Large time behavior of solutions to systems of nonlinear Klein-Gordon equations

(非線型 Klein-Gordon 方程式系の解の長時間挙動)

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Contents

Chapter 1

Introduction

The present thesis is concerned with the large time behavior of solutions to the Cauchy problem with small initial data for second order nonlinear hyperbolc systems of Klein-Gordon type:

$$(\Box + m_i^2)u_i = F_i(u, \partial u, \partial\nabla u), \quad (t,x) \in (0,\infty)\times\mathbb{R}^n,\ 1 \le i \le N, \tag{1.0.1}$$

$$(u_i, \partial_t u_i)|_{t=0} = (\varepsilon f_i, \varepsilon g_i), \qquad x \in \mathbb{R}^n,\ 1 \le i \le N. \tag{1.0.2}$$

Here $\Box = \partial_t^2 - \Delta_x$, $\partial = (\partial_0, \partial_1, \cdots, \partial_n) = (\partial_t, \nabla)$, $\nabla = (\partial_{x_1}, \cdots, \partial_{x_n})$, $\varepsilon > 0$ is a small parameter, $m_i > 0$ is a constant, f_i, g_i are real valued functions which belong to $C_0^\infty(\mathbb{R}^n)$, u_i is a unknown function and $u = (u_1, \cdots, u_N)$. Throughout this thesis, the nonlinear term $F = (F_1, \cdots, F_N)$ is supposed to be a smooth function of $(u, \partial u, \partial\nabla u)$ in its argument. We also assume that F vanishes to p-th order near the origin with some integer $p \ge 2$ in the sense that there exist some constants C and $\varsigma > 0$ such that

$$\sum_{i=1}^N |F_i(u,v,w)| \le C(|u|^p + |v|^p + |w|^p) \quad \text{if} \quad |u| + |v| + |w| \le \varsigma,$$

where the variables

$$v = (v_{aj})_{\substack{0 \le a \le n \\ 1 \le j \le N}} \in \mathbb{R}^{(1+n)\times N} \quad \text{and} \quad w = (w_{aij})_{\substack{0 \le a \le n \\ 1 \le i \le n;\, 1 \le j \le N}} \in \mathbb{R}^{(1+n)\times n\times N}$$

correspond to

$$(\partial_a u_j)_{\substack{0 \le a \le n \\ 1 \le j \le N}} \quad \text{and} \quad (\partial_a \partial_i u_j)_{\substack{0 \le a \le n \\ 1 \le i \le n;\, 1 \le j \le N}},$$

respectively. We will often use the abbreviation

$$F(u, \partial u, \partial\nabla u) = O(|u|^p + |\partial u|^p + |\partial\nabla u|^p)$$

if there is no confusion. To ensure the system (1.0.1) being quasilinear and hyperbolic, we further assume that $F = (F_1, \cdots, F_N)$ admits the following decomposition:

$$F_j(u,v,w) = \sum_{a=0}^n \sum_{i=1}^n F_{aij}^{(\mathrm{Q})}(u,v) w_{aij} + F_j^{(\mathrm{S})}(u,v) \tag{1.0.3}$$

with some

$$F^{(\mathrm{Q})}_{aij}(u,v) = O(|u|^{p-1} + |v|^{p-1}) \quad \text{and} \quad F^{(\mathrm{S})}_{j}(u,v) = O(|u|^{p} + |v|^{p})$$

for each $j \in \{1, \cdots, N\}$. In other words, each F_j is independent of $\partial\nabla u_k$ ($k \neq j$) and affine with respect to $\partial\nabla u_j$.

First of all, let us consider the case where $F \equiv 0$ (we refer to this as "*free*" in what follows). In this case, it is well known that the the energy of the solution stays bounded for all time and that pointwise decay rate is $O(t^{-n/2})$ as $t \to \infty$. Next, let us assume these properties are still valid in the nonlinear case. Then we will obtain

$$\int_0^\infty \|F^{(\mathrm{S})}_j(u,\partial u)\|_{L^2} dt \lesssim \int_0^\infty (1+t)^{-n(p-1)/2} dt < \infty$$

when $n(p-1)/2 > 1$. Also, when $(u, \partial u)$ is small enough, it is natural to expect the operator

$$\partial_t^2 - \sum_{j,k=1}^{n} \left(\delta_{jk} + F^{(\mathrm{Q})}_{jkl}(u,\partial u)\right)\partial_j\partial_k - \sum_{k=1}^{n} F^{(\mathrm{Q})}_{0kl}(u,\partial u)\partial_t\partial_k + m_l^2$$

is close to $\partial_t^2 - \Delta_x + m_l^2$ in a suitable sense. These observations suggest (1.0.1) should be regarded as a perturbation of the free Klein-Gordon equations when $p > 1+2/n$ and when the initial data is sufficiently small. According to the earlier results due to Klainerman–Ponce [20], Shatah [27], [28], Klainerman [18], etc., the above heuristic argument leads to correct prediction for (1.0.1)–(1.0.2). More precisely, when $p > 1 + 2/n$, the Cauchy problem (1.0.1)–(1.0.2) have a unique global classical solution which tends to the free solution as $t \to \infty$ for sufficiently small ε. So, of our main interest is the case where $p \leq 1 + 2/n$, i.e., $(n,p) = (2,2)$, $(1,3)$ or $(1,2)$. In this case, the situation becomes more delicate. From the heuristic point of view the nonlinear term decays no faster than $O(t^{-1})$ in the sense of L^2, hence the nonlinearity should be considered as a long range perturbation. Over the last two decades, a great deal of effort has been put into this case. For the scalar case ($N = 1$), several results have been already obtained ([11], [21], [9], [29], [26], [24], [25], [14], [10], [16], [2], [3], etc.), whereas, quite little is known about the coupled case ($N \geq 2$).

The purpose of the present thesis is to develop the understanding for coupled systems of critical nonlinear Klein-Gordon equations with possibly different masses. The main goal is to reveal the influence of the combinations of masses and the nonlinearity on the large time behavior of solutions to (1.0.1)–(1.0.2) in the case where $p = 1 + 2/n$, $N \geq 2$. The organization is as follows. In Chapter 2, we will study a sufficient conditon under which a unique classical solution exists globally in time and it tends to the solution of the corresponding free Klein-Gordon equations. We will call it the *nonresonance* condition. Chapter 3 and 4 will be devoted to the study of the *resonant* case. We will construct some examples of resonant, critical nonlinear Klein-Gordon systems whose solutions do not behave like free solutions in the large time. Finally, two examples on small data blow-up will be exhibited in Chapter 5. This thesis is based on the author's works [30], [31], [32], [33] with some revisions.

Before closing this introductory chapter, we make an observation which will be the point which we start from. Put

$$\begin{aligned} \Gamma &= \{(t,x) \in (0,\infty) \times \mathbb{R}^n \mid t^2 - |x|^2 > 1\}, \\ H = \partial\Gamma &= \{(t,x) \in (0,\infty) \times \mathbb{R}^n \mid t^2 - |x|^2 = 1\} \end{aligned}$$

and let us consider

$$\begin{cases} (\Box + m^2)u = v^p, \\ (\Box + \mu^2)v = h, \end{cases} \qquad (t,x) \in \Gamma \tag{1.0.4}$$

with a suitable function $h(t,x)$ and $p = 1 + 2/n$. We associate $(t,x) \in \Gamma$ with $(\tau,\omega) \in (1,\infty) \times H$ by

$$\tau = \sqrt{t^2 - |x|^2}, \quad \omega = \left(\frac{t}{\tau}, \frac{x}{\tau}\right).$$

Then it follows that

$$\begin{aligned} \Box + m^2 &= \frac{\partial^2}{\partial\tau^2} + \frac{n}{\tau}\frac{\partial}{\partial\tau} + m^2 - \frac{1}{\tau^2}\Delta_H \\ &= \tau^{-n/2}\left(\frac{\partial^2}{\partial\tau^2} + m^2\right)\tau^{n/2} - \frac{1}{\tau^2}\left(\Delta_H + \frac{n(n-2)}{4}\right) \\ &\approx \tau^{-n/2}\left(\frac{\partial^2}{\partial\tau^2} + m^2\right)\tau^{n/2}, \end{aligned}$$

where

$$\Delta_H = \sum_{j=1}^{n}(x_j\partial_t + t\partial_j)^2 - \sum_{j<k}(x_j\partial_k - x_k\partial_j)^2,$$

which is the Laplace-Beltrami operator on the hyperboloid H with respect to the metric $-dt^2 + |dx|^2$. Next, we put

$$\begin{aligned} a(\tau,y) &= e^{-im\tau}\left(1 + \frac{1}{im}\frac{\partial}{\partial\tau}\right)\left(\tau^{n/2}u(\tau\omega(y))\right), \\ b(\tau,y) &= e^{-i\mu\tau}\left(1 + \frac{1}{i\mu}\frac{\partial}{\partial\tau}\right)\left(\tau^{n/2}v(\tau\omega(y))\right) \end{aligned}$$

with

$$\omega(y) = \left(\frac{1}{\sqrt{1-|y|^2}}, \frac{y}{\sqrt{1-|y|^2}}\right), \quad |y| < 1.$$

Then, for $(t,x) \in \Gamma$, we can write the solution $u(t,x)$ of (1.0.4) in terms of $a(\tau,y)$, that is,

$$u(t,x) = \frac{1}{\tau^{n/2}} \operatorname{Re}\left[a(\tau,x/t)e^{im\tau}\right]. \tag{1.0.5}$$

Also we have

$$\begin{aligned}\frac{\partial a}{\partial \tau}(\tau,y) &= \frac{e^{-im\tau}}{im}\left(\frac{\partial^2}{\partial\tau^2}+m^2\right)\left(\tau^{n/2}u\right)\\ &\approx \frac{e^{-im\tau}}{im}\tau^{n/2}(\Box+m^2)u\\ &= \frac{e^{-im\tau}}{im\tau^{n(p-1)/2}}\left(\tau^{n/2}v\right)^p.\end{aligned} \tag{1.0.6}$$

Substituting the relation

$$\left(\tau^{n/2}v\right)^p = \left(\operatorname{Re}\left(be^{i\mu\tau}\right)\right)^p = \frac{1}{2^p}\sum_{k=0}^{p}\begin{pmatrix} p \\ k \end{pmatrix} e^{i(p-2k)\mu\tau}\, b^{p-k}\, \overline{b}^{\,k}$$

into (1.0.6), we have

$$\frac{\partial a}{\partial \tau}(\tau,y) \approx \sum_{k=0}^{p} \gamma_k \frac{e^{i\zeta_k\tau}}{\tau}\left(b(\tau,y)\right)^{p-k}\left(\overline{b(\tau,y)}\right)^k, \tag{1.0.7}$$

where

$$\gamma_k = \frac{1}{2^p im}\begin{pmatrix} p \\ k \end{pmatrix}, \quad \zeta_k = (p-2k)\mu - m.$$

Now, suppose that $\lim_{\tau\to\infty} a(\tau,y)$ exists in a suitable sense. Then it follows from (1.0.5) that

$$u(t,x) \approx \frac{1}{t^{n/2}} \operatorname{Re}\left[A(x/t)e^{im(t^2-|x|^2)^{1/2}}\right], \quad (t,x)\in\Gamma$$

with

$$A(y) = (1-|y|^2)^{-n/4}\lim_{\tau\to\infty} a(\tau,y),$$

which implies u behaves like a free solution in the large time. We can see that this corresponds to the case where

$$m \neq (p-2k)\mu \quad \text{for each} \quad k \in \{0,1,\cdots,[p/2]\},$$

when we remember (1.0.7) and the fact that

$$\int_1^\infty \frac{e^{i\zeta\tau}}{\tau}d\tau < \infty \quad \text{if and only if} \quad \zeta \neq 0.$$

On the other hand, if $a(\tau, y)$ grows or oscillates when $\tau \to \infty$, then some nonlinear character will appear. This corresponds to the case where

$$m = (p - 2k)\mu \quad \text{for some} \quad k \in \{0, 1, \cdots, [p/2]\}.$$

The above heuristics will naturally leads to the notion of *resonance*, which is the key word of this thesis.

Chapter 2

Nonresonance and global existence of small solution with free profile

2.1 Introduction

This chapter is taken from the author's work [30] with a few improvement. The purpose of this chapter is to study a sufficient condition under which the Cauchy problem (1.0.1)–(1.0.2) have a unique global classical solution which tends to the corresponding free solution as $t \to \infty$ even in the critical nonlinear case where $p = 1 + 2/n$.

As we have observed in the preceding chapter, the oscillating structure caused by masses is expected to play the key role in the critical case. So we are led to the following condition. In order to state it precisely, let us define $\mathcal{I}_i^{(p)} \subset \{1, \cdots, N\}^p$ by

$$\mathcal{I}_i^{(p)} = \left\{ (j_1, \cdots, j_p) \;\middle|\; m_i \neq \sum_{k=1}^{p} \lambda_k m_{j_k} \quad \text{for all} \quad \lambda_k \in \{\pm 1\} \right\}$$

for $i \in \{1, \cdots, N\}$, $p \in \mathbb{N}$ and given $(m_j)_{1 \le j \le N}$.

Definition 2.1.1 *We say that the system (1.0.1) satisfies the nonresonance condition if the following holds true:*
(1) When $(n, p) = (1, 3)$, the nonlinear term $F = (F_i)_{1 \le i \le N}$ admits the decomposition

$$F_i(u, \partial u, \partial \nabla u) = \sum_{(j,k,l) \in \mathcal{I}_i^{(3)}} G_{ijkl}(u, \partial u, \partial \nabla u) + H_i(u, \partial u, \partial \nabla u)$$

for each $i \in \{1, \cdots, N\}$, where

$$H_i(u, \partial u, \partial \nabla u) = O(|u|^4 + |\partial u|^4 + |\partial \nabla u|^4)$$

and

$$G_{ijkl}(u, \partial u, \partial \nabla u) = \sum_{|\alpha|, |\beta|, |\gamma| \le 2} C^{ijkl}_{\alpha\beta\gamma} (\partial^\alpha u_j)(\partial^\beta u_k)(\partial^\gamma u_l)$$

with real constants $C^{ijkl}_{\alpha\beta\gamma}$.
(2) When $(n,p)=(2,2)$, *the nonlinear term* $F=(F_i)_{1\le i\le N}$ *can be written as*

$$F_i(u,\partial u,\partial\nabla u)=\sum_{(j,k)\in\mathcal{I}_i^{(2)}} G_{ijk}(u,\partial u,\partial\nabla u)+H_i(u,\partial u,\partial\nabla u)$$

for each $i\in\{1,\cdots,N\}$, *where* H_i *vanishes of third order near the origin and* G_{ijk} *consists of a linear combination of* $(\partial^\alpha u_j)(\partial^\beta u_k)$ *with* $|\alpha|,|\beta|\le 2$.

Remark. Roughly speaking, this condition allows us to get rid of the characteristic oscillations in the nonlinear terms of (1.0.1) (we should keep the observation of the preceding chapter in mind). This is the reason why we call it the *nonresonance* condition. Note that any cubic (resp. quadratic) nonlinear terms $(F_i)_{1\le i\le N}$ can be written as

$$F_i(u,\partial u,\partial\nabla u)=\sum_{j,k,l=1}^{N} G_{ijkl}(u,\partial u,\partial\nabla u)+H_i(u,\partial u,\partial\nabla u),$$

where G_{ijkl} and H_i are as above (resp.

$$F_i(u,\partial u,\partial\nabla u)=\sum_{j,k=1}^{N} G_{ijk}(u,\partial u)+H_i(u,\partial u,\partial\nabla u),$$

where G_{ijk} and H_i are as above) by using the Taylor expansion.

Now, we are going to state the main result in this chapter. This asserts that the nonresonance condition introcuced above is a sufficient condition which ensures the solution exists globally and behaves like the free solution as $t\to\infty$. More precisely, we will prove the following:

Theorem 2.1.1 *Let* $p=1+2/n$. *Suppose that the system (1.0.1) satisfies the nonresonance condition and the hyperbolicity assumption (1.0.3). Then there exists a unique global classical solution to the Cauchy problem (1.0.1)-(1.0.2) for sufficiently small* ε. *Furthermore, we have*

$$\lim_{t\to\infty}\sum_{i=1}^{N}\|u_i(t)-u_i^+(t)\|_{E(m_i)}=0$$

for some free solution $u^+=(u_i^+)_{1\le i\le N}$, *i.e. the function satisfying* $(\Box+m_i^2)u_i^+=0$.

Notation. Throughout this thesis, we will use the notation

$$\|\phi(t)\|_{E(m)}=\left(\int_{\mathbb{R}^n}\left|\partial_t\phi(t,x)\right|^2+\left|\nabla\phi(t,x)\right|^2+m^2\left|\phi(t,x)\right|^2dx\right)^{1/2}$$

for smooth function $\phi(t,x)$ and $m > 0$. Note that $\|\cdot\|_{E(m)}$ is equivalent to $\|\cdot\|_{E(1)}$ as a norm for any $m > 0$, so we will sometimes write $\|\cdot\|_E$ instesd of $\|\cdot\|_{E(m)}$.

As a consequence of Theorem 2.1.1, we can see the following example: Let m, μ be positive constants with $(m-3\mu)(m-\mu)(3m-\mu) \neq 0$. Consider the Cauchy problem

$$\begin{cases} (\Box + m^2)u = F(v, \partial v) \\ (\Box + \mu^2)v = G(u, \partial u) \end{cases}$$

in $\mathbb{R}_+ \times \mathbb{R}$, where F and G are arbitrary smooth functions of degree 3 in their arguments. If the initial data is small and smooth, the solution exists globally and it behaves like the free solution as $t \to \pm\infty$. We emphasize that in this case we need not any structual restrictions, such as the null condition, on F and G except they depend only on $(v, \partial v)$ and $(u, \partial u)$ respectively. It seems to be a remarkable contrast with the scalar one dimensional Klein-Gordon equation with cubic nonlinearity since some structual restriction is necessary for the scalar equation (see [3], [10], [14], [16], [24] etc.).

This result should also be compared with that for coupled nonlinear systems of wave equations with multiple speeds

$$(\partial_t^2 - c_i^2\Delta)u_i = F_i(u, \partial u, \partial\nabla u), \quad i = 1, \cdots, N.$$

Let us mention the case where $N = 2$ and $F_1 = F_2 = (\partial_t u_1)(\partial_t u_2)$ in three space dimensions, which is a typical case of Kovalyov's work [22]. He has proved that if $c_1 \neq c_2$, the system possesses a unique global smooth solution, whereas the solution of this system blows up in finite time if $c_1 = c_2$ (cf. [13]). Similar results have been proved for more general cases (See e.g., [23] for recent development in this direction).

Remark. The nonresonance condition is certainly not a necessary condition. Indeed, according to [24], [14], [3], there are some classes of scalar cubic nonlinear Klein-Godon equations in one space dimension which have a unique global classical solution which behaves like the free solution, although any scalar cubic nonlinear Klein-Godon equations do not satisfy our condition. (In contrast, scalar quadratic nonlinear Klein-Gordon equations are always nonresonant because $m \neq \lambda_1 m + \lambda_2 m$ for any $\lambda_1, \lambda_2 \in \{\pm 1\}$, $m > 0$.) We will discuss some relations between the results of [24], [14], [3] and ours in Section 2.5.

2.2 Preliminaries

To begin with, we introduce several notations which are used throughout this chapter. We put $x_0 = -t$, $x = (x_1, \cdots, x_n)$, $\partial_0 = \partial_t$, $\partial_j = \partial/\partial x_j$ $(1 \leq j \leq n)$, $\Omega_{ab} = x_a\partial_b - x_b\partial_a$ $(0 \leq a, b \leq n)$, and introduce the Klainerman vector fields

$$Z = (Z_1, \cdots, Z_{K_n}) = (\partial_a, \Omega_{bc}; 0 \leq a \leq n, 0 \leq b < c \leq n), \quad K_n = (n+1)(n+2)/2.$$

Note that the following commutation relations hold:

$$[\Box + m^2, Z_j] = 0, \tag{2.2.1}$$

$$[\Omega_{ab}, \partial_c] = \eta_{bc}\partial_a - \eta_{ca}\partial_b, \tag{2.2.2}$$

$$[\Omega_{ab}, \Omega_{cd}] = \eta_{ad}\Omega_{bc} + \eta_{bc}\Omega_{ad} - \eta_{ac}\Omega_{bd} - \eta_{bd}\Omega_{ac}, \tag{2.2.3}$$

for $m \in \mathbb{R}$, $1 \le j \le K_n$, $0 \le a, b, \le n$. Here $[\cdot, \cdot]$ denotes the commutator of linear operators and $(\eta_{ab})_{0\le a,b\le n} = \mathrm{diag}(-1, \underbrace{1, \cdots, 1}_{\mathrm{n}})$. For a smooth function $\phi(t,x)$ and for a non-negative integer s, we define

$$|\phi(t,x)|_s := \sum_{|\alpha|\le s} |Z^\alpha \phi(t,x)|$$

and

$$\|\phi(t)\|_s := \sum_{|\alpha|\le s} \|Z^\alpha \phi(t,\cdot)\|_{L^2(\mathbb{R}^n)},$$

where $\alpha = (\alpha_1, \cdots, \alpha_{K_n})$ is a multi-index, $Z^\alpha = Z_1^{\alpha_1} \cdots Z_{K_n}^{\alpha_{K_n}}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_{K_n}$.

