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1. Introduction. Let $G$ be a finite group, $F$ and $P$ be fixed finite $G$-complexes. In this paper we shall define a general obstruction theory for extending a 1-connected cellular $G$-map $f: F \to P$ to a cellular $G$-map $\phi: X \to P$ which is a homology or homotopy equivalence, where $X$ is a finite $G$-complex.

Define a $G$-resolution of $f: F \to P$ to be an $n$-connected cellular $G$-map $\phi: X \to P$, $n = \dim X \geq \dim P$, which extends $f(n \geq 2)$ so that $G$ acts freely outside $F$, $F$ a $G$-subcomplex of $X$ and $H_{n+1}(\phi)$ is a projective $Z(G)$-module. The obstruction $\gamma_G(f, \phi)$ of a $G$-resolution $\phi$ of $f$ is defined as

$$\gamma_G(f, \phi) = (-1)^{n+1}[H_{n+1}(\phi)] \in K_0(Z(G)),$$

where $[H_{n+1}(\phi)]$ denotes the class of $H_{n+1}(\phi)$ in the projective class group $K_0(Z(G))$. Let $c: pt \to pt$ be the constant map and define

$$B(G) = \{\gamma_G(c, \phi): \phi \text{ is a } G\text{-resolution of } c\}.$$

Then $B(G)$ can be proved to be a subgroup of $K_0(Z(G))$. Assume the map $f: F \to P$ satisfies the following extension property:

(EP): Let $\phi_i: X_i \to P$ be any two $G$-resolutions of $f$, $i=1, 2$. Then

$$\phi_1 \cup \phi_2: X_1 \cup P \to P$$

extends to a $G$-resolution of $f$.

We shall show that for any such $\phi_i$, $i=1, 2$,

$$\gamma_G(f, \phi_1) - \gamma_G(f, \phi_2) \in B(G),$$

hence if we let $[\gamma_G(f, \phi)]$ to be the equivalence class of $\gamma_G(f, \phi)$ for any $G$-resolution $\phi$ of $f$, then we can define the obstruction of $f$ by

$$\gamma_G(f) = [\gamma_G(f, \phi)] \in \tilde{K}_0(Z(G))/(B(G)).$$

We will verify that the invariant $\gamma_G(f)$ is exactly the obstruction to extending
f to a homology or homotopy equivalence of G-map if the fixed point set $F^G$ is not empty.

We shall also see that the extension property (EP) holds for many G-maps. Moreover we have the following equality

$$\gamma_c(f_2f_1) = \gamma_c(f_1) + \gamma_c(f_2).$$

Now let us consider the following:

Converse to the Smith fixed point set problem. Suppose the finite group $G$ acts on a finite Poincaré complex $P$ with $P^G = \bigcup P_i(P)$ is a component of $P^G$, and $F = \bigcup F_i$ a finite complex with $F_i \sim P_i$ (i.e., $H^*(F_i; L) \approx H^*(P_i; L)$) for all $i$. Can we find a finite complex $X$ such that $G$ acts on $X$ with $X^G = F$ and $X \sim P$?

We are able to apply the obstruction theory to study this general problem. In particular we will prove the following results.

**Theorem 1.1.** Let $f: F \to P^G \subset P$ be a cellular map such that $G$ acts semifreely on $P$ and $f_*: H_*(F; Z) \approx H_*(P^G; Z)$. Suppose $f: F \to P$ is 1-connected. Then there is a finite complex $X$ which is homology equivalent to $P$, and $G$ acts semifreely on $X$ with $X^G = F$. Moreover, if $F$ is simply connected, $X$ is homotopy equivalent to $P$.

**Theorem 1.2** [4]. Let $G = \mathbb{Z}_q$ ($q$ is not necessarily prime) and $F$ a finite $\mathbb{Z}_q$-acyclic complex. Then there is a finite contractible $G$-complex $X$ such that $G$ acts semifreely on $X$ with $X^G = F$.

**Theorem 1.3.** Let $G$ be a finite group of order $q$ with periodic cohomology of period $n$, and let $d = (q, \phi(q))$, where $\phi$ is the Euler $\phi$-function (i.e., $\phi(q)$ is the number of positive integers $< q$ that are prime to $q$). Suppose $F$ is a simply connected $r$-dimensional integral homology $r$-sphere. Then there exists a finite $G$-complex $X$ which is homotopy equivalent to $S^{r+dn}$. Moreover, $G$ acts semifreely on $X$ with $X^G = F$.

In Swan [8], he proved that $G$ can act freely on some finite complex $X$ which is homotopy equivalent to $S^{dn-1}$, where $G$, $d$ and $n$ are as in 1.3.

We also can define another obstruction theory by using the $G$-resolution $\hat{\phi}$ which is defined exactly as a $G$-resolution of $f$ with $X^G = F^G$ except that $G$ is not necessarily acting freely on $X - F$. Set

$$\hat{B}(G) = \{ \hat{\phi}: \hat{\phi} \text{ is a } G\text{-resolution of } c: pt \to pt \}$$

which is again a subgroup of $K_0(Z(G))$. Thus if $f$ satisfies (EP) for $\hat{G}$-resolutions, $\hat{\gamma}_c(f) = [\hat{\gamma}_c(f, \hat{\phi})] \in K_0(Z(G))/\hat{B}(G)$ is well defined so that the obstruction
theory can be established as in $\gamma_G$-theory. The $\tilde{\sigma}_G(f)$ was defined by Oliver in [6] for the case $\chi(F) = 1 \mod m(G)$, $G$ acts trivially on $F$ and $P$ a point.

