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OBSTRUCTION THEORY FOR FINITE GROUP ACTIONS

Dedicated to Professor D. Montgomery on His 70th Birthday

HSU-TUNG KU AND MEI-CHIN KU

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1. Introduction. Let G be a finite group, F and P be fixed finite G-complexes. In this paper we shall define a general obstruction theory for extending a 1-connected cellular G-map f: $F\to P$ to a cellular G-map ϕ: $X\to P$ which is a homology or homotopy equivalence, where X is a finite G-complex.

Define a G-resolution of f: $F\to P$ to be an n-connected cellular G-map ϕ: $X\to P$, $n=\dim X\geq\dim P$, which extends $f(n\geq 2)$ so that G acts freely outside F, F a G-subcomplex of X and $H_{n+1}(\phi)$ is a projective $Z(G)$-module. The *obstruction* $\gamma_G(f, \phi)$ of a G-resolution ϕ of f is defined as

$$\gamma_G(f, \phi)=(-1)^{n+1}[H_{n+1}(\phi)]\in\tilde{K}_0(Z(G)),$$

where $[H_{n+1}(\phi)]$ denotes the class of $H_{n+1}(\phi)$ in the projective class group $\tilde{K}_0(Z(G))$. Let c: $pt\to pt$ be the constant map and define

$$B(G)=\{\gamma_G(c, \phi)\colon \phi \text{ is a } G\text{-resolution of } c\}.$$

Then $B(G)$ can be proved to be a subgroup of $\tilde{K}_0(Z(G))$. Assume the map f: $F\to P$ satisfies the following extension property:

(EP): *Let ϕ_i: $X_i\to P$ be any two G-resolutions of f, $i=1, 2$. Then*

$$\phi_1\cup\phi_2\colon X_1\underset{F}{\cup}X_2\to P$$

extends to a G-resolution of f.

We shall show that for any such ϕ_i, $i=1, 2$,

$$\gamma_G(f, \phi_1)-\gamma_G(f, \phi_2)\in B(G),$$

hence if we let $[\gamma_G(f, \phi)]$ to be the equivalence class of $\gamma_G(f, \phi)$ for any G-resolution ϕ of f, then we can define the *obstruction* of f by

$$\gamma_G(f)=[\gamma_G(f, \phi)]\in\tilde{K}_0(Z(G))/B(G).$$

We will verify that the invariant $\gamma_G(f)$ is exactly the obstruction to extending

f to a homology or homotopy equivalence of G-map if the fixed point set F^G is not empty.

We shall also see that the extension property (EP) holds for many G-maps. Moreover we have the following equality

$$\gamma_G(f_2 f_1) = \gamma_G(f_1) + \gamma_G(f_2) .$$

Now let us consider the following:

Converse to the Smith fixed point set problem. Suppose the finite group G acts on a finite Poincré complex P with $P^G = \bigcup_i P_i$ (P_i is a component of P^G), and $F = \bigcup_i F_i$ a finite complex with $F_i \underset{L}{\sim} P_i$ (i.e., $H^*(F_i; L) \approx H^*(P_i: L)$) for all i. Can we find a finite complex X such that G acts on X with $X^G = F$ and $X \underset{L}{\sim} P$?

We are able to apply the obstruction theory to study this general problem. In particular we will prove the following results.

Theorem 1.1. *Let $f: F \to P^G \subset P$ be a cellular map such that G acts semifreely on P and $f_*: H_*(F; Z) \approx H_*(P^G; Z)$. Suppose $f: F \to P$ is 1-connected. Then there is a finite complex X which is homology equivalent to P, and G acts semifreely on X with $X^G = F$. Moreover, if F is simply connected, X is homotopy equivalent to P.*

Theorem 1.2 [4]. *Let $G = Z_q$ (q is not necessarily prime) and F a finite Z_q-acyclic complex. Then there is a finite contractible G-complex X such that G acts semifreely on X with $X^G = F$.*

Theorem 1.3. *Let G be a finite group of order q with periodic cohomology of period n, and let $d = (q, \phi(q))$, where ϕ is the Euler ϕ-function* (i.e., *$\phi(q)$ is the number of positive integers $< q$ that are prime to q*). *Suppose F is a simply connected r-dimensional integral homology r-sphere. Then there exists a finite G-complex X which is homotopy equivalent to S^{r+dn}. Moreover, G acts semifreely on X with $X^G = F$.*

In Swan [8], he proved that G can act freely on some finite complex X which is homotopy equivalent to S^{dn-1}, where G, d and n are as in 1.3.

We also can define another obstruction theory by using the $\tilde{G}$-resolution $\tilde{\phi}$ which is defined exactly as a G-resolution of f with $X^G = F^G$ except that G is not necessarily acting freely on $X - F$. Set

$$\tilde{B}(G) = \{\tilde{\phi} \colon \tilde{\phi} \text{ is a } G\text{-resolution of } c \colon pt \to pt\}$$

which is again a subgroup of $\tilde{K}_0(Z(G))$. Thus if f satisfies (EP) for $\tilde{G}$-resolutions, $\tilde{\gamma}_G(f) = [\tilde{\gamma}_G(f, \tilde{\phi})] \in \tilde{K}_0(Z(G))/\tilde{B}(G)$ is well defined so that the obstruction theory can be established as in γ_G-theory. The $\tilde{\gamma}_G(f)$ was defined by Oliver in [6] for the case $\chi(F)=1 \bmod m(G)$, G acts trivially on F and P a point.

2. Preliminaries. Let X and W be finite G complexes, G finite, and $f\colon X\to W$ be an equivariant cellular map. There is an exact sequence of the map f:

$$\cdots\to \pi_{n+1}(X)\to \pi_{n+1}(W)\to \pi_{n+1}(f)\to \pi_n(X)\to\cdots$$

The element of $\pi_{n+1}(f)$ is represented by a pair (β, α) of cellular maps so that the diagram below is commutative:

$$\begin{array}{ccc} S^n & \xrightarrow{\alpha} & X \\ j\downarrow & & \downarrow f \\ D^{n+1} & \xrightarrow{\beta} & W \end{array}$$

Define $\bar{\alpha}\colon G\times S^n\to X$ by $\bar{\alpha}(g, y)=g\alpha(y)$ for $(g, y)\in G\times S^n$, and let

$$Y = X\underset{\bar{\alpha}}{\cup} G\times D^{n+1} = X\underset{\bar{\alpha}}{\cup}\Big(\bigcup_{g\in G} g\times D^{n+1}\Big) = X\cup\Big(\bigcup_{g\alpha, g\in G} D^{n+1}\Big).$$

