. \quad (\text{A.0.11})$$

Although One should perform the appropriate calculations, the phase space integrations for Eq. (A.0.3) are rather complicate because of the existence of the Fermi distribution functions.

We develop two different calculation methods for analyzing the direct URCA emissivities. One is the 'relativistic method for two-body processes' which employs the manipulations of the Lorentz transformations, and the other is the 'phase space constraints method' which analyzes the possible regions of the phase space directly. Each of the methods has no approximations of the matrix elements or angular integrations like the Bruenn's static calculations [45]. The formulations of these method are explained in the next sections. We confirmed that the emissivities of these two methods are in agreement with each other within 5%. The emissivities for the electron capture on proton and the positron capture on neutron in chapter4.2 are evaluated with those methods.

A.1 Relativistic method for two body processes

The emissivity one should calculate is the following form

$$Q = \int \frac{d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3 d\mathbf{p}_4}{E_1 E_2 E_3 E_4} \delta^{(4)}(p_1 + p_2 - p_3 - p_4) M(s, t, u) \Xi(p_1, p_2, p_3, p_4). \quad (\text{A.1.1})$$

Comparing eqs.(A.0.3) and (A.0.6), the matrix element $M(s, t, u)$ is found that

$$M(s, t, u) = \frac{4}{(2\pi)^8} \left(\frac{G_F V_{ud}}{\sqrt{2}} \right)^2 E_4 X(s, t, u). \quad (\text{A.1.2})$$

Note that

$$\frac{d^3 p_i}{E_i} \quad (\text{A.1.3})$$

$$\delta^{(4)}(p_1 + p_2 - p_3 - p_4) \quad (\text{A.1.4})$$

$$M(s, t, u) \quad (\text{A.1.5})$$

$$s = (p_1 + p_2)^2, t = (p_1 - p_3)^2 \quad (\text{A.1.6})$$

are the Lorentz invariant. $E_i = \sqrt{m_i^2 + p_i^2}$ is the energy of the relativistic form. We treat the initial and final variables as the laboratory frame(Lab) and the center of mass frame(CM), respectively. At the same time, we separate the (A.1.1) frame as

$$\left[\int \frac{d\mathbf{p}_1 d\mathbf{p}_2}{E_1 E_2} \right]_{\text{Lab}} \left\{ \int \frac{d\mathbf{p}_{3c} d\mathbf{p}_{4c}}{E_{3c} E_{4c}} \delta^4(p_{1c} + p_{2c} - p_{3c} - p_{4c}) M(s, t, u) \right\}_{\text{CM}} \left[\Xi(p_1, p_2, p_3, p_4) \right]_{\text{Lab}}. \quad (\text{A.1.7})$$

where p_i, E_i and p_{ic}, E_{ic} denote the Lab frame and the CM frame variables.

A.1.1 Lab frame and CM frame relations

To perform this manipulations, one should obtain the relations between the Lab variables and the CM ones. Let us define the two-body 4-momentum on the Lab frame as $P_L^\mu = (E = E_1 + E_2, \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2)$ and that is the CM frame as $P_C^\mu = (W, \mathbf{0})$. $W = \sqrt{E^2 - \mathbf{P}^2}$ is the invariant mass. These two valuables are related by Lorentz transformations

$$P_C^\mu = \Lambda^\mu{}_\nu P_L^\nu \quad (\text{A.1.8})$$

$$P_L^\mu = \Lambda_\nu{}^\mu P_C^\nu, \quad (\text{A.1.9})$$

where Λ is the Lorentz transformation operator

$$\Lambda^\mu{}_\nu = \begin{bmatrix} \frac{E}{W} & \frac{\mathbf{P}_j}{W} \\ \frac{\mathbf{P}_j}{W} & \delta^i{}_j + \frac{\mathbf{P}_i \mathbf{P}_j}{W(E + W)} \end{bmatrix} \quad (\text{A.1.10})$$

$$(\Lambda^{-1})^\mu{}_\nu = \Lambda_\nu{}^\mu = \begin{bmatrix} \frac{E}{W} & -\frac{\mathbf{P}_j}{W} \\ -\frac{\mathbf{P}_j}{W} & \delta^i{}_j + \frac{\mathbf{P}_i \mathbf{P}_j}{W(E + W)} \end{bmatrix}. \quad (\text{A.1.11})$$

Let us assume that A_L^μ and A_C^μ are arbitrary four-vector of the Lab frame and CM frame respectively. Using Lorentz transformation operators (A.1.10) and (A.1.11), one can obtain the relations of the time and space components as

$$A_C^0 = \frac{1}{W}(EA_L^0 - \mathbf{P} \cdot \mathbf{A}_L) \quad (\text{A.1.12})$$

$$\mathbf{A}_C = \mathbf{A}_L + \frac{\mathbf{P}}{W} \left(-A_L^0 + \frac{\mathbf{P} \cdot \mathbf{A}_L}{W + E} \right). \quad (\text{A.1.13})$$