2. Preliminaries. Let $X$ and $W$ be finite $G$ complexes, $G$ finite, and $f: X \to W$ be an equivariant cellular map. There is an exact sequence of the map $f$:

$$\cdots \to \pi_{n+1}(X) \to \pi_{n+1}(W) \to \pi_{n+1}(f) \to \pi_n(f) \to \cdots$$

The element of $\pi_{n+1}(f)$ is represented by a pair $(\beta, \alpha)$ of cellular maps so that the diagram below is commutative:

$$
\begin{array}{ccc}
S^n & \xrightarrow{\alpha} & X \\
\downarrow j & & \downarrow f \\
D^{n+1} & \xrightarrow{\beta} & W \\
\end{array}
$$

Define $\alpha: G \times S^n \to X$ by $\alpha(g, y) = g\alpha(y)$ for $(g, y) \in G \times S^n$, and let

$$Y = X \cup G \times D^{n+1} = X \cup (\bigcup_{s \in S} g \times D^{n+1}) = X \cup (\bigcup_{s \in S} \bigcup_{e \in G} D^{n+1})$$

Then there is a naturally induced action of $G$ on $Y$ defined by

$$h(y) = \begin{cases} 
hx & \text{if } y = x \in X \\
(hg, x) & \text{if } y = (g, x) \in G \times D^{n+1}, \end{cases}$$

where $h \in G$. Thus $Y$ is a $G$-complex which is obtained from $X$ by adding free orbits of $(n+1)$-cells. Define an equivariant map $f: Y \to W$ by

$$f(y) = \begin{cases} 
f(x) & \text{if } y = x \in X \\
g\beta(z) & \text{if } y = (g, z) \in G \times D^{n+1}. \end{cases}$$

By construction, $f$ is an equivariant map, and

$$\pi_i(f) = \pi_i(f) \quad \text{for } i \leq n$$

$$\pi_{n+1}(f) = \pi_{n+1}(f)/K,$$

where $K$ is the normal subgroup containing the $Z(G)$-submodule generated by the class $(\beta, \alpha)$ in $\pi_{n+1}(f)$.

**Lemma 2.1.** Let $f: X \to Y$ be an $(n-1)$-connected equivariant map with $H_i(f) = N \oplus M$, where $M$ and $N$ are $Z(G)$-modules, $n \geq 2$. Then we can add $n$-cells of free orbits to kill off $M$ to produce an $(n-1)$-connected equivariant map $\tilde{f}: \tilde{X} \to Y$ such that

1. $H_i(\tilde{f}) = H_i(f)$, $i \geq n+2$,
(2) $0 \to H_{n+1}^*(f) \to H_{n+1}(\hat{f}) \to \text{Ker } \partial_{n+1} \to 0$ and $0 \to \text{Ker } \partial_{n+1} \to H_n(\hat{X}, X) \to M \to 0$ exact.

(3) $H_n(\hat{f}) = N$.

Moreover if $M$ is $Z(G)$-projective (or $\text{Ker } \partial_{n+1}$ is $Z(G)$-projective) then

(4) $H_{n+1}(\hat{f}) = H_{n+1}(f) \oplus \text{Ker } \partial_{n+1}$.

Furthermore if $M$ is a free $Z(G)$-module, $\partial_{n+1}: H_n(\hat{X}, X) \cong M$ and

(5) $H_n(f) = H_n(f)$, $i \geq n+1$.

Proof. Since $f$ is $(n-1)$-connected, $\pi_n(f) \to H_n(f)$ is surjective by the Hurewicz Theorem. Hence there exist pairs of cellular maps $(\beta_i, \alpha_i)$, $\alpha_i: S^{n-1} \to X$, $1 \leq i \leq v$ such that $\sum_{i=1}^v (\beta_i, \alpha_i)_* \{H_n(D^n, S^{n-1})\}$ generates $M$ as $Z(G)$-module. Let

\[
\hat{X} = X \cup \{ \cup G \times D^i \}.
\]

By the construction above $f$ extends to an equivariant map $\hat{f}: \hat{X} \to Y$. There is an exact sequence

\[
0 \to H_{n+1}^*(f) \to H_{n+1}(\hat{f}) \to H_n(\hat{X}, X) \xrightarrow{\partial_{n+1}} H_n(f) \xrightarrow{\hat{j}_*} H_n(\hat{f}) \to 0.
\]

This can be easily obtained by looking at the algebraic mapping cones of $f_* : C_*(X) \to C_*(Y)$ and $\hat{f}_* : C_*(\hat{X}) \to C_*(\hat{Y})$. Now $\text{Ker } \hat{j}_* = M$ by construction, hence $H_n(\hat{f}) \cong H_n(f)/M \cong N$, and

\[
0 \to H_{n+1}^*(f) \to H_{n+1}(\hat{f}) \to H_n(\hat{X}, X) \xrightarrow{\partial_{n+1}} M \to 0
\]

is exact. The results follow easily.

**Lemma 2.2.** Let $G$ be a finite group of order $q$, and $f: M \to P \oplus T$ be a surjective map of $Z(G)$-modules. Suppose $M$ is $Z(G)$-free, $P$ $Z(G)$-projective and $T$ torsion prime to $q$. Then $\text{Ker } f$ is $Z(G)$-projective.

Proof. Let $L$ be any Sylow subgroup of $G$. Then both $M$ and $P$ are $Z(L)$-projective. Since we have exact sequence of $Z(L)$-modules

\[
0 \to \text{Ker } f \to M \to P \oplus T \to 0,
\]

we obtain an exact sequence

\[
\hat{H}(L: M) = 0 \to \hat{H}(L: T) \cong \hat{H}(L: P \oplus T) \to \hat{H}^{i+1}(L: \text{Ker } f) \to \hat{H}^{i+1}(L: M) = 0,
\]

where $\hat{H}(L: M) = \hat{H}^{i+1}(L: M) = 0$ and $\hat{H}(L: P \oplus T) \cong \hat{H}(L: T)$ because $M$ and $P$ are both $Z(L)$-projective. Since $T$ is torsion prime to $|L|$, the map $|L|: T \to T$ which is a multiplication by $|L|$ is an isomorphism. Thus it
induces an isomorphism $|L|: \hat{H}^i(L; T) \rightarrow \hat{H}^i(L; T)$. But $|L| \hat{H}^i(L; T) = 0$, hence $\hat{H}^i(L; T) = 0$, and so

$$\hat{H}^{i+1}(L; \text{Ker} f) = 0,$$

for any Sylow subgroup $L$ of $G$ and any $i$. It follows from [7] that Ker $f$ is cohomologically trivial. Therefore Ker $f$ is $Z(G)$-projective by [7]. The proof extends the idea due to Oliver (cf. [3]).

