

| | |
|--------------|---|
| Title | Carleson type conditions and weighted inequalities for harmonic functions |
| Author(s) | Choe, Boo Rim; Koo, Hyungwoon; Yi, Heungsu |
| Citation | Osaka Journal of Mathematics. 39(4) P.945-P.962 |
| Issue Date | 2002-12 |
| Text Version | publisher |
| URL | https://doi.org/10.18910/5054 |
| DOI | 10.18910/5054 |
| rights | |
| Note | |

Osaka University Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

Osaka University

CARLESON TYPE CONDITIONS AND WEIGHTED INEQUALITIES FOR HARMONIC FUNCTIONS

BOO RIM CHOE, HYUNGWOON KOO, and HEUNGSU YI

(Received March 5, 2001)

1. Introduction

Let D be the unit disc of the complex plane and dA be the normalized area measure on D . Littlewood and Paley [6] proved the following theorem.

Theorem (Littlewood-Paley). *Let $2 \leq p < \infty$. If $f \in L^p(\partial D)$ and if F is the Poisson integral of f , then*

$$\int_D |\nabla F(z)|^p (1 - |z|^2)^{p-1} dA(z) \leq C^p \int_0^{2\pi} |f(e^{i\theta})|^p \frac{d\theta}{2\pi}$$

where C is a constant independent of f and p .

In relation with this theorem, the following problem has been extensively studied (see [3], [4], [5], [6], [7], [10], [11] and references therein): Let Ω be a domain in \mathbf{R}^n . Given p, q and a differential monomial ∂^m of order m , find (locally finite) positive Borel measures $d\mu$ and $d\nu$ such that the inequality

$$(1.1) \quad \left(\int_{\Omega} |\partial^m f|^q d\mu \right)^{1/q} \leq C \left(\int_{\partial\Omega} |f|^p d\nu \right)^{1/p}$$

holds for all f harmonic on $\overline{\Omega}$. In the case where $d\nu$ is given by the Lebesgue measure on the boundary, complete characterizations have been known either on the ball or on the upper half-space. The case $2 \leq p = q < \infty$ was solved by Shirokov [10, 11] on the disc and the case $0 < p < q < \infty$ is solved by Luecking [5] on the upper half-space. All those characterizations are given in terms of Carleson type criterion. For other cases where $0 < p = q < 2$ or $0 < q < p < \infty$, characterizations are given in terms of the so-called “tent” spaces ([5]) or “balayées” conditions ([3]) on the upper half-space. In [3] Gu actually studied the case where $d\nu$ is given by an A_p -weight, but only for $m = 0$.

More recently, on the unit ball of \mathbf{C}^n with an A_p -weight given on the boundary, Kang and Koo [7] considered holomorphic functions and their ordinary, normal and

complex tangential derivatives of all orders. In this paper, we continue investigating the problem in that direction for harmonic functions on the ball. Here, we confine ourselves to the cases $2 \leq p = q < \infty$ and $1 < p < q < \infty$.

Fix an integer $n \geq 2$ and let $B = B_n$ be the unit ball of \mathbf{R}^n . In this paper we take $\Omega = B$, consider various derivatives of all orders, and characterize locally finite positive Borel measures $d\mu$ which satisfies (1.1) for all harmonic functions, in case $d\nu$ is given by an A_p -weight. To state our results, let us introduce some notations. For $\zeta \in \partial B$ and $\delta > 0$, define balls $S_\delta(\zeta)$ and their ‘‘tents’’ $\widehat{S}_\delta(\zeta)$ by

$$(1.2) \quad \begin{aligned} S_\delta(\zeta) &= \{\eta \in \partial B : |\zeta - \eta| < \delta\}, \\ \widehat{S}_\delta(\zeta) &= \{z \in B : |\zeta - z| < \delta\}. \end{aligned}$$

Also, let $\mathcal{D}f$ denote the radial derivative of f and let $T^\alpha f$ denote tangential derivatives of f (see Section 2). For an A_p -weight ω on ∂B (simply $\omega \in A_p$), we write $h^p(\omega)$ for the harmonic Hardy space with weight ω . For simplicity we let

$$\omega(S) = \int_S \omega d\sigma$$

for a Borel set $S \subset \partial B$. Here, $d\sigma$ denotes the surface area measure on ∂B .

The following is our main result. As expected, weighted inequalities are characterized by weighted Carleson type conditions of measures under consideration. Here, we use the conventional multi-index notation.

Main Theorem. *Assume $2 \leq p = q < \infty$ or $1 < p < q < \infty$. Let $\omega \in A_p$ and α be a multi-index with $|\alpha| = m \geq 1$. Then, for a locally finite positive Borel measure $d\mu$ on B , the following are equivalent.*

- (1) $\mu[\widehat{S}_\delta(\zeta)] \leq C \omega[S_\delta(\zeta)]^{q/p} \delta^{mq}$ for all $\zeta \in \partial B$ and $\delta > 0$.
- (2) $\|\mathcal{D}^m f\|_{L^q(\mu)} \leq C \|f\|_{h^p(\omega)}$ for all $f \in h^p(\omega)$.
- (3) $\sum_{|\beta|=m} \|T^\beta f\|_{L^q(\mu)} \leq C \|f\|_{h^p(\omega)}$ for all $f \in h^p(\omega)$.
- (4) $\|\partial^\alpha f\|_{L^q(\mu)} \leq C \|f\|_{h^p(\omega)}$ for all $f \in h^p(\omega)$.

As mentioned above, the case $m = 0$ (on the upper half-space) is contained in [3]. On the other hand, our results extend those of [7] concerning holomorphic functions. Proofs are divided into two cases. See Section 3 for $2 \leq p = q < \infty$ and Section 4 for $1 < p < q < \infty$. In Section 5, we prove the ‘‘little oh’’ version of our main theorem.

NOTATION. Throughout the paper we use the same letter C (often with subscripts) for various constants which may depend on given measures and some parameters such as n, p, q and m , but it will always be independent of particular functions, balls or points, etc. Also, we use the abbreviated notation $A \lesssim B$ if there exists an inessential positive constant C such that $A \leq CB$. Thus, $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

2. Preliminaries

For a given multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ with each α_j a nonnegative integer, we use notations $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\alpha! = \alpha_1! \dots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$ where ∂_j denotes the differentiation with respect to j -th variable.

For a function $f \in C^1(B)$, we let $\mathcal{D}f$ denote the radial derivative of f . More explicitly, we let

$$\mathcal{D}f(x) = \sum_{j=1}^n x_j \partial_j f(x) \quad (x \in B).$$

Note that if f is harmonic, then so is $\mathcal{D}f$.