Next, we introduce the following quadratic forms

$$Q_{ab}(\phi, \psi) = (\partial_a\phi)(\partial_b\psi) - (\partial_b\phi)(\partial_a\psi) \qquad (0 \le a, b \le n), \tag{2.2.4}$$

$$Q_0(\phi, \psi) = (\partial_t\phi)(\partial_t\psi) - (\nabla_x\phi)\cdot(\nabla_x\psi) = -\sum_{a,b=0}^{n} \eta_{ab}(\partial_a\phi)(\partial_b\psi). \tag{2.2.5}$$

They are often called *the null forms*, which have a certain compatibility with $\Box$ (see e.g. [19]). Note that Q_{ab} is also compatible with $(\Box + m^2)$, but Q_0 is sometimes not so (see [7]). Here we state well known properties of Q_{ab}:

Lemma 2.2.1

$$|Q_{ab}(\phi,\psi)| \le \frac{C}{(1+t+|x|)}\left(|\phi|_1|\partial\psi| + |\partial\phi||\psi|_1\right)$$

with some constant $C > 0$ independent of (t,x), and

$$Z^\alpha Q_{ab}(\phi,\psi) = \sum_{c,d\in\{0,\cdots,n\}} \sum_{|\beta|+|\gamma|\le|\alpha|} C^{cd}_{\beta\gamma} Q_{cd}(Z^\beta\phi, Z^\gamma\psi)$$

for any multi-indices α with appropriate constants $C^{cd}_{\beta\gamma}$.

Proof: Simple calculation yields

$$\begin{aligned}
x_c Q_{ab}(\phi,\psi) &= (\Omega_{ca}\phi)(\partial_b\psi) - (\Omega_{cb}\phi)(\partial_a\psi) + (\partial_c\phi)(\Omega_{ab}\psi), \\
\partial_c Q_{ab}(\phi,\psi) &= Q_{ab}(\partial_c\phi,\psi) + Q_{ab}(\phi,\partial_c\psi), \\
\Omega_{cd} Q_{ab}(\phi,\psi) &= Q_{ab}(\Omega_{cd}\phi,\psi) + Q_{ab}(\phi,\Omega_{cd}\psi) \\
&\quad + \eta_{ac}Q_{bd}(\phi,\psi) + \eta_{bd}Q_{ac}(\phi,\psi) - \eta_{ad}Q_{bc}(\phi,\psi) - \eta_{bc}Q_{a\nu}(\phi,\psi).
\end{aligned}$$

The lemma follows immediately from them. ∎

2.3 Decomposition of nonlinear terms

This section is devoted to the following important lemma. Similar approach can be found in the previous works by Kosecki [21], Katayama [14], Y. Tsutsumi [34], etc. and they are closely related to Shatah's method of normal forms [28] (see also [24], [25], [26], etc.).

Lemma 2.3.1 *Suppose that the system (1.0.1) satisfy the nonresonance condition and let $u(t,x)$ be a solution of it. Then $F_i(u, \partial u, \partial\nabla u)$ is decomposed into*

$$F_i(u, \partial u, \partial\nabla u) = (\Box + m_i^2)\Phi_i + \Psi_i + R_i,$$

where Φ_i, Ψ_i and R_i satisfy the following:
(1) *When $(n,p) = (1,3)$,*

$$\Phi_i = O\big(|(\partial^\alpha u)_{|\alpha|\le 3}|^3\big), \quad R_i = O\big(|(\partial^\alpha u)_{|\alpha|\le 5}|^4\big)$$

and

$$\Psi_i \text{ consists of a linear combination of } (\partial^\alpha u_j)Q_{01}(\partial^\beta u_k, \partial^\gamma u_l)$$

with $|\alpha|, |\beta|, |\gamma| \le 3$, $j,k,l \in \{1,\cdots,N\}$.
(2) *When $(n,p) = (2,2)$,*

$$\Phi_i = O\big(|(\partial^\alpha u)_{|\alpha|\le 3}|^2\big), \quad R_i = O\big(|(\partial^\alpha u)_{|\alpha|\le 5}|^3\big)$$

and

$$\Psi_i \text{ consists of a linear combination of } Q_{ab}(\partial^\alpha u_j, \partial^\beta u_k)$$

with $|\alpha|, |\beta| \le 3$, $j,k \in \{1,\cdots,N\}$, $0 \le a,b \le 2$.

This lemma is an immediate consequence of the following lemma.

Lemma 2.3.2 *Let m_i be positive constants, v_i be smooth functions of $(t,x) \in \mathbb{R}^{1+n}$, and put $h_i = (\Box + m_i^2)v_i$ for $i = 1,2,3$.*
(1) *If $m \in \mathbb{R}$ satisfy*

$$m \ne \lambda_1 m_1 + \lambda_2 m_2 + \lambda_3 m_3 \quad \text{for all} \quad \lambda_1, \lambda_2, \lambda_3 \in \{\pm 1\}, \tag{2.3.1}$$

then we have

$$v_1 v_2 v_3 = (\Box + m^2)\Phi + \Psi + R, \tag{2.3.2}$$

where

$$\Phi \text{ is a homogeneous polynomial of degree 3 with respect to } (\partial^\alpha v_i)_{i=1,2,3;|\alpha|\le 1}$$

and

$$\Psi \text{ consists of a linear combination of } \quad (\partial^\alpha v_i) Q_{ab}(\partial^\beta v_j, \partial^\gamma v_k), \tag{2.3.3}$$
$$R \text{ consists of a linear combination of } \quad v_i h_j h_k \quad \text{or} \quad (\partial^\alpha v_i)(\partial^\beta v_j)(\partial^\gamma h_k), \tag{2.3.4}$$

with $|\alpha|, |\beta|, |\gamma| \le 1$, $a, b \in \{0, \cdots, n\}$, $i, j, k \in \{1, 2, 3\}$.
(2) *If* $m \in \mathbb{R}$ *satisfy*

$$m \neq \lambda_1 m_1 + \lambda_2 m_2 \quad \text{for all} \quad \lambda_1, \lambda_2 \in \{\pm 1\},$$

then we have

$$v_1 v_2 = (\Box + m^2)\Phi + \Psi + R, \tag{2.3.5}$$

where

$$\Phi \text{ is a homogeneous polynomial of degree 2 with respect to } (\partial^\alpha v_i)_{i=1,2;|\alpha|\le 1}$$

and

$$\Psi \text{ consists of a linear combination of } \quad Q_{ab}(\partial^\beta v_j, \partial^\gamma v_k), \tag{2.3.6}$$
$$R \text{ consists of a linear combination of } \quad h_j h_k \quad \text{or} \quad (\partial^\alpha v_i)(\partial^\beta h_j), \tag{2.3.7}$$

with $|\alpha|, |\beta| \le 1$, $a, b \in \{0, \cdots, n\}$, $i, j \in \{1, 2, 3\}$.

Proof of Lemma 2.3.1: In this proof, we denote by Ψ (resp. R) the terms satisfying (2.3.3) (resp. (2.3.4)), which may be different line by line.

We put

$$\phi_0 := v_1 v_2 v_3, \ \phi_1 := \frac{-v_1 Q_0(v_2, v_3)}{m_2 m_3}, \ \phi_2 := \frac{-v_2 Q_0(v_3, v_1)}{m_3 m_1}, \ \phi_3 := \frac{-v_3 Q_0(v_1, v_2)}{m_1 m_2},$$

where Q_0 is defined by (2.2.5). Noting that

$$-\Box = \sum_{a,b=0}^{n} \eta_{ab} \partial_a \partial_b,$$

we have

$$-\Box\phi_0 = (m_1^2 + m_2^2 + m_3^2)\phi_0 + 2m_2 m_3 \phi_1 + 2m_3 m_1 \phi_2 + 2m_1 m_2 \phi_3 + R, \tag{2.3.8}$$
$$-\Box\phi_1 = 2m_2 m_3 \phi_0 + (m_1^2 + m_2^2 + m_3^2)\phi_1 + 2m_1 m_2 \phi_2 + 2m_3 m_1 \phi_3 + \Psi + R, \tag{2.3.9}$$
$$-\Box\phi_2 = 2m_3 m_1 \phi_0 + 2m_1 m_2 \phi_1 + (m_1^2 + m_2^2 + m_3^2)\phi_2 + 2m_2 m_3 \phi_3 + \Psi + R, \tag{2.3.10}$$
$$-\Box\phi_3 = 2m_1 m_2 \phi_0 + 2m_3 m_1 \phi_1 + 2m_2 m_3 \phi_2 + (m_1^2 + m_2^2 + m_3^2)\phi_3 + \Psi + R. \tag{2.3.11}$$

Indeed, simple calculation leads to

$$\begin{aligned}-\Box(v_1v_2v_3) &= -(\Box v_1)v_2v_3 - v_1(\Box v_2)v_3 - v_1v_2(\Box v_3)\\ &\quad - 2v_1Q_0(v_2,v_3) - 2v_2Q_0(v_3,v_1) - 2v_3Q_0(v_1,v_2)\\ &=(m_1^2+m_2^2+m_3^2)v_1v_2v_3\\ &\quad - 2v_1Q_0(v_2,v_3) - 2v_2Q_0(v_3,v_1) - 2v_3Q_0(v_1,v_2)\\ &\quad - h_1v_2v_3 - h_2v_3v_1 - h_3v_1v_2.\end{aligned}$$

We have (2.3.8) from this. Concerning (2.3.9), simple calculation shows that

$$\begin{aligned}&\Box\big(v_1Q_0(v_2,v_3)\big) = (\Box v_1)Q_0(v_2,v_3) + v_1Q_0(\Box v_2,v_3) + v_1Q_0(v_2,\Box v_3)\\ &\quad -2\sum_{a,b=0}^{n}\eta_{ab}\Big\{(\partial_a v_1)Q_0(\partial_b v_2,v_3) + v_1Q_0(\partial_a v_2,\partial_b v_3) + (\partial_b v_1)Q_0(v_2,\partial_a v_3)\Big\}.\end{aligned}$$

We also note the following identities:

$$\begin{aligned}&-\sum_{a,b=0}^{n}\eta_{ab}Q_0(\partial_a w_1,\partial_b w_2) = (\Box w_1)(\Box w_2) + \sum_{a,b,c,d=0}^{n}\eta_{ab}\eta_{cd}Q_{cb}(\partial_a w_1,\partial_d w_2),\\ &-\sum_{a,b=0}^{n}\eta_{ab}(\partial_a w_1)Q_0(\partial_b w_2,w_3) = (\Box w_2)Q_0(w_3,w_1) + \sum_{a,b,c,d=0}^{n}\eta_{ab}\eta_{cd}(\partial_d w_3)Q_{ac}(w_1,\partial_b w_2).\end{aligned}$$

These identities combined with the relation $-\Box v_i = m_i^2 v_i - h_i$ yield (2.3.9). We obtain (2.3.10) and (2.3.11) in the same way.

From (2.3.8)–(2.3.11), it follows that

$$\sum_{j=0}^{3} b_j\phi_j = (\Box + m^2)\Big(\sum_{j=0}^{3} a_j\phi_j\Big) + \Psi + R$$

for any $a_j \in \mathbb{R}$ $(j = 0,1,2,3)$, where

$$\begin{pmatrix} b_0\\ b_1\\ b_2\\ b_3\end{pmatrix} = \begin{pmatrix} P_1 & P_2\\ P_2 & P_1\end{pmatrix}\begin{pmatrix} a_0\\ a_1\\ a_2\\ a_3\end{pmatrix},$$

$$P_1 = \begin{pmatrix} m^2 - m_1^2 - m_2^2 - m_3^2 & -2m_2m_3\\ -2m_2m_3 & m^2 - m_1^2 - m_2^2 - m_3^2\end{pmatrix},$$

$$P_2 = \begin{pmatrix} -2m_3m_1 & -2m_1m_2\\ -2m_1m_2 & -2m_3m_1\end{pmatrix}.$$

Note that

$$\det\begin{pmatrix} P_1 & P_2 \\ P_2 & P_1 \end{pmatrix} = \det(P_1+P_2)\det(P_1-P_2)$$
$$= \prod_{\lambda_1,\lambda_2,\lambda_3\in\{\pm1\}} (m-\lambda_1 m_1-\lambda_2 m_2-\lambda_3 m_3),$$

which implies (2.3.1) and $\det\begin{pmatrix} P_1 & P_2 \\ P_2 & P_1 \end{pmatrix} \neq 0$ are equivalent. Therefore, one can choose $\Phi = \sum_{j=0}^{3} a_j\phi_j$ with

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} P_1 & P_2 \\ P_2 & P_1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

so that (2.3.2) holds. This completes the proof of (1).

The proof of (2) is analogous. The key identity corresponding to (2.3.8)–(2.3.11) is

$$(\Box + m^2)\overrightarrow{\phi} = \begin{pmatrix} m^2-m_1^2-m_2^2 & -2m_1m_2 \\ -2m_1m_2 & m^2-m_1^2-m_2^2 \end{pmatrix}\overrightarrow{\phi} + \overrightarrow{\Psi} + \overrightarrow{R},$$

where

$$\overrightarrow{\phi} = {}^t\!\big(v_1v_2,\ -Q_0(v_1,v_2)/m_1m_2\big),$$

and $\overrightarrow{\Psi}$, $\overrightarrow{R}$ consist of a linear combination of the terms like (2.3.6), (2.3.7) respectively. ∎

2.4 Proof of Theorem 2.1.1

Now, we are ready to prove Theorem 2.1.1. Since the local existence is well known (see [17], [12] etc.), what we have to do is to get some a priori estimate. We prove only the one-dimensional, cubic nonlinear case since the other one follows in the same way. From now on, we suppose that the system (1.0.1) satisfy the nonresonance condition and let $u = (u_i)_{i=1,\cdots,N}$ be a solution to the Cauchy problem (1.0.1)–(1.0.2) for $t \in [0,T[$. We define

$$E_s(T;u) := \sup_{0\le t<T}\Big[(1+t)^{-\rho}(\|u(t,\cdot)\|_s + \|\partial u(t,\cdot)\|_s) \tag{2.4.1}$$
$$+ \|u(t,\cdot)\|_{s-4} + \|\partial u(t,\cdot)\|_{s-4} + \sup_{x\in\mathbb{R}}\{(1+t+|x|)^{1/2}|u(t,x)|_{s-11}\}\Big],$$

where $s \ge 27$ and $\rho \in]0,1/2[$. Note that $\|u(t,\cdot)\|_\sigma + \|\partial u(t,\cdot)\|_\sigma$ is equivalent to

$$\sum_{|\alpha|\le\sigma}\sum_{i=1}^{N} \|Z^\alpha u_i(t)\|_E$$

because of the commutation relations (2.2.2) and (2.2.3). In order to prove the global existence, it suffices to show the following.

Proposition 2.4.1 *For any ε and $\delta \in]0,1]$,*

$$E_s(T;u) \leq \delta \quad \textit{implies} \quad E_s(T;u) \leq C_s(\varepsilon + \delta^3),$$

where C_s is a positive constant depending on s, but independent of $\varepsilon, \delta \in]0,1]$ and $T(>0)$.

Once we have this proposition, we can obtain global existence in the following way. If we choose δ and ε_0 so that

$$C_s\delta^2 \leq 1/4, \quad C_s\varepsilon_0 \leq \delta/4 \quad \text{and} \quad E_s(0;u) < \delta,$$

then it follows that $E_s(T;u) \leq \delta$ implies $E_s(T;u) \leq \delta/2$ for any $\varepsilon \in]0,\varepsilon_0]$. Then, by the continuity arguments, we can show that $E_s(t;u) \leq \delta$ holds as long as the solution exists, provided that $\varepsilon \in]0,\varepsilon_0]$. The global existence is an immediate consequence of this estimate and the local existence theorem.

Remark. In the case where $(n,p) = (2,2)$, it suffices to control the quantity

$$\sup_{0\leq t<T}\Big[\ (1+t)^{-\rho}\big(\|u(t,\cdot)\|_s + \|\partial u(t,\cdot)\|_s\big) \\ + \|u(t,\cdot)\|_{s-4} + \|\partial u(t,\cdot)\|_{s-4} + \sup_{x\in\mathbb{R}^2}\big\{(1+t+|x|)|u(t,x)|_{s-12}\big\}\ \Big]$$

for $s \geq 29$, $\rho \in]0,1[$, instead of (2.4.1).

Proof of Proposition 2.4.1: In what follows, we denote by C_s various positive constants which are independent of ε, δ and T, and may change line by line.

We assume that $E_s(T;u) \leq \delta$. From Lemma 2.3.1 and the commutation relation (2.2.1), we have

$$(\Box + m_i^2)Z^\alpha(u_i - \Phi_i) = Z^\alpha(\Psi_i + R_i) \tag{2.4.2}$$

for $t \in [0,T[$ and for any multi-index α. Note that it follows from Lemma 2.2.1 that

$$|Z^\alpha\Psi_i(t,x)| \leq \frac{C_\alpha}{1+t+|x|}|u|^2_{[|\alpha|/2]+4}\big(|u|_{|\alpha|+3} + |\partial u|_{|\alpha|+3}\big) \tag{2.4.3}$$

Also we have

$$\begin{aligned}|Z^\alpha R_i(t,x)| &\leq C_\alpha\big|(\partial^\beta u)_{|\beta|\leq 5}\big|^3_{[|\alpha|/2]}\big|(\partial^\beta u)_{|\beta|\leq 5}\big|_{|\alpha|} \\ &\leq C_\alpha|u|^3_{[|\alpha|/2]+5}\big(|u|_{|\alpha|+4} + |\partial u|_{|\alpha|+4}\big).\end{aligned} \tag{2.4.4}$$

Here we denote by $[\sigma]$ the largest integer which does not exceed σ.