The inverse forms are

$$A_L^0 = \frac{1}{W}(EA_C^0 + \mathbf{P} \cdot \mathbf{A}_C) \quad (\text{A.1.14})$$

$$\mathbf{A}_L = \mathbf{A}_C + \frac{\mathbf{P}}{W} \left(A_C^0 + \frac{\mathbf{P} \cdot \mathbf{A}_C}{W + E} \right). \quad (\text{A.1.15})$$

Now one can get the relations of energy and momentum of those frames. The initial state variables are treated as the Lab frame such as

$$p_{1L}^\mu = (E_1, \mathbf{p}_1), E_1 = \sqrt{m_1^2 + \mathbf{p}_1^2} \quad (\text{A.1.16})$$

$$p_{2L}^\mu = (E_2, \mathbf{p}_2), E_2 = \sqrt{m_2^2 + \mathbf{p}_2^2}. \quad (\text{A.1.17})$$

The CM frame variables are obtained from Eq. (A.1.12) and (A.1.13) as

$$p_{1C}^\mu = \Lambda^\mu{}_\nu p_{1L}^\nu = (E_{1C}, \mathbf{q}) \quad (\text{A.1.18})$$

$$p_{2C}^\mu = \Lambda^\mu{}_\nu p_{2L}^\nu = (E_{2C}, -\mathbf{q}) \quad (\text{A.1.19})$$

$$E_{1C} = \sqrt{m_1^2 + \mathbf{q}^2} = \frac{1}{W}(EE_1 - \mathbf{P} \cdot \mathbf{p}_1) \quad (\text{A.1.20})$$

$$E_{2C} = \sqrt{m_2^2 + \mathbf{q}^2} = \frac{1}{W}(EE_2 - \mathbf{P} \cdot \mathbf{p}_2) \quad (\text{A.1.21})$$

The CM momentum can be also obtained as

$$|\mathbf{q}| = q = \sqrt{\left(\frac{W^2 + m_2^2 - m_1^2}{2W} \right)^2 - m_2^2}. \quad (\text{A.1.22})$$

For the final state the variables are treated as the CM frame:

$$p_{3C}^\mu = (E_{3C}, \mathbf{q}'), E_{3C} = \sqrt{m_3^2 + \mathbf{q}^2} \quad (\text{A.1.23})$$

$$p_{4C}^\mu = (E_{4C}, -\mathbf{q}'), E_{4C} = \sqrt{m_4^2 + \mathbf{q}^2}. \quad (\text{A.1.24})$$

The transformations of these variables into the Lab frame can be obtained in the same

way as follows

$$p_{3L}^\mu = \Lambda_\mu{}^\nu p_{3C}^\mu = (E_3, \mathbf{p}_3) \quad (\text{A.1.25})$$

$$p_{4L}^\mu = \Lambda_\mu{}^\nu p_{4C}^\mu = (E_4, \mathbf{p}_4) \quad (\text{A.1.26})$$

$$E_3 = \frac{1}{W}(EE_{3C} + \mathbf{P} \cdot \mathbf{q}'), \mathbf{p}_3 = \sqrt{E_3^2 - m_3^2} \quad (\text{A.1.27})$$

$$E_4 = \frac{1}{W}(EE_{4C} - \mathbf{P} \cdot \mathbf{q}'), \mathbf{p}_4 = \sqrt{E_4^2 - m_4^2} \quad (\text{A.1.28})$$

$$|\mathbf{q}'| = q' = \sqrt{\left(\frac{W^2 + m_4^2 - m_3^2}{2W}\right) - m_4^2} \quad (\text{A.1.29})$$

Then one can perform the emissivity calculations using these variables, as explained in the following sections.

A.1.2 Exact form

The integration of the final state including the delta functions can be performed as below

$$\int \frac{d\mathbf{p}_{3C} d\mathbf{p}_{4C}}{E_{3C} E_{4C}} \delta^{(4)}(p_{1C} + p_{2C} - p_{3C} - p_{4C}) \quad (\text{A.1.30})$$

$$= \int \frac{d\mathbf{p}_{3C} d\mathbf{p}_{4C}}{E_{3C} E_{4C}} \delta(W - E_{3C} - E_{4C}) \delta^{(3)}(\mathbf{0} - \mathbf{p}_{3C} - \mathbf{p}_{4C}) \quad (\text{A.1.31})$$

$$= \int \frac{d\mathbf{q}'}{E_{3C} E_{4C}} \delta(W - E_{3C} - E_{4C}) \quad (\text{A.1.32})$$

$$= \int d\Omega_{q'} \frac{q'}{W}. \quad (\text{A.1.33})$$

where $d\Omega_{q'}$ represents the angular integral of $\mathbf{q}'$. Therefore Eq. (A.1.1) can be rewritten as

$$Q = \int \frac{d\mathbf{p}_1 d\mathbf{p}_2}{E_1 E_2} \int d\Omega_{q'} \frac{q'}{W} M(s, t, u) \Xi(p_1, p_2, p_3, p_4) \quad (\text{A.1.34})$$

$$= \int dp_1 p_1^2 \int dp_2 p_2^2 \int d\Omega_{p_1} d\Omega_{p_2} d\Omega_{q'} \frac{q'}{W} M(s, t, u) \Xi(p_1, p_2, p_3, p_4). \quad (\text{A.1.35})$$