Since there is no smooth nonvanishing tangential vector field on ∂B for $n > 2$, we define tangential derivatives by means of a family of tangential vector fields generating all the tangent vectors. We define tangential derivatives $\mathcal{T}_{ij}f$ of $f \in C^1(B)$ by

$$\mathcal{T}_{ij}f(x) = (x_i \partial_j - x_j \partial_i)f(x) \quad (x \in B)$$

for $1 \leq i < j \leq n$. As in the case of radial derivatives, tangential derivatives of harmonic functions are again harmonic. Given a nontrivial multi-index α , we abuse the notation $\mathcal{T}^\alpha = \mathcal{T}_{i_1 j_1}^{\alpha_1} \dots \mathcal{T}_{i_n j_n}^{\alpha_n}$ for any choice of i_1, \dots, i_n and j_1, \dots, j_n .

By the mean value property of harmonic functions and Cauchy's estimates, we have the following lemma. See [1, Chapter 8] for a proof. Here and in what follows, dV denotes the Lebesgue measure on \mathbf{R}^n .

Proposition 2.1. *Let $1 \leq p < \infty$ and α be a multi-index. Suppose f is harmonic on a domain Ω in \mathbf{R}^n . Then, we have*

$$|\partial^\alpha f(x)|^p \leq \frac{C^p}{d(x, \partial\Omega)^{n+p|\alpha|}} \int_{\Omega} |f|^p dV \quad (x \in \Omega)$$

where $d(x, \partial\Omega)$ denotes the distance from x to $\partial\Omega$. The constant C depends only on n and α .

Let $1 < p < \infty$ and ω be a weight function on ∂B . We say $\omega \in A_p$ if ω satisfies the A_p condition of Muckenhoupt (see [9]), that is, there exists a constant C such that

$$\omega(S) \left(\int_S \omega^{-1/(p-1)} d\sigma \right)^{p-1} < C |S|^p$$

for all $S = S_\delta(\zeta)$. Here, $|S| = \sigma(S)$. Note that A_p -weights are doubling measures by Hölder's inequality. Namely, to each $\omega \in A_p$ there corresponds a "doubling" constant C_ω such that

$$(2.1) \quad \omega[S_{2\delta}(\zeta)] \leq C_\omega \omega[S_\delta(\zeta)]$$

for any $\delta > 0$ and $\zeta \in \partial B$.

For $\omega \in A_p$, let $L^p(\omega) = L^p(\omega d\sigma)$. The weighted harmonic Hardy space $h^p(\omega)$ is then the space of all harmonic functions f on B for which $\mathcal{N}f \in L^p(\omega)$ and define $\|f\|_{h^p(\omega)} = \|\mathcal{N}f\|_{L^p(\omega)}$. Here, $\mathcal{N}f$ denotes the nontangential maximal function of f defined by

$$\mathcal{N}f(\zeta) = \sup_{x \in \Gamma(\zeta)} |f(x)|, \quad \zeta \in \partial B$$

where $\Gamma(\zeta)$ is the nontangential approach region

$$\Gamma(\zeta) = \{x \in B : |x - \zeta| < 2(1 - |x|)\}.$$

By the local Fatou theorem every $f \in h^p(\omega)$ has nontangential limit, which we again denote by f , at almost all boundary points. Note that $f \in L^p(\omega)$ for $f \in h^p(\omega)$, because $|f| \leq \mathcal{N}f$ on ∂B . It is well known that $\|f\|_{h^p(\omega)} \approx \|f\|_{L^p(\omega)}$ (see Lemma 3.1 below). Also, note that $L^p(\omega) \subset L^1(\sigma)$ for $\omega \in A_p$. Thus, for each $f \in h^p(\omega)$, the Poisson integral of its boundary function is well defined. Moreover, it is not hard to see that each $f \in h^p(\omega)$ is recovered by the Poisson integral of $f \in L^p(\omega)$.

3. The Case $2 \leq p = q < \infty$

This section is devoted to the proof of the main theorem for the case $2 \leq p = q < \infty$. The proof will be completed in the following order:

- (1) \implies (4), (1) \implies (2) + (3)
- (2) \implies (1), (3) \implies (1), (4) \implies (1).

Our proof of (1) \implies (4) depends on the weighted inequalities for the nontangential operator and the so-called area integral operator. For $x \in B$, put

$$r(x) = 1 - |x|.$$

For a function f harmonic on B , the area integral function $\mathcal{S}f$ is then defined by

$$\mathcal{S}f(\zeta) = \left(\int_{\Gamma(\zeta)} |\nabla f|^2 r^{2-n} dV \right)^{1/2}$$

for $\zeta \in \partial B$. For the operators \mathcal{S} and \mathcal{N} , the weighted inequalities below with respect to A_p -weights are well known. In fact, the first inequality below is proved on the upper half-space in [12, Theorem 2 of Chapter VI], and one may use a similar argument to obtain the same on the ball. On the other hand, since A_p -weights are precisely those ones with respect to which the standard Hardy-Littlewood maximal operator satisfies weighted inequalities, the second inequality below is a consequence of the fact

that the nontangential maximal operator is dominated by the Hardy-Littlewood maximal operator (see [8, Theorem 3] or [1, Theorem 6.23]).

Lemma 3.1. *For $1 < p < \infty$ and $\omega \in A_p$, the inequalities*

$$\|\mathcal{S}f\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}, \quad \|\mathcal{N}f\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}$$

hold for functions $f \in h^p(\omega)$.

We also need relations between various balls. For $x \in B$, let $B(x)$ be the ball centered at x with radius $r(x)/4$. Note $r(x) \approx r(y)$ for $y \in B(x)$ or $x \in B(y)$.

Lemma 3.2. *Let $x \in B$ and $y \in B(x)$. Put $y = |y|\eta$ where $\eta \in \partial B$. Then the following hold.*

- (1) $B(x) \subset \widehat{S}_{2r(y)}(\eta)$.
- (2) $y \in \Gamma(\zeta)$ for any $\zeta \in S_{r(y)}(\eta)$.