**Lemma 2.3.** Let $G = Z_q$, $(n, q) = 1$, $M$ and $N$ be free $Z(G)$-modules and $P: N \oplus Z_n \rightarrow Z_n$ the projection. Suppose $h: M \rightarrow N \oplus Z_n$ is an epimorphism such that $p_h: M \rightarrow Z_n$ is an augmentation map. Then Ker $h$ is cohomologically trivial. Therefore Ker $h$ is $Z(G)$-projective by [7], The proof extends the idea due to Oliver (cf. [3]).

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3. An obstruction theory for finite group actions

**Lemma 3.1.** Let $\psi: Y \rightarrow P$ be an equivariant map and $n = \text{dim} Y \geq P$ such that $\psi|F = f$ and $\psi$ is 1-connected. Assume that $H_i(\psi)$ is $Z(G)$-projective for $i \geq 2$. Then $\psi$ can be embedded in a $G$-resolution $\phi: X \rightarrow P$ of $f$ such that

$$\gamma_G(f, \phi) = \sum_{i+1}^n (-1)^i[H_i(\psi)].$$

**Proof.** Let $k$ be the smallest integer such that $H_{k+1}(\psi) \neq 0$. If $k = n$, $\phi = \psi$ is a $G$-resolution of $f$.

Suppose $k < n$. Since the Hurewitz homomorphism $h: \pi_{k+1}(\psi) \rightarrow H_{k+1}(\psi)$ is an epimorphism by the Hurewitz theorem, we can add free orbits of $(k+1)$-cells to kill off $H_{k+1}(\psi)$. This creates an equivariant map $f_k: X_k \rightarrow P$. According to 2.1 we have

$$H_i(f_k) = 0, \quad i \leq k+1.$$  
$$H_i(f_k) = H_i(\psi), \quad i \geq k+3.$$  
$$H_{k+2}(f_k) = H_{k+2}(\psi) \oplus M, \quad M = \text{Ker} \partial_{k+2}, \quad \text{and}$$  
$$H_{k+1}(X_k, Y) = M \oplus H_{k+1}(\psi),$$

where $H_{k+1}(X_k, Y)$ is $Z(G)$-free. Thus $[M] = -[H_{k+1}(\psi)] \in \tilde{K}_n((Z(G))$. It follows that
\[ \sum_{i=1}^{\delta+2} (-1)^i[H_i(f_3)] = \sum_{i=1}^{\delta+1} (-1)^i[H_i(\phi_2)]. \]

By repeating this process, eventually we will produce a \( G \)-resolution \( \phi \) of \( f \) with
\[ \gamma_\phi(f, \phi) = (-1)^{\delta+1}[H_\delta(\phi)] = \sum_{i=1}^{\delta+1} (-1)^i[H_i(\phi)]. \]

**Lemma 3.2.** Let \( B(G) = \{ \gamma_\phi(c, \phi) : \phi \) is a \( G \)-resolution of \( c \} \), where \( c : pt \to pt \) is the constant map. Then \( B(G) \) is a subgroup of \( K_0(Z(G)) \).

**Proof.** Let \( \gamma_\phi(c, \phi_i) \in B(G) \), where \( \phi_i : X_i \to pt \) are two \( G \)-resolutions of \( c \), and \( \dim X_i = n_i \), \( i = 1, 2 \). Then
\[ H_{j+1}(\phi_1 \vee \phi_2) = H_{j+1}(\phi_1) \oplus H_{j+1}(\phi_2), \quad \text{for all } j, \]
where \( \phi_1 \vee \phi_2 : X_1 \vee X_2 \to pt \). Thus \( H_\phi(\phi_1 \vee \phi_2) \) is \( Z(G) \)-projective. It follows from this and 3.1 that there is a \( G \)-resolution \( \phi \) of \( c \) such that
\[ \gamma_\phi(c, \phi) = (-1)^{\delta+1}[H_{\delta+1}(\phi_1 \vee \phi_2)] + (-1)^{\delta+1}[H_{\delta+1}(\phi_1 \vee \phi_2)] = (-1)^{\delta+1}[H_{\delta+1}(\phi_1)] + (-1)^{\delta+1}[H_{\delta+1}(\phi_2)], \]
that is, \( \gamma_\phi(c, \phi_1) + \gamma_\phi(c, \phi_2) \in B(G) \).

Suppose now that \( \gamma_\phi(c, \phi) \in B(G) \), \( \phi : X \to pt \) is a \( G \)-resolution of \( c \) (\( \dim X = n \)). Then \( \phi : \Sigma X \to pt \) is also a \( G \)-resolution of \( c \), where \( \Sigma X \) denotes the reduced suspension of \( X \). This implies that
\[ -\gamma_\phi(c, \phi) = (-1)^{\delta+2}[H_{\delta+2}(\phi)] = (-1)^{\delta+2}[H_{\delta+2}(\phi)] = \gamma_\phi(c, \phi) \in B(G). \]

**Proposition 3.3.** Suppose the map \( f : F \to P \) satisfies (EP). Then for any two \( G \)-resolutions \( \phi_1 \) and \( \phi_2 \), \( \gamma_\phi(f, \phi_1) - \gamma_\phi(f, \phi_2) \in B(G) \).

**Proof.** By hypothesis (EP), \( \phi_1 \cup \phi_2 : X_1 \cup X_2 \to P \) extends to a \( G \)-resolution \( \phi : X \to P \) of \( f \). Let \( \dim X = m \) \( \dim X_i = n_i \), \( i = 1, 2 \). We can assume that \( m \geq n_i + 2 \) and \( n_i > \dim P \), \( i = 1, 2 \).

By assumptions, \( \phi_1 : H_j(X) \approx H_j(P) \), \( j < m \); and \( \phi_2 : H_j(X_i) \approx H_j(P) \) for \( j < n_i \), \( i = 1, 2 \), hence from the homology exact sequences of the pairs \( (X, X_i) \) we have \( H_m(X) \approx H_m(X_i) \), \( H_{n+1}(X/X_i) \approx H_{n+1}(X) \) and \( H_j(X/X_i) = 0 \) otherwise. For instance if \( j < n_i \),
\[ H_j(X) \approx H_j(X \to P) \approx H_j(X/X_i) \approx H_j(X_{i-1}) \approx H_{j-i}(X). \]
From the homology exact sequences of \( \phi \) and \( \phi_i \), we obtain \( H_{m+1}(\phi) \approx H_m(X) \) and \( H_{n+1}(\phi_i) \approx H_n(X_i) \).