Then there is a naturally induced action of G on Y defined by

$$h(y) = \begin{cases} hx & \text{if } y = x\in X \\ (hg, x) & \text{if } y=(g, x)\in G\times D^{n+1}, \end{cases}$$

where $h\in G$. Thus Y is a G-complex which is obtained from X be adding free orbits of $(n+1)$-cells. Define an equivariant map $\bar{f}\colon Y\to W$ by

$$\bar{f}(y) = \begin{cases} f(x) & \text{if } y = x\in X \\ g\beta(z) & \text{if } y = (g, z)\in G\times D^{n+1}. \end{cases}$$

By construction, $\bar{f}$ is an equivariant map, and

$$\pi_i(\bar{f}) = \pi_i(f) \qquad \text{for } i\leq n$$
$$\pi_{n+1}(\bar{f}) = \pi_{n+1}(f)/K,$$

where K is the normal subgroup containing the $Z(G)$-submodule generated by the class (β, α) in $\pi_{n+1}(f)$.

Lemma 2.1. *Let $f\colon X\to Y$ be an $(n-1)$-connected equivariant map with $H_n(f)=N\oplus M$, where M and N are $Z(G)$-modules, $n\geq 2$. Then we can add n-cells of free orbits to kill off M to produce an $(n-1)$-connected equivariant map $\hat{f}\colon \hat{X}\to Y$ such that*

(1) $H_i(\hat{f})=H_i(f)$, $i\geq n+2$,

(2) $0 \to H_{n+1}(f) \to H_{n+1}(\hat{f}) \to \operatorname{Ker} \partial_{n+1} \to 0$ *and*
$0 \to \operatorname{Ker} \partial_{n+1} \to H_n(\hat{X}, X) \to M \to 0$ *exact.*
(3) $H_n(\hat{f}) = N$.
Moreover if M *is* $Z(G)$*-projective* (*or* $\operatorname{Ker} \partial_{n+1}$ *is* $Z(G)$*-projective*) *then*
(4) $H_{n+1}(\hat{f}) = H_{n+1}(f) \oplus \operatorname{Ker} \partial_{n+1}$.
Furthermore if M *is a free* $Z(G)$*-module,* ∂_{n+1}: $H_n(\hat{X}, X) \approx M$ *and*
(5) $H_i(\hat{f}) = H_i(f)$, $i \geq n+1$.

Proof. Since f is $(n-1)$-connected, $\pi_n(f) \to H_n(f)$ is surjective by the Hurewicz Theorem. Hence there exist pairs of cellular maps (β_i, α_i), α_i: $S^{n-1} \to X$, $1 \leq i \leq v$ such that $\sum_{i=1}^{v} (\beta_i, \alpha_i)_* \{H_n(D^n, S^{n-1})\}$ generates M as $Z(G)$-module. Let

$$\hat{X} = X \underset{\cup \bar{\alpha}_i}{\cup} \{\underset{i}{\cup} G \times D_i^n\} .$$

By the construction above f extends to an equivariant map $\hat{f}$: $\hat{X} \to Y$. There is an exact sequence

$$0 \to H_{n+1}(f) \to H_{n+1}(\hat{f}) \to H_n(\hat{X}, X) \xrightarrow{\partial_{n+1}} H_n(f) \xrightarrow{j_*} H_n(\hat{f}) \to 0 .$$

This can be easily obtained by looking at the algebraic mapping cones of f_*: $C_*(X) \to C_*(Y)$ and $\hat{f}_*$: $C_*(\hat{X}) \to C_*(Y)$. Now $\operatorname{Ker} j_* = M$ by construction, hence $H_n(\hat{f}) \approx H_n(f)/M \approx N$, and

$$0 \to H_{n+1}(f) \to H_{n+1}(\hat{f}) \to H_n(\hat{X}, X) \xrightarrow{\partial_{n+1}} M \to 0$$

is exact. The results follow easily.

Lemma 2.2. *Let* G *be a finite group of order* q, *and* f: $M \to P \oplus T$ *be a surjective map of* $Z(G)$*-modules. Suppose* M *is* $Z(G)$*-free,* P $Z(G)$*-projective and* T *torsion prime to* q. *Then* $\operatorname{Ker} f$ *is* $Z(G)$*-projective.*

Proof. Let L be any Sylow subgroup of G. Then both M and P are $Z(L)$-projective. Since we have exact sequence of $Z(L)$-modules

$$0 \to \operatorname{Ker} f \to M \to P \oplus T \to 0 ,$$

we obtain an exact sequence

$$\hat{H}^i(L\colon M) = 0 \to \hat{H}^i(L\colon T) \approx \hat{H}^i(L\colon P \oplus T) \to \hat{H}^{i+1}(L\colon \operatorname{Ker} f) \to \hat{H}^{i+1}(L\colon M) = 0 ,$$

where $\hat{H}^i(L\colon M) = \hat{H}^{i+1}(L\colon M) = 0$ and $\hat{H}^i(L\colon P \oplus T) \approx \hat{H}^i(L\colon T)$ because M and P are both $Z(L)$-projective. Since T is torsion prime to $|L|$, the map $|L|$: $T \to T$ which is a multiplication by $|L|$ is an isomorphism. Thus it induces an isomorphism $|L|: \hat{H}^i(L; T) \to \hat{H}^i(L; T)$. But $|L|\hat{H}^i(L; T)=0$, hence $\hat{H}^i(L; T)=0$, and so

$$\hat{H}^{i+1}(L; \operatorname{Ker} f) = 0 ,$$

for any Sylow subgroup L of G and any i. It follows from [7] that Ker f is cohomologically trivial. Therefore Ker f is $Z(G)$-projective by [7]. The proof extends the idea due to Oliver (cf. [3]).

