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Note on Chain Conditions in Free Groups

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§ 1. The purpose of this paper is to see what types of chain conditions hold in free groups.

In a previous paper[1] the author proved that the *maximum condition* holds on such subgroups of a free group F that are all generated by a finite number r of elements, for any prescribed natural number r. That is, if

$$H_1 \subseteqq H_2 \subseteqq \dots \quad \subseteqq H_n \subseteqq \dots$$

is a sequence of subgroups in a free group F and every H_n is generated by r elements, then the sequence is finite. This fact was proved also by H. Higman independently.[2]

Of course, in a free group the minimum condition on subgroups of the same type does not hold in general. But we can prove that a type of *restricted minimum condition* holds (Theorem 3). This can be proved as an immediate consequence of a theorem (Theorem 2) which is useful in some researches for sequences of subgroups in a free group.

The results obtained by M. Hall in his recent paper[3], on descending sequences of subgroups in a free group, are slightly generalized in ours (Theorem 4), and this generalization enables us to prove the result of F. Levi[4] too, from which the Hall's results can be derived immediately.

§ 2. Let F be a free group with a set $X=\{x_i\}$ of free generators. Any element f of F is uniquely expressible in its normal form $x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \dots x_{i_\lambda}^{\varepsilon_\lambda}$, where the x_i are elements out of X, $\varepsilon = \pm 1$, and two relations

$$x_{i_\nu} = x_{i_{\nu+1}} \quad \text{and} \quad \varepsilon_\nu + \varepsilon_{\nu+1} = 0$$

do not hold simultaneously for any $\nu = 1, 2, \dots, \lambda-1$. The number $\lambda = \lambda(f)$ is called the *length* of f. The cardinal number of X is called the *rank* of F. The rank is determined uniquely by the group F, not depending on the choice of its set of free generators.

1) M. Takahasi: Note on locally free groups, Journ. of the Inst. of Polytechnics, Osaka City Univ. 1 (1950), 65-70.

2) H. Higman: A finitely related group with an isomorphic factor group, Journ. London Math. Soc. 26 (1951), 59-61.

3) M. Hall: A Topology for free groups and related groups, Ann. of Math. 52 (1950), 127-139.

4) F. Levi: Über die Untergruppen der freien Gruppen, Math. Zeitschr. 37 (1933), 90-97.

To prove the theorem that any subgroup U of F is also a free group, F. Levi[5] constructed a set of free generators of U by a special well-ordering of the elements. His set of free generators has an important property as in the following Theorem 1. It is explicitly pointed out by H. Federer and B. Jonsson in their recent paper[6].

We can formulate those results in the following

Theorem 1. *Let U be an arbitrary subgroup of a free group F with a set X of free generators. There exists a set Y of free generators of U with the following property:*

If $u = y_{j_1}^{\delta_1} y_{j_2}^{\delta_2} \dots y_{j_\mu}^{\delta_\mu}$ is the normal form of an element u of U in terms of Y, then

$$\lambda(u) \geqq \lambda(y_{j_\nu}) \quad for \quad \nu = 1, \dots, \mu,$$

where λ denotes the length of element in terms of X.

This theorem is the starting point of our investigation in the following, but the proof of the theorem will be ommitted here, because the proof in the original paper of F. Levi gives the statement almost immediately.

§ **3.** First we prove

Theorem 2. *Let F be a free group of any (finite or infinite) rank. If H is a subgroup of finite rank in F, then there exist at most a finite number of subgroups of F, which contain H but any of whose proper free factors does not contain H.*

Proof. Take a subgroup U, if exists, which contains H but any of whose proper free factors does not contain H.