This form includes three angular variables where

$$\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2 = \cos \theta_{12} \quad (\text{A.1.36})$$

$$\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{q}}' = \cos \theta_{q'} \quad (\text{A.1.37})$$

$$\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{q}}' = \cos \theta_{12} \cos \theta_{q'} + \sin \theta_{12} \sin \theta_{q'} \cos \phi_{q'}. \quad (\text{A.1.38})$$

Then one can organize the angle integrations like

$$Q = 8\pi^2 \int dp_1 \int dp_2 \frac{p_1^2 p_2^2}{E_1 E_2} \int_{-1}^1 d \cos \theta_{12} \frac{q'}{W} \int_{-1}^1 d \cos \theta_{q'} \int_0^{2\pi} d\phi_{q'} M(s, t, u) \Xi(p_1, p_2, p_3, p_4). \quad (\text{A.1.39})$$

The matrix element of the electron capture is treated in the CM frame

$$M(s, t, u) = \frac{4}{(2\pi)^8} \left(\frac{G_F V_{ud}}{\sqrt{2}} \right)^2 E_{4C} X(s, t, u), \quad (\text{A.1.40})$$

$$(\text{A.1.41})$$

and these variables are given as follows

$$W = \sqrt{(E_1 + E_2)^2 - p_1^2 - p_2^2 - 2p_1 p_2 \cos \theta_{12}} \quad (\text{A.1.42})$$

$$s = W^2 \quad (\text{A.1.43})$$

$$t = (E_{1C} - E_{3C})^2 - q^2 - q'^2 + 2\mathbf{q} \cdot \mathbf{q}' \quad (\text{A.1.44})$$

$$\mathbf{q} \cdot \mathbf{q}' = q' \hat{\mathbf{q}}' \cdot \left[\mathbf{p}_1 + \frac{\mathbf{P}}{W} \left(-E_1 + \frac{\mathbf{P} \cdot \mathbf{p}_1}{W + E} \right) \right] \quad (\text{A.1.45})$$

$$\mathbf{p}_1 \cdot \hat{\mathbf{q}}' = p_1 \cos \theta_{q'} \quad (\text{A.1.46})$$

$$\mathbf{p}_2 \cdot \hat{\mathbf{q}}' = p_2 (\cos \theta_{12} \cos \theta_{q'} + \sin \theta_{12} \sin \theta_{q'} \cos \phi_{q'}). \quad (\text{A.1.47})$$

A.2 Phase space constraints method

Here we denote the initial and final nucleon 4-momenta as p, p' and the masses as M, M' . The lepton ones are labeled as q, q' and m, m' . We assume the neutrino mass to be $m' = 0$.

The emissivity is following form

$$I = \int d\mathbf{p} d\mathbf{q} d\mathbf{p}' d\mathbf{q}' \delta^4(p + q - p' - q') X' \quad (\text{A.2.1})$$

X' contains the matrix element and Fermi distribution functions. One can perform $d\mathbf{p}'$ integral

$$I = \int d\mathbf{p} d\mathbf{q} d\mathbf{q}' \delta(E_p + \epsilon_q - E'_{\mathbf{p}+\mathbf{q}-\mathbf{q}'} - q') X'. \quad (\text{A.2.2})$$

Here we define the initial energy and momentum as

$$E \equiv E_p + \epsilon_q \quad (\text{A.2.3})$$

$$\mathbf{P} \equiv \mathbf{p} + \mathbf{q} \quad (\text{A.2.4})$$

then the invariant mass has the condition as follows

$$S = E^2 - \mathbf{P}^2 > M + m. \quad (\text{A.2.5})$$

Next one can proceed the dq' integral

$$I = \int d\mathbf{p} d\mathbf{q} q'^2 d\Omega_{q'} \frac{1}{\left|1 + \frac{dE'}{dq'}\right|} X' \quad (\text{A.2.6})$$

$$\frac{dE'}{dq'} = \frac{q' - \mathbf{P} \cdot \hat{\mathbf{q}}'}{E'} \quad (\text{A.2.7})$$

$$E' = \sqrt{M'^2 + (\mathbf{P} - \mathbf{q}')^2} \quad (\text{A.2.8})$$

q' is taken by solving the energy delta function

$$q' = \frac{S - M'^2}{2(E - \mathbf{P} \cdot \hat{\mathbf{q}}')}. \quad (\text{A.2.9})$$