Lemma 2.3. *Let $G=Z_q$, $(n, q)=1$, M and N be free $Z(G)$-modules and $P_1: N \oplus Z_n \to Z_n$ the projection. Suppose $h: M \to N \oplus Z_n$ is an epimorphism such that $p_1 h: M \to Z_n$ is an augmentation map. Then* Ker *h is stably $Z(G)$-free.*

Proof. Let $p: N \oplus Z_n \to N$ be the projection. Since N is $Z(G)$-free, there is a $Z(G)$-homomorphism $\phi: N \to M$ such that $(ph)\phi=1$. Thus $M=K\oplus\phi(N)$ as $Z(G)$-modules and $h|\phi(N): \phi(N) \approx N$, where $K=\operatorname{Ker}(ph)$. It follows that h induces an epimorphism $\bar{h}: K \to Z_n$ with Ker $\bar{h}$=Ker h. Note that we have exact sequence

$$0 \to \operatorname{Ker} h \oplus \phi(N) \to M = K \oplus \phi(N) \xrightarrow{p_1 h = (\bar{h},\, 0)} Z_n \to 0 .$$

Hence Ker $h\oplus\phi(N)$ is $Z(G)$-free by applying [4, Lemma 1.1] inductively on the rank of M. As $\phi(N)$ is $Z(G)$-free, Ker h is stably $Z(G)$-free.

3. An obstruction theory for finite group actions

Lemma 3.1. *Let $\psi: Y \to P$ be an equivariant map and $n=\dim Y \geq P$ such that $\psi|F=f$ and ψ is 1-connected. Assume that $H_i(\psi)$ is $Z(G)$-projective for $i \geq 2$. Then ψ can be embedded in a G-resolution $\phi: X \to P$ of f such that*

$$\gamma_G(f, \phi) = \textstyle\sum_{i=2}^{n+1} (-1)^i [H_i(\psi)] .$$

Proof. Let k be the smallest integer such that $H_{k+1}(\psi) \neq 0$. If $k=n$, $\phi=\psi$ is a G-resolution of f.

Suppose $k<n$. Since the Hurewitz homomorphism $h: \pi_{k+1}(\psi) \to H_{k+1}(\psi)$ is an epimorphism by the Hurewitz theorem, we can add free orbits of $(k+1)$-cells to kill off $H_{k+1}(\psi)$. This creates an equivariant map $f_k: X_k \to P$. According to 2.1 we have

$$\begin{aligned}
&\tilde{H}_i(f_k) = 0 , \qquad i \leq k+1 . \\
&H_i(f_k) = H_i(\psi) , \qquad i \geq k+3 . \\
&H_{k+2}(f_k) = H_{k+2}(\psi) \oplus M , \qquad M = \operatorname{Ker} \partial_{k+2} , \text{ and} \\
&H_{k+1}(X_k, Y) = M \oplus H_{k+1}(\psi) ,
\end{aligned}$$

where $H_{k+1}(X_k, Y)$ is $Z(G)$-free. Thus $[M] = -[H_{k+1}(\psi)] \in \tilde{K}_0[(Z(G)$. It follows that

$$\sum_{i=k+2}^{n+1}(-1)^i[H_i(f_k)] = \sum_{i=2}^{n+1}(-1)^i[H_i(\psi)]\,.$$

By repeating this process, eventually we will produce a G-resolutiont ϕ of f with

$$\gamma_G(f,\phi) = (-1)^{n+1}[H_{n+1}(\phi)] = \sum_{i=2}^{n+1}(-1)^i[(H_i(\psi)]\,.$$

Lemma 3.2. *Let $B(G)=\{\gamma_G(c,\phi)\colon \phi$ is a G-resolution of $c\}$, where $c\colon pt\to pt$ is the constant map. Then $B(G)$ is a subgroup of $\tilde{K}_0(Z(G))$.*

Proof. Let $\gamma_G(c,\phi_i)\in B(G)$, where $\phi_i\colon X_i\to pt$ are two G-resolutions of c, and $\dim X_i=n_i$, $i=1, 2$. Then

$$H_{j+1}(\phi_1\vee\phi_2) = H_{j+1}(\phi_1)\oplus H_{j+1}(\phi_2)\,, \qquad \text{for all } j,$$

where $\phi_1\vee\phi_2\colon X_1\underset{pt}{\vee}X_2\to pt$. Thus $\tilde{H}_*(\phi_1\vee\phi_2)$ is $Z(G)$-projective. It follows from this and 3.1 that there is a G-resolution ϕ of c such that

$$\begin{aligned}\gamma_G(c,\phi) &= (-1)^{n_1+1}[H_{n_1+1}(\phi_1\vee\phi_2)]+(-1)^{n_2+1}[H_{n_2+1}(\phi_1\vee\phi_2)]\\ &= (-1)^{n_1+1}[H_{n_1+1}(\phi_1)]+(-1)^{n_2+1}[H_{n_2+1}(\phi_2)]\,,\end{aligned}$$

that is, $\gamma_G(c,\phi_1)+\gamma_G(c,\phi_2)\in B(G)$.

Suppose now that $\gamma_G(c,\phi)\in B(G)$, $\phi\colon X\to pt$ is a G-resolution of c ($\dim X=n$). Then $\tilde{\phi}\colon \Sigma X\to pt$ is also a G-resolution of c, where ΣX denotes the reduced suspension of X. This implies that

$$-\gamma_G(c,\phi) = (-1)^{n+2}[H_{n+1}(\phi)] = (-1)^{n+2}[H_{n+2}(\tilde{\phi})] = \gamma_G(c,\tilde{\phi})\in B(G)\,.$$

Proposition 3.3. *Suppose the map $f\colon F\to P$ satisfies* (EP). *Then for any two G-resolutions ϕ_1 and ϕ_2, $\gamma_G(f,\phi_1)-\gamma_G(f,\phi_2)\in B(G)$.*

Proof. By hypothesis (EP), $\phi_1\cup\phi_2\colon X_1\underset{F}{\cup}X_2\to P$ extends to a G-resolution $\phi\colon X\to P$ of f. Let $\dim X=m$ $\dim X_i=n_i$, $i=1, 2$. We can assume that $m\geq n_i+2$ and $n_i>\dim P$, $i=1, 2$.

By assumptions, $\phi_*\colon H_j(X)\approx H_j(P)$, $j<m$; and $\phi_{i*}\colon H_j(X_i)\approx H_j(P)$ for $j<n_i$, $i=1, 2$, hence from the homology exact sequences of the pairs (X, X_i) we have $H_m(X)\approx H_m(X/X_i)$, $H_{n_i+1}(X/X_i)\approx H_{n_i}(X_i)$ and $\tilde{H}_j(X/X_i)=0$ otherwise. For instance if $j<n_j$,

$$\begin{array}{ccccccccc} H_j(X_i) & \xrightarrow{\approx} & H_j(X) & \to & H_j(X/X_i) & \to & H_{j-1}(X_i) & \xrightarrow{\approx} & H_{j-1}(X)\\ & \approx\searrow & \swarrow\approx & & & & & \approx\searrow & \swarrow\approx \\ & H_j(P) & & & & & & H_{j-1}(P) & \end{array}$$

From the homology exact sequences of ϕ and ϕ_i we obtain $H_{m+1}(\phi)\approx H_m(X)$ and $H_{n_i+1}(\phi_i)\approx H_{n_i}(X_i)$.