Now, we recall the following L^∞–L^2 estimate due to Georgiev [8] (see also Chapter 7 of Hörmander [12]):

Lemma 2.4.1 *Let $m > 0$ be a constant and $w(t,x)$ be a solution of the inhomogeneous linear Klein-Gordon equation $(\Box + m^2)w = h$ for $t \geq 0$, $x \in \mathbb{R}^n$, $n = 1, 2$. Then we have*

$$\begin{aligned}&(1+t+|x|)^{n/2}|w(t,x)|\\ &\leq C\sum_{j=0}^{\infty}\sum_{|\beta|\leq[n/2]+3}\sup_{\tau\in[0,t]}\varphi_j(\tau)\big\|(1+\tau+|y|)Z^\beta h(\tau,y)\big\|_{L^2_y}\\ &\quad+C\sum_{j=0}^{\infty}\sum_{|\beta|\leq[n/2]+4}\big\|(1+|y|)^{1/2}\varphi_j(|y|)Z^\beta w(0,y)\big\|_{L^2_y},\end{aligned}$$

if the norms in the right-hand side are finite. Here $\{\varphi_j\}_{j=0}^\infty$ is a Littlewood-Payley partition of unity, i.e.,

$$\begin{aligned}&\sum_{k=0}^{\infty}\varphi_k(\tau)=1\quad(\tau\geq 0);\quad \varphi_j\in C_0^\infty(\mathbb{R}),\ \varphi_j\geq 0 \text{ for } j\geq 0;\\ &\operatorname{supp}\varphi_j\subset[2^{j-1},2^{j+1}] \text{ for } j\geq 1,\quad \operatorname{supp}\varphi_0\cap\mathbb{R}_+\subset[0,2].\end{aligned}$$

Applying this estimate with $w = Z^\alpha(u_i - \Phi_i)$, $|\alpha| \leq s - 11$, we have

$$\begin{aligned}&(1+t+|x|)^{1/2}|Z^\alpha(u_i-\Phi_i)(t,x)|\\ &\qquad\leq C_s\varepsilon+C_s\sum_{j=0}^{\infty}\sum_{|\beta|\leq s-8}\sup_{\tau\in[0,t]}\varphi_j(\tau)\big\|(1+\tau+|\cdot|)Z^\beta\{\Psi_i(\tau,\cdot)+R_i(\tau,\cdot)\}\big\|_{L^2}.\end{aligned}$$

Using (2.4.3) and (2.4.4), we have

$$\begin{aligned}&\big\|(1+\tau+|\cdot|)Z^\beta\{\Psi_i(\tau,\cdot)+R_i(\tau,\cdot)\}\big\|_{L^2}\\ &\leq\frac{C_s}{1+\tau}\big(\|u\|_{s-5}+\|\partial u\|_{s-5}\big)\sup_{(\sigma,y)\in[0,\tau[\times\mathbb{R}}\Big((1+\sigma+|y|)^{1/2}|u(\sigma,y)|_{[(s-8)/2]+4}\Big)^2\\ &\quad+\frac{C_s}{(1+\tau)^{1/2}}\big(\|u\|_{s-4}+\|\partial u\|_{s-4}\big)\sup_{(\sigma,y)\in[0,\tau[\times\mathbb{R}}\Big((1+\sigma+|y|)^{1/2}|u(\sigma,y)|_{[(s-8)/2]+5}\Big)^3\\ &\leq C_s\delta^3(1+\tau)^{-1/2}\end{aligned}$$

for $|\beta| \le s-8$. Here we have used $[(s-8)/2]+5 \le s-11$ for $s \ge 23$. So we have

$$\begin{aligned}
&(1+t+|x|)^{1/2}|Z^\alpha(u_i-\Phi_i)(t,x)| \\
&\quad \le C_s\left(\varepsilon+\delta^3\sum_{j=0}^{\infty}\sup_{\tau\in[0,t]}\varphi_j(\tau)(1+\tau)^{-1/2}\right) \\
&\quad \le C_s\left(\varepsilon+\delta^3\Big(1+\sum_{j=1}^{\infty}(1+2^{j-1})^{-1/2}\Big)\right) \\
&\quad \le C_s(\varepsilon+\delta^3).
\end{aligned}$$

Since Φ is a homogeneous polynomial of degree 3 with respect to $(\partial^\beta u_i)_{i=1,\cdots,N;|\beta|\le 3}$, the Sobolev embedding implies

$$\begin{aligned}
|\Phi_i(t,x)|_{s-11} &\le C_s|u(t,x)|^2_{[(s-11)/2]+3}|u(t,x)|_{s-8} \\
&\le C_s|u(t,x)|^2_{s-11}\|u(t,\cdot)\|_{s-7} \\
&\le C_s(1+t+|x|)^{-1}\delta^3.
\end{aligned}$$

Summing up, we obtain

$$(1+t+|x|)^{1/2}|u(t,x)|_{s-11} \le C_s(\varepsilon+\delta^3) \tag{2.4.5}$$

for $t \in [0,T[$.

Next, let $|\alpha| \le s-4$ in (2.4.2). Applying the standard energy estimate for $\Box + m_i^2$, we obtain

$$\begin{aligned}
&\|Z^\alpha(u_i-\Phi_i)(t,\cdot)\|_{L^2}+\|\partial Z^\alpha(u_i-\Phi_i)(t,\cdot)\|_{L^2} \\
&\qquad \le C_s\left(\varepsilon+\int_0^t\big\|\Psi_i(\tau,\cdot)\big\|_{s-4}+\big\|R_i(\tau,\cdot)\big\|_{s-4}d\tau\right).
\end{aligned}$$

Using (2.4.3) and (2.4.4) again, we have

$$\begin{aligned}
&\big\|\Psi_i(\tau,\cdot)\big\|_{s-4}+\big\|R_i(\tau,\cdot)\big\|_{s-4} \\
&\le \frac{C_s}{(1+\tau)^2}\big(\|u\|_{s-1}+\|\partial u\|_{s-1}\big)\sup_{(\sigma,y)\in[0,\tau[\times\mathbb{R}}\Big((1+\sigma+|y|)^{1/2}\big|u(\sigma,y)\big|_{[(s-4)/2]+4}\Big)^2 \\
&\quad +\frac{C_s}{(1+\tau)^{3/2}}\big(\|u\|_s+\|\partial u\|_s\big)\sup_{(\sigma,y)\in[0,\tau[\times\mathbb{R}}\Big((1+\sigma+|y|)^{1/2}\big|u(\sigma,y)\big|_{[(s-4)/2]+5}\Big)^3 \\
&\le C_s\delta^3(1+\tau)^{\rho-3/2}\in L^1(0,\infty)
\end{aligned}$$

since $\rho-3/2 < -1$ and $[(s-4)/2]+5 \le s-11$ for $s \ge 27$. Also, it follows from $\Phi_i = O\left(|(\partial^\alpha u)_{|\alpha|\le 3}|^3\right)$ that

$$\begin{aligned}
\|\Phi_i(t,\cdot)\|_{s-4}+\|\partial\Phi_i(t,\cdot)\|_{s-4} &\le C_s|u(t,x)|^2_{[(s-4)/2]+3}\big(\|u(t,\cdot)\|_{s-2}+\|\partial u(t,\cdot)\|_{s-2}\big) \\
&\le C_s\delta^3(1+t)^{\rho-1} \qquad\qquad (2.4.6)\\
&\le C_s\delta^3.
\end{aligned}$$

Thus we have

$$\|u(t,\cdot)\|_{s-4} + \|\partial u(t,\cdot)\|_{s-4} \le C_s(\varepsilon + \delta^3) \tag{2.4.7}$$

for $t \in [0, T[$.

Finally, let $|\alpha| \le s$. We rewrite (1.0.1) as

$$\Big(\square + m_i^2 - \sum_{a=0}^{1} F_{a1i}^{(Q)}(u, \partial u)\partial_a\partial_1\Big) Z^\alpha u_i = Z^\alpha F_i^{(S)}(u, \partial u) + \sum_{a=0}^{1}\Big[Z^\alpha, F_{a1i}^{(Q)}(u, \partial u)\partial_a\partial_1\Big] u_i. \tag{2.4.8}$$

Using the commutation relations, we can estimate the L^2–norm of the right hand side of (2.4.8) by $C\delta^3(1+t)^{\rho-1}$ for $t \in [0, T[$. Therefore, we deduce from the energy estimate for the perturbed Klein-Gorodn operators (see § 2.5) that

$$\begin{aligned}\|u(t,\cdot)\|_s + \|\partial u(t,\cdot)\|_s &\le C_s\left(\varepsilon + \delta^3 \int_0^t (1+\tau)^{\rho-1} d\tau\right)\\ &\le C_s(\varepsilon + \delta^3)(1+t)^\rho \end{aligned}\tag{2.4.9}$$

for $t \in [0, T[$.

From (2.4.5), (2.4.7) and (2.4.9), we have

$$E_s(T; u) \le C_s(\varepsilon + \delta^3),$$

which completes the proof of Proposition 2.4.1.

Now we prove the existence of a free profile. It follows from Lemma 2.3.1 that

$$(\square + m_i^2)(u_i - \Phi_i) = \Psi_i + R_i.$$

Also, since $E_s(\infty; u) \le \delta$, we can show as in the proof of Proposition 2.4.1 that

$$\|\Psi_i(t,\cdot) + R_i(t,\cdot)\|_{L^2} \le C_s\delta^3(1+t)^{\rho-3/2} \in L^1(0,\infty).$$

Note that $0 < \rho < 1/2$. Therefore, by the standard argument as in [14], [24], [26], $u_i - \Phi_i$ has a free profile, i.e., there exists $(\phi_i, \psi_i) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ such that

$$\|\big((u_i - \Phi_i) - u_i^+\big)(t)\|_{E(m_i)} \to 0$$

as $t \to \infty$, where u_i^+ is a solution of the free Klein-Gordon equation

$$(\square + m_i^2)u_i^+ = 0 \quad t > 0,\ x \in \mathbb{R}$$

with the initial data

$$u_i^+(0,x) = \phi_i(x),\ \partial_t u_i^+(0,x) = \psi_i(x).$$

Since we can see from (2.4.6) that

$$\|\Phi_i(t)\|_{E(m_i)} \le C\big(\|\Phi_i(t,\cdot)\|_{H^1} + \|\partial_t\Phi_i(t,\cdot)\|_{L^2}\big) \le C_s\delta^3(1+t)^{\rho-1} \to 0$$

as $t \to \infty$, we obtain

$$\|\big(u_i - u_i^+\big)(t)\|_{E(m_i)} \to 0$$

as $t \to \infty$. This completes the proof of Theorem 2.1.1. ∎

2.5 Notes

(1) **Energy estimate for the perturbed Klein-Gordon operators.**

Here we state the energy estimate for the perturbed Klein-Gordon operators which we have made use of in the proof of Proposition 2.4.1. Let us consider

$$\big(\partial_t^2 - \Delta_x + m^2\big)v + \sum_{a=0}^{n}\sum_{k=1}^{n} g_{ak}(t,x)\partial_a\partial_k v = h(t,x) \tag{2.5.1}$$

in $(t,x) \in [0,T[\times\mathbb{R}^n$, where $g_{ak} \in \mathcal{B}^2([0,T[\times\mathbb{R}^2)$ and $h \in C([0,T[;L^2(\mathbb{R}^n))$. We assume that there exist positive constants c_1 and c_2 such that

$$c_1|\xi|^2 \le \sum_{j,k=1}^{n} \tilde{g}_{jk}(t,x)\xi_j\xi_k \le c_2|\xi|^2$$

for any $(t,x) \in]0,T[\times\mathbb{R}^2$ and any $\xi \in \mathbb{R}^n$, where

$$\tilde{g}_{jk}(t,x) = \delta_{jk} + g_{jk}(t,x), \quad j,k = 1,\cdots,n$$

and δ_{jk} is the Kronecker symbol. We define the norm $\|\cdot\|_{\tilde{E}(m)}$ as follows:

$$\|\phi(t)\|_{\tilde{E}(m)}^2 = \int_{\mathbb{R}^n} \big(\partial_t\phi\big)^2 + \sum_{j,k=1}^{n} \tilde{g}_{jk}(t,x)\big(\partial_j\phi\big)\big(\partial_k\phi\big) + m^2\phi^2 dx.$$

Note that $\|\cdot\|_{\tilde{E}(m)}$ is equivalent to $\|\cdot\|_{E(m)}$ as a norm. Under these notations and assumptions, we have the following enrgy estimate, whose proof can be found in [17], [12], etc..

Proposition 2.5.1 *Suppose that $v(t,x)$ satisfy (2.5.1) in $(t,x) \in [0,T[\times\mathbb{R}^n$. Then we have*

$$\frac{d}{dt}\|v(t)\|_{\tilde{E}(m)} \le C\Big(\max_{a,b,k}\|\partial_b g_{ak}(t,\cdot)\|_{L^\infty}\Big)\|v(t)\|_{\tilde{E}(m)} + \|h(t,\cdot)\|_{L^2},$$

where C a positive constant.

In the author's original paper [30], only the semilinear case has been treated. However, as is pointed out in [34] and [4], one can easily extend the result of [30] to the quasilinear case without any essential difficulty if one impose the hyperbolicity assumption (1.0.3). Only a difference is the energy estimate just mentioned above. This is the reason why we have treated the quasilinear case in this chapter revising the argument of [30]. It is worth noting that the Maxwell-Higgs equation studied in [34], which can be regarded as a system of nolinear Klein-Godon equations with several masses under a suitable gauge condition, does not satisfy (1.0.3).

(2) **Remarks on scalar cubic nonlinear Klein-Gordon equations in one space dimension.**

As we have mentioned at the end of Section 2.1, the nonresonance condition considered in this chapter is not optimal because of the results due to Moriyama [24], Katayama [14], Delort [3] concerning scalar nonlinear Klein-Gordon equations in one space dimension. Here we would like to discuss some relations between their results and ours.

Let us consider scalar Klein-Gordon equations with cubic nonlinearity in one space dimension:

$$(\Box + 1)u = F(u, \partial u, \partial\partial_x u), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R};$$
$$F(u, v, w) = O(|u|^3 + |v|^3 + |w|^3).$$

It is obvious they do not satisfy our condition since $1 = 1 + 1 - 1$. Nevertheless, the same conclusion as Theorem 2.1.1 holds true for some classes of cubic nonlinearity. In fact, Moriyama [24] showed global existence of small data solution and its convergence to some free solution as $t \to \infty$ if F is written as a linear combination of the following $F^{(1)}$–$F^{(7)}$:

$$\begin{aligned}
F^{(1)} &= 3uu_t^2 - 3uu_x^2 - u^3,\\
F^{(2)} &= 3u_t^2u_x - u_x^3 - 3u^2u_x + 6uu_tu_{tx},\\
F^{(3)} &= uu_xu_{xx} - u^2u_x + u_t^2u_x + 2uu_tu_{tx}\\
F^{(4)} &= (u_t^2 - u_x^2 - u^2)u_{xx} - 2uu_x^2,\\
F^{(5)} &= (u_t^2 - u_x^2 - u^2)u_{tx} - 2uu_tu_x,\\
F^{(6)} &= u_t^3 - 3u_x^2u_t - 3u^2u_t - 6uu_xu_{tx},\\
F^{(7)} &= u_tu_x^2 + uu_tu_{xx} + 2uu_xu_{tx}.
\end{aligned}$$

Later, Katayama [14] considered another sufficient condition. The class of nonlinear terms he found is the following: *F admits the decomposition*

$$F(u, \partial u, \partial\partial_x u) = \sum_{j=1}^{10} C_j G^{(j)}(u, \partial u, \partial\partial_x u) + N(u, \partial u, \partial\partial_x u) + H(u, \partial u, \partial\partial_x u),$$

where $C_j \in \mathbb{R}$ $(j = 1, \cdots, 10)$ *and*
(i) $G^{(j)}$ $(1 \le j \le 10)$ *are given by*

$$
\begin{aligned}
G^{(1)} &= u(-u^2 + 3u_t^2 - 3u_x^2),\\
G^{(2)} &= u_t(-3u^2 + u_t^2 - u_x^2) + 2u(u_t u_{xx} - u_x u_{tx}),\\
G^{(3)} &= u_t(-u^2 + u_t^2 - u_x^2) + 2u(u_t u_{tx} - u_x u_{xx}),\\
G^{(4)} &= u^3 - 2u^2 u_{xx} - 3uu_t^2 + 2u_t^2 u_{xx} - 2u_t u_x u_{tx} - u(u_{tx}^2 - u_{xx}^2),\\
G^{(5)} &= (-u^2 + u_t^2 - u_x^2)u_{tx} - 2uu_t u_x,\\
G^{(6)} &= -uu_x^2 + 2u_x(u_t u_{tx} - u_x u_{xx}) + u(u_{tx}^2 - u_{xx}^2),\\
G^{(7)} &= 3u^2 u_t - 6uu_t u_{xx} - u_t^3 - 3u_t(u_{tx}^2 - u_{xx}^2),\\
G^{(8)} &= u^2 u_x - 2uu_t u_{tx} - 2uu_x u_{xx} - u_t^2 u_x - u_x(u_{tx}^2 - u_{xx}^2),\\
G^{(9)} &= -2uu_x u_{tx} - u_t u_x^2 + u_t(u_{tx}^2 - u_{xx}^2),\\
G^{(10)} &= -u_x^3 + 3u_x(u_{tx}^2 - u_{xx}^2),
\end{aligned}
$$

(ii) N *is of the form*

$$
\begin{aligned}
N(u, \partial u, \partial\partial_x u) =&(u_t u_{tx} - u_x u_{xx} + uu_x)P_1(u, \partial u, \partial\partial_x u)\\
&+ (u_t u_{xx} - u_x u_{tx})P_2(u, \partial u, \partial\partial_x u)\\
&+ (u_{tx}^2 - u_{xx}^2 + uu_{xx})P_3(u, \partial u, \partial\partial_x u),
\end{aligned}
$$

with some $P_j = O(|u| + |\partial u| + |\partial\partial_x u|)$ $(j = 1, 2, 3)$,

(iii) $$H(u, \partial u, \partial\partial_x u) = O(|u|^4 + |\partial u|^4 + |\partial\partial_x u|^4).$$

Actually, Katayama's class is wider than Moriyama's one because we can verify by straightforard calculations that

$$
\begin{aligned}
F^{(1)} &= G^{(1)},\\
F^{(2)} &= \big(3G^{(3)} - 3G^{(8)} - G^{(10)}\big)/2,\\
F^{(3)} &= \big(G^{(3)} - 3G^{(8)} - G^{(10)}\big)/4,\\
F^{(4)} &= \big(G^{(1)} + G^{(4)} + G^{(6)}\big)/2,\\
F^{(5)} &= G^{(5)},\\
F^{(6)} &= \big(3G^{(2)} + G^{(7)} + 3G^{(9)}\big)/2,\\
F^{(7)} &= \big(-G^{(2)} - G^{(7)} - 3G^{(9)}\big)/2.
\end{aligned}
$$

Also, using the relation $\Box u = -u + F$, we see that

$$N(u, \partial u, \partial\partial_x u) = Q_{01}(u, \partial_0 u)P_1 + Q_{01}(u, \partial_1 u)P_2 - Q_{01}(\partial_0 u, \partial_1 u)P_3 + u_x F P_1 + u_{xx} F P_3$$

and that each $G^{(j)}$ $(j = 1, \cdots, 10)$ satisfies

$$2G^{(j)} - (\Box + 1)\Phi^{(j)} = O\big(|\partial^\alpha u|_{|\alpha|\le 3}^5\big),$$

where

$$\Phi^{(1)} = u^3,\ \ \Phi^{(2)} = u^2 u_t,\ \ \Phi^{(3)} = u^2 u_x,\ \ \Phi^{(4)} = u u_t^2,\ \ \Phi^{(5)} = u u_t u_x,$$
$$\Phi^{(6)} = u u_x^2,\ \ \Phi^{(7)} = u_t^3,\ \ \Phi^{(8)} = u_t^2 u_x,\ \ \Phi^{(9)} = u_t u_x^2,\ \ \Phi^{(10)} = u_x^3.$$

Therefore, we have the following:

Proposition 2.5.2 *If a cubic nonlinear term F satisfy Moriyama's or Katayama's condition, then it can be rewritten as*

$$F(u, \partial u, \partial\partial_x u) = (\Box + 1)\Phi + \Psi + R, \tag{2.5.2}$$

where Φ, Ψ, R are as in Lemma 2.3.1.

Finally, we would like to mention the works of Delort [2]–[3]. He considered another criterion from a different viewpoint. His approach is based on a certain asymptotic analysis. Here we do not state his result precisely but only point out the fact that if a nonlinear term F is written of the form (2.5.2), then one can check that it satisfy his condition. We close this section mentioning that he also succeeded in finding some class of nonlinear terms which admits a global solution which does *not* behave like the free solution.

Chapter 3

Large time asymptotics of solutions in the resonant case

3.1 Introduction

In the previous chapter, it has been shown that the Cauchy problem (1.0.1)–(1.0.2) admits a unique global smooth solution which tends to a free solution as $t \to \infty$ under the nonresonance condition.

From now on, we turn our attentions to the opposite direction. We raise the following question: *How does the solution behave as* $t \to \infty$ *in the resonant case?* In this chapter, we restrict ourselves to the simplest example

$$\begin{cases} (\Box + m_1^2)u_1 = 0 \\ (\Box + m_2^2)u_2 = 0 \\ (\Box + m_3^2)u_3 = u_1 u_2 \end{cases} \tag{3.1.1}$$

in two space dimensions under the assumption

$$m_3 = \lambda_1 m_1 + \lambda_2 m_2 \quad \text{with some} \quad \lambda_1, \lambda_2 \in \{\pm 1\}. \tag{3.1.2}$$

Following the author's paper [31], we will show that the large time behavior of u_3 is quite different from that of the free solution. More precisely, we will construct a solution whose energy grows like $O(\log t)$ as $t \to \infty$. The main result in this chapter is the following:

Theorem 3.1.1 *Suppose masses satisfy (3.1.2). Then the Cauchy problem for (3.1.1) possesses a solution which satisfy*

$$C_1 \log t \geq \|u_3(t)\|_E \geq C_2 \log t \quad \textit{for} \quad t \geq T$$

when the initial data is appropriately chosen in $\mathcal{S}(\mathbb{R}^2)$. *Here* $T > 1$ *and* $C_1 \geq C_2 > 0$ *are constants which depend on the initial data.*

Remark. As we will see in the proof, it suffices to choose the Cauchy data so that

$$\int u_j(0,x)dx \neq 0 \quad \text{and} \quad \int \partial_t u_j(0,x)dx = 0 \quad (j=1,2)$$

to obtain the above assertion.