Now \( G \) acts on \( X/X_i \) with \( (X/X_i)^G = pt \). Apply 3.1 to the constant maps
\[ f_i: X/X_i \to pt, \] since \( H_{m+1}(f_i) = H_m(X/X_i) = H_{m+1}(\phi) \) and \( H_{n+i}(f_i) = H_{n+i}(X/X_i) = H_{n+i}(\phi_i) \) are \( Z(G) \)-projective and \( H_{j}(f_i) = 0 \) otherwise, there exist \( G \)-resolutions \( \psi_i: Y_i \to pt \) of \( c \) such that

\[
\gamma_0(c, \psi_i) = (-1)^{n_i+2}(H_{n+i}(f_i)) + (-1)^{m+1}[H_{m+1}(f_i)] \\
= (-1)^{n_i+2}[H_{n+i}(\phi_i)] + (-1)^{m+1}[H_{m+1}(\phi)] \\
= -\gamma_0(f, \phi_i) + \gamma_0(f, \phi), \quad i = 1, 2.
\]

Clearly this implies that \( \gamma_0(f, \phi_1) = \gamma_0(f, \phi_2) \in B(G) \).

**Theorem 3.4.** If \( \gamma_0(f) = 0 \) and \( F^G = \phi \), then \( G \) acts on some finite complex \( X \) and there is an equivariant map \( \phi: X \to P \) which is a homology equivalence, \( \phi|_F = f \) and \( G \) acting freely on \( X - F \). If \( F \) is simply connected, then \( \phi \) is a homotopy equivalence.

**Proof.** Let \( \gamma_0(f) = [\gamma_0(f, \phi)] = 0 \), where \( \phi: X \to P \) is \( G \)-resolution of \( f \). Then \( \gamma_0(f, \phi) \in B(G) \). Hence there is a \( G \)-resolution \( \psi: Y \to pt \) of \( c: pt \to pt \) such that \( \gamma_0(f, \phi) = -\gamma_0(c, \psi) \). Consider

\[
\phi \vee \psi: X \vee Y \to P \vee pt = P
\]

where the \( pt \in Y \) is joined to any fixed point of \( X \). Thus we have

\[
H_*(\phi \vee \psi) = H_*(\phi) \oplus H_*(\psi)
\]

which is \( Z(G) \)-projective. By 3.1, \( \phi \vee \psi \) can be embedded in a \( G \)-resolution \( \bar{\phi}: \bar{X} \to P \) of \( f \), and

\[
\gamma_0(f, \bar{\phi}) = \sum_i (-1)^{i}[H_i(\phi \vee \psi)] = \gamma_0(f, \phi) + \gamma_0(c, \psi) = 0.
\]

Thus \( \gamma_0(f, \bar{\phi}) \) is stably \( Z(G) \)-free. It follows from 2.1 that \( \bar{\phi} \) can be extended to an equivariant map \( \bar{\phi}: \bar{X} \to P \) which is a homology equivalence.

**Proposition 3.5.** Suppose \( G \) acts semifreely on both \( F \) and \( P \) and \( \gamma_0(f) = 0 \). Then

\[
f^G_\sharp: H_*(F^G; Z_p) \cong H_*(P^G; Z_p),
\]

for every prime \( p \) such that \( p \mid |G| \), where \( f^G: F^G \to P^G \) is the restriction of \( f \). Equivalently, \( H_*(f^G) \) is torsion prime to \( |G| \), \( |G| = \text{order of } G \).

**Proof.** Let \( \phi: X \to P \) be an equivariant map which is a homology equivalence and \( \phi|_F = f \). For every prime \( p \), \( p \mid |G| \), there is a subgroup \( Z_p \) of \( G \) of order \( p \) and

\[
X^{Z_p} = F^{Z_p} = F^G, \quad P^{Z_p} = P^G.
\]
Since the group \(Z_p\) acts semifreely on the mapping cone \(C_\phi\) which is \(Z_p\)-acyclic with \(C_\phi^2 = C_\phi G\), hence \(C_\psi G\) is \(Z_p\)-acyclic by the Smith fixed point theorem, that is, \(H_\ast(f^G; Z_p) = 0\), or
\[
f^G_\ast: H_\ast(F^G; Z_p) \approx H_\ast(P^G; Z_p)
\]
for every \(p | |G|\).

4. Existence of G-resolutions. The obstruction theory which we have defined depends on the existence of \(G\)-resolutions satisfying (EP). In this section we shall investigate the existence of the equivariant maps \(f: F \to P\) satisfying (EP).

Lemma 4.1. Let \(|G| = q\) and \(\phi: X \to P\) be an \(n\)-connected equivariant cellular map, \(n \geq 2\). Suppose \(H_\ast(\phi^G)\) is torsion prime to \(q\) (resp. \(H_\ast(\phi^H) = 0\)) for every \(\phi^G = \phi|X^G: X^G \to P^G\), where \(e \neq H \subset G\), \(\dim X^G < n = \dim X\) and \(\dim P \leq \dim X\). Then \(H_{n+1}(\phi)\) is \(Z(G)\)-projective (resp. stably \(Z(G)\)-free).

Proof. Let \(\hat{X} = U \{X^H: e \neq H \subset G\}\) and \(\hat{P} = U \{P^H: e \neq H \subset G\}\). By the Mayer-Vietoris sequence and induction, it is easy to verify that \(H_\ast(\hat{\phi})\) is torsion prime to \(q\), where \(\hat{\phi} = \phi|\hat{X}: \hat{X} \to \hat{P}\). Since \(\phi\) is \(n\)-connected, it induces an equivariant map \(\hat{\phi}: X/\hat{X} \to P/\hat{P}\) with \(H_i(\hat{\phi}) \approx H_i(\phi)\) for \(i \leq n\). Hence \(H_i(\hat{\phi})\) is torsion prime to \(q\) for every \(i \leq n\). We may assume that both \(X/\hat{X}\) and \(P/\hat{P}\) are simply connected. Otherwise we simply consider the suspension map \(\Sigma \hat{\phi}: \Sigma(X/\hat{X}) \to \Sigma(P/\hat{P})\). Add free orbits to \(X/\hat{X}\) inductively to kill off \(H_i(\hat{\phi})\), \(i \leq n\). By 2.1 this creates equivariant \(i\)-connected cellular maps \(\hat{\phi}_i: X_i \to P/\hat{P}\), \(2 \leq i \leq n\), with

\[
\begin{align*}
H_{i+1}(\phi_i) &= H_{i+1}(\phi_{i-1}) \oplus \text{Ker } \partial_{i+1} \\
H_j(\phi_i) &= H_j(\phi_{i-1}), & j \geq i+2,
\end{align*}
\]

where \(\text{Ker } \partial_{i+1}\) is \(Z(G)\)-projective for every \(i\) by 2.2.