Proof. For $y \in B(x)$, we have

$$|r(x) - r(y)| \leq |x - y| < \frac{r(x)}{4}$$

and thus $r(x) < 2r(y)$. It follows that, for $z \in B(x)$,

$$|\eta - z| \leq |\eta - y| + |y - z| < r(y) + \frac{r(x)}{2} < 2r(y).$$

This shows the first part of the lemma. Next, assume $\zeta \in S_{r(y)}(\eta)$. Then

$$|\zeta - y| \leq |\zeta - \eta| + |\eta - y| = |\zeta - \eta| + (1 - |y|) < 2r(y)$$

and therefore $y \in \Gamma(\zeta)$. □

Proof of (1) \implies (4). Assume (1) holds. Let $f \in h^p(\omega)$. First, note that we have by Proposition 2.1

$$(3.1) \quad |\partial^\alpha f(x)|^p \lesssim r(x)^{-pm+p-n} \int_{B(x)} |\nabla f|^p dV \quad (x \in B).$$

Also, for any $y \in B$, we have by assumption and doubling property

$$(3.2) \quad \mu \left[\widehat{S}_{2r(y)}(\eta) \right] \lesssim \omega \left[S_{2r(y)}(\eta) \right] r(y)^{mp} \lesssim C_\omega \omega \left[S_{r(y)}(\eta) \right] r(y)^{mp}$$

where $y = |y|\eta$, $\eta \in \partial B$.

Now, integrate both sides of (3.1) against the measure $d\mu$, interchange the order of integrations using Lemma 3.2, and then apply (3.2). What we have is

$$\begin{aligned} \int_B |\partial^\alpha f(x)|^p d\mu(x) &\lesssim \int_B r(x)^{-mp+p-n} \int_{B(x)} |\nabla f(y)|^p dy d\mu(x) \\ &\lesssim \int_B |\nabla f(y)|^p r(y)^{-mp-n+p} \mu \left[\widehat{S}_{2r(y)} \left(\frac{y}{|y|} \right) \right] dy \\ &\lesssim C_\omega \int_B |\nabla f(y)|^p r(y)^{-n+p} \omega \left[S_{r(y)} \left(\frac{y}{|y|} \right) \right] dy. \end{aligned}$$

Here and elsewhere, $dy = dV(y)$. Thus, interchanging the order of integrations once more, we have

$$\begin{aligned} \int_B |\partial^\alpha f(x)|^p d\mu(x) &\lesssim \int_B |\nabla f(y)|^p r(y)^{-n+p} \int_{S_{r(y)}(y/|y|)} \omega(\zeta) d\sigma(\zeta) dy \\ (3.3) \qquad \qquad \qquad &\lesssim \int_{\partial B} \int_{\Gamma(\zeta)} |\nabla f(y)|^p r(y)^{-n+p} dy \omega(\zeta) d\sigma(\zeta) \end{aligned}$$

where the second inequality holds by Lemma 3.2. Note that

$$\begin{aligned} \int_{\Gamma(\zeta)} |\nabla f|^p r^{-n+p} dV &\leq \left(\sup_{y \in \Gamma(\zeta)} r(y) |\nabla f(y)| \right)^{p-2} \int_{\Gamma(\zeta)} |\nabla f|^2 r^{-n+2} dV \\ &\lesssim |\mathcal{N}f(\zeta)|^{p-2} |\mathcal{S}f(\zeta)|^2 \end{aligned}$$

for all $\zeta \in \partial B$. The second inequality of the above holds by Proposition 2.1. Inserting the above into (3.3) and then applying Hölder’s inequality, we finally have

$$\begin{aligned} \int_B |\partial^\alpha f|^p d\mu &\lesssim \int_{\partial B} |\mathcal{N}f|^{p-2} |\mathcal{S}f|^2 \omega d\sigma \\ &\lesssim \left(\int_{\partial B} |\mathcal{N}f|^p \omega d\sigma \right)^{1-2/p} \left(\int_{\partial B} |\mathcal{S}f|^p \omega d\sigma \right)^{2/p} \\ &\lesssim \int_{\partial B} |f|^p \omega d\sigma. \end{aligned}$$

Here, the last inequality follows from Lemma 3.1. This completes the proof. □

Proof of (1) \implies (2)+(3). Assume (1) holds. Then, for each nonnegative integer $k \leq m$, we have

$$\mu[\widehat{S}_\delta(\zeta)] \lesssim \omega[S_\delta(\zeta)] \delta^{kp}$$

for all $\zeta \in \partial B$ and $\delta > 0$. Thus, since we already have (1) \implies (4), the above yields

$$\sum_{|\alpha| \leq m} \int_B |\partial^\alpha f|^p d\mu \lesssim \int_{\partial B} |f|^p \omega d\sigma$$

for all $f \in h^p(\omega)$, which trivially implies (2) and (3). The proof is complete. \square

Before proceeding to the proofs of other implications, we first introduce some notations. Let φ be the Newtonian potential. i.e.,

$$(3.4) \quad \varphi(x) = \begin{cases} \log |x| & \text{for } n = 2 \\ |x|^{-n+2} & \text{for } n > 2. \end{cases}$$

By induction one may verify that, to each multi-index $\gamma \neq 0$, there corresponds a (harmonic) homogeneous polynomial g_γ of degree $|\gamma|$ such that

$$(3.5) \quad \partial^\gamma \varphi(x) = g_\gamma(x) |x|^{-(n+2|\gamma|-2)}.$$

Now, for $\delta > 0$ and $\zeta, \eta \in \partial B$, define $\varphi_{\delta, \zeta, \eta}$ by

$$\varphi_{\delta, \zeta, \eta}(x) = \varphi(x - \zeta - \delta\eta).$$

Note that we have by (3.5) and homogeneity of g_γ

$$(3.6) \quad |\partial^\gamma \varphi_{\delta, \zeta, \eta}(x)| \leq \frac{\|g_\gamma\|_{L^\infty(\sigma)}}{|x - \zeta - \delta\eta|^{n+|\gamma|-2}}.$$

We will use these functions as test functions in the proofs of all other remaining implications. We first prove some properties of those functions. For simplicity we write $S_\delta(\zeta) = \mathcal{S}_\delta$ and $\widehat{S}_\delta(\zeta) = \widehat{\mathcal{S}}_\delta$. Recall that C_ω is the ‘‘doubling’’ constant of $\omega \in A_p$ introduced in (2.1). In what follows $\zeta \cdot \eta$ denotes the euclidean inner product on \mathbf{R}^n .

Lemma 3.3. *Let $\omega \in A_p$ and γ be a multi-index such that $C_\omega \leq 2^{|\gamma|}$. Then, there exists a constant $C_{p, \gamma}$ such that*

$$\int_{\partial B} |\partial^\gamma \varphi_{\delta, \zeta, \eta}|^p \omega d\sigma \leq C_{p, \gamma} \frac{\omega(\mathcal{S}_\delta)}{(\delta \zeta \cdot \eta)^{p(n+|\gamma|-2)}}$$

for all $0 < \delta < 1$ and $\zeta, \eta \in \partial B$ with $\zeta \cdot \eta > 0$.