Remark. If u_3 had a free profile in the sense of Theorem 2.1.1, the energy should stay bounded, i.e., $\|u_3(t)\|_E \leq C$ for any $t > 0$ with some positive constant C. So u_3 does not have a free profile in the usual sense when $m_3 = \lambda_1 m_1 + \lambda_2 m_2$ for some $\lambda_1, \lambda_2 \in \{\pm 1\}$. The asymptotic profile of u_3 will be obtained in § 3.4 (see Theorem 3.4.1 below).

Remark. On the other hand, when $m_3 \neq \lambda_1 m_1 + \lambda_2 m_2$ for any $\lambda_1, \lambda_2 \in \{\pm 1\}$, u_3 has a free profile in the sense of Theorem 2.1.1 since the nonresonance condition is satisfied.

Remark. We can obtain the similar result in the same way for the system

$$\begin{cases} (\Box + m_1^2)u_1 = 0 \\ (\Box + m_2^2)u_2 = 0 \\ (\Box + m_3^2)u_3 = 0 \\ (\Box + m_4^2)u_4 = u_1 u_2 u_3 \end{cases}$$

in one space dimension when $m_4 = \lambda_1 m_1 + \lambda_2 m_2 + \lambda_3 m_3$ for some $\lambda_1, \lambda_2, \lambda_3 \in \{\pm 1\}$.

The rest part of this chapter is devoted to the proof of Theorem 3.1.1. The proof is divided into 4 steps. In the first step, we will consider large time asymptotics of the solutions to free Klein-Gordon equations using the result from §7.2 of [12]. In the next step, we will prepare some lemma related to the normal form argument (cf. §2.1 of [2]). In the third step, we will obtain the asymptotic profile for u_3 in the system (3.1.1), and we will reach the desired conclusion in the final step. Since we are interested in the large time behavior, we always suppose that $t \gg 1$ in what follows.

3.2 Large time asymptotics of the free solution

We first investigate the asymptotics as $t \to \infty$ of the oscillatory integral

$$I(t,x) = \int e^{i(x\xi + t\langle\xi\rangle)} h(\xi) d\xi,$$

where $h \in \mathcal{S}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. The following lemma is due to Hörmander [12], though we state a slightly modified version here.

Lemma 3.2.1 *If $h \in \mathcal{S}(\mathbb{R}^n)$, then the oscillatory integral $I(t,x)$ can be written in the form*

$$I(t,x) = \frac{e^{i\varphi(t,x)}}{t^{n/2}} A(t,x) + R(t,x)$$

where $A,\ R \in C^\infty(]1,\infty[\times\mathbb{R}^n)$ and $\varphi(t,x) = (t^2 - |x|^2)_+^{1/2}$. The function A has the asymptotic expansion

$$A(t,x) \sim \sum_{j\geq 0} t^{-j} a_j\left(\frac{x}{t}\right),$$

where $a_j \in C^\infty(\mathbb{R}^n;\mathbb{C})$, $j = 0,1,2,\cdots$, satisfy

$$|\partial_y^\alpha a_j(y)| \leq C_{j,\alpha,N}(1-|y|^2)_+^N \tag{3.2.1}$$

with some positive constant $C_{j,\alpha,N}$ for any multi-indices α and $N \in \mathbb{N}$. In particular, the leading term a_0 is given by

$$a_0(y) = \begin{cases} \dfrac{e^{i\pi/4}}{(2\pi)^{n/2}}(1-|y|^2)^{-(n+2)/4} h\Big(\dfrac{-y}{\sqrt{1-|y|^2}}\Big) & \text{if } |y| < 1, \\ 0 & \text{if } |y| \geq 1. \end{cases}$$

Concerning the function R, for any multi-indices α and $N \in \mathbb{N}$, there exists a positive constant $C_{\alpha,N}$ such that

$$\left|\partial_{t,x}^\alpha R(t,x)\right| \leq C_{\alpha,N}(t+|x|)^{-N}.$$

Remark. In the above statement, we have used the notation $\sim$ in the following sense: We write

$$p(t,x) \sim \sum_{j\geq 0} p_j(t,x)$$

if for any multi-indices α and $N \in \mathbb{N}$ there exists a positive constant $C_{\alpha,N}$ such that

$$\left|\partial_{t,x}^\alpha\Big\{p(t,x) - \sum_{j=0}^{N-1} p_j(t,x)\Big\}\right| \leq C_{\alpha,N}\big(t+|x|\big)^{-N-|\alpha|}.$$

The proof of Lemma 3.2.1 can be found in Section 7.2 of [12] except the estimate (3.2.1), so we shall only check it. According to Hörmander's original argument, $a_j(y)$ is given of the form

$$a_j(y) = \begin{cases} b_j\Big(\frac{-y}{\sqrt{1-|y|^2}}\Big) & \text{if } |y| < 1, \\ 0 & \text{if } |y| \geq 1 \end{cases}$$

with appropriate $b_j \in \mathcal{S}(\mathbb{R}^n)$. So it is easy to see that

$$|a_j(y)| \le C_{j,0,N}(1-|y|^2)_+^N.$$

Here we note that $1-|y|^2 = \langle\xi\rangle^{-2}$ when $\xi = -y/\sqrt{1-|y|^2}$. Also, $\partial a_j/\partial y_k$ can be written

$$\frac{\partial a_j}{\partial y_k}(y) = \begin{cases} (\Theta_k b_j)\left(\frac{-y}{\sqrt{1-|y|^2}}\right) & \text{if } |y|<1, \\ 0 & \text{if } |y|\ge 1 \end{cases}$$

with

$$\Theta_k = -\langle\xi\rangle \sum_{l=1}^{n} (\delta_{kl} + \xi_k\xi_l)\frac{\partial}{\partial \xi_l}.$$

Since $\Theta_k b_j \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\max_{|\alpha|=1} |\partial_y^\alpha a_j(y)| \le C_{j,1,N}(1-|y|^2)_+^N.$$

By induction we obtain (3.2.1). ∎

Now, we put

$$h(\xi) = (2\pi)^{-n}\left(\hat{f}(\xi) - i\frac{\hat{g}(\xi)}{\langle\xi\rangle}\right),$$

where $\hat{f}$ stands for the Fourier transform of f. Since the solution v of the Cauchy problem

$$(\Box + 1)v = 0, \quad v(0,x) = f(x), \quad \partial_t v(0,x) = g(x) \tag{3.2.2}$$

is given by

$$v(t,x) = \frac{1}{2}\sum_{\lambda\in\{\pm1\}} \int e^{i(x\xi+\lambda t\langle\xi\rangle)}\left(\hat{f}(\xi) - i\lambda\frac{\hat{g}(\xi)}{\langle\xi\rangle}\right)\frac{d\xi}{(2\pi)^n} = \mathrm{Re}\left[\int e^{i(x\xi+t\langle\xi\rangle)}h(\xi)d\xi\right],$$

we can apply Lemma 3.2.1 to obtain

$$\begin{aligned} v(t,x) &= \mathrm{Re}\Big[\frac{e^{i\varphi(t,x)}}{t^{n/2}}A(t,x) + R(t,x)\Big] \\ &= \mathrm{Re}\Big[\frac{e^{i\varphi(t,x)}}{t^{n/2}}a_0(x/t)\Big] + \mathrm{Re}\Big[\frac{e^{i\varphi(t,x)}}{t^{n/2+1}}t\Big\{A(t,x) - a_0(x/t)\Big\} + R(t,x)\Big], \end{aligned}$$

where A, R, a_0 are as in Lemma 3.2.1 with above h. Summing up, we have the following:

Corollary 3.2.1 *For the solution v of (3.2.2), we have*

$$|v(t,x)| \le C_{N_1,N_2}\Big\{t^{-n/2}(1-|x/t|^2)_+^{N_1} + (t+|x|)^{-N_2}\Big\}$$

and

$$\left|v(t,x) - \frac{1}{2}\sum_{\lambda\in\{\pm 1\}} \frac{e^{i\lambda\varphi(t,x)}}{t^{n/2}} a^{(\lambda)}\left(\frac{x}{t}\right)\right| \le C_{N_1,N_2}\left\{t^{-(n/2+1)}(1-|x/t|^2)_+^{N_1} + (t+|x|)^{-N_2}\right\}$$

with some constant C_{N_1,N_2} *for any* $N_1, N_2 \in \mathbb{N}$. *Here* $a^{(\lambda)} : \mathbb{R}^n \to \mathbb{C}$ *is given by* $a^{(+1)}(y) = a_0(y)$ *and* $a^{(-1)}(y) = \overline{a_0(y)}$.

3.3 Key lemma

The goal of the second step is to show the following:

Lemma 3.3.1 *Let* $a(y)$ *be a* $\mathbb{C}$*-valued smooth function supported in the unit ball* $\{y \in \mathbb{R}^n | \ |y| \le 1\}$, *and* m, μ *be real constants. Also let* $\varphi(t,x) = (t^2 - |x|^2)_+^{1/2}$. *Then we have*

$$\frac{e^{i\mu\varphi(t,x)}}{t^{1+n/2}} a\left(\frac{x}{t}\right) - (\Box + m^2)\left[\frac{e^{i\mu\varphi(t,x)}}{(m^2-\mu^2)t^{1+n/2}} a\left(\frac{x}{t}\right)\right] \in L^1(1,\infty; L^2(\mathbb{R}^n))$$

if $|m| \ne |\mu|$, *while*

$$\frac{e^{i\mu\varphi(t,x)}}{t^{1+n/2}} a\left(\frac{x}{t}\right) - (\Box + m^2)\left[\frac{e^{i\lambda m\varphi(t,x)}\log t}{2i\lambda m t^{n/2}}\left(1 - \left|\frac{x}{t}\right|^2\right)_+^{1/2} a\left(\frac{x}{t}\right)\right] \in L^1(1,\infty; L^2(\mathbb{R}^n))$$

if $m = \lambda\mu$, $\lambda \in \{\pm 1\}$.

Remark. For non-trivial $a \in L^2(\mathbb{R}^n)$, $e^{i\mu\varphi(t,x)}t^{-\nu}a(x/t)$ belongs to $L^1(1,\infty; L^2(\mathbb{R}^n))$ if and only if $\nu > 1 + n/2$ because

$$\left\|\frac{e^{i\mu\varphi(t,\cdot)}}{t^\nu} a\left(\frac{\cdot}{t}\right)\right\|_{L^2_x} = \frac{1}{t^{\nu-n/2}}\left\{\int \left|a\left(\frac{x}{t}\right)\right|^2 \frac{dx}{t^n}\right\}^{1/2} = t^{n/2-\nu}\|a\|_{L^2_y}.$$

Lemma 3.3.1 is a consequence of the following lemma, which appears in [2] in somewhat different form.

Lemma 3.3.2 *Let m, μ, φ be as in the Lemma 3.3.1 and let A be a $\mathbb{C}$-valued smooth function of $(s, y) \in]0, \infty[\times\mathbb{R}^n$ which vanishes when $|y| \geq 1$. Then, for $\nu \in \mathbb{R}$, we have*

$$\begin{aligned}(\Box + m^2)&\left[\frac{e^{i\mu\varphi(t,x)}}{t^\nu}A\Big(\log t, \frac{x}{t}\Big)\right]\\ &=\frac{e^{i\mu\varphi(t,x)}}{t^\nu}(m^2-\mu^2)A\Big(\log t, \frac{x}{t}\Big)\\ &+\frac{e^{i\mu\varphi(t,x)}}{t^{\nu+1}}2i\mu\left[(1-|y|^2)_+^{-1/2}\Big(\frac{\partial}{\partial s}+\frac{n}{2}-\nu\Big)A\right]\Big(\log t, \frac{x}{t}\Big)\\ &+\frac{e^{i\mu\varphi(t,x)}}{t^{\nu+2}}(P_\nu A)\Big(\log t, \frac{x}{t}\Big),\end{aligned}$$

where

$$P_\nu = \left(\frac{\partial}{\partial s} - \sum_{j=1}^n y_j\frac{\partial}{\partial y_j} - \nu - 1\right)\left(\frac{\partial}{\partial s} - \sum_{j=1}^n y_j\frac{\partial}{\partial y_j} - \nu\right) - \Delta_y.$$

Remark. The auxiliary variable $s = \log t$ in this lemma is often called the *slow time*, which is often used in the theory of blowup for nonlinear hyperbolic equations (see [1]).

Proof of Lemma 3.3.1 via Lemma 3.3.2: When we choose $A(s, y) = (m^2 - \mu^2)^{-1}a(y)$ and $\nu = 1 + n/2$ in Lemma 3.3.2, we have

$$\begin{aligned}&\frac{e^{i\mu\varphi(t,x)}}{t^{1+n/2}}a(x/t) - (\Box + m^2)\left[\frac{e^{i\mu\varphi(t,x)}}{(m^2-\mu^2)t^{1+n/2}}a(x/t)var\right]\\ &= \frac{2i\mu e^{i\mu\varphi(t,x)}}{(m^2-\mu^2)t^{2+n/2}}\big(1-|x/t|^2\big)_+^{-1/2}a(x/t) - \frac{e^{i\mu\varphi(t,x)}}{(m^2-\mu^2)t^{3+n/2}}(P_\nu a)(x/t)\end{aligned}$$

if $|m| \neq |\mu|$. Since both $(1-|y|^2)_+^{-1/2}a(y)$ and $(P_\nu a)(y)$ are smooth functions with their support in the unit ball if $a(y)$ is so, we obtain the first half of Lemma 3.3.1. When $m = \lambda\mu$, $\lambda \in \{\pm 1\}$, we have

$$\begin{aligned}\frac{e^{i\mu\varphi(t,x)}}{t^{1+n/2}}a(x/t) - (\Box + m^2)&\left[\frac{e^{i\lambda m\varphi(t,x)}\log t}{2i\lambda m t^{n/2}}\big(1-|x/t|^2\big)_+^{1/2}a(x/t)\right]\\ &= -\frac{e^{i\lambda m\varphi(t,x)}}{t^{2+n/2}}(P_\nu A)(\log t, x/t)\end{aligned}$$

by choosing $A(s, y) = (2i\mu)^{-1}s(1-|y|^2)_+^{1/2}a(y)$ and $\nu = n/2$ in Lemma 3.3.2. Since $(P_\nu A)(s, y)$ is majorized by $(1 + s)b(y)$ with appropriate $b \in L^2(\mathbb{R}^n)$, we have

$$\left\|\frac{e^{i\lambda m\varphi(t,\cdot)}}{t^{2+n/2}}(P_\nu A)(\log t, \,\cdot\,/t)\right\|_{L^2_x} \leq \frac{(1+\log t)}{t^2}\|b\|_{L^2_y} \in L^1(1, \infty),$$

which yields the latter half of Lemma 3.3.1. ∎

Proof of Lemma 3.3.2: First we note that we may assume $|x| < t$ since both left and right hand sides vanish when $|x| \geq t$.

Let us introduce

$$X_\nu = \frac{\partial}{\partial s} - \sum_{j=1}^{n} y_j \frac{\partial}{\partial y_j} - \nu$$

so that

$$\frac{\partial}{\partial t}\left[\frac{1}{t^\nu} A(\log t, x/t)\right] = \frac{1}{t^{\nu+1}}(X_\nu A)(\log t, x/t).$$

Then it follows that

$$\begin{aligned}
&(\Box + m^2)\left[\frac{e^{i\mu\varphi(t,x)}}{t^\nu} A(\log t, x/t)\right] \\
=&\frac{e^{i\mu\varphi}}{t^\nu}\Big\{m^2 - \mu^2\Big[(\partial_t\varphi)^2 - |\nabla_x\varphi|^2\Big]\Big\} A(\log t, x/t) \\
&+ \frac{e^{i\mu\varphi}}{t^{\nu+1}} 2i\mu\Big[(\partial_t\varphi)X_\nu - (\nabla_x\varphi)\cdot\nabla_y + \frac{t}{2}(\Box\varphi)\Big] A(\log t, x/t) \\
&+ \frac{e^{i\mu\varphi}}{t^{\nu+2}}\Big(X_{\nu+1}X_\nu - \Delta_y\Big) A(\log t, x/t).
\end{aligned}$$

Since φ satisfies

$$(\partial_t\varphi)^2 - |\nabla_x\varphi|^2 = 1,$$
$$\Box\varphi = \frac{n}{(t^2 - |x|^2)^{1/2}}$$

and

$$\left(t\frac{\partial}{\partial x_j} + x_j\frac{\partial}{\partial t}\right)\varphi = 0,$$

we have

$$\begin{aligned}
(\partial_t\varphi)X_\nu - (\nabla_x\varphi)\cdot\nabla_y + \frac{t}{2}(\Box\varphi) &= (\partial_t\varphi)\Big[X_\nu + \frac{x}{t}\cdot\nabla_y\Big] + \frac{n}{2}\Big(1 - |x/t|^2\Big)_+^{-1/2} \\
&= (1 - |y|^2)_+^{-1/2}\Big(\frac{\partial}{\partial s} - \nu + \frac{n}{2}\Big)\Big|_{(s,y)=(\log t, x/t)},
\end{aligned}$$

which completes the proof. ∎

3.4 Construction of the asymptotic profile

In this step, we are going to find the large time asymptotic profile for u_3 in the system (3.1.1). The goal is Theorem 3.4.1 below.

First, we put $v_j(t,x) = u_j(t/m_j, x/m_j)$ and

$$r_j(t,x) = u_j(t,x) - \sum_{\lambda\in\{\pm1\}} \frac{e^{i\lambda m_j\varphi(t,x)}}{m_j t} a_j^{(\lambda)}\Big(\frac{x}{t}\Big)$$

for u_j in the system (3.1.1) so that $(\Box+1)v_j = 0$, $v_j(0,x) = u_j(0,x/m_j)$, $(\partial_t v_j)(0,x) = m_j^{-1}(\partial_t u_j)(0,x/m_j)$ and

$$r_j(t,x) = \Big[v_j(t',x') - \frac{1}{2}\sum_{\lambda\in\{\pm1\}} \frac{e^{i\varphi(t',x')}}{t'} a_j^{(\lambda)}\Big(\frac{x'}{t'}\Big)\Big]_{(t',x')=(m_j t, m_j x)}.$$

Here $j = 1, 2$ and $a_j^{(\lambda)}$ is given by

$$\begin{aligned} a_j^{(+1)}(y) =& \frac{m_j^2 e^{i\pi/4}}{8\pi^3}(1-|y|^2)_+^{-1}\widehat{u}_j\big(0, \frac{-m_j y}{\sqrt{1-|y|^2}}\big) \\ &- \frac{m_j e^{i3\pi/4}}{8\pi^3}(1-|y|^2)_+^{-1/2}\big(\widehat{\partial_t u_j}\big)\big(0, \frac{-m_j y}{\sqrt{1-|y|^2}}\big), \\ a_j^{(-1)}(y) =& \overline{a_j^{(+1)}(y)}. \end{aligned}$$

Then it follows from Corollary 3.2.1 that

$$|u_j(t,x)| \le C_{N_1,N_2}\Big\{t^{-1}(1-|x/t|^2)_+^{N_1} + (t+|x|)^{-N_2}\Big\}$$

and

$$|r_j(t,x)| \le C_{N_1,N_2}\Big\{t^{-2}(1-|x/t|^2)_+^{N_1} + (t+|x|)^{-N_2}\Big\}$$

for $j = 1, 2$ and any $N_1, N_2 \in \mathbb{N}$ with some positive constant C_{N_1,N_2}. Thus, putting

$$R_1 := u_1 u_2 - (u_1 - r_1)(u_2 - r_2),$$

we have

$$\begin{aligned} |R_1(t,x)| &\le |u_1||r_2| + |u_2||r_1| + |r_1||r_2| \\ &\le C'_{N_1,N_2}\Big\{t^{-3}(1-|x/t|^2)_+^{N_1} + (t+|x|)^{-N_2}\Big\} \end{aligned}$$

for any $N_1, N_2 \in \mathbb{N}$ with some positive constant C'_{N_1,N_2}. In particular we have $R_1 \in L^1(1,\infty; L^2(\mathbb{R}^2))$.