Now \(G\) acts semifreely on both \(X_n\) and \(P/\hat{P}\) with exactly one fixed point, say \(x_0\) and \(p_0\) respectively. Let \(h = \phi_*|x_0\). Then \(C_* = C_* (C_{\phi_*}, C_n)\) is a free \(Z(G)\)-module with \(H_i(C_\ast) \approx H_i(\phi_n)\) for all \(i \geq 0\). As \(\phi_*\) is \(n\)-connected, there is an exact sequence

\[
0 \to H_{n+1}(\phi_n) \to C_{n+1} \to \cdots \to C_0 \to 0.
\]

This implies that if we let \(N = C_\ast \oplus C_{\ast-2} \oplus \cdots\), then \(H_{n+1}(\phi_n) \oplus N\) is \(Z(G)\)-free. Let \(M = \text{Ker } \partial_{n+1}\). According to (1)

\[
H_{n+1}(\phi_n) = H_{n+1}(\phi_{n-1}) \oplus M = \cdots = H_{n+1}(\hat{\phi}) \oplus M.
\]

Since the sequence

\[
0 \to H_{n+1}(\phi) \to H_{n+1}(\hat{\phi}) \to H_n(\hat{\phi}) \to 0
\]
is exact, we obtain an exact sequence

$$0 \to H_{n+1}(\phi) \oplus M \oplus N \to H_{n+1}(\phi) \oplus M \oplus N = H_{n+1}(\phi) \oplus M \oplus N \to H_n(\phi) \to 0.$$ 

But $H_n(\phi)$ is torsion prime to $q$, and $H_{n+1}(\phi) \oplus M \oplus N$ is $Z(G)$-free, hence $H_{n+1}(\phi) \oplus M \oplus N$ is $Z(G)$-projective by 2.2. It follows that $H_{n+1}(\phi)$ is $Z(G)$-projective.

Suppose now that $\mathcal{H}_n(\phi^u)=0$ for every $H \neq e$. Then $\mathcal{H}_i(\phi)=0$ for $i \leq n$. Hence $\phi$ is $n$-connected and (2) holds for $\phi$. Thus $H_{n+1}(\phi)$ is stably $Z(G)$-free.

But $H_n(\phi)=0$ and so $H_{n+1}(\phi)=H_{n+1}(\tilde{\phi})$ by (3). This proves that $H_{n+1}(\phi)$ is stably $Z(G)$-free.

**Theorem 4.2 (Existence theorem I).** Let $f: F \to P$ be an 1-connected equivariant cellular map.

1. Suppose $\mathcal{H}_n(f^u)$ is torsion prime to $q=|G|$ for every $f^u=f|F^u: F^u \to P^u$, where $e \neq H \subset G$. Then $\gamma_n(f)$ is well defined, i.e., $f$ satisfies (EP).

2. If $\mathcal{H}_n(f^u)=0$ for every $f^u$, $e \neq H \subset G$, then $\gamma_n(f)=0$.

Proof. Add free orbits of cells of dimensions $\leq n$ inductively to get an $n$-connected cellular map $\phi: X \to P$ with $n=\dim X \geq \dim P$. Now $\phi^u=f^u$ for every $e \neq H \subset G$ by construction, hence by 4.1 $H_{n+1}(\phi)$ is $Z(G)$-projective, i.e., $\phi$ is a $G$-resolution of $f$.

If $\phi_i: X_i \to P$ are two $G$-resolutions of $f$, $i=1, 2$, add free orbits of cells to $X_1 \cup X_2$ extending $\phi_1 \cup \phi_2: X_1 \cup X_2 \to P$ to an $n$-connected equivariant cellular map $\phi: X \to P$. Again $\phi$ is a $G$-resolution of $f$ which extends both $\phi_1$ and $\phi_2$, i.e., $f$ satisfies (EP). Hence $\gamma_n(f)$ is well defined.

If $\mathcal{H}_n(f^u)=0$ for every $G \supset H \neq e$, then $H_{n+1}(\phi)$ is stably $Z(G)$-free. Hence $\gamma_n(f)=0$.

**Theorem 4.3 (Existence theorem II).** Suppose $f$ is 1-connected and $H_i(f)=M_i \oplus T_i$ for all $i \geq 2$, where $M_i$'s are $Z(G)$-projective and $T_i$'s torsion prime to $q=|G|$. Then the $G$-resolutions of $f$ always exist and $f$ satisfying (EP). Moreover if $H_i(f)$ is $Z(G)$-free for every $i \geq 2$, then $\gamma_i(f)=0$.

Proof. We shall construct inductively $s$-connected equivariant cellular maps $f_s: X_s \to P$ satisfying the following:

1. $H_{s+i}(f_s)=H_{s+i}(f) \oplus$ a $Z(G)$-projective module, $i=1, 2$.

2. $H_i(f_s)=H_i(f)$, $i \geq s+3$.

Assume that $f_s$ has been constructed. By 2.1. add free orbits of $(s+1)$-cells to kill off $H_{s+1}(f_s)$ to obtain a $(s+1)$-connected cellular map $g_{s+1}: Y_{s+1} \to P$ such that

1. $H_i(g_{s+1})=H_i(f)$, hence $H_i(g_{s+1})=H_i(f)$, $i \geq s+3$ by (2).