Proof. Assume $\delta < 1$ and $\zeta, \eta \in \partial B$ with $\zeta \cdot \eta > 0$. Let $N = |\gamma|$. Note $\sqrt{1 + 2\delta \zeta \cdot \eta} < |\zeta + \delta\eta| < 2$. Thus, for $\xi \in \partial B$, we have

$$|\xi - \zeta - \delta\eta| \geq |\zeta + \delta\eta| - 1 \gtrsim \delta \zeta \cdot \eta$$

and thus, by (3.6),

$$(3.7) \quad |\partial^\gamma \varphi_{\delta, \zeta, \eta}(\xi)| \lesssim \frac{\|g_\gamma\|_{L^\infty(\sigma)}}{(\delta \zeta \cdot \eta)^{n+N-2}}, \quad \xi \in \partial B.$$

For $\xi \notin S_{2^k \delta}$, $k \geq 1$, we have

$$|\xi - \zeta - \delta \eta| \geq |\xi - \zeta| - \delta \geq 2^k \delta - \delta \gtrsim 2^k \delta$$

and thus, by (3.6),

$$(3.8) \quad |\partial^\gamma \varphi_{\delta, \zeta, \eta}(\xi)| \lesssim \frac{\|g_\gamma\|_{L^\infty(\sigma)}}{(2^k \delta)^{n+N-2}}, \quad \xi \in S_{2^{k+1} \delta} \setminus S_{2^k \delta}$$

for all $k \geq 1$. Also, since $C_\omega < 2^N$, we have by doubling property

$$\omega(S_{2^{k+1} \delta}) \leq C_\omega^{k+1} \omega(S_\delta) \leq 2^{N(k+1)} \omega(S_\delta)$$

for each $k \geq 0$. Thus, it follows from (3.7) and (3.8) that

$$\begin{aligned} \int_{\partial B} |\partial^\gamma \varphi_{\delta, \zeta, \eta}|^p \omega \, d\sigma &= \int_{S_{2\delta}} |\partial^\gamma \varphi_{\delta, \zeta, \eta}|^p \omega \, d\sigma + \sum_{k=1}^\infty \int_{S_{2^{k+1} \delta} \setminus S_{2^k \delta}} |\partial^\gamma \varphi_{\delta, \zeta, \eta}|^p \omega \, d\sigma \\ &\lesssim \|g_\gamma\|_{L^\infty(\sigma)}^p \frac{2^N \omega(S_\delta)}{(\delta \zeta \cdot \eta)^{p(n+N-2)}} \sum_{k=0}^\infty 2^{-k[p(n+N-2)-N]} \\ &= C_{p, \gamma} \frac{\omega(S_\delta)}{(\delta \zeta \cdot \eta)^{p(n+N-2)}} \end{aligned}$$

as desired. The proof is complete. □

For $\xi \in \partial B$, let D_ξ denote the differentiation in the direction of ξ .

Lemma 3.4. *For each positive integer m there exists a constant $C_m \neq 0$ such that*

$$D_\xi^m \varphi(x) = \frac{C_m}{|x|^{n+m-2}} \left[1 + O \left(1 - \left| \xi \cdot \frac{x}{|x|} \right| \right) \right]$$

for all $\xi \in \partial B$ and $x \neq 0$ such that $\xi \cdot x < 0$. The constant involved in $O(1 - |\xi \cdot x / |x||)$ depends only on n and m .

Proof. Let m be a positive integer. Let $\xi \in \partial B$, $x \neq 0$ and assume $\xi \cdot x < 0$. Put $\eta = x/|x|$. A simple calculation yields

$$D_\xi (|x|^{-k}) = -k \frac{\xi \cdot x}{|x|^{k+2}}, \quad D_\xi [(\xi \cdot x)^k] = k(\xi \cdot x)^{k-1}$$

for integers $k \geq 0$. Thus, by induction, one can show that there are coefficients $c_j = c_j(m)$ such that

$$\begin{aligned} D_\xi^m \varphi(x) &= \sum_{0 \leq j \leq m/2} c_j \frac{(\xi \cdot x)^{m-2j}}{|x|^{n-2+2m-2j}} \\ &= (-1)^m |x|^{-(n+m-2)} \sum_{0 \leq j \leq m/2} c_j |\xi \cdot \eta|^{m-2j}. \end{aligned}$$

Note $\sum c_j = D_\xi^m \varphi(\xi)$ which is a nonzero (by a direct calculation) constant depending only on n and m . Thus, letting $C_m = \sum c_i \neq 0$, we have

$$\sum_{0 \leq j \leq m/2} c_j |\xi \cdot \eta|^{m-2j} = C_m [1 + O(1 - |\xi \cdot \eta|)]$$

where the constant involved in $O(1 - |\xi \cdot \eta|)$ is easily seen to depend only on n and m . The proof is complete. □

Proof of (2) \implies (1). Assume (2) holds. First, fix a large positive integer N such that $C_\omega < 2^N$. Let $\zeta \in \partial B$. For $0 < \delta < 1$, put $f_\delta = \mathcal{D}^N \varphi_\delta$ where $\varphi_\delta = \varphi_{\delta, \zeta, \zeta}$. Then f_δ is harmonic on \overline{B} . We have by assumption and Lemma 3.3

$$(3.9) \quad \int_B |\mathcal{D}^m f_\delta|^p d\mu \lesssim \int_{\partial B} |f_\delta|^p \omega d\sigma \lesssim \frac{\omega(\mathcal{S}_\delta)}{\delta^{p(n+N-2)}}.$$

Now, consider $y \in \widehat{S}_{\epsilon\delta}$ where $\epsilon < 1/2$ is a small positive number to be chosen in a moment. Note that

$$D_{y/|y|}^m f_\delta(y) = |y|^{-m} \sum_{|\gamma|=m} \frac{m!}{\gamma!} y^\gamma \partial^\gamma f_\delta(y).$$

Also, note

$$\mathcal{D}^m f_\delta(y) = \sum_{|\gamma|=m} \frac{m!}{\gamma!} y^\gamma \partial^\gamma f_\delta(y) + E_m f_\delta(y)$$

for some differential operator E_m of order $(m - 1)$ with smooth coefficients. Therefore, we have

$$(3.10) \quad \mathcal{D}^m f_\delta(y) = |y|^m D_{y/|y|}^m f_\delta(y) + E_m f_\delta(y).$$