The neutrino energy q' must be positive ($q' > 0$). The denominator of Eq. (A.2.9) has the condition of $E - \mathbf{P} \cdot \hat{\mathbf{q}}' > 0$, because

$$E^2 = E_p^2 + \epsilon_q^2 + 2E_p\epsilon_q \quad (\text{A.2.10})$$

$$= M^2 + p^2 + m^2 + q^2 + 2E_p\epsilon_q \quad (\text{A.2.11})$$

$$> M^2 + (p + q)^2 \quad (\text{A.2.12})$$

$$> (\mathbf{P} \cdot \hat{\mathbf{q}}')^2. \quad (\text{A.2.13})$$

Thus the condition $q' > 0$ is rewritten as

$$S > M' \quad (\text{A.2.14})$$

The integrand of Eq. (A.2.6) includes three angles

$$\hat{\mathbf{p}} \cdot \hat{\mathbf{q}}, \hat{\mathbf{p}} \cdot \hat{\mathbf{q}}', \hat{\mathbf{q}} \cdot \hat{\mathbf{q}}', \quad (\text{A.2.15})$$

then one can set $\hat{\mathbf{p}}$ as z axis and perform the angular integral of (A.2.6), given by

$$I = 8\pi^2 \int dp dq d(\cos \theta_q) \int_{-1}^1 d(\cos \theta_{q'}) \int_0^{2\pi} d(\phi_q - \phi_{q'}) \frac{p^2 q^2 q'^2}{\left|1 + \frac{q' - \mathbf{P} \cdot \hat{\mathbf{q}}'}{E'}\right|} X' \quad (\text{A.2.16})$$

$$\hat{\mathbf{p}} \cdot \hat{\mathbf{q}} = \cos \theta_q \quad (\text{A.2.17})$$

$$\hat{\mathbf{p}} \cdot \hat{\mathbf{q}}' = \cos \theta_{q'} \quad (\text{A.2.18})$$

$$\hat{\mathbf{q}} \cdot \hat{\mathbf{q}}' = \cos \theta_q \cos \theta_{q'} + \sin \theta_q \sin \theta_{q'} \cos(\phi_q - \phi_{q'}) \quad (\text{A.2.19})$$

The remaining problem is to find the region of the integral $\int dp dq d(\cos \theta_q)$ satisfying the constrains (A.2.14). We exhibit the condition (A.2.14) again

$$S = (E_p + \epsilon_q)^2 - (\mathbf{p} + \mathbf{q})^2 > M'^2. \quad (\text{A.2.20})$$

This can be rewritten in terms of $\cos \theta_q$

$$\cos \theta_q \equiv x < \frac{1}{pq}(E_p \epsilon_q - \Delta^2) \quad (\text{A.2.21})$$

$$\Delta^2 \equiv \frac{1}{2}(M'^2 - M^2 - m^2) \quad (\text{A.2.22})$$

Then if $\Delta^2 < 0$, all $[p, q, x]$ region is allowed because

$$\frac{E_p \epsilon_q}{pq} > 1. \quad (\text{A.2.23})$$

Therefore, one should consider the behavior of the function $F(p, q)$ defined as

$$F(p, q) \equiv \frac{1}{pq}(E_p \epsilon_q - \Delta^2) \text{ for } \Delta^2 > 0. \quad (\text{A.2.24})$$

Taking the limits of $p \rightarrow 0, \infty$, the function $F(p, q)$ behaves as follows

$$\lim_{p \rightarrow 0} F = \frac{1}{p}(M \epsilon_q - \Delta^2) \begin{cases} > 0 & \text{for } M \epsilon_q / \Delta^2 > 1 \\ < 0 & \text{for } M \epsilon_q / \Delta^2 < 1 \end{cases} \quad (\text{A.2.25})$$

$$\lim_{p \rightarrow \infty} F = \frac{\epsilon_q}{q} > 1. \quad (\text{A.2.26})$$

Moreover, the p derivative is

$$\frac{\partial F}{\partial p} = \frac{1}{qp^2 E_p}(E_p \Delta^2 - M^2 \epsilon_q) \quad (\text{A.2.27})$$

$$= \begin{cases} > 0 & \text{for } \frac{E_p}{M} > \frac{M \epsilon_q}{\Delta^2} \\ < 0 & \text{for } \frac{E_p}{M} < \frac{M \epsilon_q}{\Delta^2} \end{cases}. \quad (\text{A.2.28})$$

Since $E_q/M > 1$ the p dependence of the function $F(p, q)$ can be written as Figs. A.1 and A.2. For $M \epsilon_q / \Delta^2 > 1$, $F(p, q)$ has a minimum

$$F_{\min} = \frac{1}{pq}(E_p \epsilon_q - \Delta^2) \text{ with } E_p = \frac{M^2 \epsilon_q}{\Delta^2} \quad (\text{A.2.29})$$

$$= \frac{\Delta^2}{pq} \left[\left(\frac{M \epsilon_q}{\Delta^2} \right)^2 - 1 \right] > 0 \quad (\text{A.2.30})$$

so if $F_{\min} > 1$ all region is allowed. When $0 < F_{\min} < 1$ there are three reasonable regions separated by p_a, p_b

$$F(p_a, q) = F(p_b, q) = 1, \quad p_a < p_b. \quad (\text{A.2.31})$$

To solving this, one can lead the momentum $p_{a,b}$

$$p_{a,b} = \frac{1}{m^2} \left[q \Delta^2 \mp \sqrt{(q \Delta^2)^2 + (\Delta^4 - M^2 \epsilon_q^2) m^2} \right]. \quad (\text{A.2.32})$$