If F is of infinite rank, H is contained in a proper free factor F_1 of finite rank of F, since H is generated by a finite number of elements. Let $F = F_1 * F_2$ and $F_1 \supseteqq H$. According to the subgroup-theorem[7] in free products of groups, we see

$$U = (U \frown F_1) * U_2,$$

where $U \frown F_1 \supseteqq H \neq 1$. If $U_2 \neq 1$, the proper free facter $U \frown F_1$ of U contains H. This is contrary to the assumption. Hence $U_2 = 1$ and $U \subseteqq F_1$.

Therefore every subgroup U subject to the condition is contained

5) F. Levi: Über die Untergruppen der freien Gruppen, Math. Zeitschr. 32 (1930), 315-318.

6) H. Federer and B. Jonsson: Some properties of free groups, Trans. A.M.S. 68 (1950), 1-27.

7) M. Takahasi: Bemerkungen über den Untergruppensatz in freien Produkte, Proc. Imp. Acad. Tokyo, 20 (1945), 589-594,

in a free subgroup F_1 of finite rank, and we may restrict our attention to the case when F is of finite rank.

Now, according to theorem 1, there can be chosen a set Y of free generators of U, such that $u = y_{j_1}^{\delta_1} y_{j_2}^{\delta_2} \dots y_{j_\mu}^{\delta_\mu}$ implies $\lambda(u) \geqq \lambda(y_{j_\nu})$, $\nu = 1, \dots, \mu$, whenever the expression of u is the normal form in terms of Y.

We take a set Z of free generators of H. Since the rank of H is finite, Z is a finite set: $Z = \{z_1, \dots, z_r\}$.

If some y_j in Y does not appear in the normal form of z_k, $k=1, \dots, r$, then H is contained in a proper free factor of U, which is generated by the set $Y - y_j$. Hence every y_j in Y must occur at least in some one of the normal forms of z_k, $k = 1, \dots, r$.

Therefore we see that the length of every y_j in Y, in terms of X, is equal to or shorter than the length of some one z_k in $Z = \{z_1, \dots, z_r\}$;

$$\lambda(z_k) \geqq \lambda(y_j).$$

If we denote by m the maximum value of $\lambda(z_1), \dots, \lambda(z_r)$, every y_j in Y is of length shorter than m.

Since X is a finite set, the number of elements of length shorter than m is, of course, finite. Hence the number of subgroups U is also finite.

In a free group holds the maximum condition on subgroups of any prescribed finite rank r. That is, if

$$H_1 \subseteqq H_2 \subseteqq \dots \quad \subseteqq H_n \subseteqq \dots$$

is an ascending sequence of subgroups and if every H_n is of rank r, then the sequence is finite. Accordingly there exists a *maximal* subgroup of rank r containing any given r elements, in the sense that it can be contained in no greater subgroup of rank r.

The minimum condition does not hold in general. This is shown by the trivial example:

$$(a) \supset (a^{n_1}) \supset (a^{n_1 n_2}) \supset \dots, \quad n_1 \neq 1,\ n_2 \neq 1, \dots .$$

But we see that the *restricted minimum condition* holds as follows.

Theorem 3. *Let F be a free group of any (finite or infinite) rank, and let H_0 be a subgroup of rank r $(r \neq 0)$, which can not be contained in any subgroup of rank less than r. If*

$$H_1 \supseteqq H_2 \supseteqq \dots \quad \supseteqq H_n \supseteqq \dots$$

is a descending sequence of subgroups and if every H_n in it is of rank r and contains H_0, then the sequence is finite.

Proof. Since no subgroup of rank less than r contains H_0, any proper free factor of every H_n does not contain H_0, for every H_n is of rank r and its proper free factor is necessarily of rank less than r. Hence the validity of the statement follows immediately from Theorem 2.

§ 4. We next consider, in general, descending sequences of subgroups in a free group F. We have

Theorem 4. *Let F be any free group. If*

$$H_1 \supset H_2 \supset \dots \quad \supset H_n \supset \dots$$

is an infinite descending sequence of subgroups in F, then the intersection of all H_n is not equal to 1 if, and only if, there exists a common free factor of almost all subgroups H_n.