We first estimate the first term of the right side of (3.10). Put $z = y - \zeta - \delta\zeta$ and $\xi = y/|y| \in S_{\epsilon\delta}$. Then, $\delta(1 - \epsilon) < |z| < \delta(1 + \epsilon)$ and thus $|z| \approx \delta$. Therefore we have

$$\xi \cdot z = (\xi - \zeta) \cdot (\xi - \delta\zeta) - \delta < \delta(2\epsilon - 1) < 0$$

and

$$\left| \xi \cdot \frac{z}{|z|} \right| \gtrsim 1 - 2\epsilon.$$

Note $\mathcal{D}_\xi^k \varphi_\delta(y) = D_\xi^k \varphi(y - \zeta - \delta\zeta)$ for any integer $k \geq 1$. Hence, by Lemma 3.4, we have

$$D_\xi^m f_\delta(y) = D_\xi^{N+m} \varphi(y - \zeta - \delta\zeta) = \frac{C_1}{\delta^{n+N+m-2}} [1 + O(\epsilon)].$$

Recall that $C_1 \neq 0$ is a constant depending only on n and $m + N$ and the same is true for the constant involved in $O(\epsilon)$. Next, for the second term of the right side of (3.10), it is straightforward to see from (3.6) that

$$(3.11) \quad |E_m f_\delta(y)| \lesssim \delta^{-(n+N+m-3)}.$$

Since $|y| \approx 1$, combining these estimates, we have by (3.10)

$$|\mathcal{D}^m f_\delta(y)| \approx \frac{1 + O(\epsilon) + O(\delta)}{\delta^{n+N+m-2}}$$

so that we can fix ϵ and δ_0 sufficiently small such that

$$(3.12) \quad |\mathcal{D}^m f_\delta(y)| \approx \frac{1}{\delta^{n+N+m-2}}, \quad y \in \widehat{S}_{\epsilon\delta}, \quad \delta \leq \delta_0,$$

which is a uniform estimate independent of ζ and δ . Thus, for $\delta \leq \delta_0$, we obtain from (3.12) and (3.9)

$$\frac{\mu(\widehat{S}_{\epsilon\delta})}{\delta^{p(n+N+m-2)}} \approx \int_{\widehat{S}_{\epsilon\delta}} |\mathcal{D}^m f_\delta|^p d\mu \lesssim \frac{\omega(S_\delta)}{\delta^{p(n+N-2)}}$$

so that

$$\mu(\widehat{S}_{\epsilon\delta}) \lesssim \delta^{mp} \omega(S_\delta) \lesssim \delta^{mp} \omega(S_{\epsilon\delta})$$

where the second inequality holds by doubling property. Consequently, for $\delta \leq \epsilon\delta_0$, we have

$$(3.13) \quad \mu(\widehat{S}_\delta) \lesssim \delta^{mp} \omega(S_\delta)$$

and this estimate is independent of ζ . Note that the above argument shows that $\mu(\widehat{S}_{\epsilon\delta_0}) < \infty$ for all $\zeta \in \partial B$. Since $d\mu$ is locally finite, it follows that $d\mu$ is a finite measure. Thus, for $\delta > \epsilon\delta_0$, we also have (3.13) by doubling property. This completes the proof. □

Proof of (3) \implies (1). Assume (3) holds. As above, fix a large positive integer N such that $C_\omega < 2^N$. Let $\zeta \in \partial B$ and choose $\xi \in \partial B$ such that $\zeta \cdot \xi = 0$. Let $\eta = \epsilon\zeta + \sqrt{1 - \epsilon^2}\xi$ where $\epsilon < 1/2$ is a small positive number to be chosen later. For $0 < \delta < 1$, this time we let $f_\delta = D_\xi^N \varphi_\delta$ where $\varphi_\delta = \varphi_{\delta, \zeta, \eta}$. Then f_δ is harmonic on \overline{B} . Since $\zeta \cdot \eta = \epsilon$, we have by assumption and Lemma 3.3

$$(3.14) \quad \int_B |D^m f_\delta|^p d\mu \lesssim \int_{\partial B} |f_\delta|^p \omega d\sigma \lesssim \frac{\omega(\mathcal{S}_\delta)}{(\epsilon\delta)^{p(n+N-2)}}.$$

Consider $y \in \widehat{S}_{\epsilon\delta}$ and put $z = y - \zeta - \delta\eta$. Then, $\delta(1 - \epsilon) < |z| < \delta(1 + \epsilon)$ and thus $|z| \approx \delta$. Since $\eta \cdot \xi = \sqrt{1 - \epsilon^2} > 1 - \epsilon$, we have

$$\xi \cdot z = (y - \zeta) \cdot \xi - \delta\eta \cdot \xi < \delta(2\epsilon - 1) < 0$$

and

$$\left| \xi \cdot \frac{z}{|z|} \right| \gtrsim 1 - 2\epsilon.$$

Therefore, by Lemma 3.4, we have

$$(3.15) \quad D_\xi^m f_\delta(y) = D_\xi^{N+m} \varphi(y - \zeta - \delta\eta) = \frac{C_1}{\delta^{n+N+m-2}} [1 + O(\epsilon)]$$

where $C_1 \neq 0$ is a constant depending only on n and $m + N$ and the same is true for the constant involved in $O(\epsilon)$. Now, since $\zeta \cdot \xi = 0$, we can find coefficients $c_{ij} = c_{ij}(\zeta, \xi)$ such that

$$D_\xi f_\delta(\zeta) = T f_\delta(\zeta), \quad T = \sum c_{ij} \mathcal{T}_{ij}.$$

Moreover, as in the proof of Proposition 5.2 of [2], we may find those coefficients in such a way that $\sup_{\zeta, \xi} |c_{ij}(\zeta, \xi)| < \infty$. Let

$$a_j(x, \xi) = \sum_{i < j} c_{ij} x_i - \sum_{i > j} c_{ji} x_i$$

for each j . Put $a = (a_1, \dots, a_n)$. Then, we have

$$\begin{aligned} T^m f_\delta(y) &= \left(\sum a_j(y, \xi) \partial_j \right)^m f_\delta(y) \\ &= \sum_{|\beta|=m} \frac{m!}{\beta!} a^\beta(y, \xi) \partial^\beta f_\delta(y) + E_m f_\delta(y) \\ &= D_\xi^m f_\delta(y) + \sum_{|\beta|=m} \frac{m!}{\beta!} [a^\beta(y, \xi) - \xi^\beta] \partial^\beta f_\delta(y) + E_m f_\delta(y) \end{aligned}$$

where E_m is a differential operator of order $(m - 1)$ with smooth coefficients. Note $a(\zeta, \xi) = \xi$. Thus, since c_{ij} 's are uniformly bounded, we have

$$|a(y, \xi) - \xi| \leq C_2|y - \zeta| < C_2\epsilon\delta, \quad y \in \widehat{S}_{\epsilon\delta}$$

for some constant C_2 depending only on n . Also, by (3.6) we have

$$\sum_{|\beta|=m} |\partial^\beta f_\delta(y)| \leq \frac{C_3}{\delta^{n+N+m-2}}, \quad y \in \widehat{S}_{\epsilon\delta}$$

where C_3 is a constant depending only on n and $m + N$. Thus, for $y \in \widehat{S}_{\epsilon\delta}$, we have

$$T^m f_\delta(y) = D_\xi^m f_\delta(y) + \frac{O(\epsilon\delta)}{\delta^{n+N+m-2}} + E_m f_\delta(y).$$