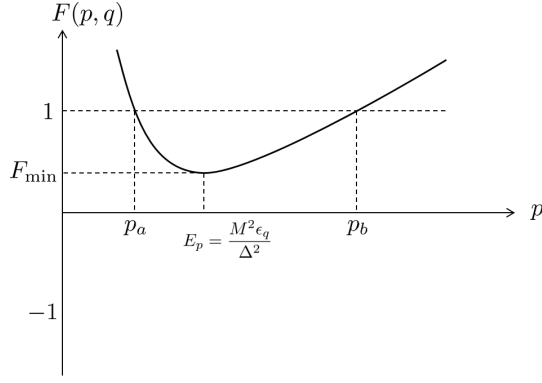


Figure A.1: $M\epsilon_q/\Delta^2 > 1, F_{\min} < 1$.

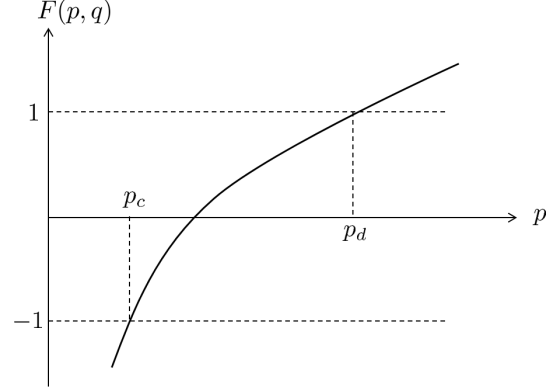


Figure A.2: $M\epsilon_q/\Delta^2 < 1$

The conditions of x in each regions are below

$$0 < p < p_a \rightarrow -1 < x < 1 \quad (\text{A.2.33})$$

$$p_a < p < p_b \rightarrow -1 < x < F(p, q) \quad \text{for} \quad \frac{M\epsilon_q}{\Delta^2} > 1, F_{\min} < 1. \quad (\text{A.2.34})$$

$$p_b < p \rightarrow -1 < x < 1 \quad (\text{A.2.35})$$

Similarly, for $M\epsilon_q/\Delta^2 < 1$ there are three reasonable regions separated by p_c, p_d

$$F(p_c, q) = -1 \quad (\text{A.2.36})$$

$$F(p_d, q) = 1, \quad p_c < p_d \quad (\text{A.2.37})$$

$$\therefore p_{c,d} = \frac{1}{m^2} \left[\mp q \Delta^2 + \sqrt{(q \Delta^2)^2 + (\Delta^4 - M^2 \epsilon_q^2) m^2} \right]. \quad (\text{A.2.38})$$

The conditions of x are

$$0 < p < p_c \rightarrow \text{none} \quad (\text{A.2.39})$$

$$p_c < p < p_d \rightarrow -1 < x < F(p, q), \quad \text{for} \quad \frac{M\epsilon_q}{\Delta^2} < 1. \quad (\text{A.2.40})$$

$$p_d < p \rightarrow -1 < x < 1 \quad (\text{A.2.41})$$

A.2.1 Summary for the phase space constraint method

We summarize the results for the phase space constraint method as below.

- Emissivity

$$Q = \frac{1}{(2\pi)^8} 8\pi^2 \left[\int dp dq dx \right] \int_{-1}^1 d(\cos \theta_{q'}) \int_0^{2\pi} d\phi_{q-q'} \frac{p^2 q^2 q'^2}{1 + \frac{q' - \mathbf{P} \cdot \hat{\mathbf{q}}'}{E'}} q' \sum_{\text{spin}} |M|^2 \Xi \quad (\text{A.2.42})$$

- Matrix element (upper sign: $e^-p \rightarrow n\nu_e$, lower: $e^+n \rightarrow p\bar{\nu}_e$)

$$\sum_{\text{spin}} |M|^2 = 4 \left(\frac{G_F V_{ud}}{\sqrt{2}} \right)^2 \frac{X(s, t)}{E_p \epsilon_q E' q'} \quad (\text{A.2.43})$$

$$\begin{aligned} X(s, t) &= \frac{m_1 m_3 (g_A^2 - g_V^2)}{2} (t - m^2) \\ &+ \frac{|g_A \pm g_V|^2}{4} (s - M^2 - m^2)(s - M'^2) \\ &+ \frac{|g_A \mp g_V|^2}{4} (s + t - m^2 - M'^2)(s + t - M^2), \end{aligned} \quad (\text{A.2.44})$$