Next, let us introduce

$$\begin{aligned} \Lambda_+ &= \Big\{(\lambda_1,\lambda_2) \in \{\pm1\}^2 \;\Big|\; \lambda_1 m_1 + \lambda_2 m_2 = m_3\Big\}, \\ \Lambda_- &= \Big\{(\lambda_1,\lambda_2) \in \{\pm1\}^2 \;\Big|\; \lambda_1 m_1 + \lambda_2 m_2 = -m_3\Big\}, \\ \Lambda_0 &= \Big\{(\lambda_1,\lambda_2) \in \{\pm1\}^2 \;\Big|\; |\lambda_1 m_1 + \lambda_2 m_2| \ne m_3\Big\} \end{aligned}$$

and

$$w_+(t,x) = \frac{e^{im_3\varphi(t,x)}\log t}{8im_1m_2m_3t}\Big(1-\Big|\frac{x}{t}\Big|^2\Big)_+^{1/2} \sum_{(\lambda_1,\lambda_2)\in\Lambda_+} a_1^{(\lambda_1)}\Big(\frac{x}{t}\Big)a_2^{(\lambda_2)}\Big(\frac{x}{t}\Big),$$

$$w_-(t,x) = \frac{e^{-im_3\varphi(t,x)}\log t}{-8im_1m_2m_3t}\Big(1-\Big|\frac{x}{t}\Big|^2\Big)_+^{1/2} \sum_{(\lambda_1,\lambda_2)\in\Lambda_-} a_1^{(\lambda_1)}\Big(\frac{x}{t}\Big)a_2^{(\lambda_2)}\Big(\frac{x}{t}\Big),$$

$$w_0(t,x) = \sum_{(\lambda_1,\lambda_2)\in\Lambda_0} \frac{e^{i(\lambda_1m_1+\lambda_2m_2)\varphi(t,x)}}{4m_1m_2\{m_3^2-(\lambda_1m_1+\lambda_2m_2)^2\}t^2} a_1^{(\lambda_1)}\Big(\frac{x}{t}\Big)a_2^{(\lambda_2)}\Big(\frac{x}{t}\Big).$$

Then it follows from Lemma 3.3.1 that

$$\begin{aligned} R_2(t,x) := & \sum_{(\lambda_1,\lambda_2)\in\{\pm1\}^2} \frac{e^{i(\lambda_1m_1+\lambda_2m_2)\varphi(t,x)}}{4m_1m_2t^2} a_1^{(\lambda_1)}\Big(\frac{x}{t}\Big)a_2^{(\lambda_2)}\Big(\frac{x}{t}\Big) \\ & - (\Box+m_3^2)\Big[w_+(t,x)+w_-(t,x)+w_0(t,x)\Big] \in L^1(1,\infty;L^2(\mathbb{R}^2)). \end{aligned}$$

Now, we put $w_1(t,x) = w_+(t,x)+w_-(t,x) = 2\,\mathrm{Re}\big[w_+(t,x)\big]$. Since

$$R_1+R_2 = u_1u_2 - (\Box+m_3^2)\big(w_0+w_1\big),$$

we have

$$(\Box+m_3^2)\big(u_3-w_0-w_1\big) = R_1+R_2 \in L^1(1,\infty;L^2(\mathbb{R}^2)).$$

Therefore there exists a solution $w_2(t,x)$ of the free Klein-Gordon equation $(\Box+m_3^2)w=0$ such that

$$\big\|\{u_3(t)-w_0(t)-w_1(t)\}-w_2(t)\big\|_{E(m_3)} \to 0 \quad \text{as} \quad t\to\infty.$$

Furthermore, since

$$\|w_0(t)\|_{E(m_3)} \to 0 \quad \text{as} \quad t\to\infty,$$

$w_3 := u_3 - w_1 - w_2$ has the same decay property. Summing up, we obtain the following decomposition:

Theorem 3.4.1 *For u_3 in the system (3.1.1), we have*

$$u_3 = w_1+w_2+w_3,$$

where

$$w_1 = \mathrm{Re}\left[\frac{e^{im_3\varphi(t,x)}}{m_3t}A\Big(\frac{x}{t}\Big)\log t\right]$$

with

$$A(y) = \begin{cases} \dfrac{1}{i4m_1m_2}(1-|y|^2)_+^{1/2} \displaystyle\sum_{(\lambda_1,\lambda_2)\in\Lambda_+} a_1^{(\lambda_1)}(y)a_2^{(\lambda_2)}(y) & \text{if} \quad \Lambda_+\neq\emptyset, \\ 0 & \text{if} \quad \Lambda_+=\emptyset, \end{cases}$$

w_2 is a solution of the free Klein-Gordon equation $(\Box+m_3^2)w=0$ and w_3 satisfies $\|w_3(t)\|_{E(m_3)}\to 0$ as $t\to\infty$.

3.5 The end of the proof

Now, we are in position to finish the proof of Theorem 3.1.1. Since the upper bound follows immediately from Theorem 3.4.1, we omit the proof of it and we prove the lower bound here.

By Theorem 3.4.1, we have

$$\|u_3(t)\|_{E(m_3)} \geq \|w_1(t)\|_{E(m_3)} - \|w_2(t)\|_{E(m_3)} - \|w_3(t)\|_{E(m_3)}$$

and $\|w_2(t)\|_{E(m_3)} = \text{const.}$, $\|w_3(t)\|_{E(m_3)} \to 0$ as $t \to \infty$. Also, since

$$\begin{aligned}\partial_t w_1(t,x) &= m_3\big(\partial_t\varphi(t,x)\big)\frac{\log t}{t}\,\mathrm{Im}\left[e^{im_3\varphi(t,x)}A(x/t)\right]\\ &\quad+\frac{1+\log t}{t^2}\,\mathrm{Re}\left[e^{im_3\varphi(t,x)}A(x/t)\right]\\ &\quad+\frac{\log t}{t^2}\,\mathrm{Re}\left[e^{im_3\varphi(t,x)}\big(y\cdot\nabla_y A\big)_{y=x/t}\right]\end{aligned}$$

and

$$|\partial_t\varphi(t,x)|^2 = \frac{t^2}{t^2-|x|^2} \geq 1 \quad \text{for} \quad |x| < t,$$

we have

$$|\partial_t w_1(t,x)|^2 \geq m_3^2\left|\frac{\log t}{t}\,\mathrm{Im}\left[e^{im_3\varphi(t,x)}A(x/t)\right]\right|^2 - \frac{(\log t)^2}{t^3}\big|B(x/t)\big|^2$$

with some smooth function $B(y)$ which vanishes when $|y| \geq 1$. Thus, we have

$$\begin{aligned}\|w_1(t)\|^2_{E(m_3)} &\geq \int |\partial_t w_1(t,x)|^2 + m_3^2|w_1(t,x)|^2 dx\\ &\geq m_3^2\frac{(\log t)^2}{t^2}\int |A(x/t)|^2 dx - \frac{(\log t)^2}{t^3}\int |B(x/t)|^2 dx\\ &= \Big(\|A\|_{L^2} m_3 \log t\Big)^2 - \frac{(\log t)^2}{t}\|B\|^2_{L^2}.\end{aligned}$$

Therefore we can choose $T > 1$ such that

$$\|u_3(t)\|_{E(m_3)} \geq \frac{1}{2}\|A\|_{L^2} m_3 \log t \qquad \text{for} \quad t \geq T,$$

provided that $\|A\|_{L^2}$ is strictly positive. In order that $\|A\|_{L^2}$ be strictly positive, it is sufficient to choose the initial data so that

$$\int u_j(0,x)dx \neq 0 \quad \text{and} \quad \int (\partial_t u_j)(0,x)dx = 0$$

for $j = 1, 2$. Indeed we can choose $\delta \in]0,1[$ so that $|A(y)| \geq |A(0)|/2 > 0$ for $|y| \leq \delta$ since $A(0) = c\widehat{u}_1(0,0)\widehat{u}_2(0,0)$ with some $c \in \mathbb{C}\backslash\{0\}$ then. Therefore we have

$$\|A\|_{L^2} \geq \left\{ \int_{|y|\leq\delta} |A(0)|^2/4 \, dy \right\}^{1/2} = \delta\pi^{1/2}|A(0)|/2 > 0,$$

which completes the proof. ∎

3.6 A remark on the subcritical case

In one space dimensional quadratic nonlinear case, we can prove a result similar to Theorem 3.1.1 when $O(\log t)$ is replaced by $O(t^{1/2})$. More generally, we can obtain the following result:

Proposition 3.6.1 *Let us consider*

$$\begin{cases} (\Box + {m_0}^2)u_0 = 0, \\ (\Box + {m_1}^2)u_1 = {u_0}^2, \\ (\Box + {m_2}^2)u_2 = {u_1}^2, \\ \quad\vdots \\ (\Box + {m_N}^2)u_N = {u_{N-1}}^2 \end{cases} \tag{3.6.1}$$

in $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^1$ *with the initial condition*

$$u_j(0,x) = f_j(x), \ \partial_t u_j(0,x) = g_j(x), \quad j = 0, 1, \cdots, N.$$

Suppose that $m_j = 2m_{j-1}$ *for each* $j \in \{1, \cdots, N\}$ *and that* f_0 *or* g_0 *does not identically vanish. Then there exist positive constants* $C_1 \geq C_2 > 0$ *and* $T > 0$ *such that*

$$C_1 t^{(2^j-1)/2} \geq \|u_j(t)\|_{E(m_j)} \geq C_2 t^{(2^j-1)/2} \tag{3.6.2}$$

for any $t \geq T$, $j \in \{0, 1, \cdots, N\}$.

The above result allows us to obtain some small data blow-up examples for systems of nolinear Klein-Gordon equations, which will be mentioned in Chapter 5 (see also the author's forthcoming paper [33] for more detail).

Chapter 4

Large time asymptotics of solutions in the resonant case, continued

4.1 Introduction

In this chapter, we continue to study the resonant case following the author's work [32]. The examples considered in the preceding chapter are decoupled and we can use the explicit representations of the solutions. In the next chapter, we will obtain the similar result for some coupled nonlinear systems which can be regarded as a perturbation of the system like (3.1.1) instead of that of the free equations.

Although several generalizations are possible, we mainly treat the following very simple example so that the essential idea becomes most clear. Let us consider

$$\begin{cases} (\Box + m^2)u = \alpha v^4, & \\ (\Box + \mu^2)v = \beta u^3, & t > 0,\ x \in \mathbb{R}, \\ (u, \partial_t u, v, \partial_t v)\,\big|_{t=0} = (\varepsilon u_0, \varepsilon u_1, \varepsilon v_0, \varepsilon v_1), & x \in \mathbb{R}, \end{cases} \tag{4.1.1}$$

where $\alpha, \beta \in \mathbb{R}$, $\varepsilon > 0$ is a small parameter, and u_0, u_1, v_0, $v_1 \in C_0^\infty(\mathbb{R})$. We will show that, as $t \to \infty$, the amplitude of v is modulated by the long range interaction when $\mu = m$ or $\mu = 3m$, whereas, when $\mu \neq m$, $\mu \neq 3m$, the influence of nonlinearity disappears eventually and v behaves like a free solution in the large time. More precisely, we will prove the following:

Theorem 4.1.1 *For any u_0, u_1, v_0, $v_1 \in C_0^\infty(\mathbb{R})$, there exists $\varepsilon_0 > 0$ such that (4.1.1) admits a unique global classical solution if $\varepsilon \in]0, \varepsilon_0]$. Moreover, the following asymptotics is valid as $t \to \infty$, uniformly with respect to $x \in \mathbb{R}$:*

$$u(t,x) = \mathrm{Re}\left[\frac{e^{im(t^2-|x|^2)_+^{1/2}}}{m\sqrt{t}} a(x/t)\right] + O(t^{-1+\delta}),$$

$$v(t,x) = \mathrm{Re}\left[\frac{e^{i\mu(t^2-|x|^2)_+^{1/2}}}{\mu\sqrt{t}} \Big\{A(x/t)\log t + b(x/t)\Big\}\right] + O(t^{-1+\delta}).$$

Here $(\,\cdot\,)_+$ *stands for* $\max\{\,\cdot\,,0\}$, $i=\sqrt{-1}$, δ *is an arbitrary small positive number,* $a(y)$, $b(y)$ *are* $\mathbb{C}$*-valued smooth functions which vanish when* $|y|\geq 1$, *and* $A(y)$ *is given by*

$$A(y)=\begin{cases} \dfrac{\beta}{i8m^3}(1-|y|^2)_+^{1/2}a(y)^3 & \text{if } \mu=3m,\\ \dfrac{3\beta}{i8m^3}(1-|y|^2)_+^{1/2}|a(y)|^2a(y) & \text{if } \mu=m,\\ 0 & \text{if } \mu\neq 3m, \mu\neq m. \end{cases}$$

Remark. It is worth comparing this result with the corresponding one to the scalar case

$$(\Box+1)w=\beta w^3, \qquad t>0,\ x\in\mathbb{R}. \tag{4.1.2}$$

This has been studied by Delort [2] in much more general situations including quasi-linear case. According to his result, w has the following asymptotics:

$$w(t,x)=\operatorname{Re}\left[\frac{1}{\sqrt{t}}e^{i\{(t^2-|x|^2)_+^{1/2}+\varphi(x/t)\log t\}}a(x/t)\right]+O(t^{-1+\delta}), \quad t\to\infty$$

with

$$\varphi(y)=-\frac{3\beta}{8}(1-|y|^2)_+^{1/2}|a(y)|^2.$$

Roughly speaking, this shows that the long range character of nonlinearity appears at the level of the *phase* of oscillation of the solution for the scalar equation (4.1.2), while the above result claims that the long range character appears at the level of the *amplitude* of the solution for the system (4.1.1).

The same proof is available for a bit more general systems, such as

$$\begin{cases} (\Box+m_1^2)u_1=F_1(u,\partial u),\\ (\Box+m_2^2)u_2=F_2(u,\partial u),\\ (\Box+m_3^2)u_3=F_3(u,\partial u),\\ (\Box+m_4^2)u_4=\gamma u_1u_2u_3+F_4(u,\partial u), \end{cases} \qquad t>0,\ x\in\mathbb{R}, \tag{4.1.3}$$

with the initial data

$$(u_j,\partial_t u_j)\big|_{t=0}=(\varepsilon u_{0j},\varepsilon u_{1j}), \quad j=1,2,3,4. \tag{4.1.4}$$

Here $u=(u_j)_{1\leq j\leq 4}$, $\gamma\in\mathbb{R}$ and $F_j(u,\partial u)=O(|u|^4+|\partial u|^4)$. When we put

$$\Lambda=\left\{(\lambda_1,\lambda_2,\lambda_3)\in\{\pm 1\}^3 \;\middle|\; m_4=\lambda_1m_1+\lambda_2m_2+\lambda_3m_3\right\},$$

the corresponding result to Theorem 4.1.1 is stated as follows:

Theorem 4.1.2 *For any u_{0j}, $u_{1j} \in C_0^\infty(\mathbb{R})$, there exists $\varepsilon_0 > 0$ such that (4.1.3)–(4.1.4) admits a unique global C^∞ solution if $\varepsilon \in]0, \varepsilon_0]$. Moreover, the following asymptotics is valid as $t \to \infty$, uniformly with respect to $x \in \mathbb{R}$:*

$$u_j(t,x) = \mathrm{Re}\left[\frac{e^{im_j(t^2-|x|^2)_+^{1/2}}}{m_j\sqrt{t}}a_j(x/t)\right] + O\left(t^{-1+\delta}\right), \quad j = 1,2,3,$$

$$u_4(t,x) = \mathrm{Re}\left[\frac{e^{im_4(t^2-|x|^2)_+^{1/2}}}{m_4\sqrt{t}}\left\{A(x/t)\log t + a_4(x/t)\right\}\right] + O\left(t^{-1+\delta}\right).$$

Here, δ is an arbitrary small positive number, a_j ($j = 1,2,3$) are $\mathbb{C}$-valued smooth functions which vanish when $|y| \geq 1$, and $A(y)$ is given by

$$A(y) = \begin{cases} \dfrac{\gamma}{i8m_1m_2m_3}(1-|y|^2)_+^{1/2} \displaystyle\sum_{(\lambda_1,\lambda_2,\lambda_3)\in\Lambda} a_1^{(\lambda_1)}(y)a_2^{(\lambda_2)}(y)a_3^{(\lambda_3)}(y) & \text{if } \Lambda \neq \emptyset, \\ 0 & \text{if } \Lambda = \emptyset, \end{cases}$$

where $a_j^{(+1)}(y) = a_j(y)$, $a_j^{(-1)}(y) = \overline{a_j(y)}$.

4.2 Reduction of the problem

In this section, we perform some reduction following the idea developed by [3], [4], [5] (see also [16]). In what follows, we fix $B > 0$ so that

$$\mathrm{supp}(u_0, u_1, v_0, v_1) \subset \left\{x \in \mathbb{R} \;\middle|\; |x| \leq B\right\}.$$

Also we fix $\tau_0 > \max\{1, 2B\}$. We begin with the fact that without loss of generality we may treat the problem as if the Cauchy data is given on the upper branch of the hyperbola

$$\left\{(t,x) \in \mathbb{R}^{1+1} \;\middle|\; (t+2B)^2 - |x|^2 = \tau_0^2,\ t > 0\right\}$$

and it is sufficiently small, smooth, and compactly supported. (This is a consequence of the classical local existence theorem and the finite speed of propagation. See [3, Proposition 1.4], [4], [5] and [18], [12, Chapter 7] for detail.) Next, as in [3], [18], we introduce the hyperbolic coordinates (τ, z) in the interior of the light cone, i.e.,

$$t + 2B = \tau\cosh z,\ x = \tau\sinh z, \quad \text{for} \quad |x| < t + 2B$$

so that

$$\begin{cases} \partial_t = (\cosh z)\partial_\tau - \frac{1}{\tau}(\sinh z)\partial_z, \\ \partial_x = -(\sinh z)\partial_\tau + \frac{1}{\tau}(\cosh z)\partial_z, \end{cases}$$

$$\Box + m^2 = \frac{\partial^2}{\partial\tau^2} + \frac{1}{\tau}\frac{\partial}{\partial\tau} + m^2 - \frac{1}{\tau^2}\frac{\partial^2}{\partial z^2}$$

and

$$\tau = \sqrt{(t+2B)^2 - |x|^2}, \quad z = \tanh^{-1}\left(\frac{x}{t+2B}\right).$$

We also introduce the new unknowns $(\tilde{u}, \tilde{v})$ as follows:

$$u(t,x) = \frac{\tilde{u}(\tau,z)}{\tau^{1/2}\cosh\kappa z}, \quad v(t,x) = \frac{\tilde{v}(\tau,z)}{\tau^{1/2}\cosh\kappa z},$$

where $\kappa > 0$ is a parameter which is determined later. Roughly speaking, κ measures the decay of the solution outside the light cone because $(\cosh\kappa z)^{-1} \approx \left((1-|x/t|)_+ + 1/t\right)^{\kappa/2}$. Now, let us derive the equations which $(\tilde{u}, \tilde{v})$ satisfies when (u, v) is a solution of (4.1.1). Since

$$v^4 = \tau^{-2}(\cosh\kappa z)^{-4}\tilde{v}^4$$

and

$$(\Box + m^2)u = \tau^{-1/2}(\cosh\kappa z)^{-1}(\tilde{\Box} + m^2)\tilde{u},$$

we have

$$(\tilde{\Box} + m^2)\tilde{u} = \frac{\alpha}{\tau^{3/2}(\cosh\kappa z)^3}\tilde{v}^4,$$

where

$$\tilde{\Box} = \frac{\partial^2}{\partial\tau^2} - \frac{1}{\tau^2}\frac{\partial^2}{\partial z^2} + \frac{2\kappa\tanh\kappa z}{\tau^2}\frac{\partial}{\partial z} + \frac{1}{\tau^2}\left(\frac{1}{4} + \kappa^2(1 - 2(\tanh\kappa z)^2)\right).$$

In the same way, we see that $\tilde{v}$ satisfies

$$(\tilde{\Box} + \mu^2)\tilde{v} = \frac{\beta}{\tau(\cosh\kappa z)^2}\tilde{u}^3.$$

Summing up, the original problem (4.1.1) is reduced to the following Cauchy problem:

$$\begin{cases} (\tilde{\Box} + m^2)\tilde{u} = \dfrac{\alpha}{\tau^{3/2}(\cosh\kappa z)^3}\tilde{v}^4, \\ (\tilde{\Box} + \mu^2)\tilde{v} = \dfrac{\beta}{\tau(\cosh\kappa z)^2}\tilde{u}^3, \end{cases} \quad \tau > \tau_0, \ z \in \mathbb{R} \tag{4.2.1}$$

with the initial data

$$\begin{cases} (\tilde{u}, \partial_\tau\tilde{u})|_{\tau=\tau_0} = (\varepsilon\tilde{u}_0, \varepsilon\tilde{u}_1) \\ (\tilde{v}, \partial_\tau\tilde{v})|_{\tau=\tau_0} = (\varepsilon\tilde{v}_0, \varepsilon\tilde{v}_1). \end{cases} \tag{4.2.2}$$

Our strategy is to prove global exitence and uniqueness of the solution to (4.2.1)–(4.2.2) (see Proposition 4.2.1 below), find the asymptotics for $(\tilde{u}, \tilde{v})$ as $\tau \to \infty$ (see (4.4.2), (4.4.3) in § 4.4), and finally return to the solution of the original problem.