Therefore, by (3.15) and (3.11), we can fix $\epsilon > 0$ and $\delta_0 > 0$ such that

$$(3.16) \quad |T^m f_\delta(y)| \approx \frac{1}{\delta^{n+N+m-2}}, \quad y \in \widehat{S}_{\epsilon\delta}, \quad \delta \leq \delta_0.$$

Now, for the rest of the proof, one may proceed as in the proof of (2) \implies (1) by using (3.14) and (3.16). The proof is complete. \square

Proof of (4) \implies (1). Assume (4) holds. By compactness of ∂B , it suffices to give local estimates. So, fix $\eta \in \partial B$ and assume $\zeta \in \partial B, \zeta \cdot \eta > 1/2$. Since φ is harmonic and not a polynomial, we can choose $\beta = \beta(\eta)$ such that $\partial^\alpha \partial^\beta \varphi(-\eta) \neq 0$ and $C_\omega < 2^{|\beta|}$. Let $N = |\beta|$. For $0 < \delta < 1$, put $f_\delta = \partial^\beta \varphi_\delta$ where $\varphi_\delta = \varphi_{\delta, \zeta, \eta}$ $f_\delta = \partial^\beta \varphi_{\delta, \zeta, \eta}$. Then f_δ is harmonic on \overline{B} . Since $\zeta \cdot \eta > 1/2$, we have by assumption and Lemma 3.3

$$(3.17) \quad \int_B |\partial^\gamma f_\delta|^p d\mu \lesssim \int_{\partial B} |f_\delta|^p \omega d\sigma \leq C_1 \frac{\omega(S_\delta)}{\delta^{p(n+N-2)}}$$

where $C_1 = C_1(p, \eta)$ is a constant independent of ζ and δ .

Consider $y \in \widehat{S}_{\epsilon\delta}$ where $\epsilon = \epsilon(\eta) < 1/2$ is a small positive constant to be chosen in a moment. Put $z = (y - \zeta)/\delta$ and $\gamma = \alpha + \beta$. Then, by (3.5), we have

$$(3.18) \quad \partial^\alpha f_\delta(y) = \partial^\gamma \varphi(\delta z - \delta\eta) = \delta^{-(n+N+m-2)} \frac{g_\gamma(z - \eta)}{|z - \eta|^{n+2N+2m-2}}.$$

Note $g_\gamma(z - \eta) = g_\gamma(-\eta) + O(|z|)$ and $|z| < \epsilon$. Also, $g_\gamma(-\eta) \neq 0$, because $\partial^\gamma \varphi(-\eta) \neq 0$. Thus,

$$g_\gamma(z - \eta) = g_\gamma(-\eta)[1 + O(\epsilon)], \quad y \in \widehat{S}_{\epsilon\delta}.$$

Here, the constant involved in $O(\epsilon)$ is independent of ζ and δ . Note $|z - \eta| \approx 1$. Thus, by (3.18), we may fix ϵ sufficiently small such that

$$(3.19) \quad |\partial^\alpha f_\delta(y)| \approx \frac{1}{\delta^{n+N+m-2}} \quad y \in \widehat{S}_{\epsilon\delta}, \quad \delta < 1$$

and this estimate is independent of ζ and δ .

Now, using (3.17), (3.19) and imitating the argument of (2) \implies (1), we obtain

$$\mu(\widehat{S}_\delta) \leq C_\eta \delta^{mp} \omega(S_\delta), \quad y \in \widehat{S}_\delta, \quad \delta > 0$$

which is an estimate independent of ζ and δ . The proof is complete. \square

4. The case $1 < p < q < \infty$

In this section we give a proof of the main theorem for the case $1 < p < q < \infty$. Except for the implication (1) \implies (4), the arguments of other implications of the previous section are easily modified and thus details are left to the readers. For the implication (1) \implies (4), we make use of the idea of [7].

Lemma 4.1. *For $a > 1$ and $\omega \in A_p$, let $d\tau_a$ be a measure on B defined by*

$$d\tau_a(y) = r(y)^{-n} \omega \left[S_{r(y)} \left(\frac{y}{|y|} \right) \right]^a dy.$$

Then, we have

$$(4.1) \quad \tau_a[\widehat{S}_\delta(\zeta)] \leq C \omega[S_\delta(\zeta)]^a$$

for all $\zeta \in \partial B$ and $\delta > 0$.

The analogue of the above lemma is proved in [7, Lemma 2.3] on the unit ball of the complex n -space. Their idea is to use reverse Hölder's inequality for A_p -weights. More precisely, to each $\omega \in A_p$ there corresponds a constant $b > 1$ such that

$$(4.2) \quad \left(\frac{1}{|S|} \int_S \omega^a d\sigma \right)^{1/a} \lesssim \frac{1}{|S|} \int_S \omega d\sigma$$

holds for all $S = S_\delta(\zeta)$ and $0 < a \leq b$.

Proof. Let $S_\delta(\zeta)$ be given. First, assume $1 < a \leq b$ where b is chosen so that (4.2) holds for ω . Then, by integrating in polar coordinates, we have

$$(4.3) \quad \tau_a[\widehat{S}_\delta(\zeta)] \lesssim \int_0^\delta t^{-n} \int_{S_\delta(\zeta)} \omega[S_t(\eta)]^a d\sigma(\eta) dt.$$

Note that, for $\eta \in \mathcal{S}_\delta(\zeta)$ and $0 < t < \delta$, we have $\mathcal{S}_t(\eta) \subset \mathcal{S}_{2\delta}(\zeta)$. Thus, letting $|\mathcal{S}_t| = |\mathcal{S}_t(\xi)|$ for any $\xi \in \partial B$, we have by Hölder’s inequality, reverse Hölder’s inequality and doubling property

$$\begin{aligned} \int_{\mathcal{S}_\delta(\zeta)} \omega[\mathcal{S}_t(\eta)]^a d\sigma(\eta) &\leq |\mathcal{S}_t|^{a-1} \int_{\mathcal{S}_{2\delta}(\zeta)} \int_{\mathcal{S}_t(\eta)} \omega(\xi)^a d\sigma(\xi) d\sigma(\eta) \\ &\leq |\mathcal{S}_t|^a \int_{\mathcal{S}_{2\delta}(\zeta)} \omega^a d\sigma \\ &\lesssim |\mathcal{S}_t|^a |\mathcal{S}_{2\delta}|^{1-a} \omega[\mathcal{S}_{2\delta}(\zeta)]^a \\ &\lesssim |\mathcal{S}_t|^a |\mathcal{S}_\delta|^{1-a} \omega[\mathcal{S}_\delta(\zeta)]^a. \end{aligned}$$

Now, inserting this estimate into (4.3), we obtain (4.1).