Now G acts on X/X_i with $(X/X_i)^G=pt$. Apply 3.1 to the constant maps

$f_i: X/X_i \to pt$, since $H_{m+1}(f_i) = H_m(X/X_i) = H_{m+1}(\phi)$ and $H_{n_i+2}(f_i) = H_{n_i+1}((X/X_i) = H_{n_i+1}(\phi_i)$ are $Z(G)$-projective and $\tilde{H}_j(f_i) = 0$ otherwise, there exist G-resolutions $\psi_i: Y_i \to pt$ of c such that

$$\begin{aligned}\gamma_G(c, \psi_i) &= (-1)^{n_i+2}[H_{n_i+2}(f_i)] + (-1)^{m+1}[H_{m+1}(f_i)] \\ &= -(-1)^{n_i+1}[H_{n_i+1}(\phi_i)] + (-1)^{m+1}[H_{m+1}(\phi)] \\ &= -\gamma_G(f, \phi_i) + \gamma_G(f, \phi), \qquad i = 1, 2\,.\end{aligned}$$

Clearly this implies that $\gamma_G(f, \phi_1) - \gamma_G(f, \phi_2) \in B(G)$.

Theorem 3.4. *If $\gamma_G(f)=0$ and $F^G \neq \phi$, then G acts on some finite complex $\hat{X}$ and there is an equivariant map $\hat{\phi}: \hat{X} \to P$ which is a homology equivalence, $\hat{\phi}\,|\,F = f$ and G acting freely on $\hat{X} - F$. If F is simply connected, then $\hat{\phi}$ is a homotopy equivalence.*

Proof. Let $\gamma_G(f) = [\gamma_G(f, \phi)] = 0$, where $\phi: X \to P$ is G-resolution of f. Then $\gamma_G(f, \phi) \in B(G)$. Hence there is a G-resolution $\psi: Y \to pt$ of $c: pt \to pt$ such that $\gamma_G(f, \phi) = -\gamma_G(c, \psi)$. Consider

$$\phi \vee \psi: X \vee Y \to P \vee pt = P$$

where the $pt \in Y$ is joined to any fixed point of X. Thus we have

$$\tilde{H}_*(\phi \vee \psi) = \tilde{H}_*(\phi) \oplus \tilde{H}_*(\psi)$$

which is $Z(G)$-projective. By 3.1, $\phi \vee \psi$ can be embedded in a G-resolution $\tilde{\phi}: \tilde{X} \to P$ of f, and

$$\gamma_G(f, \tilde{\phi}) = \sum_i (-1)^i [H_i(\phi \vee \psi)] = \gamma_G(f, \phi) + \gamma_G(c, \psi) = 0\,.$$

Thus $\gamma_G(f, \tilde{\phi})$ is stably $Z(G)$-free. It follows from 2.1 that $\tilde{\phi}$ can be extended to an equivariant map $\hat{\phi}: \hat{X} \to P$ which is a homology equivalence.

Proposition 3.5. *Suppose G acts semifreely on both F and P and $\gamma_G(f)=0$. Then*

$$f_*^G: H_*(F^G; Z_p) \xrightarrow{\approx} H_*(P^G; Z_p)\,,$$

for every prime p such that $p \,|\, |G|$, where $f^G: F^G \to P^G$ is the restriction of f. Equivalently, $\tilde{H}_*(f^G)$ is torsion prime to $|G|$, $|G|$ = order of G.

Proof. Let $\phi: X \to P$ be an equivariant map which is a homology equivalence and $\phi\,|\,F = f$. For every prime p, $p \,|\, |G|$, there is a subgroup Z_p of G of order p and

$$X^{Z_p} = F^{Z_p} = F^G,\ P^{Z_p} = P^G\,.$$

Since the group Z_p acts semifreely on the mapping cone C_ϕ which is Z_p-acyclic with $C_\phi^{Z_p}=C_fG$, hence C_fG is Z_p-acyclic by the Smith fixed point theorem, that is, $\tilde{H}_*(f^G; Z_p)=0$, or

$$f_*^G: H_*(F^G; Z_p)\approx H_*(P^G; Z_p) \quad \text{for every } p\,|\,|G|.$$

4. Existence of G-resolutions. The obstruction theory which we have defined depends on the existence of G-resolutions satisfying (EP). In this section we shall investigate the existence of the equivariant maps $f: F\to P$ satisfying (EP).

Lemma 4.1. *Let $|G|=q$ and $\phi: X\to P$ be an n-connected equivariant cellular map, $n\geq 2$. Suppose $\tilde{H}_*(\phi^H)$ is torsion prime to q (resp. $\tilde{H}_*(\phi^H)=0$) for every $\phi^H=\phi|X^H: X^H\to P^H$, where $e\neq H\subset G$, $\dim X^H<n=\dim X$ and $\dim P\leq\dim X$. Then $H_{n+1}(\phi)$ is $Z(G)$-projective (resp. stably $Z(G)$-free).*

Proof. Let $\hat{X}=U\{X^H: e\neq H\subset G\}$ and $\hat{P}=\cup\{P^H: e\neq H\subset G\}$. By the Mayer-Vietoris sequence and induction, it is easy to verify that $\tilde{H}_*(\hat{\phi})$ is torsion prime to q, where $\hat{\phi}=\phi|\hat{X}: \hat{X}\to\hat{P}$. Since ϕ is n-connected, it induces an equivariant map $\bar{\phi}: X/\hat{X}\to P/\hat{P}$ with $H_i(\bar{\phi})\approx\tilde{H}_{i-1}(\hat{\phi})$ for $i\leq n$. Hence $\tilde{H}_i(\bar{\phi})$ is torsion prime to q for every $i\leq n$. We may assume that both $X/\hat{X}$ and $P/\hat{P}$ are simply connected. Otherwise we simply consider the suspension map $\Sigma\bar{\phi}: \Sigma(X/\hat{X})\to\Sigma(P/\hat{P})$. Add free orbits to $X/\hat{X}$ inductively to kill off $\tilde{H}_i(\bar{\phi})$, $i\leq n$. By 2.1. this creates equivariant i-connected cellular maps $\phi_i: X_i\to P/\hat{P}$, $2\leq i\leq n$, with

$$(1)\qquad \begin{cases} H_{i+1}(\phi_i) = H_{i+1}(\phi_{i-1})\oplus\operatorname{Ker}\partial_{i+1} \\ H_j(\phi_i) = H_j(\phi_{i-1}), \qquad j\geq i+2, \end{cases}$$

where Ker ∂_{i+1} is $Z(G)$-projective for every i by 2.2.