In the next section we shall prove the following proposition. In what follows, we denote by $H^s(\mathbb{R})$ the standard Sobolev space for $s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Proposition 4.2.1 *Let $\kappa \geq 0$ and let σ be an integer larger than $1 + 4\kappa$. For any $(\tilde{u}_0, \tilde{u}_1), (\tilde{v}_0, \tilde{v}_1) \in H^{2\sigma}(\mathbb{R}_z) \times H^{2\sigma-1}(\mathbb{R}_z)$, there exists $\varepsilon_0 > 0$ such that the Cauchy problem (4.2.1)-(4.2.2) admits a unique global solution*

$$(\tilde{u}, \tilde{v}) \in \bigcap_{j=0}^{1} C^j(\tau_0, \infty; H^{2\sigma-j}(\mathbb{R}_z)) \times \bigcap_{j=0}^{1} C^j(\tau_0, \infty; H^{2\sigma-j}(\mathbb{R}_z))$$

if $\varepsilon \in]0, \varepsilon_0]$. Moreover, we have

$$\|\partial_\tau^{\ell_1} \partial_z^{j+\ell_2} \tilde{u}(\tau, \cdot)\|_{H^\sigma(\mathbb{R}_z)} \leq C\tau^{\frac{j}{4}+\ell_2},$$
$$\|\partial_\tau^{\ell_1} \partial_z^{j+\ell_2} \tilde{v}(\tau, \cdot)\|_{H^\sigma(\mathbb{R}_z)} \leq C\tau^{\delta+\frac{j}{4}+\ell_2}$$

for each $0 \leq j \leq \sigma - 1$, $0 \leq \ell_1 + \ell_2 \leq 1$, and for arbitaray small $\delta > 0$. Here C is a positive constant independent of τ.

Remark. Consequently, we have

$$\|\tilde{u}(\tau, \cdot)\|_{H^\sigma(\mathbb{R}_z)} \leq C, \tag{4.2.3}$$
$$\|\tilde{v}(\tau, \cdot)\|_{H^\sigma(\mathbb{R}_z)} \leq C\tau^\delta \tag{4.2.4}$$

for any $\tau \geq \tau_0$.

4.3 Proof of Proposition 4.2.1

This section is devoted to the proof of the Proposition 4.2.1. The proof is done by means of the contraction mapping principle. In order to do so, we prepare a version of the energy estimate for $(\tilde{\Box} + M^2)$, which is essentially due to Delort–Fang–Xue [4] (see also [5]). We state and prove it here with a minor modification.

Let us define

$$\|\phi(\tau)\|_{e(M)} := \left(\int_{z\in\mathbb{R}} |\partial_\tau \phi(\tau, z)|^2 + \frac{1}{\tau^2} |\partial_z \phi(\tau, z)|^2 + M^2 |\phi(\tau, z)|^2 dz \right)^{1/2}$$

for smooth function ϕ and positive constant M. We begin with the following basic estimates.

Lemma 4.3.1 *For $\kappa \geq 0$, $M > 0$, $s \in \mathbb{N}_0$ and for smooth function $\phi(\tau, z)$, we have*

$$\frac{d}{d\tau} \|\partial_z^s \phi(\tau)\|_{e(M)}^2 \leq \frac{2\kappa}{\tau} \|\partial_z^s \phi(\tau)\|_{e(M)}^2 + \frac{C}{\tau^2} \sum_{j=0}^{s} \|\partial_z^j \phi(\tau)\|_{e(M)}^2 + C \|\partial_z^s \phi(\tau)\|_{e(M)} \left\| \partial_z^s (\tilde{\Box} + M^2) \phi(\tau) \right\|_{L^2(\mathbb{R}_z)} \tag{4.3.1}$$

and

$$\frac{d}{d\tau}\|\partial_z^s\phi(\tau)\|_{e(M)}^2 \le \frac{C}{\tau^2}\sum_{j=0}^{s+1}\|\partial_z^j\phi(\tau)\|_{e(M)}^2 + C\|\partial_z^s\phi(\tau)\|_{e(M)}\big\|\partial_z^s(\tilde{\Box}+M^2)\phi(\tau)\big\|_{L^2(\mathbb{R}_z)}, \tag{4.3.2}$$

provided that the right hand side is finite. Here C is a positive constant depending only on κ, M, s.

Proof: In what follows, we denote by C various positive constants which might be different line by line. We first show the case where $s = 0$. As in the standard energy integral method, we start from the following calculations

$$\begin{aligned}
\frac{d}{d\tau}\|\phi(\tau)\|_{e(M)}^2 =&2\int_{\mathbb{R}}(\partial_\tau\phi)(\partial_\tau^2\phi) + \frac{1}{\tau^2}(\partial_z\phi)(\partial_\tau\partial_z\phi) + M^2\phi(\partial_\tau\phi) - \frac{1}{\tau^3}|\partial_z\phi|^2dz\\
\le&2\int_{\mathbb{R}}(\partial_\tau\phi)\left(\partial_\tau^2\phi - \frac{1}{\tau^2}\partial_z^2\phi + M^2\phi\right)dz\\
=&2\int_{\mathbb{R}}\frac{-2\kappa\tanh\kappa z}{\tau^2}(\partial_\tau\phi)(\partial_z\phi)\\
&-\frac{1}{\tau^2}\left\{\frac{1}{4}+\kappa^2-2\kappa^2(\tanh\kappa z)^2\right\}\phi(\partial_\tau\phi) + (\partial_\tau\phi)(\tilde{\Box}+M^2)\phi dz\\
\le&\frac{4\kappa}{\tau^2}\int_{\mathbb{R}}|\partial_\tau\phi||\partial_z\phi|dz + \frac{C}{\tau^2}\|\phi(\tau)\|_{e(M)}^2 + C\|\phi(\tau)\|_{e(M)}\big\|(\tilde{\Box}+M^2)\phi\big\|_{L^2}.
\end{aligned}$$

The inequalities $(4.3.1)_{s=0}$ and $(4.3.2)_{s=0}$ follow from the fact that the first term in the right hand side is dominated by

$$\frac{2\kappa}{\tau}\|\phi(\tau)\|_{e(M)}^2 \quad\text{and}\quad \frac{2\kappa}{\tau^2}\left(\|\phi(\tau)\|_{e(M)}^2 + \frac{1}{M^2}\|\partial_z\phi(\tau)\|_{e(M)}^2\right),$$

respectively. Next, we consider the case where $s \ge 1$. Using $(4.3.1)_{s=0}$ with ϕ replaced by $\partial_z^s\phi$, we have

$$\begin{aligned}
\frac{d}{d\tau}\|\partial_z^s\phi(\tau)\|_{e(M)}^2 \le&\frac{2\kappa}{\tau}\|\partial_z^s\phi(\tau)\|_{e(M)}^2 + \frac{C}{\tau^2}\|\partial_z^s\phi(\tau)\|_{e(M)}^2 + C\|\partial_z^s\phi(\tau)\|_{e(M)}\big\|\partial_z^s(\tilde{\Box}+M^2)\phi(\tau)\big\|_{L^2}\\
&+C\|\partial_z^s\phi(\tau)\|_{e(M)}\big\|[(\tilde{\Box}+M^2),\partial_z^s]\phi(\tau)\big\|_{L^2}.
\end{aligned}$$

On the other hand, we have the following commutation relation:

$$[\tilde{\Box}+M^2,\partial_z^s] = \frac{1}{\tau^2}\sum_{j=0}^{s}\gamma_{j,s}(z)\partial_z^j \tag{4.3.3}$$

with appropriate coefficients $\gamma_{j,s}(z)$ satisfying $\|\gamma_{j,s}\|_{L^\infty} < \infty$, from which it follows that

$$\big\|[(\tilde{\Box}+M^2),\partial_z^s]\phi(\tau)\big\|_{L^2} \le \frac{1}{\tau^2}\sum_{j=0}^{s}\|\gamma_{j,s}\|_{L^\infty}\|\partial_z^j\phi(\tau)\|_{L^2} \le \frac{C}{\tau^2}\sum_{j=0}^{s}\|\partial_z^j\phi(\tau)\|_{e(M)}.$$

Summing up, we obtain (4.3.1). In the same way (4.3.2) follows. ∎

Next, we show the following energy inequality, which is the main tool for the proof of Proposition 4.2.1.

Proposition 4.3.1 *Let ϕ be a smooth function of $(\tau, z) \in [\tau_0, \infty[\times\mathbb{R}$, and let $\kappa \geq 0$, $M > 0$, $\nu \geq 0$, $s_1, s_2 \in \mathbb{N}_0$. If $s_1 \geq 4\kappa$, we have*

$$\sup_{\tau\geq\tau_0}\Big(\sum_{j_1=0}^{s_1}\sum_{j_2=0}^{s_2}\tau^{-(\nu+j_1/4)}\left\|\partial_z^{j_1+j_2}\phi(\tau)\right\|_{e(M)}\Big) \leq C\Big(\|\phi(\tau_0)\|_{H^{s_1+s_2+1}} + \|\partial_\tau\phi(\tau_0)\|_{H^{s_1+s_2}}\Big)$$
$$+C\sum_{j_1=0}^{s_1}\sum_{j_2=0}^{s_2}\int_{\tau_0}^{\infty}\tau^{-(\nu+j_1/4)}\|\partial_z^{j_1+j_2}(\tilde{\Box}+M^2)\phi(\tau)\|_{L^2(\mathbb{R}_z)}d\tau,$$

provided that the right hand side is finite. Here C is a positive constant independent of ν, τ_0.

Proof: We first note that we can choose some constant $C_s \geq 1$ so that

$$C_s^{-1}\mathcal{E}_s(\tau) \leq \sum_{j=0}^{s}\tau^{-(\nu+j/4)}\left\|\partial_z^j\phi(\tau)\right\|_{e(M)} \leq C_s\mathcal{E}_s(\tau)$$

holds, where

$$\mathcal{E}_s(\tau) = \Big(\sum_{j=0}^{s}\tau^{-(2\nu+j/2)}\left\|\partial_z^j\phi(\tau)\right\|_{e(M)}^2\Big)^{1/2}.$$

Straightforward calculation yields

$$\frac{d}{d\tau}\mathcal{E}_{s_1}(\tau)^2 = \sum_{j=0}^{s_1}\Big\{\tau^{-(2\nu+j/2)}\frac{d}{d\tau}\left\|\partial_z^j\phi(\tau)\right\|_{e(M)}^2 - (2\nu+j/2)\tau^{-(2\nu+j/2)-1}\left\|\partial_z^j\phi(\tau)\right\|_{e(M)}^2\Big\}$$
$$\leq \sum_{j=0}^{s_1-1}\tau^{-(2\nu+j/2)}\frac{d}{d\tau}\left\|\partial_z^j\phi(\tau)\right\|_{e(M)}^2 + \tau^{-(2\nu+s_1/2)}\frac{d}{d\tau}\left\|\partial_z^{s_1}\phi(\tau)\right\|_{e(M)}^2$$
$$- (2\nu+s_1/2)\tau^{-(2\nu+s_1/2)-1}\left\|\partial_z^{s_1}\phi(\tau)\right\|_{e(M)}^2.$$

Using Lemma 4.3.1 and the relation $-(2\nu + s_1/2) \le -2\kappa$, we have

$$
\begin{aligned}
&\frac{d}{d\tau}\mathcal{E}_{s_1}(\tau)^2 \\
&\le \sum_{j=0}^{s_1-1} \tau^{-(2\nu+j/2)} \left\{ \frac{C}{\tau^2} \sum_{\ell=0}^{j+1} \|\partial_z^\ell \phi(\tau)\|_{e(M)}^2 + C\|\partial_z^j \phi(\tau)\|_{e(M)} \left\|\partial_z^j (\widetilde{\Box} + M^2)\phi\right\|_{L^2} \right\} \\
&\quad + \tau^{-(2\nu+s_1/2)} \left\{ \frac{2\kappa}{\tau} \|\partial_z^{s_1}\phi\|_{e(M)}^2 + \frac{C}{\tau^2} \sum_{\ell=0}^{s_1} \|\partial_z^\ell \phi(\tau)\|_{e(M)}^2 + C\|\partial_z^{s_1}\phi(\tau)\|_{e(M)} \left\|\partial_z^{s_1}(\widetilde{\Box} + M^2)\phi\right\|_{L^2} \right\} \\
&\quad - 2\kappa\tau^{-(2\nu+s_1/2)-1} \|\partial_z^{s_1}\phi(\tau)\|_{e(M)}^2 \\
&= \frac{C}{\tau^2} \left\{ \sum_{j=0}^{s_1-1} \sum_{\ell=0}^{j+1} \tau^{-(2\nu+j/2)} \|\partial_z^\ell \phi(\tau)\|_{e(M)}^2 + \sum_{\ell=0}^{s_1} \tau^{-(2\nu+s_1/2)} \|\partial_z^\ell \phi(\tau)\|_{e(M)}^2 \right\} \\
&\quad + C \sum_{j=0}^{s_1} \tau^{-(2\nu+j/2)} \|\partial_z^j \phi(\tau)\|_{e(M)} \left\|\partial_z^j (\widetilde{\Box} + M^2)\phi\right\|_{L^2} \\
&=: I_1 + I_2.
\end{aligned}
$$

To estimate I_1, we note the following relation:

$$
\begin{aligned}
\sum_{j=0}^{s_1-1} \sum_{\ell=0}^{j+1} \tau^{-(2\nu+j/2)} \|\partial_z^\ell \phi(\tau)\|_{e(M)}^2 &= \tau^{1/2} \sum_{j=0}^{s_1-1} \sum_{\ell=0}^{j+1} \tau^{-\frac{j+1-\ell}{2}} \tau^{-(2\nu+\ell/2)} \|\partial_z^\ell \phi(\tau)\|_{e(M)}^2 \\
&\le \tau^{1/2} s_1 \sum_{\ell=0}^{s_1} \tau^{-(2\nu+\ell/2)} \|\partial_z^\ell \phi(\tau)\|_{e(M)}^2 \\
&= s_1 \tau^{1/2} \mathcal{E}_{s_1}(\tau)^2.
\end{aligned}
$$

This relation gives

$$
I_1 \le \frac{C}{\tau^2} \left\{ s_1 \tau^{1/2} \mathcal{E}_{s_1}(\tau)^2 + \mathcal{E}_{s_1}(\tau)^2 \right\} \le \frac{C}{\tau^{3/2}} \mathcal{E}_{s_1}(\tau)^2.
$$

As for I_2, the Cauchy-Schwarz inequality implies

$$
\begin{aligned}
I_2 &\le C \left\{ \sum_{j=0}^{s_1} \tau^{-(2\nu+j/2)} \|\partial_z^j \phi(\tau)\|_{e(M)}^2 \right\}^{1/2} \left\{ \sum_{j=0}^{s_1} \tau^{-(2\nu+j/2)} \left\|\partial_z^j (\widetilde{\Box} + M^2)\phi(\tau)\right\|_{L^2}^2 \right\}^{1/2} \\
&\le C\mathcal{E}_{s_1}(\tau) \sum_{j=0}^{s_1} \tau^{-(\nu+j/4)} \left\|\partial_z^j (\widetilde{\Box} + M^2)\phi(\tau)\right\|_{L^2}.
\end{aligned}
$$

Summing up, we obtain

$$
\frac{d\mathcal{E}_{s_1}}{d\tau}(\tau) \le \frac{C}{\tau^{3/2}} \mathcal{E}_{s_1}(\tau) + C \sum_{j=0}^{s_1} \tau^{-(\nu+j/4)} \left\|\partial_z^j (\widetilde{\Box} + M^2)\phi(\tau)\right\|_{L^2},
$$

which implies

$$\mathcal{E}_{s_1}(\tau) \leq e^{-C/\tau^{1/2}+C/\tau_0^{1/2}}\mathcal{E}_{s_1}(\tau_0) + C\sum_{j=0}^{s_1}\int_{\tau_0}^{\tau} e^{-C/\tau^{1/2}+C/\rho^{1/2}}\rho^{-(\nu+j/4)}\big\|\partial_z^j(\widetilde{\Box}+M^2)\phi(\rho)\big\|_{L^2}d\rho.$$

Using the fact that $1 \leq e^{C/\tau^{1/2}} \leq e^{C/\tau_0^{1/2}} \leq e^C$ for any $\tau \geq \tau_0 \geq 1$, we obtain the desired inequality with $s_2 = 0$. Concerning the case of $s_2 \geq 1$, we have only to use the commutation relation (4.3.3) and the Gronwall inequality. ∎

Now, let us introduce the function space

$$\begin{aligned} Y^{\sigma,\delta} :=\Big\{ \boldsymbol{\phi} = (\phi_1,\phi_2) \in C^0(\tau_0,\infty; H^{2\sigma}(\mathbb{R};\mathbb{R}^2)) \cap C^1(\tau_0,\infty; H^{2\sigma-1}(\mathbb{R};\mathbb{R}^2)) \ \Big| \\ 0 \leq^{\forall} j \leq 2\sigma-1,\ {}^{\exists}C_j > 0 \text{ s.t.} \\ \|\partial_z^j\phi_1(\tau)\|_{e(m)} \leq C_j\tau^{\frac{1}{4}(j-\sigma)_+},\ \|\partial_z^j\phi_2(\tau)\|_{e(\mu)} \leq C_j\tau^{\delta+\frac{1}{4}(j-\sigma)_+}\Big\} \end{aligned}$$

equipped with the norm

$$\|\boldsymbol{\phi}\|_{Y^{\sigma,\delta}} = \sup_{\tau\geq\tau_0}\sum_{j_1=0}^{\sigma-1}\sum_{j_2=0}^{\sigma}\Big(\tau^{-j_1/4}\big\|\partial_z^{j_1+j_2}\phi_1(\tau)\big\|_{e(m)} + \tau^{-(\delta+j_1/4)}\big\|\partial_z^{j_1+j_2}\phi_2(\tau)\big\|_{e(\mu)}\Big).$$

Here, $\delta \in]0, 1/10]$ and $\sigma \geq 1+4\kappa$. We denote by $Y^{\sigma,\delta}(r)$ the closed ball in $Y^{\sigma,\delta}$ of radius r centered at the origin, i.e.,

$$Y^{\sigma,\delta}(r) := \big\{\boldsymbol{\phi} \in Y^{\sigma,\delta} \ \big|\ \|\boldsymbol{\phi}\|_{Y^{\sigma,\delta}} \leq r\big\}.$$

For $\boldsymbol{\phi} = (\phi_1,\phi_2) \in Y^{\sigma,\delta}$, let $S(\boldsymbol{\phi})$ be the solution $\boldsymbol{\psi} = (\psi_1,\psi_2)$ to the Cauchy problem

$$\begin{cases} (\widetilde{\Box}+m^2)\psi_1 = \dfrac{\alpha}{\tau^{3/2}(\cosh\kappa z)^3}\phi_2^4, & \\ & \tau > \tau_0,\ z \in \mathbb{R}, \\ (\widetilde{\Box}+\mu^2)\psi_2 = \dfrac{\beta}{\tau(\cosh\kappa z)^2}\phi_1^3, & \\ (\psi_1,\psi_2,\partial_\tau\psi_1,\partial_\tau\psi_2)\big|_{\tau=\tau_0} = (\varepsilon\tilde{u}_0,\varepsilon\tilde{v}_0,\varepsilon\tilde{u}_1,\varepsilon\tilde{v}_1), & z \in \mathbb{R}. \end{cases}$$

We shall show that S becomes a contraction mapping on $Y^{\sigma,\delta}(r)$ when we choose ε_0, r appropriately. Then we can apply the fixed point theorem to obtain Proposition 4.2.1.