2. $0 \to H_{s+1}(f) \to H_{s+1}(g_{s+1}) \to \ker \partial_{s+2} \to 0$ and

$$0 \to \ker \partial_{s+2} \to H_{s+1}(Y_s, X_s) \xrightarrow{\partial_{s+2}} H_{s+1}(f_s) \to 0 \text{ exact},$$
where \( \text{Ker} \partial_{s+2} \) is \( Z(G) \)-projective 2.2. Thus

\[
H_{s+2}(g_{s+1}) = H_{s+2}(f_s) \oplus \text{Ker} \partial_{s+2}.
\]

Again by 2.1. add \((s+2)\)-cells to \( Y_s \) to kill off \( \text{ker} \delta_{s+2} \) to produce a \((s+1)\)-connected equivariant cellular map \( f_{s+1}: X_{s+1} \to P \) such that

\[
\begin{align*}
(5) & \quad H_{s+2}(f_{s+1}) = H_{s+2}(f_s), \text{ hence } H_{s+2}(f_{s+1}) = H_{s+2}(f) \oplus a \ Z(G) \text{-projective module by (1).} \\
(6) & \quad H_i(f_{s+1}) = H_i(g_{s+1}) = H_i(f), \ i \geq s+4 \ (\text{by (3)}) \\
(7) & \quad 0 \to H_{s+3}(g_{s+1}) \to H_{s+3}(f_{s+1}) \to \text{Ker} \delta_{s+2} \to 0 \text{ and } 0 \to \text{Ker} \tilde{\delta}_{s+2} \to H_{s+2}(X_{s+1}, Y_s) \tilde{\delta}_{s+2} \to 0 \text{ exact, where } \text{Ker} \tilde{\delta}_{s+2} \text{ is } Z(G) \text{-projective. Thus}
\end{align*}
\]

This completes the inductive construction.

Now if \( s > \max \{ \dim F, \dim P \} \), then \( H_i(f) = 0 \) for \( i \geq s+1 \) and \( \dim X_s = s+1 \). It follows from (1) and (2) that we have an \( s \)-connected equivariant cellular map \( f_s: X_s \to P \) with \( H_i(f_s) \ Z(G) \)-projective for \( i = s+1 \) and \( s+2 \). By 2.1, we can add \((s+1)\)-cells to \( X \) to kill off \( H_{s+1}(f_s) \) to obtain a \( G \)-resolution \( \phi\): \( X \to P \) of \( f \) with

\[
H_{s+2}(\phi) = H_{s+2}(f_s) \oplus \text{Ker} \partial_{s+2}
\]

which is \( Z(G) \)-projective.

Next, let \( \phi_i: X_i \to P \) be any two \( G \)-resolutions of \( f_i, i = 1, 2 \). We may assume that \( \dim X_i = n_i > \max \{ \dim F, \dim P \} \). Clearly we have an exact sequence

\[
\cdots \to H_{i+1}(f) \to H_{i+1}(\phi_1) \oplus H_{i+1}(\phi_2) \to H_{i+1}(\phi_1 \cup \phi_2) \to H_i(f) \to \cdots
\]

But \( H_{i+1}(\phi_j) = 0 \) for \( i \neq n_j, j = 1, 2 \). Thus

\[
H_{i+1}(\phi_1 \cup \phi_2) \approx H_i(f) \quad \text{for } i \neq n_1, n_2,
\]

\[
H_{i+1}(\phi_1 \cup \phi_2) = \begin{cases} 
H_{n_1+1}(\phi_1) \oplus H_{n_2+1}(\phi_2), & \text{if } i = n_1 = n_2 \\
H_{n_1+1}(\phi_1) & \text{if } i = n_1, n_1 \neq n_2 \\
H_{n_2+1}(\phi_2) & \text{if } i = n_2, n_1 \neq n_2. 
\end{cases}
\]

Hence \( \phi_1 \cup \phi_2 \) can be embedded in a \( G \)-resolution of \( f \). This proves that \( f \) satisfies (EP).

If \( H_i(f) \) is \( Z(G) \)-free for every \( i \geq 2 \), it is not difficult to see from the proof that \( H_{s+2}(\phi) \) is stably \( Z(G) \)-free, hence \( \gamma_G(f) = 0 \).

**Proposition 4.4.** (1) If \( \gamma_G(f) \) is well defined for \( f: F \to P \), then \( \gamma_G(\Sigma f) = -\gamma_G(f) \).
(2) Suppose that \( f_1: F \to P_1 \) and \( f_2: P_1 \to P_2 \) both satisfy 4.2 \( \gamma_c(1) \) or 4.3 so that \( \gamma_c(f_i) \) are well defined, \( i = 1, 2 \). Then
\[
\gamma_c(f_2 f_1) = \gamma_c(f_1) + \gamma_c(f_2).
\]

Proof. (1) Obvious.

(2) Let \( \phi_i: X_i \to P_i \) be a \( G \)-resolution of \( f_1 \) with \( n_i = \dim X_i > \max \{ \dim F, \dim P_1, \dim P_2 \} \). We can see that
\[
H_{i+1}(f_2 \phi_i) = H_{i+1}(f_2), \quad i < n_1, \\
H_{n_i+1}(f_2 \phi_i) = H_{n_i}(X_i) = H_{n_i+1}(\phi_i), \text{ a } Z(G)\text{-projective module,} \\
H_{i+1}(f_2 \phi_i) = 0, \quad i \geq n_i + 1.
\]

For example, if \( i < n_1 \), we have
\[
\cdots \to H_{i+1}(f_2 \phi_i) = H_{i+1}(f_2) \to H_i(X_i) (f_2 \phi_i)_* \xrightarrow{\phi_i *} H_i(P_2) \to \cdots
\]

Now let \( \psi_2: M_2 \to H_2(f_2 \phi_1) = H_2(f_2) \) be a surjective map with \( M_2 \) a \( Z(G)\)-free module. Adding cells to both \( X_1 \) and \( P_1 \) to kill off both \( H_2(f_2 \phi_1) \) and \( H_2(f_2) \) realizing \( \psi_2 \). This creates 2-connected equivariant cellular maps \( g_2: Y_2 \to P_2 \) and \( h_2: W_2 \to P_2 \) extending \( f_2 \phi_1 \) and \( f_2 \) respectively. By 2.1 we can see that \( H_3(g_2) = H_3(h_2) \). Continuing this construction, eventually we will get a \( G \)-resolution \( \phi = h_{n_2}: W_{n_2} \to P_2 \) of \( f_2 \) \( \dim W_{n_2} = n_2 \) and an \( n_2 \)-connected equivariant map \( g_{n_2}: Y_{n_2} \to P_2 \) extending \( f_2 \phi_1 \) such that \( H_{n_2+1}(h_{n_2}) = H_{n_2+1}(g_{n_2}) \) and \( H_i(g_{n_2}) = H_i(f_2 \phi_1) \) for \( i \geq n_2 + 2 \). We may assume that \( n_i > n_2 \). By construction, it is easy to see that
\[
H_{i+1}(g_{n_2}) = \begin{cases} 
H_{n_j+1}(\phi_j), & \text{if } i = n_j, j = 1, 2 \text{ (hence } Z(G)\text{-projective)} \\
0, & \text{otherwise}.
\end{cases}
\]