Now G acts semifreely on both X_n and $P/\hat{P}$ with exactly one fixed point, say x_0 and p_0 respectively. Let $h=\phi_n|x_0$. Then $C_*=C_*(C_{\phi_n}, C_h)$ is a free $Z(G)$-module with $H_i(C_*)=\tilde{H}_i(\phi_n)$ for all $i\geq 0$. As ϕ_n is n-connected, there is an exact sequence

$$(2)\qquad 0\to H_{n+1}(\phi_n)\to C_{n+1}\to\cdots\to C_0\to 0\,.$$

This implies that if we let $N=C_n\oplus C_{n-2}\oplus\cdots$, then $H_{n+1}(\phi_n)\oplus N$ is $Z(G)$-free. Let $M=\operatorname{Ker}\partial_{n+1}$. According to (1)

$$H_{n+1}(\phi_n) = H_{n+1}(\phi_{n-1})\oplus M = \cdots = H_{n+1}(\bar{\phi})\oplus M\,.$$

Since the sequence

$$(3)\qquad 0\to H_{n+1}(\phi)\to H_{n+1}(\bar{\phi})\to H_n(\hat{\phi})\to 0$$

is exact, we obtain an exact sequence

$$0 \to H_{n+1}(\phi)\oplus M\oplus N \to H_{n+1}(\bar{\phi})\oplus M\oplus N = H_{n+1}(\phi)\oplus N \to H_n(\hat{\phi}) \to 0\,.$$

But $H_n(\hat{\phi})$ is torsion prime to q, and $H_{n+1}(\phi_n)\oplus N$ is $Z(G)$-free, hence $H_{n+1}(\phi)\oplus M\oplus N$ is $Z(G)$-projective by 2.2. It follows that $H_{n+1}(\phi)$ is $Z(G)$-projective.

Suppose now that $\tilde{H}_*(\phi^H)=0$ for every $H \neq e$. Then $\tilde{H}_i(\bar{\phi})=0$ for $i\leq n$. Hence $\bar{\phi}$ is n-connected and (2) holds for $\bar{\phi}$. Thus $H_{n+1}(\bar{\phi})$ is stably $Z(G)$-free. But $H_n(\hat{\phi})=0$ and so $H_{n+1}(\phi)=H_{n+1}(\bar{\phi})$ by (3). This proves that $H_{n+1}(\phi)$ is stably $Z(G)$-free.

Theorem 4.2 (Existence theorem I). *Let $f\colon F\to P$ be an 1-connected equivariant cellular map.*

(1) *Suppose $\tilde{H}_*(f^H)$ is torsion prime to $q=|G|$ for every $f^H=f\,|\,F^H\colon F^H\to P^H$, where $e\neq H\subset G$. Then $\gamma_G(f)$ is well defined, i.e., f satisfies* (EP).

(2) If $\tilde{H}_*(f^H)=0$ for every f^H, $e\neq H\subset G$, then $\gamma_G(f)=0$.

Proof. Add free orbits of cells of dimensions $\leq n$ inductively to get an n-connected cellular map $\phi\colon X\to P$ with $n=\dim X\geq \dim P$. Now $\phi^H=f^H$ for every $e\neq H\subset G$ by construction, hence by 4.1 $H_{n+1}(\phi)$ is $Z(G)$-projective, i.e., ϕ is a G-resolution of f.

If $\phi_i\colon X_i\to P$ are two G-resolutions of f, $i=1, 2$, add free orbits of cells to $X_1\cup_F X_2$ extending $\phi_1\cup\phi_2\colon X_1\cup_F X_2\to P$ to an n-connected equivariant cellular map $\phi\colon X\to P$. Again ϕ is a G-resolution of f which extends both ϕ_1 and ϕ_2, i.e., f satisfies (EP). Hence $\gamma_G(f)$ is well defined.

If $\tilde{H}_*(f^H)=0$ for every $G\supset H\neq e$, then $H_{n+1}(\phi)$ is stably $Z(G)$-free. Hence $\gamma_G(f)=0$.

Theorem 4.3 (Existence theorem II). *Suppose f is 1-connected and $H_i(f)=M_i\oplus T_i$ for all $i\geq 2$, where M_i's are $Z(G)$-projective and T_i's torsion prime to $q=|G|$. Then the G-resolutions of f always exist and f satisfying* (EP). *Moreover if $H_i(f)$ is $Z(G)$-free for every $i\geq 2$, then $\gamma_G(f)=0$.*

Proof. We shall construct inductively s-connected equivariant cellular maps $f_s\colon X_s\to P$ satisfying the following:

(1) $H_{s+i}(f_s)=H_{s+i}(f)\oplus$ a $Z(G)$-projective module, $i=1, 2$.

(2) $H_i(f_s)=H_i(f)$, $i\geq s+3$.

Assume that f_s has been constructed. By 2.1. add free orbits of $(s+1)$-cells to kill off $H_{s+1}(f_s)$ to obtain a $(s+1)$-connected cellular map $g_{s+1}\colon Y_s\to P$ such that

(3) $H_i(g_{s+1})=H_i(f_s)$, hence $H_i(g_{s+1})=H_i(f)$, $i\geq s+3$ by (2).

(4) $0\to H_{s+2}(f_s)\to H_{s+2}(g_{s+1})\to \operatorname{Ker}\partial_{s+2}\to 0$ and

$$0\to \operatorname{Ker}\partial_{s+2}\to H_{s+1}(Y_s, X_s)\xrightarrow{\partial_{s+2}} H_{s+1}(f_s)\to 0 \text{ exact,}$$

where Ker ∂_{s+2} is $Z(G)$-projective 2.2. Thus

$$H_{s+2}(g_{s+1}) = H_{s+2}(f_s)\oplus\text{Ker }\partial_{s+2}\,.$$

Again by 2.1. add $(s+2)$-cells to Y_s to kill off $\ker\partial_{s+2}$ to produce a $(s+1)$-connected equivariant cellular map f_{s+1}: $X_{s+1}\to P$ such that

(5) $H_{s+2}(f_{s+1})=H_{s+2}(f_s)$, hence
$H_{s+2}(f_{s+1})=H_{s+2}(f)\oplus a$ $Z(G)$-projective module by (1).