Let $\boldsymbol{\phi} = (\phi_1,\phi_2) \in Y^{\sigma,\delta}(r)$. It follows from Proposition 4.3.1 that

$$\|S(\boldsymbol{\phi})\|_{Y^{\sigma,\delta}} \leq C\varepsilon + C\int_{\tau_0}^{\infty} G(\tau)d\tau,$$

where

$$G(\tau) = \sum_{j_1=0}^{\sigma-1}\sum_{j_2=0}^{\sigma}\Big(\tau^{-(\delta+1+j_1/4)}\,\big\|\partial_z^{j_1+j_2}\{\phi_1(\tau)^3\}\big\|_{L^2(\mathbb{R}_z)} + \tau^{-(3/2+j_1/4)}\,\big\|\partial_z^{j_1+j_2}\{\phi_2(\tau)^4\}\big\|_{L^2(\mathbb{R}_z)}\Big)$$

Since $[(j_1+j_2)/2]+1 \le [\sigma-1/2]+1=\sigma$, the Leibniz formula and the Sobolev imbedding yield

$$\begin{aligned}\left\|\partial_z^{j_1+j_2}\{\phi_1(\tau)^3\}\right\|_{L^2} &\le C\|\phi_1(\tau)\|^2_{W^{[(j_1+j_2)/2],\infty}}\sum_{\ell=0}^{j_1+j_2}\|\partial_z^\ell\phi_1(\tau)\|_{L^2}\\ &\le C\|\phi_1(\tau)\|^2_{H^\sigma}\sum_{\ell=0}^{j_1+j_2} r\tau^{\frac14(\ell-j_2)_+}\\ &\le Cr^3\tau^{j_1/4}.\end{aligned}$$

Here $[\,\cdot\,]$ stands for the integer part. Similarly we have

$$\begin{aligned}\left\|\partial_z^{j_1+j_2}\{\phi_2(\tau)^4\}\right\|_{L^2} &\le C\|\phi_2(\tau)\|^3_{H^\sigma}\sum_{\ell=0}^{j_1+j_2} r\tau^{\delta+\frac14(\ell-j_2)_+}\\ &\le Cr^4\tau^{4\delta+j_1/4}.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\|S(\boldsymbol{\phi})\|_{Y^{\sigma,\delta}} &\le C\varepsilon + C\int_{\tau_0}^\infty r^3\tau^{-(1+\delta)}+r^4\tau^{-(3/2-4\delta)}d\tau\\ &\le C\varepsilon + C(1+r)r^3\int_{\tau_0}^\infty \tau^{-(1+\delta)}d\tau\\ &\le C\varepsilon + C(1+r)r^3/\delta.\end{aligned}$$

Note that $1+\delta\le 3/2-4\delta$ since $\delta\le 1/10$. When we take $\varepsilon_0 := r/2C$ and choose $r>0$ so small that $C(1+r)r^2\le \delta/2$, we have

$$\|S(\boldsymbol{\phi})\|_{Y^{\sigma,\delta}} \le r,$$

provided that $\varepsilon \in]0,\varepsilon_0]$.

Next, we put $\tilde{\boldsymbol{\psi}} = S(\boldsymbol{\phi})-S(\boldsymbol{\phi}')$ for $\boldsymbol{\phi}=(\phi_1,\phi_2), \boldsymbol{\phi}'=(\phi_1',\phi_2')\in Y^{\sigma,\delta}(r)$. Then $\tilde{\boldsymbol{\psi}}=(\tilde{\psi}_1,\tilde{\psi}_2)$ satisfies

$$\begin{cases}(\tilde{\Box}+m^2)\tilde{\psi}_1 = \dfrac{\alpha}{\tau^{3/2}(\cosh\kappa z)^3}(\phi_2^4-\phi_2'^4),\\ (\tilde{\Box}+\mu^2)\tilde{\psi}_2 = \dfrac{\beta}{\tau(\cosh\kappa z)^2}(\phi_1^3-\phi_1'^3)\end{cases}$$

with the initial data

$$\tilde{\psi}_1=\tilde{\psi}_2=\partial_\tau\tilde{\psi}_1=\partial_\tau\tilde{\psi}_2=0$$

at $\tau=\tau_0$. Using Proposition 4.3.1 again, we have

$$\|\tilde{\boldsymbol{\psi}}\|_{Y^{\sigma,\delta}} \le C\sum_{j_1=0}^{\sigma-1}\sum_{j_2=0}^{\sigma}\int_{\tau_0}^\infty G_{j_1j_2}(\tau)d\tau,$$

where

$$\begin{aligned} G_{j_1j_2}(\tau) =& \tau^{-(\delta+1+j_1/4)} \left\| \partial_z^{j_1+j_2} \{ \phi_1(\tau)^3 - \phi'_1(\tau)^3 \} \right\|_{L^2(\mathbb{R}_z)} \\ &+ \tau^{-(3/2+j_1/4)} \left\| \partial_z^{j_1+j_2} \{ \phi_2(\tau)^4 - \phi'_2(\tau)^4 \} \right\|_{L^2(\mathbb{R}_z)}. \end{aligned}$$

In the same way as before, we have

$$\begin{aligned} G_{j_1j_2}(\tau) \leq& Cr^2 \tau^{-(1+\delta+j_1/4)} \sum_{\ell=0}^{j_1+j_2} \left\| \partial_z^\ell \{ \phi_1(\tau) - \phi'_1(\tau) \} \right\|_{L^2} \\ &+ Cr^3 \tau^{-(3/2-3\delta+j_1/4)} \sum_{\ell=0}^{j_1+j_2} \left\| \partial_z^\ell \{ \phi_2(\tau) - \phi'_2(\tau) \} \right\|_{L^2} \\ \leq& Cr^2 \tau^{-(1+\delta)} \sum_{\ell=0}^{j_1+j_2} \tau^{\frac{1}{4}\{-j_1+(\ell-j_2)_+\}} \|\phi - \phi'\|_{Y^{\sigma,\delta}} \\ &+ Cr^3 \tau^{-(3/2-4\delta)} \sum_{\ell=0}^{j_1+j_2} \tau^{\frac{1}{4}\{-j_1+(\ell-j_2)_+\}} \|\phi - \phi'\|_{Y^{s,\delta}} \\ \leq& C(1+r) r^2 \tau^{-(1+\delta)} \|\phi - \phi'\|_{Y^{\sigma,\delta}}. \end{aligned}$$

Therefore, if r is chosen so small that $\sigma(\sigma+1)C(1+r)r^2 \leq \delta/2$ holds, then we have

$$\begin{aligned} \|S(\phi) - S(\phi')\|_{Y^{\sigma,\delta}} &\leq \frac{\delta}{2} \left(\int_{\tau_0}^\infty \tau^{-(1+\delta)} d\tau \right) \|\phi - \phi'\|_{Y^{\sigma,\delta}} \\ &\leq \frac{1}{2} \|\phi - \phi'\|_{Y^{\sigma,\delta}}. \end{aligned}$$

This completes the proof of Proposition 4.2.1. ∎

4.4 Proof of Theorem 4.1.1

In what follows, we only treat the case where $\mu = 3m$. The other cases can be treated in the same manner.

First, we rewrite (4.2.1) as

$$\begin{cases} (\partial_\tau^2 + m^2)\tilde{u} = \dfrac{1}{\tau^{3/2-4\delta}} R_1, \\ (\partial_\tau^2 + \mu^2)\tilde{v} = \dfrac{\beta}{\tau(\cosh \kappa z)^2} \tilde{u}^3 + \dfrac{1}{\tau^{2-\delta}} R_2, \end{cases}$$

where

$$R_1 = \frac{\alpha}{(\cosh \kappa z)^3} \left(\tau^{-\delta} \tilde{v} \right)^4 + \frac{1}{\tau^{1/2+4\delta}} \mathcal{L}\tilde{u}, \quad R_2 = \tau^{-\delta} \mathcal{L}\tilde{v}$$

with

$$\mathcal{L} = \tau^2(\partial_\tau^2 - \widetilde{\square}) = \partial_z^2 - 2\kappa(\tanh\kappa z)\partial_z - \frac{1}{4} - \kappa^2 + 2\kappa^2(\tanh\kappa z)^2.$$

It follows from (4.2.3), (4.2.4) and the Sobolev imbedding that

$$\sup_{\tau\geq\tau_0}\Big(\|R_1(\tau,\cdot)\|_{L^\infty(\mathbb{R}_z)} + \|R_2(\tau,\cdot)\|_{L^\infty(\mathbb{R}_z)}\Big) < \infty.$$

Next, we put

$$\tilde{a}_\pm = e^{\mp im\tau}(m \mp i\partial_\tau)\tilde{u},$$
$$\tilde{b}_\pm = e^{\mp i\mu\tau}(\mu \mp i\partial_\tau)\tilde{v}.$$

Note that $\tilde{a}_\pm$ and $\tilde{b}_\pm$ satisfy

$$\partial_\tau\tilde{a}_\pm = \mp ie^{\mp im\tau}(\partial_\tau^2 + m^2)\tilde{u} = \frac{\pm e^{\mp im\tau}}{i\tau^{3/2-4\delta}}R_1$$

and

$$\begin{aligned}\partial_\tau\tilde{b}_\pm =& \frac{\pm e^{\mp i\mu\tau}\beta}{i\tau(\cosh\kappa z)^2}\left(\frac{e^{+im\tau}\tilde{a}_+ + e^{-im\tau}\tilde{a}_-}{2m}\right)^3 + \frac{\pm e^{\mp i\mu\tau}}{i\tau^{2-\delta}}R_2\\ =& \frac{\pm\beta}{i8m^3(\cosh\kappa z)^2}\sum_{\ell=0}^{3}\binom{3}{\ell}\frac{e^{i\{(3-2\ell)m\mp\mu\}\tau}}{\tau}(\tilde{a}_+)^{3-\ell}(\tilde{a}_-)^\ell + \frac{\pm e^{\mp i\mu\tau}}{i\tau^{2-\delta}}R_2.\end{aligned} \tag{4.4.1}$$

We are going to find the asymptotics of $\tilde{a}_\pm, \tilde{b}_\pm$ as $\tau \to \infty$. It is easy to do it for $\tilde{a}_\pm$. Indeed, since

$$\int_\tau^\infty |\partial_\tau\tilde{a}_\pm(\rho,z)|\,d\rho \leq C\int_\tau^\infty \rho^{-3/2+4\delta}d\rho \leq C\tau^{-1/2+4\delta},$$

we have

$$\tilde{a}_\pm(\tau,z) = \tilde{a}_\pm^\infty(z) + O(\tau^{-1/2+4\delta}), \tag{4.4.2}$$

where

$$\tilde{a}_\pm^\infty(z) := e^{\mp im\tau_0}\{m\tilde{u}_0(z) \mp i\tilde{u}_1(z)\} + \int_{\tau_0}^\infty \frac{\pm e^{\mp im\rho}}{i\rho^{3/2-4\delta}}R_1(\rho,z)d\rho.$$

To get the asymptotics of $\tilde{b}_\pm$, we use the following lemma.

Lemma 4.4.1 *Let $\nu \in \mathbb{R}$ and let $\psi_j(\tau,z)$ $(j = 1,2,\cdots,N)$ be smooth functions which satisfy*

$$|\psi_j(\tau,z)| \leq C_0, \quad |\partial_\tau\psi_j(\tau,z)| \leq C_0\tau^{-\nu}$$

for some constant $C_0 \geq 0$. *Then we have*

$$\frac{e^{i\omega\tau}}{\tau}\prod_{j=1}^{N}\psi_j(\tau,z) = \frac{\partial}{\partial\tau}\Big\{\frac{e^{i\omega\tau}}{i\omega\tau}\prod_{j=1}^{N}\psi_j(\tau,z)\Big\} + O\left(\tau^{-\min\{2,1+\nu\}}\right)$$

for $\omega \in \mathbb{R}\backslash\{0\}$, *while*

$$\frac{1}{\tau}\prod_{j=1}^{N}\psi_j(\tau,z) = \frac{\partial}{\partial\tau}\Big\{(\log\tau)\prod_{j=1}^{N}\psi_j(\tau,z)\Big\} + O\left(\tau^{-\nu}\log\tau\right).$$

Proof is quite simple. Indeed, using the relation

$$\frac{e^{i\omega\tau}}{\tau}\prod_{j=1}^{N}\psi_j = \frac{\partial}{\partial\tau}\Big\{\frac{e^{i\omega\tau}}{i\omega\tau}\prod_{j=1}^{N}\psi_j\Big\} - \frac{e^{i\omega\tau}}{i\omega}\frac{\partial}{\partial\tau}\left(\frac{1}{\tau}\prod_{j=1}^{N}\psi_j\right),$$

we have

$$\begin{aligned}\left|\frac{e^{i\omega\tau}}{\tau}\prod_{j=1}^{N}\psi_j - \frac{\partial}{\partial\tau}\Big\{\frac{e^{i\omega\tau}}{i\omega\tau}\prod_{j=1}^{N}\psi_j\Big\}\right| &\leq \frac{1}{|\omega|\tau^2}\prod_{j=1}^{N}|\psi_j| + \frac{1}{|\omega|\tau}\sum_{k=1}^{N}|\partial_\tau\psi_k|\prod_{j\neq k}|\psi_j| \\ &\leq \frac{C_0^N}{|\omega|\tau^2} + \frac{NC_0^N}{|\omega|\tau^{1+\nu}}.\end{aligned}$$

The other one follows similarly from the relation

$$\frac{1}{\tau}\prod_{j=1}^{N}\psi_j = \frac{\partial}{\partial\tau}\Big\{(\log\tau)\prod_{j=1}^{N}\psi_j\Big\} - (\log\tau)\frac{\partial}{\partial\tau}\Big(\prod_{j=1}^{N}\psi_j\Big).$$

∎

Applying the above lemma to (4.4.1), we obtain

$$\partial_\tau(\tilde{b}_\pm - \Phi_\pm) = \Psi_\pm + \frac{\pm e^{\mp i\mu\tau}}{i\tau^{2-\delta}}R_2,$$

where

$$\Phi_+(\tau,z) = (\log\tau)\frac{\beta\{\tilde{a}_+(\tau,z)\}^3}{i8m^3(\cosh\kappa z)^2} + O(\tau^{-1}), \quad \Phi_-(\rho,z) = \overline{\Phi_+(\tau,z)},$$

and

$$\Psi_+(\tau,z) = O(\tau^{-3/2+4\delta}\log\tau), \quad \Psi_-(\tau,z) = \overline{\Psi_+(\tau,z)}.$$

From this it follows that

$$\left|\tilde{b}_\pm(\tau,z) - \Phi_\pm(\tau,z) - \tilde{b}_\pm^\infty(z)\right| \leq \int_\tau^\infty |\Psi_\pm(\rho,z)| + \left|\frac{R_2(\rho,z)}{\rho^{2-\delta}}\right| d\rho \leq C\tau^{-1/2+5\delta}$$

where

$$\tilde{b}_{\pm}^{\infty}(z) := e^{\mp i\mu\tau_0}\{\mu\tilde{v}_0(z) \mp i\tilde{v}_1(z)\} - \Phi_{\pm}(\tau_0, z) + \int_{\tau_0}^{\infty} \Psi_{\pm}(\rho, z) + \frac{\pm e^{\mp i\mu\rho}}{i\rho^{2-\delta}} R_2(\rho, z)d\rho.$$

Therefore we obtain

$$\tilde{b}_{\pm}(\tau, z) = \pm(\log\tau)\frac{\beta\left(\tilde{a}_{\pm}^{\infty}(z)\right)^3}{i8m^3(\cosh\kappa z)^2} + \tilde{b}_{\pm}^{\infty}(z) + O(\tau^{-1/2+5\delta}) \tag{4.4.3}$$

as $\tau \to \infty$, uniformly with respect to $z \in \mathbb{R}$.

Now, we return to the original variables. Remember that

$$u(t,x) = \frac{e^{im\tau}\tilde{a}_+(\tau,z) + e^{-im\tau}\tilde{a}_-(\tau,z)}{2m\tau^{1/2}\cosh\kappa z},$$
$$v(t,x) = \frac{e^{i\mu\tau}\tilde{b}_+(\tau,z) + e^{-i\mu\tau}\tilde{b}_-(\tau,z)}{2\mu\tau^{1/2}\cosh\kappa z},$$

$$\tau = \sqrt{(t+2B)^2 - |x|^2}, \quad z = \tanh^{-1}\left(\frac{x}{t+2B}\right)$$

and $t \gg 1$, $|x| < t + 2B$. Using the relations $\tilde{a}_-^{\infty} = \overline{\tilde{a}_+^{\infty}}$, $\tilde{b}_-^{\infty} = \overline{\tilde{b}_+^{\infty}}$ and

$$\begin{aligned}\frac{\tau^{-\nu}}{\cosh\kappa z} &\le \frac{Ct^{-\nu}}{(\cosh z)^{\kappa}}\left(\frac{t+2B}{\tau}\right)^{\nu} \\ &= Ct^{-\nu}(1 - |x/(t+2B)|^2)^{\kappa-\nu} \\ &\le Ct^{-\nu}\end{aligned}$$

for $\kappa \ge \nu \ge 0$, we have

$$u(t,x) = \mathrm{Re}\left[\frac{e^{im\tau(t,x)}}{m\sqrt{t+2B}}\underline{a}\left(\tanh^{-1}\left(\frac{x}{t+2B}\right)\right)\right] + O\left(t^{-1+4\delta}\right), \tag{4.4.4}$$

$$\begin{aligned}v(t,x) = \mathrm{Re}&\left[\frac{e^{i\mu\tau(t,x)}}{\mu\sqrt{t+2B}}\left\{\frac{\beta}{i8m^3}\left\{\underline{a}\left(\tanh^{-1}\left(\frac{x}{t+2B}\right)\right)\right\}^3\frac{\log\tau(t,x)}{\cosh z(t,x)} + \underline{b}\left(\tanh^{-1}\left(\frac{x}{t+2B}\right)\right)\right\}\right] \\ &+ O\left(t^{-1+5\delta}\right),\end{aligned} \tag{4.4.5}$$

where

$$\underline{a}(z) = \frac{(\cosh z)^{1/2}\tilde{a}_+^{\infty}(z)}{\cosh\kappa z}, \quad \underline{b}(z) = \frac{(\cosh z)^{1/2}\tilde{b}_+^{\infty}(z)}{\cosh\kappa z}.$$

Next, we put

$$a(y) = \begin{cases} e^{i2Bm\sqrt{1-|y|^2}}\underline{a}(\tanh^{-1}(y)) & \text{if } |y| < 1, \\ 0 & \text{if } |y| \ge 1, \end{cases}$$

$$A(y) = \frac{\beta}{i8m^3}(1-|y|^2)_+^{1/2} a(y)^3,$$

$$b(y) = \begin{cases} e^{i2B\mu\sqrt{1-|y|^2}} \underline{b}(\tanh^{-1}(y)) + A(y)\log\sqrt{1-|y|^2} & \text{if} \quad |y|<1, \\ 0 & \text{if} \quad |y| \geq 1. \end{cases}$$

Note that the following estimates are valid (cf. [3, p.58–59]):

$$|\partial_y^j a(y)| \leq C(1-|y|)_+^{\kappa/2-1/4-j},$$

$$|a(x/(t+2B)) - a(x/t)| \leq Ct^{-1}\{(1-|x/t|)_+ + 1/t\}^{\kappa/2-5/4},$$

$$1_{|x|<t}\left| e^{imt\sqrt{1-|x/(t+2B)|^2}} - e^{imt\sqrt{1-|x/t|^2}} \right| \leq Ct^{-1}\{(1-|x/t|)_+ + 1/t\}^{-1/2}.$$