- Fermi distribution function

$$\Xi = f_N f_e (1 - f_{N'}) (1 - f_{\nu_e}) \quad (\text{A.2.45})$$

$$f_i = \frac{1}{1 + e^{\beta(E_i - \mu_i)}} \quad (\text{A.2.46})$$

$$1 - f_{\nu_e} \sim 1 \quad (\text{A.2.47})$$

- Valuables

$$\hat{\mathbf{p}} \cdot \hat{\mathbf{q}} = \cos \theta_q \quad (\text{A.2.48})$$

$$\hat{\mathbf{p}} \cdot \hat{\mathbf{q}}' = \cos \theta_{q'} \quad (\text{A.2.49})$$

$$\hat{\mathbf{q}} \cdot \hat{\mathbf{q}}' = \cos \theta_q \cos \theta_{q'} + \sin \theta_q \sin \theta_{q'} \cos \phi_{q-q'} \quad (\text{A.2.50})$$

$$q' = \frac{S - M'^2}{2(E - \mathbf{P} \cdot \hat{\mathbf{q}}')} \quad (\text{A.2.51})$$

$$E' = \sqrt{M'^2 + (\mathbf{P} - \mathbf{q}')^2} \quad (\text{A.2.52})$$

$$s = M^2 + m^2 + 2(E_p \epsilon_q - \mathbf{p} \cdot \mathbf{q}) \quad (\text{A.2.53})$$

$$t = m^2 - 2(\epsilon_q q' - \mathbf{q} \cdot \mathbf{q}') \quad (\text{A.2.54})$$

- $[\int dp dq dx]$ integration range (for $\Delta^2 > 0$)

$$(1) \quad \epsilon_q > \frac{\Delta^2}{M} \begin{cases} F_{\min} > 1 \cdots \text{all region allowed} \\ F_{\min} < 1, \begin{cases} 0 < p < p_a \cdots -1 < x < 1 \\ p_a < p < p_b \cdots -1 < x < F(p, q) \\ p_b < p \cdots -1 < x < 1 \end{cases} \end{cases} \quad (\text{A.2.55})$$

$$(2) \quad \epsilon_q < \frac{\Delta^2}{M} \begin{cases} 0 < p < p_c \cdots \text{none} \\ p_c < p < p_d \cdots -1 < x < F(p, q) \\ p_d < p \cdots -1 < x < 1 \end{cases} \quad (\text{A.2.56})$$

$$F(p, q) \equiv \frac{1}{pq}(E_p \epsilon_q - \Delta^2) \quad (\text{A.2.57})$$

$$\Delta^2 \equiv \frac{1}{2}(M'^2 - M^2 - m^2) \quad (\text{A.2.58})$$

$$p_{a,b} = \frac{1}{m^2} \left[q\Delta^2 \mp \sqrt{(q\Delta^2)^2 + (\Delta^4 - M^2\epsilon_q^2)m^2} \right] \quad (\text{A.2.59})$$

$$p_{c,d} = \frac{1}{m^2} \left[\mp q\Delta^2 + \sqrt{(q\Delta^2)^2 + (\Delta^4 - M^2\epsilon_q^2)m^2} \right] \quad (\text{A.2.60})$$

Appendix B

Summation of Matsubara frequency

In this Appendix we give the technique for a summation of the function depending on Matsubara frequency. Consider the following contour integral

$$I = -\beta \oint_C \frac{dz}{2\pi i} f(z)h(z), \quad f(z) = \frac{1}{e^{\beta(z-\mu)} + 1}. \quad (\text{B.0.1})$$

where $f(z)$ is the Fermi distribution function involving poles at

$$z = z_\nu = \frac{\nu\pi}{-i\beta} + \mu, \quad \nu = 1, 3, 5, \dots \quad (\text{B.0.2})$$

with the residue $(-1)^\nu/\beta$. The function $h(z)$ is an arbitrary function involving i 's simple poles at position $z = a_i$. We suppose that the pole positions of $f(z)$ and $h(z)$ are different. To take the contour C as surrounding $f(z)$ poles, B.0.1 is consistent with the Matsubara sum of $h(z)$

$$I = \sum_{\nu} h(z_\nu). \quad (\text{B.0.3})$$

B.1 Single pole

Consider the function $h(z)$ has a simple pole at position $z = a$. We suppose that the pole positions of $f(z)$ and $h(z)$ are different. Because the function $f(z)h(z)$ is analytic without those poles exhibited, the contour integral C can be continued analytically in C' , which consists of surrounding $h(z)$ pole and infinite circle, as shown in the Fig. B.1. The contribution of infinitesimally closed transverse lines are canceled each other. If the