(6) $H_i(f_{s+1})=H_i(g_{s+1})=H_i(f)$, $i\geq s+4$ (by (3))

(7) $0\to H_{s+3}(g_{s+1})\to H_{s+3}(f_{s+1})\to\text{Ker }\tilde\partial_{s+2}\to 0$ and

$$0\to\text{Ker }\tilde\partial_{s+2}\to H_{s+2}(X_{s+1},\ Y_s)\xrightarrow{\tilde\partial_{s+2}}\text{Ker }\partial_{s+2}\to 0$$

exact, where Ker $\tilde\partial_{s+2}$ is $Z(G)$-projective. Thus

$$\begin{aligned} H_{s+3}(f_{s+1}) &= H_{s+3}(g_{s+1})\oplus\text{Ker }\tilde\partial_{s+2} \\ &= H_{s+3}(f)\oplus\text{Ker }\tilde\partial_{s+2} \qquad \text{(by (3))}\,. \end{aligned}$$

This completes the inductive construction.

Now if $s>\max\{\dim F,\ \dim P\}$, then $H_i(f)=0$ for $i\geq s+1$ and $\dim X_s=s+1$. It follows from (1) and (2) that we have an s-connected equivariant cellular map f_s: $X_s\to P$ with $H_i(f_s)$ $Z(G)$-projective for $i=s+1$ and $s+2$. By 2.1, we can add $(s+1)$-cells to X to kill off $H_{s+1}(f_s)$ to obtain a G-resolution ϕ: $X\to P$ of f with

$$H_{s+2}(\phi) = H_{s+2}(f_s)\oplus\text{Ker }\partial_{s+2}$$

which is $Z(G)$-projective.

Next, let ϕ_i: $X_i\to P$ be any two G-resolutions of f, $i=1, 2$. We may assume that $\dim X_i=n_i>\max\{\dim F,\ \dim P\}$. Clearly we have an exact sequence

$$\cdots\to H_{i+1}(f)\to H_{i+1}(\phi_1)\oplus H_{i+1}(\phi_2)\to H_{i+1}(\phi_1\cup\phi_2)\to H_i(f)\to\cdots\,.$$

But $H_{i+1}(\phi_j)=0$ for $i\neq n_j$, $j=1,2$. Thus

$$H_{i+1}(\phi_1\cup\phi_2)\approx H_i(f) \qquad \text{for } i\neq n_1,\ n_2\,,$$

$$H_{i+1}(\phi_1\cup\phi_2)=\begin{cases} H_{n_1+1}(\phi_1)\oplus H_{n_2+1}(\phi_2)\,, & \text{if } i=n_1=n_2 \\ H_{n_1+1}(\phi_1) & \text{if } i=n_1,\ n_1\neq n_2 \\ H_{n_2+1}(\phi_2) & \text{if } i=n_2,\ n_1\neq n_2\,. \end{cases}$$

Hence $\phi_1\cup\phi_2$ can be embedded in a G-resolution of f. This proves that f satisfies (EP).

If $H_i(f)$ is $Z(G)$-free for every $i\geq 2$, it is not difficult to see from the proof that $H_{s+2}(\phi)$ is stably $Z(G)$-free, hence $\gamma_G(f)=0$.

Proposition 4.4. (1) *If $\gamma_G(f)$ is well defined for f: $F\to P$, then $\gamma_G(\Sigma f)=-\gamma_G(f)$.*

(2) *Suppose that* $f_1\colon F\to P_1$ *and* $f_2\colon P_1\to P_2$ *both satisfy* 4.2 $\gamma_G(1)$ *or* 4.3 *so that* $\gamma_G(f_i)$ *are well defined,* $i=1, 2$. *Then*

$$\gamma_G(f_2f_1)=\gamma_G(f_1)+\gamma_G(f_2)\,.$$

Proof. (1) Obvious.

(2) Let $\phi_1\colon X_1\to P_1$ be a G-resolution of f_1 with $n_1=\dim X_1>\max\{\dim F, \dim P_1, \dim P_2\}$. We can see that

$$\begin{aligned}&H_{i+1}(f_2\phi_1)=H_{i+1}(f_2)\,, \qquad i<n_1\,,\\ &H_{n_1+1}(f_2\phi_1)=H_{n_1}(X_1)=H_{n_1+1}(\phi),\ \text{a } Z(G)\text{-projective module},\\ &H_{i+1}(f_2\phi_1)=0\,, \qquad i\geq n_1+1\,.\end{aligned}$$

For example, if $i<n_1$, we have

$$\cdots\to H_{i+1}(f_2\phi_1)=H_{i+1}(f_2)\to H_i(X_1)\xrightarrow{(f_2\phi_1)_*} H_i(P_2)\to\cdots$$
$$\phi_{1*}\searrow\approx \quad \nearrow f_{2*}$$
$$H_i(P_1)$$

Now let $\psi_2\colon M_2\to H_2(f_2\phi_1)=H_2(f_2)$ be a surjective map with M_2 a $Z(G)$-free module. Adding cells to both X_1 and P_1 to kill off both $H_2(f_2\phi_1)$ and $H_2(f_2)$ realizing ψ_2. This creates 2-connected equivariant cellular maps $g_2\colon Y_2\to P_2$ and $h_2\colon W_2\to P_2$ extending $f_2\phi_1$ and f_2 respectively. By 2.1 we can see that $H_3(g_2)=H_3(h_2)$. Continuing this construction, eventually we will get a G-resolution $\phi_2=h_{n_2}\colon W_{n_2}\to P_2$ of f_2, $\dim W_{n_2}=n_2$ and an n_2-connected equivariant map $g_{n_2}\colon Y_{n_2}\to P_1$ extending $f_2\phi_1$ such that $H_{n_2+1}(h_{n_2})=H_{n_2+1}(g_{n_2})$ and $H_i(g_{n_2})=H_i(f_2\phi_1)$ for $i\geq n_2+2$. We may assume that $n_1>n_2$. By construction, it is easy to see that

$$H_{i+1}(g_{n_2})=\begin{cases} H_{n_j+1}(\phi_j)\,, & \text{if } i=n_j,\ j=1, 2 \text{ (hence } Z(G)\text{-projective)}\\ 0\,, & \text{otherwise.}\end{cases}$$

By 3.1 g_{n_2} can be embedded in a G-resolution ϕ of $f_2\phi_1$ which is also a G-resolution of f_2f_1 such that

$$\begin{aligned}\gamma_G(f_2f_1, \phi)&=\gamma_G(f_2\phi_1, \phi)=(-1)^{n_1+1}[H_{n_1+1}(\phi_1)]+(-1)^{n_2+1}[H_{n_2+1}(\phi_2)]\\ &=\gamma_G(f_1, \phi_1)+\gamma_G(f_2, \phi_2)\,.\end{aligned}$$

Hence $\gamma_G(f_2f_1)=\gamma_G(f_1)+\gamma_G(f_2)$ as required.