These relations give us

$$\begin{aligned} &\left| \frac{e^{imt\sqrt{1-|x/(t+2B)|^2}}}{\sqrt{t+2B}} a\Big(\frac{x}{t+2B}\Big) - \frac{e^{im\sqrt{t^2-|x|^2}}}{\sqrt{t}} a\Big(\frac{x}{t}\Big) \right| \\ &\leq \frac{1}{\sqrt{t+2B}} \left| a\Big(\frac{x}{t+2B}\Big) - a\Big(\frac{x}{t}\Big) \right| + \left| \frac{1}{\sqrt{t+2B}} - \frac{1}{\sqrt{t}} \right| \left| a\Big(\frac{x}{t}\Big) \right| \\ &\quad + \frac{1}{\sqrt{t}} \left| e^{imt\sqrt{1-|x/(t+2B)|^2}} - e^{imt\sqrt{1-|x/t|^2}} \right| \left| a\Big(\frac{x}{t}\Big) \right| \\ &\leq Ct^{-3/2}\{(1-|x/t|)_+ + 1/t\}^{\kappa/2-5/4} + Ct^{-3/2}(1-|x/t|)_+^{\kappa/2-1/4} \\ &\quad + Ct^{-3/2}\{(1-|x/t|)_+ + 1/t\}^{\kappa/2-1/4-1/2} \\ &\leq Ct^{-3/2}, \end{aligned}$$

provided that $\kappa > 5/2$. Summing up, we have

$$\begin{aligned} \frac{e^{im\tau(t,x)}}{m\sqrt{t+2B}} \underline{a}\Big(\tanh^{-1}\Big(\frac{x}{t+2B}\Big)\Big) &= \frac{e^{imt\sqrt{1-|x/(t+2B)|^2}}}{m\sqrt{t+2B}} a\Big(\frac{x}{t+2B}\Big) \\ &= \frac{e^{im\sqrt{t^2-|x|^2}}}{m\sqrt{t}} a\Big(\frac{x}{t}\Big) + O(t^{-3/2}). \end{aligned}$$

Substituting it for the first term of the right hand side of (4.4.4), we obtain the asymptotic formula of u. In the same way, the first term of the right hand side of (4.4.5) can be written as

$$\operatorname{Re}\left[\frac{e^{i\mu\sqrt{t^2-|x|^2}}}{\mu\sqrt{t}} \left\{ A\Big(\frac{x}{t}\Big)\log t + b\Big(\frac{x}{t}\Big) \right\} \right] + O(t^{-3/2}\log t),$$

which yields the asymptotic formula of v. This completes the proof of Theorem 4.1.1. ∎

4.5 Remarks

(1) We can prove the analogous result for two-dimensional case, such as

$$\begin{cases} (\Box + m^2)v_1 = \alpha v_2^3, \\ (\Box + \mu^2)v_2 = \beta v_1^2, \end{cases} \quad t > 0,\ x \in \mathbb{R}^2, \tag{4.5.1}$$

where $\alpha, \beta \in \mathbb{R}$, or

$$\begin{cases} (\Box + m_1^2)u_1 = F_1(u, \partial u), \\ (\Box + m_2^2)u_2 = F_2(u, \partial u), \\ (\Box + m_3^2)u_3 = \gamma u_1 u_2 + F_3(u, \partial u), \end{cases} \quad t > 0,\ x \in \mathbb{R}^2, \tag{4.5.2}$$

where $u = (u_j)_{1 \le j \le 3}$, $\partial = (\partial_t, \partial_{x_1}, \partial_{x_2})$, $\gamma \in \mathbb{R}$ and $F_j(u, \partial u) = O(|u|^3 + |\partial u|^3)$. For the solution v_2 of (4.5.1) (resp. u_3 of (4.5.2)), the long range effect as in Theorem 4.1.1 (resp. Theorem 4.1.2) is observed if and only if $\mu = 2m$ (resp. $m_3 = \lambda_1 m_1 + \lambda_2 m_2$ for some $\lambda_1, \lambda_2 \in \{\pm 1\}$). See also [6] for related result.

(2) It is likely that a result similar to Theorem 4.1.1 holds true for one-dimensional cubic homogeneous case, such as

$$\begin{cases} (\Box + m^2)u = \alpha v^3, \\ (\Box + \mu^2)v = \beta u^3, \end{cases} \tag{4.5.3}$$

but it seems still open so far. The main reason is that the method of this chapter does not work well without some growth with respect to t or loss of derivaives.

Chapter 5

Examples on small data blow-up

In this chapter, we will construct two examples concerning nonexistence of global classical solutions. The first one is taken from [30], which is a system version of Yordanov's example, and the second one comes from the author's work [33].

5.1 Example (1)

Let us consider the Cauchy problem

$$\begin{cases} (\Box + m_j^2)u_j = F_j(u, \partial u), \\ (u_j, \partial_t u_j)|_{t=0} = (\varepsilon f_j, \varepsilon g_j), \quad 1 \le j \le N \end{cases} \tag{5.1.1}$$

in one space dimension. We assume there exist j_0, $k_0 \in \{1, \cdots, N\}$ such that the following (i)–(iii) hold:

(i) $\quad (\partial_x u_{j_0}) F_{k_0}(u, \partial u) + (\partial_x u_{k_0}) F_{j_0}(u, \partial u) \ge \delta \left\{ (\partial_x u_{j_0})^2 (\partial_t u_{k_0})^2 + (\partial_x u_{k_0})^2 (\partial_t u_{j_0})^2 \right\}$

for some constant $\delta > 0$,

(ii)
$$K := \int f'_{j_0}(x) g_{k_0}(x) + f'_{k_0}(x) g_{j_0}(x) dx > 0,$$

and

(iii)
$$m_{j_0} = m_{k_0}.$$

We also assume that the Cauchy data is compactly-supported, say,

(iv)
$$f_j(x) = g_j(x) = 0 \quad \text{for} \quad |x| \ge r, \quad j \in \{1, \cdots, N\}.$$

Proposition 5.1.1 *Under the assumptions (i)–(iv), the lifespan T_ε of the classical solution of (5.1.1) must be finite for any $\varepsilon > 0$. More precisely, we have*

$$T_\varepsilon \le r \exp\left(\frac{4}{K \delta \varepsilon^2} \right).$$

Proof: We first note a simple fact on ODE: If $I \in C^1([0,T])$ satisfies the differential inequality

$$\frac{dI}{dt}(t) \geq c(t+1)^{-1}I^2(t) \quad \text{for} \quad t \in [0,T] \tag{5.1.2}$$

with $I(0) > 0$, then

$$I(T) \geq \frac{I(0)}{1 - cI(0)\log(T+1)}.$$

We also note the identity

$$\frac{\partial}{\partial t}\Big\{(\partial_t u_j)(\partial_x u_k) + (\partial_t u_k)(\partial_x u_j)\Big\} + \frac{\partial}{\partial x}\Big\{m_k^2 u_j u_k - (\partial_t u_j)(\partial_t u_k) - (\partial_x u_j)(\partial_x u_k)\Big\}$$
$$= (m_k^2 - m_j^2)u_j \partial_x u_k + F_j \partial_x u_k + F_k \partial_x u_j. \tag{5.1.3}$$

Now, we put $I(t) := \int (\partial_t u_{j_0})(\partial_x u_{k_0}) + (\partial_t u_{k_0})(\partial_x u_{j_0})dx$ and we shall see that $I(t)$ satisfies (5.1.2). From (5.1.3) and the assumption (iii), we have

$$\frac{dI}{dt}(t) = \int F_{j_0}\partial_x u_{k_0} + F_{k_0}\partial_x u_{j_0} dx \geq 0$$

and $I(0) = \varepsilon^2 K > 0$ by the assumption (ii). On the other hand, since $u(t,x)$ vanishes when $|x| \geq t + r$ by the assumption (iv) and the finite speed of propagation, it follows from the Cauchy-Schwarz inequality and the assumption (i) that

$$\begin{aligned} I(t)^2 &\leq 4(t+r)\int (\partial_t u_{j_0})^2(\partial_x u_{k_0})^2 + (\partial_t u_{k_0})^2(\partial_x u_{j_0})^2 dx \\ &\leq \frac{4}{\delta}(t+r)\int F_{j_0}\partial_x u_{k_0} + F_{k_0}\partial_x u_{j_0} dx \\ &\leq \frac{4}{\delta}(t+r)\frac{dI}{dt}(t). \end{aligned}$$

Therefore we have

$$\|u_{j_0}(T)\|_E^2 + \|u_{k_0}(T)\|_E^2 \geq I(T) \geq \frac{4\varepsilon^2 K}{4 - \varepsilon^2 K\delta \log(T/r+1)} \longrightarrow \infty$$

as $T \to r\exp\big(4/\varepsilon^2 K\delta\big) - r$, which completes the proof. ∎

As a consequence of Theorem 2.1.1 and Proposition 5.1.1, we obtain the following example: Let us consider the Cauchy problem

$$\begin{cases} (\Box + m^2)u = (\partial_t u)^2(\partial_x v) \\ (\Box + \mu^2)v = (\partial_t v)^2(\partial_x u) \end{cases}$$

in $\mathbb{R}_+^{1+1}$ with C_0^∞ initial data of size $O(\varepsilon)$. If $(m - 3\mu)(m - \mu)(3m - \mu) \neq 0$, the system possesses a unique global classical solution which approaches a free solution as $t \to \infty$, whereas if $m = \mu$, the solution of this system blows up in finite time of order $\exp(O(\varepsilon^{-2}))$ under some restriction on the initial data. The case $m = 3\mu$ or $3m = \mu$ seems open so far.

5.2 Example (2)

The second example is the following:

$$\begin{cases} (\Box + m_0^2)u_0 = 0, \\ (\Box + m_1^2)u_1 = {u_0}^2, \\ (\Box + m_2^2)u_2 = {u_1}^2, \\ (\Box + m_3^2)u_3 = {u_3}^2 + G(u_2, \partial u_2) \end{cases} \tag{5.2.1}$$

in $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^1$ with

$$(u_j, \partial_t u_j)|_{t=0} = (\varepsilon f_j, \varepsilon g_j), \quad 0 \le j \le 3, \tag{5.2.2}$$

where f_j, $g_j \in C_0^\infty(\mathbb{R})$, $m_j > 0$. We assume that (v) $m_j = 2m_{j-1}$ for $j = 1, 2$,

(vi) $$G(\phi, \partial\phi) \ge C_0(|\phi|^2 + |\partial\phi|^2)$$

with some $C_0 > 0$, and (vii) f_0 or g_0 does not identically vanish. Then we have the following:

Proposition 5.2.1 *Under the assumptions (v)–(vii), there exist no global solutions to (5.2.1) – (5.2.2) for any* $\varepsilon > 0$.

Remark. Of course, u_0, u_1 and u_2 never blow up in finite time since they are decoupled. It is only u_3 that actually blows up in finite time. Note that the Cauchy problem for the scalar equation $(\Box + m^2)v = v^2$, which corresponds to the case where $G \equiv 0$, admits a unique global classical solution if the data is sufficiently small and smooth.

Proof of Proposition 5.2.1: What is important in the proof is the following lemma, which is a version of Lemma 2.2 of Keel and Tao [16] (the basic idea goes back to T.Kato [15]).

Lemma 5.2.1 *Let w be a smooth function of $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ which satisfy*

$$\|w(t)\|_E \ge C_0 t^\nu \tag{5.2.3}$$

with some $\nu > 1/2$, $C_0 > 0$, and

$$w(t,x) = 0 \quad \text{if} \quad |x| \ge t + r$$

with some $r > 0$. Suppose that $f, g \in C_0^\infty(\mathbb{R})$, $m \in \mathbb{R}$ and G satisfy (vi). Then there exists no global solution v of the Cauchy problem

$$\begin{cases} (\Box + m^2)v = v^2 + G(w, \partial w), \quad t > 0, \ x \in \mathbb{R}, \\ (v, \partial_t v)|_{t=0} = (f, g). \end{cases}$$

Once we obtain this lemma, what remains to do is to check that u_2 satisfy (5.2.3). This has been attained in Proposition 3.6.1. As a consequence, Proposition 5.2.1 follows.

The rest of this section is devoted to the proof of Lemma 5.2.1. First, we fix $R > r$ so that

$$f(x) = g(x) = 0 \quad \text{if} \quad |x| \geq R.$$

Then it follws from the finite speed of propagation that

$$v(t,x) = 0 \quad \text{if} \quad |x| \geq t + R.$$

Next, we put

$$J(t) = \int_{|x| \leq t+R} v(t,x)dx.$$

Then, from the equation, we have

$$\frac{d^2J}{dt^2}(t) + m^2 J(t) = \int_{|x|\leq t+R} |v(t,x)|^2 dx + \int_{|x|\leq t+R} G(w, \partial w)dx.$$

Also, the Cauchy–Schwarz inequality yields

$$J(t)^2 \leq 2(t+R) \int_{|x|\leq t+R} |v(t,x)|^2 dx,$$

so we have

$$\begin{aligned} \frac{d^2J}{dt^2}(t) &\geq -m^2 J(t) + \frac{1}{2(t+R)} J(t)^2 + \int G(w, \partial w)dx \\ &\geq -\frac{m^4}{2}(t+R) + C_0 \|w(t)\|_E^2. \end{aligned} \tag{5.2.4}$$

Integrating this estimate twice with respect to t, we obtain

$$\begin{aligned} \frac{dJ}{dt}(t) &\geq \frac{dJ}{dt}(0) - \frac{m^4}{4}\{(t+R)^2 - R^2\} + \int_0^t C_0 C_1 t^{2\nu} d\tau \\ &\geq -C_2 t^2 + \frac{C_0 C_1}{2\nu+1} t^{2\nu+1} \\ &> 0 \end{aligned}$$

and

$$J(t) \geq -C_3 t^3 + \frac{C_0 C_1}{(2\nu+1)(2\nu+2)} t^{2\nu+2} \geq \frac{C_0 C_1}{2(2\nu+1)(2\nu+2)} t^{2\nu+2}$$

for $t > T_0$, where C_2, $C_3 \geq 0$, $T_0 \gg 1$. Using (5.2.4) again, we have

$$\frac{d^2J}{dt^2}(t) \geq \frac{1}{4(t+R)} J(t)^2 + \left(\frac{1}{4(t+R)} J(t) - m^2 \right) J(t) \geq \frac{1}{4(t+R)} J(t)^2,$$

whence we obtain

$$\left(\frac{dJ(t)}{dt}\right)^2 \geq \frac{1}{6(t+R)}J(t)^3 - C_4 \geq \frac{1}{12(t+R)}J(t)^3$$

for $t \geq T_1$, where $C_4 \geq 0$, $T_1 \gg T_0$. Therefore we have

$$J(t) \geq \left\{J(T_1)^{-1/2} + \frac{1}{\sqrt{12}}(\sqrt{T_1+R} - \sqrt{t+R})\right\}^{-2}$$

for $t \geq T_1$. This implies $J(t)$ can not be defined globally in time, while it could be if $v(t,x)$ were a global solution. ∎

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Bibliography

[1] S. Alinhac, *Blowup for nonlinear hyperbolic equations,* Birkhäuser, Boston, 1995.

[2] J.-M. Delort, *Minoration du temps d'existence pour l'équation de Klein-Gordon non-linéaire en dimension 1 d'espace,* Ann. Inst. Henri Poincaré (Analyse non linéaire) **16** (1999), 563–591.

[3] J.-M. Delort, *Existence globale et comportement asymptotique pour l'équation de Klein-Gordon quasi linéaire à données petites en dimension* 1, Ann. Sci. Ec. Norm. Sup. 4ᵉ série **34** (2001), 1–61.

[4] J.-M. Delort, D. Fang, R. Xue, *Global existence of small solutions for quadratic quasilinear Klein-Gordon systems in two space dimensions,* Prépublications Mathematiques de l'Université Paris 13, 2002-29.

[5] D. Fang, R. Xue, *Global existence of small solutions for cubic quasilinear Klein-Gordon systems in one space dimension,* preprint, 2003.

[6] D. Fang, R. Xue, *Global existence and asymptotics behavior of solutions for a resonant Klein-Gordon systems in two space dimensions,* preprint, 2003.

[7] V. Georgiev, *Global solution of the system of wave and Klein-Gordon equations,* Math. Zeit. **203** (1990), 683–698.

[8] V. Georgiev, *Decay estimates for the Klein-Gordon equation,* Comm. Partial Differential Equations **17** (1992), 1111–1139.

[9] V. Georgiev, P. Popivanov, *Global solution to the two-dimensional Klein-Gordon equation,* Comm. Partial Differential Equations **16** (1991), 941–995.

[10] V. Georgiev, B. Yordanov, *Asymptotic behaviour of the one-dimensional Klein-Gordon equation with a cubic nonlinearity,* preprint, 1996.

[11] L. Hörmander, *Remarks on the Klein-Gordon equations,* Journées equations aux dérivées partielles Saint-Jean-Montes, Juin 1987, 1–9.

[12] L. Hörmander, *Lectures on nonlinear hyperbolic differential equations,* Springer Verlag, Berlin, 1997.

[13] F. John, *Blow-up of solutions for quasi-linear wave equations in three space dimensions,* Comm. Pure Appl. Math. **34** (1981), 29–51.

[14] S. Katayama, *A note on global existence of soluions to nonlinear Klein-Gordon equations in one space dimension,* J. Math. Kyoto Univ. **39** (1999), 203–213.

[15] T. Kato, *Blow-up of solutions of some nonlinear hyperbolic equations,* Comm. Pure Appl. Math. **33** (1980), 501–505.

[16] M. Keel, T. Tao, *Small data blow-up for semilinear Klein-Gordon equations,* Amer. J. Math. **121** (1999), 629–669.

[17] S. Klainerman, *Global existence for nonlinear wave equations,* Comm. Pure Appl. Math. **33** (1980), 43–101.

[18] S. Klainerman, *Global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space-time dimensions,* Comm. Pure Appl. Math. **38** (1985), 631–641.

[19] S. Klainerman, *The null condition and global existence to nonlinear wave equations in three spsce dimensions,* in "Lectures in Applied Mathematics", Vol. 23, Amer. Math. Soc., Providence, R.I., 1986, pp. 293–326.

[20] S. Klainerman, G. Ponce, *Global, small amplitude solutions to nonlinear evolution equations,* Comm. Pure Appl. Math. **36** (1983), 133–141.

[21] R. Kosecki, *The unit condition and global existence for a class of nonlinear Klein-Gordon equations,* J. Differential Equations **100** (1992), 257–268.

[22] M. Kovalyov, *Resonance-type behaviour in a system of nonlinear wave equations,* J. Differential Equations **77** (1989), 73–83.

[23] H. Kubo (ed.), *On global behavior of solutions for nonlinear hyperbolic systems,* RIMS Kōkyūroku **1331** (2003).

[24] K. Moriyama, *Normal forms and global existence of solutions to a class of cubic nonlinear Klein-Gordon equations in one space dimension,* Differential Integral Equations **10** (1997), 499–520.

[25] K. Moriyama, S. Tonegawa, Y. Tsutsumi, *Almost global existence of solutions for the quadratic semilinear Klein-Gordon equation in one space dimension,* Funkcial. Ekvac. **40** (1997), 313–333.

[26] T. Ozawa, K. Tsutaya, Y. Tsutsumi, *Global existence and asymptotic behavior of solutions for the Klein-Gordon equations with quadratic nonlinearity in two space dimensions,* Math. Zeit. **222** (1996), 341–362.

[27] J. Shatah, *Global existence of small solutions to nonlinear evolution equations,* J. Differential Equations **38** (1982), 409–425.

[28] J. Shatah, *Normal forms and quadratic nonlinear Klein-Gordon equations,* Comm. Pure Appl. Math. **38** (1985), 685–696.

[29] J.C.H. Simon, E. Taflin, *The Cauchy problem for non-linear Klien-Gordon euations,* Commun. Math. Phys. **152** (1993) 433–478.

[30] H. Sunagawa, *On global small amplitude solutions to systems of cubic nonlinear Klein-Gordon equations with different mass terms in one space dimension,* J. Differential Equations **192** (2003), 308–325.

[31] H. Sunagawa, *A note on the large time asymptotics for a system of Klein-Gordon equations,* Hokkaido Math. J. (to appear).

[32] H. Sunagawa, *Large time asymptotics of solutions to nonlinear Klein-Gordon systems,* Osaka J. Math. (to appear).

[33] H. Sunagawa, *A note on large time asymptotics for systems of Klein-Gordon equations, II: Applications to small data blow-up,* preprint, 2003.

[34] Y. Tsutsumi, *Stability of constant equilibrium for the Maxwell-Higgs equations,* Funkcial. Ekvac. **46** (2003), 41-62.

List of Author's Papers Cited in This Thesis

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