Next, assume $a > b$. Note that we have by doubling property

$$\frac{\omega[\mathcal{S}_{r(y)}(y/|y|)]}{\omega[\mathcal{S}_\delta(\zeta)]} \leq \frac{\omega[\mathcal{S}_{2\delta}(\zeta)]}{\omega[\mathcal{S}_\delta(\zeta)]} \lesssim 1, \quad y \in \widehat{\mathcal{S}}_\delta(\zeta)$$

so that

$$\begin{aligned} \omega[\mathcal{S}_\delta(\zeta)]^{-a} \tau_a[\widehat{\mathcal{S}}_\delta(\zeta)] &= \int_{\widehat{\mathcal{S}}_\delta(\zeta)} \left(\frac{\omega[\mathcal{S}_{r(y)}(y/|y|)]}{\omega[\mathcal{S}_\delta(\zeta)]} \right)^a r(y)^{-n} dy \\ &\lesssim \omega[\mathcal{S}_\delta(\zeta)]^{-b} \tau_b[\widehat{\mathcal{S}}_\delta(\zeta)]. \end{aligned}$$

Thus, (4.1) follows from the previous case. The proof is complete. □

The following is a special case of [3, Theorem 5.2]. In fact, Gu [3] worked on the half-space and a straightforward modification gives the same on the ball.

Lemma 4.2. *Let $1 < p \leq q < \infty$. Assume $\omega \in A_p$ and $d\tau$ is a locally finite positive Borel measure on B . Then,*

$$\tau[\widehat{\mathcal{S}}_\delta(\zeta)] \leq C \omega[\mathcal{S}_\delta(\zeta)]^{q/p} \quad \text{for all } \zeta \in \partial B \text{ and } \delta > 0$$

if and only if

$$\|f\|_{L^q(\tau)} \leq C \|f\|_{L^p(\omega)} \quad \text{for all } f \in h^p(\omega).$$

Now, we give the proof of (1) \implies (4). Assume (1) holds. Let $f \in h^p(\omega)$. One may proceed as in the case $p = q \geq 2$ to obtain the following estimates:

$$\int_B |\partial^\alpha f(x)|^q d\mu(x) \lesssim \int_B r(x)^{-mq-n} \int_{B(x)} |f(y)|^q dy d\mu(x)$$

$$\begin{aligned} &\lesssim \int_B |f(y)|^q r(y)^{-mq-n} \mu \left[\widehat{\mathcal{S}}_{2r(y)} \left(\frac{y}{|y|} \right) \right] dy \\ &\lesssim \int_B |f(y)|^q d\tau(y) \end{aligned}$$

where $d\tau = d\tau_{q/p}$ is the measure defined as in Lemma 4.1. Therefore, we conclude (4) by Lemma 4.1 and Lemma 4.2.

5. Compactness

Recall that a linear operator Λ from a Banach space into another is called compact if Λ maps bounded sets onto relatively compact sets. In the following we let $\widehat{\mu}_\delta = \widehat{\mu}_{\delta,m,p,q}$ denote the function defined by

$$\widehat{\mu}_\delta(\zeta) = \frac{\mu[\widehat{\mathcal{S}}_\delta(\zeta)]}{\omega[\mathcal{S}_\delta(\zeta)]^{q/p} \delta^{mp}}, \quad \zeta \in \partial B, \quad \delta > 0$$

and let

$$\mathcal{T}^m = \sum_{|\beta|=m} |\mathcal{T}^\beta|.$$

Theorem 5.1. *Assume $2 \leq p = q < \infty$ or $1 < p < q < \infty$. Let $\omega \in A_p$ and α be a multi-index of order $m \geq 1$. Then, for a locally finite positive Borel measure $d\mu$ on B , the following are equivalent.*

- (1) $\widehat{\mu}_\delta(\zeta) = o(1)$ uniformly in $\zeta \in \partial B$ as $\delta \rightarrow 0$.
- (2) $\mathcal{D}^m: h^p(\omega) \rightarrow L^q(\mu)$ is compact.
- (3) $\mathcal{T}^m: h^p(\omega) \rightarrow L^q(\mu)$ is compact.
- (4) $\partial^\alpha: h^p(\omega) \rightarrow L^q(\mu)$ is compact.

Proof of (1) \implies (4). Assume (1) holds. For $0 < t < 1$, let B_t be the ball centered at the origin with radius t and \mathcal{X}_t be the characteristic function of B_t . Define $\Lambda_t: h^p(\omega) \rightarrow L^q(\mu)$ by $\Lambda_t f = \mathcal{X}_t \partial^\alpha f$. First, we show that each Λ_t is compact. Let U be a given bounded set in $h^p(\omega)$ and fix $f \in U$. Then, for each $x \in B$, we have

$$f(x) = \int_{\partial B} P(x, \zeta) f(\zeta) d\sigma(\zeta)$$

where P is the Poisson kernel for B . Hence, by Hölder's inequality, we have

$$\begin{aligned} |f(x)| &\leq \int_{\partial B} P(x, \zeta) |f(\zeta)| d\sigma(\zeta) \\ &\leq \frac{1}{(1-|x|)^{n-1}} \int_{\partial B} |f| \omega^{-1} \cdot \omega d\sigma \end{aligned}$$

$$\lesssim \frac{\|f\|_{h^p(\omega)} \|\omega^{-1}\|_{L^{p'}(\omega)}}{(1 - |x|)^{n-1}}$$

where p' is conjugate index of p . Note $\omega^{-1} \in L^{p'}(\omega)$, because $\omega \in A_p$. Thus U is locally uniformly bounded and thus is a normal family. Therefore, there is a sequence $\{f_j\}$ in U which converges uniformly on every compact subset of B . Let $f = \lim f_j$. Now, since $\partial^\alpha f_j \rightarrow \partial^\alpha f$ uniformly on B_t , we have $\Lambda_t f_j \rightarrow \mathcal{X}_t \partial^\alpha f$ in $L^q(\mu)$. Hence, $\Lambda_t : h^p(\omega) \rightarrow L^q(\mu)$ is compact.