5. Converse to the Smith fixed point theorem.

The converse to the Smith fixed point theorem 1.1 is an immediate consequence of the following.

Theorem 5.1. *Let* $f\colon F\to P$ *be an equivariant* 1*-connected cellular map* *such that G acts semifreely on both F and P.*

(1) *Suppose $\tilde{H}_*(f^G; Z_p)=0$ for all p, where $p \mid |G|$ and p is prime. Then $\gamma_G(f)$ is well defined,* i.e., *f satisfies (EP).*

(2) If $\tilde{H}_*(f^G)=0$, then $\gamma_G(f)=0$.

Proof. Since G acts semifreely on F, P and X, $f^G=\phi^G$ for all $e \neq H \subset G$. Hence the result follows from 4.2.

Corollary 5.2. (1) *Let F be a Z-acyclic finite complex and G a finite group. Then there is a finite contractible complex X such that $X^G=F$ and G acts semifreely on X.*

(2) *Let F be an integral r-homology sphere $(r>0)$ and G acts semifreely on S^k with fixed point set $S^r (r\geq 2)$. Suppose there is a cellular map $f: F\to S^r$ such that f_*: $H_*(F)\approx H_*(S^r)$. Then there is a finite G-complex X which is homology equivalent to S^k, and G acts semifreely on X with $X^G=F$. Moreover X is homotopy equivalent to S^k if F is simply connected.*

Theorem 5.3. *Let $G=Z_q$ and $f: F\to P$ be a 1-connected equivariant cellular map. Suppose that $H_i(f)=M_i\oplus T_i$ for all $i\geq 2$, where M_i's are $Z(G)$-free and T_i's torsion prime to q. Moreover assume that*

$$(*)\qquad T_i = \text{torsion submodule of } H_i(f^G) \text{ for all } i\geq 2\,.$$

Then $\gamma_G(f)=0$.

Proof. According to the proof of 4.3, we have the following

$$(1)\qquad \begin{aligned} H_{s+i}(f_s) &= H_{s+i}(f)\oplus \operatorname{Ker} \partial_{s+i} \\ &= T_{s+i}\oplus M_{s+i}\oplus \operatorname{Ker} \partial_{s+i}\,, \qquad i=1, 2\,. \end{aligned}$$

By using the notation in the proof of 4.3, the following composition of maps is an augmentation map by (*)

$$H_{s+1}(Y_s, X_s) \xrightarrow{\partial_{(s+2)}} H_{s+1}(f_s) \xrightarrow{\text{proj.}} T_{s+1}\,.$$

Thus $\operatorname{Ker} \partial_{s+2}$ is a stably free $Z(G)$-module by 2.3. Hence (1) becomes (1)' $H_{s+i}(f_s)=T_{s+i}\oplus$a stably free $Z(G)$-module, $i=1, 2$. If follows that $H_i(f_s)$ is stably $Z(G)$-free for $i=s+1$ and $s+2$ if $s>\max\{\dim F, \dim P\}$. Therefore $\gamma_G(f)=0$.

Now Theorem 1.2 is a simple corollary of the following.

Theorem 5.4. *Let $G=Z_q$ and F be a finite G-complex such that $\tilde{H}_*(F; Z_q)=0$ and $i_*: H_*(F^G)\xrightarrow{\approx} H_*(F)$, where $i: F^G\to F$ is an inclusion. Then $\gamma_G(f)=0$, where $f: F\to pt$ is the constant map. Thus there exists a contractible finite G-complex X which contains F as a G-subcomplex and acts freely outside F.*

Proof. First, add free orbits of 2-cells to F to get a simply connected G-complex Y. According to 2.3 and 2.1 we have $Y^G=F^G$ and

$$H_2(Y)=H_2(F)\oplus M,\ M \text{ a free } Z(G)\text{-module},$$
$$H_i=H_i(F) \qquad \text{for } i\geq 3.$$

Let ψ: $Y\to pt$ be the constant map. Then $H_{i+1}(\psi^G)=H_{i+1}(f^G)=H_i(F^G)\approx H_i(F)$ and

$$H_i(\psi)=\tilde{H}_{i-1}(Y)=0, \qquad i=1,2.$$
$$H_3(\psi)=H_2(F)\oplus M,$$
$$H_i(\psi)=H_{i-1}(F), \qquad i\geq 4,$$

where $H_i(F)$ are torsion prime to q for $i\geq 2$. Thus the conclusion follows from 5.3 and 3.4.

6. Semifree actions of groups with periodic cohomology on homology spheres

In [8], Swan has proved that if a finite group G acts freely on a compact integral cohomology n-sphere, then G has periodic cohomology with period $n+1$. This result can be generalized for semifree actions. More precisely, we have

Theorem 6.1. *Let G be a finite group acting semifreely on a locally compact space X with $\dim_z X<\infty$ and $X\underset{z}{\sim}S^n$. Suppose $F\underset{z}{\sim}S_r$ $(n-r\geq 1)$. Then G has periodic cohomology with period $n-r$.* (Here we use the Alexander Spanier cohomology with compact supports).