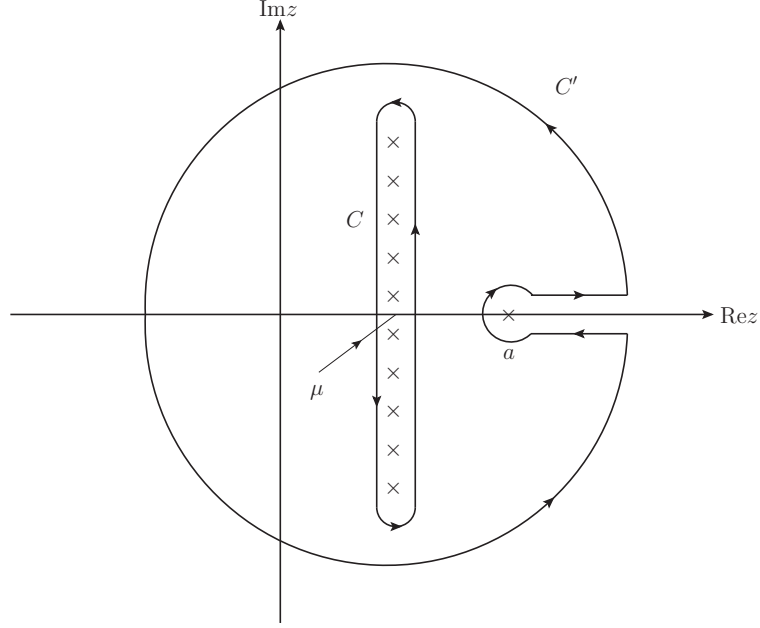


Figure B.1: Contour integral for C and C'. $f(z)$ poles are arrayed longitudinally at $\text{Re}z = \mu$, and $h(z)$ pole exists at $z=a$.

integrations of infinite circle converge, the C' integration reduces the part of a residue on $z = a$:

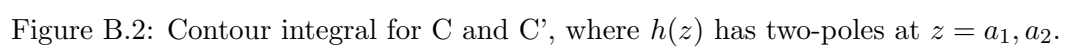
$$\begin{aligned}
 I &= -\beta \oint_C \frac{dz}{2\pi i} f(z)h(z) \\
 &= -\beta \oint'_C \frac{dz}{2\pi i} f(z)h(z) \\
 &= \beta \text{Res}_{z=a}[h(z)f(z)] \tag{B.1.1}
 \end{aligned}$$

$$= \beta \lim_{z \rightarrow a} (z - a)h(z)f(z) \tag{B.1.2}$$

B.2 Two or more poles

The case that $h(z)$ has two or more simple poles are readily obtained. When $h(z)$ has two-poles at $z = a_1, a_2$, the analytic continuations from C to C' are given by Fig. B.2. Thus the Matsubara sum reduces the contributions of two $h(z)$ residues. One can extend the case that more $h(z)$ poles exist, that is,

$$\sum_{\nu} h(z_{\nu}) = \beta \sum_i \text{Res}_{z=a_i}[h(z)f(z)]. \tag{B.2.1}$$



Bibliography

- [1] H. A. Bethe, Rev. Mod. Phys. **62** (1990) 801.
- [2] K. Kotake, K. Sato, and K. Takahashi, Rev. Prog. Phys. **69** (2006) 971.
- [3] H.-T. Janka, K. Langanke, A. Marek, G. Martínez-Pinedo and B. Muller, Phys. Rep. **442** (2007) 38.
- [4] M. Fukugita and A. Suzuki, *Physics and Astrophysics of Neutrinos*(Springer-Verlag, Tokyo, 1994).
- [5] K. Sumiyoshi and G. Röpke, Phys. Rev. C **77** (2008) 055804.
- [6] A. Arcones, G. Martínez-Pinedo, E. O'Connor, A. Schwenk, H-T. Janka, C. J. Horowitz and K. Langanke, Phys. Rev. C **78** (2008) 015806.
- [7] M. Hempel, T. Fischer, J. Schaffener-Bielich and M. Liebendörfer, Astrophys. J. **748** (2012) 70.
- [8] J. M. Lattimer and F. D. Swesty, Nucl. Phys. A **535** (1991) 331.
- [9] H. Shen, H. Toki, K. Oyamatsu and K. Sumiyoshi, Nucl. Phys. A **637** (1998) 435.
- [10] H. Shen, H. Toki, K. Oyamatsu and K. Sumiyoshi, Prog. Theor. Phys. bf 100 (1998) 1013.
- [11] W. C. Haxton, Phys. Rev. **60** (1988) 1999.
- [12] S. W. Bruenn and W. C. Haxton, Astrophys. J. **376** (1991) 678.
- [13] N. Ohnishi, K. Kotake and S. Yamada, Astrophys. J. **667** (2007) 375.
- [14] T. Suzuki, S. Chiba, T. Yoshida, T. Kajino and T. Otsuka, Phys. Rev. C **74** (2006) 034307.