For the case $p < q$, one may follow the argument (using Lemma 4.1) in the previous section to obtain

$$\int_{B \setminus B_t} |\partial^\alpha f|^q d\mu \lesssim \left(\sup_{\substack{\delta \lesssim 1-t \\ \zeta \in \partial B}} \widehat{\mu}_\delta(\zeta) \right) \left(\int_{\partial B} |f|^p d\sigma \right)^{q/p}$$

for functions $f \in h^p(\omega)$. Also, one may follow the arguments of (3.1), (3.2) and (3.3) to see the same for the case $p = q \geq 2$. Hence, in either case, we have

$$\|\Lambda_t - \partial^\alpha\| \lesssim \left(\sup_{\substack{\delta \lesssim 1-t \\ \zeta \in \partial B}} \widehat{\mu}_\delta(\zeta) \right)^{1/q} \rightarrow 0 \quad \text{as } t \rightarrow 1$$

so that ∂^α is compact, as desired. This completes the proof. □

Now, the implication (1) \implies (2) + (3) easily follows from (1) \implies (4).

Proof of (2) \implies (1). Assume (2) holds. Let $\zeta \in \partial B$. We continue using the notations defined in the proof of (2) \implies (1) of Section 3. For $\delta > 0$, let

$$h_\delta = \delta^{n+N-2} \omega(S_\delta)^{-1/p} f_\delta.$$

Note $\|h_\delta\|_{h^p(\omega)} \lesssim 1$ by Lemma 3.3. First, we show that

$$(5.1) \quad \int_B |\mathcal{D}^m h_\delta|^q d\mu \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

Suppose not. Then there exists a sequence $\delta_j \rightarrow 0$ such that

$$\inf_j \int_B |\mathcal{D}^m h_{\delta_j}|^q d\mu > 0$$

Since h_δ 's are bounded in $h^p(\omega)$, by using the compactness of \mathcal{D}^m , we may assume $\mathcal{D}^m h_{\delta_j} \rightarrow h$ in $L^q(\mu)$ for some $h \in L^q(\mu)$. Note $\|h\|_{L^q(\mu)} > 0$. On the other hand,

since f_δ 's are locally uniformly bounded by (3.6) and

$$\omega(\partial B) \leq 2^{N(\log_2 \delta^{-1+2})} \omega(S_\delta) = 2^{2N} \delta^{-N} \omega(S_\delta)$$

by doubling property, we see that h_{δ_j} converges to 0 uniformly on every compact subset of B , and so is $\mathcal{D}^m h_{\delta_j}$. It follows that $h = 0$ in $L^q(\mu)$, which is a contradiction. Thus, (5.1) holds.

Now, we have by (3.12)

$$|\mathcal{D}^m h_\delta(y)|^q \approx \delta^{-mq} \omega(S_\delta)^{-q/p}, \quad y \in \widehat{S}_{\epsilon\delta}, \quad \delta \leq \delta_0$$

and thus

$$\mu(\widehat{S}_{\epsilon\delta}) \lesssim \delta^{mq} \omega(S_\delta)^{q/p} \int_B |\mathcal{D}^m h_\delta|^q d\mu$$

for all δ sufficiently small. Thus, we have by doubling property

$$\widehat{\mu}_{\epsilon\delta}(\zeta) \lesssim \left(\frac{\omega(S_\delta)}{\omega(S_{\epsilon\delta})} \right)^{q/p} \int_B |\mathcal{D}^m h_\delta|^q d\mu \lesssim \int_B |\mathcal{D}^m h_\delta|^q d\mu$$

and this is an estimate independent of ζ and δ small. Thus, we conclude (1) by (5.1). The proof is complete. \square

Proofs of the implications (3) \implies (1) and (4) \implies (1) are also easy modifications of corresponding ones in the previous section.

References

- [1] S. Axler, P. Bourdon and W. Ramey: Harmonic function theory, Springer-Verlag, New York, 1992.
- [2] B.R. Choe, H. Koo and H. Yi: *Derivatives of harmonic Bergman and Bloch functions on the ball*, J. Math. Analysis and Applications, **260** (2001), 100–123.
- [3] D. Gu: *Two-weighted norm inequality and Carleson measure in weighted Hardy spaces*, Canad. J. Math. **44** (1992), 1206–1219.
- [4] D. Girela, M. Lorenta and M. Sarrion: *Embedding derivatives of weighted Hardy spaces into Lebesgue spaces*, Math. Proc. Camb. Phil. Soc. **116** (1994), 151–166.
- [5] D. Luecking: *Embedding derivatives of Hardy spaces into Lebesgue spaces*, Proc. London. Math. Soc. **63** (1991), 595–619.
- [6] J. Littlewood and R. Paley: *Theorems on Fourier series and power series*. II, Proc. London. Math. Soc. **42** (1936), 52–89.
- [7] H. Kang and H. Koo: *Two-weighted inequalities for the derivatives of holomorphic functions and Carleson measures on the unit ball*, Nagoya Math. J. **158** (2000), 107–131.
- [8] E.M. Stein: *Boundary Behavior of Holomorphic Functions of Several Complex variables*, Princeton University Press, Princeton, NJ, 1972.
- [9] E.M. Stein: *Harmonic analysis*, Princeton University Press, Princeton, NJ, 1993.

- [10] N.A. Shirokov: *Some generalizations of the Littlewood-Paley theorem*, Zap. Nauch. Sem. LOMI **39** (1974), 162–175; J. Soviet Mat. **8** (1977), 119–129.
- [11] N.A. Shirokov: *Some embedding theorems for spaces of harmonic functions*, Zap. Nauch. Sem. LOMI **56** (1976), 191–194; J. Soviet Mat. **14** (1980), 119–129.
- [12] J.-O. Strömberg and A. Torchinsky: *Weighted Hardy Spaces*, Lecture Notes in Math. 1381, Springer-Verlag, 1989.

Boo Rim Choe
Department of Mathematics
Korea University
Seoul 136-701, Korea
e-mail: choebr@math.korea.ac.kr

Hyungwoon Koo
Department of Mathematics
Korea University
Seoul 136-701, Korea
e-mail: koohw@math.korea.ac.kr

HeungSu Yi
Department of Mathematics
Research Institute of Basic Sciences
Kwangwoon University
Seoul 139-701, Korea
e-mail: hsyi@math.kwangwoon.ac.kr