Proof. The cohomology exact sequence of the pair (X, F), $F=X^G$, gives

$$H^n(X-F;Z)=H^{r+1}(X-F;Z)=Z \text{ and}$$
$$H^i(X-F;Z)=0 \qquad \text{for } i\neq n,\ r+1.$$

From the spectral sequence of the fibration $(X-F)\to(X-F)_G\overset{\pi_2}{\to}B_G$ (cf [1]), we have the following Gysin type exact sequence (cf. [2])

$$\cdots\to H^i((X-F)_G)\to H^{i-n}(B_G, H^n(X-F))\to H^{i-r}(B_G; H^{r+1}(X-F))$$
$$\to H^{i+1}((X-F)_G\to\cdots$$

The map π_1: $(X-F)_G\to(X-F)/G$ induces isomorphism

$$\pi_1^*\colon H^i((X-F)/G;Z)\approx H^i((X-F)_G, Z)$$

for $i>0$ by the Vietoris-Begle mapning theorem. But $H^i((X-F)/G;Z)=0$ $i>n$ by [1]. It follows that

$$H^{i-n}(B_G; Z) \approx H^{i-r}(B_G; Z) \qquad \text{for } i>n .$$

Now we shall establish the converse of this result, i.e. Theorem 1.3 which is a special case of the following:

Theorem 6.2. *Let G be a finite group of order q with periodic cohomology of period n, $d=(q, \phi(q))$, and $f\colon F \to P$ be an equivariant cellular map with F and P both simply connected. Assume*

$$H_{r+1}(f) = H_k(f) = Z , \quad k = r+dn \quad \text{and}$$
$$\tilde{H}_i(f) = 0 , \qquad i \neq r+1, k .$$

Then $\gamma_G(f)$ is well defined and $\gamma_G(f)=0$.

Proof. According to Swan [9], there is a periodic free resolution over Z of period dn, i.e., an exact sequence

$$(*) \qquad 0 \to Z \to F_k \xrightarrow{\partial} F_{k-1} \xrightarrow{\partial} \cdots \to F_{r+1} \xrightarrow{\psi} Z = H_{r+1}(f) \to 0 ,$$

with all F_is $Z(G)$-free.

Now add free orbits of $(r+1)$-cells to kill off $H_{r+1}(f)$ realizing ψ. This creates an $(r+1)$-connected equivariant cellular map $f_{r+1}\colon X_{r+1} \to P$ such that

$$0 \to H_{r+2}(f_{r+1}) \to H_{r+1}(X, F) = F_{r+1} \xrightarrow{\psi} Z \to 0$$

is exact. Hence $\operatorname{Im}\{\partial\colon F_{r+2} \to F_{r+1}\} = \operatorname{Ker} \psi = H_{r+2}(f_{r+1})$. Again, adding free orbits of cells to kill off $H_{r+2}(f_{r+1})$ and realizing $\partial\colon F_{r+2} \to H_{r+2}(f_{r+1})$. This produces an $(r+2)$-connected equivariant cellular map $f_{r+2}\colon X_{r+2} \to P$ such that $\operatorname{Im}\{\partial\colon F_{r+3} \to F_{r+2}\} = \operatorname{Ker}\{\partial\colon F_{r+2} \to H_{r+2}(f_{r+1})\} = H_{r+3}(f_{r+2})$. Repeating this procedure eventually we will get an $(k-1)$-connected equivariant cellular map $f_{k-1}\colon X_{k-1} \to P$ such that

$$0 \to H_k(f_{k-2}) = Z \to H_k(f_{k-1}) \to \operatorname{Ker} \partial \to 0$$

is exact, where $\partial\colon F_{k-1} \to H_{k-1}(f_{k-2})$. It follows from this and (*) that $H_k(f_{k-1}) = F_k$, a free $Z(G)$-module. Since both F and P are simply connected, we can add k-cells to X_{k-1} to get equivariant cellular map $\phi\colon X \to P$ which is a homotopy equivalence by 2.1.

To verify that f satisfies (EP), let $\phi_i\colon X_i \to P$ be any two G-resolutions of f, $i=1,2$. We may assume that $\dim X_i = n_i > k$. Then

$$H_{i+1}(\phi_1 \cup \phi_2) = H_i(f) , \qquad i \leq k .$$

Thus

$$H_{k+1}(\phi_1 \cup \phi_2) = Z , \quad H_{r+2}(\phi_1 \cup \phi_2) = Z \text{ and}$$
$$\tilde{H}_i(\phi_1 \cup \phi_2) = 0 \qquad \text{for } i \leq k,\ i \neq r+2 .$$

We can use the periodic free resolution

$$(**)\qquad 0 \to H_{k+1}(\phi_1 \cup \phi_2) = Z \to \tilde{F}_{k+1} \xrightarrow{\partial} \tilde{F}_k \xrightarrow{\partial} \cdots \xrightarrow{\partial} \tilde{F}_{r+2} \to Z$$
$$= H_{r+2}(\phi_1 \cup \phi_2) \to 0$$

where $\tilde{F}_{i+1}=F_i$, $r+1 \leq i \leq k$, as above to kill off all homology groups of dimensions $\leq k+1$ to get a map $\tilde{\phi}$: $\tilde{X} \to P$ such that $\tilde{H}_*(\tilde{\phi})$ is $Z(G)$-projective. By 3.1 the G-resolutions of $\tilde{\phi}$ exists. This proves that f satisfies (EP).

REMARK. We can combine (*) to get new periodic free resolutions

$$0 \to Z \to F_k \to \cdots \to F_{r+1} \to F_k \to \cdots \to F_{r+1} \to$$
$$F_k \to \cdots \to F_{r+2} \to F_{r+1} \to Z \to 0 .$$

Thus 1.3 and 6.2 also hold for $k=r+sdn$, s positive integers.

References

[1] A. Borel et al: Seminar on transformation groups, Ann. of Math. Studies, No. 46, Princeton Univ. Press., 1960.

[2] H. Cartan and S. Eilenberg: Homological algebra, Princeton Univ. Press., 1956.

[3] R.M. Dotzel: *A converse of the Borel formula,* to appear in Proc. Amer. Math. Soc.

[4] L. Jones: *The converse to the fixed point theorem of P. A. Smith,* Ann. of Math. **94** (1971), 52–68.

[5] H.T. Ku and M.C. Ku: Seminar on transformation groups, mimeographed, Univ. of Mass., Amherst, 1979.

[6] R.A. Oliver: *Fixed point set of group actions on finite acyclic complexes,* Comment. Math. Helv. **50** (1975), 155–177.

[7] D.S. Rim: *Modules over finite groups,* Ann. of Math. **69** (1959), 700-712.

[8] R.G. Swan: *A new method in fixed point theory,* Comment. Math. Helv. **34** (1960), 1–16.

[9] R.G. Swan: *Periodic resolutions for finite groups,* Ann. of Math. **72** (1960), 267–291.

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