By 3.1 \( g_{n_2} \) can be embedded in a \( G \)-resolution \( \phi \) of \( f_2 \phi_1 \) which is also a \( G \)-resolution of \( f_2 f_1 \) such that
\[
\gamma_c(f_2 f_1, \phi) = \gamma_c(f_2 \phi_1, \phi) = (-1)^{n_2+1} [H_{n_1+1}(\phi_1)] + (-1)^{n_2} [H_{n_2+1}(\phi_2)]
\]
\[
= \gamma_c(f_1, \phi_1) + \gamma_c(f_2, \phi_2).
\]

Hence \( \gamma_c(f_2 f_1) = \gamma_c(f_1) + \gamma_c(f_2) \) as required.

5. Converse to the Smith fixed point theorem. The converse to the Smith fixed point theorem 1.1 is an immediate consequence of the following.

**Theorem 5.1.** Let \( f: F \to P \) be an equivariant 1-connected cellular map
such that $G$ acts semifreely on both $F$ and $P$.

(1) Suppose $H_\ast(f^G; \mathbb{Z}_p) = 0$ for all $p$, where $p | |G|$ and $p$ is prime. Then $\gamma_c(f)$ is well defined, i.e., $f$ satisfies (EP).

(2) If $H_\ast(f^G) = 0$, then $\gamma_c(f) = 0$.

Proof. Since $G$ acts semifreely on $F$, $P$ and $X$, $f^G = \phi^G$ for all $e \neq H \subset G$. Hence the result follows from 4.2.

Corollary 5.2. (1) Let $F$ be a $\mathbb{Z}$-acyclic finite complex and $G$ a finite group. Then there is a finite contractible complex $X$ such that $X^G = F$ and $G$ acts semifreely on $X$.

(2) Let $F$ be an integral $r$-homology sphere ($r > 0$) and $G$ acts semifreely on $S^k$ with fixed point set $S^r (r \geq 2)$. Suppose there is a cellular map $f: F \rightarrow S'$ such that $f_*: H_\ast(F) \approx H_\ast(S')$. Then there is a finite $G$-complex $X$ which is homology equivalent to $S^k$, and $G$ acts semifreely on $X$ with $X^G = F$. Moreover $X$ is homotopy equivalent to $S^k$ if $F$ is simply connected.

Theorem 5.3. Let $G = \mathbb{Z}_q$ and $f: F \rightarrow P$ be a 1-connected equivariant cellular map. Suppose that $H_i(f) = M_i \oplus T_i$ for all $i \geq 2$, where $M_i$ is $Z(G)$-free and $T_i$ is torsion prime to $q$. Moreover assume that

\[ (*) \quad T_i = \text{torsion submodule of } H_i(f^G) \text{ for all } i \geq 2. \]

Then $\gamma_c(f) = 0$.

Proof. According to the proof of 4.3, we have the following

\[ H_{s+i}(f^G) = H_{s+i}(f) \oplus \ker \partial_{s+i} = T_{s+i} \oplus M_{s+i} \oplus \ker \partial_{s+i}, \quad i = 1, 2. \]

By using the notation in the proof of 4.3, the following composition of maps is an augmentation map by ($*$)

\[ H_{s+i}(Y_s, X_s) \xrightarrow{\partial_{s+i}} H_{s+i}(f^G) \xrightarrow{\text{proj.}} T_{s+i}. \]

Thus $\ker \partial_{s+2}$ is a stably free $Z(G)$-module by 2.3. Hence (1) becomes

\[ (1') \quad H_{s+i}(f^G) = T_{s+i} \oplus \text{a stably free } Z(G)\text{-module}, \quad i = 1, 2. \]

If follows that $H_i(f^G)$ is stably $Z(G)$-free for $i = s+1$ and $s+2$ if $s > \max\{ \dim F, \dim P \}$. Therefore $\gamma_c(f) = 0$.

Now Theorem 1.2 is a simple corollary of the following.

Theorem 5.4. Let $G = \mathbb{Z}_q$ and $F$ be a finite $G$-complex such that $H_\ast(F; \mathbb{Z}_q) = 0$ and $i_*: H_\ast(F^G) \xrightarrow{\approx} H_\ast(F)$, where $i: F^G \rightarrow F$ is an inclusion. Then $\gamma_c(f) = 0$, where $f: F \rightarrow \text{pt}$ is the constant map. Thus there exists a contractible finite $G$-complex $X$ which contains $F$ as a $G$-subcomplex and acts freely outside $F$. 


Proof. First, add free orbits of 2-cells to $F$ to get a simply connected $G$-complex $Y$. According to 2.3 and 2.1 we have $Y^G = F^G$ and

$$H_i(Y) = H_i(F) \oplus M, \quad M \text{ a free } Z(G)\text{-module,}$$

$$H_i = H_i(F) \quad \text{for } i \geq 3.$$  

Let $\psi: Y \to pt$ be the constant map. Then $H_{i+1}(\psi^G) = H_{i+1}(F^G) = H_i(F^G) \approx H_i(F)$ and

$$H_i(\psi) = H_{i-1}(Y) = 0, \quad i = 1, 2.$$  

$$H_3(\psi) = H_3(F) \oplus M,$$

$$H_i(\psi) = H_{i-1}(F), \quad i \geq 4,$$

where $H_i(F)$ are torsion prime to $q$ for $i \geq 2$. Thus the conclusion follows from 5.3 and 3.4.