Proof. If $\bigcap H_n \neq 1$, we take an element $h \neq 1$ out of $\bigcap H_n$. For every H_n, we consider the free factor of H_n containing h. If H'_n, H''_n are two of such free factors of H_n, then, according to the subgroup-theorem, $H'_n \bigcap H''_n$ is also a free factor of H'_n and contains h. Hence $H'_n \bigcap H''_n$ is also a free factor of H_n containing h. Either the rank of $H'_n \bigcap H''_n$ is less than the rank of H'_n, or $H'_n \bigcap H''_n = H'_n$, that is, $H'_n \subseteqq H''_n$. Therefore the least free factor H'_n of H_n which contains h is determined uniquely.

On the other hand, if we take such free factors H'_n, H'_m of H_n, H_m, for $n > m$, respectively, $H'_n \bigcap H_m$ is a free factor of H_m containing h, according to the subgroup-theorem again, and therefore $H'_n \bigcap H_m \supseteqq H'_m$. Hence

$$H'_n \supseteqq H'_m \,.$$

Now we obtain the descending sequence of subgroups

$$H'_1 \supseteqq H'_2 \supseteqq \dots \quad \supseteqq H'_n \supseteqq \dots$$

where H'_n is the least free factor of H_n containing h.

Since h is contained in no proper free factor of every H'_n, Theorem 2 gives the finiteness of the sequence. That is, there exists an integer N such that

$$n \geqq N \quad \text{implies } H'_n = H'_N \,.$$

H'_N is a common free factor of almost all H_n.

The "*if*" part is trivial.

M. Hall obtained the following theorem in his recent paper.[8]

8) Cf. M. Hall, cited in 3).

Theorem. *Let F be a free group and let $F = V_0 \supset V_1 \supset \dots$ be a descending sequence of subgroups of F. Suppose that for any set of free generators of V_n no element of V_{n+1} has the length less than 3 in terms of the generators of V_n. Then the intersection of all V's is the identity.*

Theorem 4 is a slight generalization of this theorem, in the point that the value *three* as required by the Hall's theorem may be reduced to *two* in the corresponding formulation of our theorem 4. And this point enables us to have the following theorem of F. Levi as an immediate corollary.

Theorem 5.[9] *Let F be a free group and let $F = V_0 \supset V_1 \supset \dots$ be a descending sequence of subgroups of F. Suppose that every V_{n+1} is a characteristic subgroup of the preceding V_n. Then the intersection of all V's is the identity.*

After having the theorem of F. Levi, it is immediate to prove that *the descending sequences $\{V_n\}$ of subgroups in a free group F have the identity as the intersection of all terms V's, when*

(1) *V_{n+1} is the commutator subgroup of V_n.*

(2) *V_{n+1} is the intersection of all normal subgroups of index p in V_n, where p is a prime number (equal or not equal to 2).*

(3) *V_{n+1} is the intersection of all normal subgroups of index m_n in V_n, where m_n is an integer not equal to 1.*[10]

The statement in Case (2) gives immediately the theorem of Iwasawa[11] in a sharp form as follows.

Theorem 6. *Let F be any free group and let p be an arbitrary prime number. The intersection of all normal subgroups of indices some powers of p in F is the identity.*

And from this, as is shown by M. Hall, follows the theorem of W. Magnus[12], that *the intersection of all the higher commutator subgroups of a free group F is the identity.*

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9) Cf. F. Levi. cited in 4).

10) M. Hall considered and proved the statement under the assumption that F is of finite rank, but this assumption is unnecessary in Theorem 5, therefore we obtain the statement without any assumption on the rank of F.

11) K. Iwasawa: Einige Säze über freie Gruppen, Proc. Imp. Acad. Tokyo, 19 (1943), 272-274.

12) W. Magnus, Über Beziehungen zwischen höheren Kommutatoren, Journ. für Reine und Angew. Math. 177 (1937), 105-115.