- [15] D. Gazit and N. Barnea, Phys. Rev. Lett. **98** (2007) 192501.
- [16] E. O’Conner, D. Gazit, C. J. Horowitz, A. Schwenk and N. Barnea, Phys. Rev. C **75** (2007) 055803.
- [17] S. X. Nakamura, K. Sumiyoshi, and T. Sato, Phys. Rev. C **80** (2009) 035802.
- [18] S. Furusawa, H. Nagakura, K. Sumiyoshi and S. Yamada, Astrophys. J. **774** (2013) 78.
- [19] S. Nasu, S. X. Nakamura, K. Sumiyoshi, T. Sato, F. Myhrer and K. Kubodera, astro-ph.HE/1402.0959.
- [20] S. Nakamura, T. Sato, V. Gudkov, and K. Kubodera, Phys. Rev. C **63** (2001) 034617.
- [21] S. Nakamura, T. Sato, S. Ando, T.-S. Park, F. Myhrer, V. Gudkov, and K. Kubodera, Nucl. Phys. A **707** (2002) 561.
- [22] M. Doi, T. Sato, H. Ohtsubo and M. Morita, Nucl. Phys. A **511** (1990) 507.
- [23] K. Tamura, T. Niwa, T. Sato and H. Ohtsubo, Nucl. Phys. A **536** (1992) 536.
- [24] T. Sato, T. Niwa and T. Ohtsubo, in Proceedings of the IVth International Symposium on Weak and Electromagnetic Interaction in Nuclei, ed. H. Ejiri, T. Kishimoto and T. Sato(World Scientific, Singapore), 488.
- [25] K. Kubodera and T.-S. Park, Ann. Rev. Nucl. Part. Sci. **54** (2004) 19.
- [26] R. Schiavilla, V. G. Stokes, W. Glockle, H. Kamada, A. Nogga, J. Carlson and R. Machleidt, Phys. Rev. C **58** (1998) 1263.
- [27] T. S. Park, L. E. Marcucci, R. Schiavilla, M. Viviani, A. Kievsky, S. Rosati, K. Kubodera, D.-P. Min and M. Rho, Phys. Rev. C **67** (2003) 055206.
- [28] S. Ando and K. Kubodera, Phys. Lett. B **633** (2006) 253.
- [29] M. Schmidt, G. Röpke and H. Schulz, Ann. Phys. **202** (1990) 57.
- [30] G. Röpke and A. Schnell, Prog. Part. Nucl. Phys. **42** (1999) 53.
- [31] T. Alm, G. Röpke, A. Schnell, N. H. Kwong and H. S. Köhler, Phys. Rev. C **53** (1996) 2181.

- [32] T. Frick and H. M  ther, Phys. Rev. C **68** (2003) 034310.
- [33] A. Sedrakian, Prog. Part. Nucl. Phys. **58** (2007) 168.
- [34] T. Alm, G. R  pke and M. schmidit, Phys. Rev. C **50** (1994) 31.
- [35] D. Blaschke, G. R  pke, H. Schulz, A. D. Sedrakian and D. N. Voskresensky, Mon. Not. R. Astron. Soc. **273** (1995) 596.
- [36] H. Suzuki, in Proceedings of the International Symposium on Neutrino Astrophysics: Frontiers of Neutrino Astrophysics, ed. Y. Suzuki and K. Nakamura (Universal Academy Press Inc., Tokyo), 219.
- [37] J. Beringer et al., Phys. Rev. D **86** (2012) 010001.
- [38] M. Gourdin, Phys. Rep. **11** (1974) 29.
- [39] V. Bernard, L. Elouadrhiri, and U.-G. Meißner, J. Phys. G **28** (2002) R1.
- [40] C. Amsler et al. (Particle Data Group), Phys. Lett. B **667** (2008) 1.
- [41] P. Mergell, et al., Nucl. Phys. A **596** (1996) 367.
- [42] K. Kubodera, J. Delorme and M. Rho, Phys. Rev. Lett. **40** (1978) 755.
- [43] K. Sumiyoshi, S. Yamada, H. Suzuki, H. Shen, S. Chiba and H. Toki, Astrophys. J., **629** (2005) 922.
- [44] H. Shen, H. Toki, K. Oyamatsu and K. Sumiyoshi, ApJS, **197** (2011) 20.
- [45] S. W. Bruenn, Astrophys. J. Suppl., **62** (1986) 331.
- [46] S. Furusawa, K. Sumiyoshi, S. Yamada and H. Suzuki, Astrophys. J. **772** (2013) 95.
- [47] A. Burrows, S. Reddy and T. A. Thompson, Nucl. Phys. A **777** (2006) 356.
- [48] S. E. Woosley, D. H. Hoffman and W. C. Haxton, Astrophys. J. **356** (1990) 272.
- [49] T. Yoshida, M. Terasawa, T. Kajino and K. Sumiyoshi, Astrophys. J. **600** (2004) 204.
- [50] L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (Benjamin, New York, 1962).

- [51] W. D. Kraeft, D. Kremp, W. Ebeling and G. Röpke, *Quantum Statistics of Charged Particle Systems* (Springer, New York, 1986).
- [52] A. M. Zagoskin, *Quantum Theory of Many-Body Systems* (Springer, New York, 1998).
- [53] T. Frick, Ph. D. thesis, University of Tübingen (2004), advisor: Prof. H. Mütter.
- [54] A. Rios, Ph. D. thesis, Universitat de Barcelona (2007), advisor: Prof. A. Polls and Prof. A. Ramos.
- [55] Y. Yamaguchi, Phys. Rev. **95** (1954) 1628.
- [56] R. T. Cahill and I. H. Sloan, Nucl. Phys. A **165** (1971) 161.
- [57] R. B. Wiringa, V. G. J. Stoks and R. Schiavilla, Phys. Rev. C **51** (1995) 38.