6. Semifree actions of groups with periodic cohomology on homology spheres

In [8], Swan has proved that if a finite group $G$ acts freely on a compact integral cohomology $n$-sphere, then $G$ has periodic cohomology with period $n+1$. This result can be generalized for semifree actions. More precisely, we have

**Theorem 6.1.** Let $G$ be a finite group acting semifreely on a locally compact space $X$ with $\dim X < \infty$ and $X \sim S^n$. Suppose $F \sim S_{n-r}$ $(n-r \geq 1)$. Then $G$ has periodic cohomology with period $n-r$. (Here we use the Alexander Spanier cohomology with compact supports).

**Proof.** The cohomology exact sequence of the pair $(X, F), F = X^G$, gives

$$H^*(X-F; Z) = H^{r+1}(X-F; Z) = Z$$

and

$$H^i(X-F; Z) = 0 \quad \text{for } i \neq n, r+1.$$

From the spectral sequence of the fibration $(X-F) \to (X-F)/G$, we have the following Gysin type exact sequence (cf. [2])

$$\cdots \to H^i((X-F)_G) \to H^{i+n}(B_G, H^n(X-F)) \to H^{i+n}(B_G; H^{r+1}(X-F))$$

$$\to H^{i+n}(B_G; H^{r+1}(X-F)) \to \cdots$$

The map $\pi_1: (X-F)_G \to (X-F)/G$ induces isomorphism

$$\pi_1^*: H^i((X-F)/G; Z) \approx H^i((X-F)_G, Z)$$

for $i > 0$ by the Vietoris-Begle mapping theorem. But $H^i((X-F)/G; Z) = 0$ for $i > n$ by [1]. It follows that
Now we shall establish the converse of this result, i.e. Theorem 1.3 which is a special case of the following:

**Theorem 6.2.** Let \( G \) be a finite group of order \( q \) with periodic cohomology of period \( n \), \( d=(q,\phi(q)) \), and \( f: F\to P \) be an equivariant cellular map with \( F \) and \( P \) both simply connected. Assume

\[
H_{r+1}(f) = H_k(f) = Z, \quad k = r+dn \quad \text{and} \quad \check{H}_i(f) = 0, \quad i \neq r+1, k.
\]

Then \( \gamma_G(f) \) is well defined and \( \gamma_G(f)=0 \).

**Proof.** According to Swan [9], there is a periodic free resolution over \( Z \) of period \( dn \), i.e., an exact sequence

\[
(*) \quad 0 \to Z \to F_k \xrightarrow{\partial} F_{k-1} \xrightarrow{\partial} \cdots \xrightarrow{\psi} F_{r+1} \xrightarrow{\partial} Z = H_{r+1}(f) \to 0,
\]

with all \( F_i \)'s \( Z(G) \)-free.

Now add free orbits of \((r+1)\)-cells to kill off \( H_{r+1}(f) \) realizing \( \psi \). This creates an \((r+1)\)-connected equivariant cellular map \( f_{r+1}: X_{r+1} \to P \) such that

\[
0 \to H_{r+2}(f_{r+1}) \to H_{r+1}(X, F) = F_{r+1} \xrightarrow{\psi} Z \to 0
\]

is exact. Hence \( \text{Im} \{ \partial: F_{r+2} \to F_{r+1} \} = \text{Ker} \psi = H_{r+2}(f_{r+1}) \). Again, adding free orbits of cells to kill off \( H_{r+2}(f_{r+1}) \) and realizing \( \partial: F_{r+2} \to H_{r+2}(f_{r+1}) \). This produces an \((r+2)\)-connected equivariant cellular map \( f_{r+2}: X_{r+2} \to P \) such that \( \text{Im} \{ \partial: F_{r+3} \to F_{r+2} \} = \text{Ker} \{ \partial: F_{r+2} \to H_{r+2}(f_{r+1}) \} = H_{r+3}(f_{r+2}) \). Repeating this procedure eventually we will get an \((k-1)\)-connected equivariant cellular map \( f_{k-1}: X_{k-1} \to P \) such that

\[
0 \to H_{k}(f_{k-2}) = Z \to H_{k}(f_{k-1}) \to \text{Ker} \partial \to 0
\]

is exact, where \( \partial: F_{k-1} \to H_{k-1}(f_{k-2}) \). It follows from this and (*) that \( H_{k}(f_{k-1}) = F_{k} \), a free \( Z(G) \)-module. Since both \( F \) and \( P \) are simply connected, we can add \( k \)-cells to \( X_{k-1} \) to get equivariant cellular map \( \phi: X \to P \) which is a homotopy equivalence by 2.1.

To verify that \( f \) satisfies (EP), let \( \phi_i: X_i \to P \) be any two \( G \)-resolutions of \( f \), \( i=1,2 \). We may assume that \( \text{dim } X_i = n_i > k \). Then

\[
H_{i+1}(\phi_1 \cup \phi_2) = H_i(f), \quad i \leq k.
\]

Thus

\[
H_{k+1}(\phi_1 \cup \phi_2) = Z, \quad H_{r+2}(\phi_1 \cup \phi_2) = Z \quad \text{and} \quad \check{H}_i(\phi_1 \cup \phi_2) = 0 \quad \text{for } i \leq k, i \neq r+2.
\]

We can use the periodic free resolution
\((**\)\) \[ 0 \to H_{k+1}(\phi_1 \cup \phi_2) = Z \to \overline{F}_k \to \overline{F}_{k+1} \to \cdots \to \overline{F}_{r+2} \to Z \]
\[ = H_{r+2}(\phi_1 \cup \phi_2) \to 0 \]

where \( \overline{F}_{i+1} = F_i, r+1 \leq i \leq k, \) as above to kill off all homology groups of dimensions \( \leq k+1 \) to get a map \( \overline{\phi}: \overline{X} \to P \) such that \( \overline{H}_k(\overline{\phi}) \) is \( Z(G) \)-projective. By 3.1 the \( G \)-resolutions of \( \overline{\phi} \) exists. This proves that \( f \) satisfies (EP).

**Remark.** We can combine (*) to get new periodic free resolutions
\[ 0 \to Z \to F_k \to \cdots \to F_{r+1} \to F_k \to \cdots \to F_{r+1} \to 0. \]
Thus 1.3 and 6.2 also hold for \( k = r + sdn, s \) positive integers.

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**References**


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