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## **OBSTRUCTION THEORY FOR FINITE GROUP ACTIONS**

Dedicated to Professor D. Montgomery on His 70th Birthday

HSU-TUNG KU AND MEI-CHIN KU

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**1.** Introduction. Let G be a finite group, F and P be fixed finite G-complexes. In this paper we shall define a general obstruction theory for extending a 1-connected cellular G-map  $f: F \rightarrow P$  to a cellular G-map  $\phi: X \rightarrow P$  which is a homology or homotopy equivalence, where X is a finite G-complex.

Define a G-resolution of  $f: F \to P$  to be an *n*-connected cellular G-map  $\phi: X \to P$ ,  $n = \dim X \ge \dim P$ , which extends  $f(n \ge 2)$  so that G acts freely outside F, F a G-subcomplex of X and  $H_{n+1}(\phi)$  is a projective Z(G)-module. The obstruction  $\gamma_G(f, \phi)$  of a G-resolution  $\phi$  of f is defined as

$$\gamma_{G}(f, \phi) = (-1)^{n+1} [H_{n+1}(\phi)] \in \tilde{K}_{0}(Z(G)),$$

where  $[H_{n+1}(\phi)]$  denotes the class of  $H_{n+1}(\phi)$  in the projective class group  $\tilde{K}_0(Z(G))$ . Let  $c: pt \to pt$  be the constant map and define

 $B(G) = \{\gamma_G(c, \phi): \phi \text{ is a } G \text{-resolution of } c\}$ .

Then B(G) can be proved to be a subgroup of  $\tilde{K}_0(Z(G))$ . Assume the map  $f: F \to P$  satisfies the following extension property:

(EP): Let  $\phi_i: X_i \rightarrow P$  be any two G-resolutions of f, i=1, 2. Then

$$\phi_1 \cup \phi_2 \colon X_1 \bigcup_F X_2 \to P$$

extends to a G-resolution of f.

We shall show that for any such  $\phi_i$ , i=1, 2,

$$\gamma_G(f, \phi_1) - \gamma_G(f, \phi_2) \in B(G)$$
,

hence if we let  $[\gamma_c(f, \phi)]$  to be the equivalence class of  $\gamma_c(f, \phi)$  for any G-resolution  $\phi$  of f, then we can define the *obstruction* of f by

$$\gamma_{G}(f) = [\gamma_{G}(f, \phi)] \in \tilde{K}_{0}(Z(G))/B(G)$$

We will verify that the invariant  $\gamma_{c}(f)$  is exactly the obstruction to extending

f to a homology or homotopy equivalence of G-map if the fixed point set  $F^{c}$  is not empty.

We shall also see that the extension property (EP) holds for many G-maps. Moreover we have the following equality

$$egin{aligned} &\gamma_{\scriptscriptstyle G}(f_2f_1) = \gamma_{\scriptscriptstyle G}(f_1) + \gamma_{\scriptscriptstyle G}(f_2) \,. \end{aligned}$$

Now let us consider the following:

Converse to the Smith fixed point set problem. Suppose the finite group G acts on a finite Poincré complex P with  $P^{G} = \bigcup_{i} P_{i}(P_{i} \text{ is a component of } P^{G})$ , and  $F = \bigcup_{i} F_{i}$  a finite complex with  $F_{i} \underset{L}{\sim} P_{i}$  (i.e.,  $H^{*}(F_{i}; L) \approx H^{*}(P_{i}; L)$ ) for all *i*. Can we find a finite complex X such that G acts on X with  $X^{G} = F$  and  $X \underset{L}{\sim} P$ ?

We are able to apply the obstruction theory to study this general problem. In particular we will prove the following results.

**Theorem 1.1.** Let  $f: F \rightarrow P^c \subset P$  be a cellular map such that G acts semifreely on P and  $f_*: H_*(F; Z) \approx H_*(P^c; Z)$ . Suppose  $f: F \rightarrow P$  is 1-connected. Then there is a finite complex X which is homology equivalent to P, and G acts semifreely on X with  $X^c = F$ . Moreover, if F is simply connected, X is homotopy equivalent to P.

**Theorem 1.2** [4]. Let  $G=Z_q$  (q is not necessarily prime) and F a finite  $Z_q$ -acyclic complex. Then there is a finite contractible G-complex X such that G acts semifreely on X with  $X^{c}=F$ .

**Theorem 1.3.** Let G be a finite group of order q with periodic cohomology of period n, and let  $d=(q, \phi(q))$ , where  $\phi$  is the Euler  $\phi$ -function (i.e.,  $\phi(q)$  is the number of positive integers < q that are prime to q). Suppose F is a simply connected r-dimensional integral homology r-sphere. Then there exists a finite G-complex X which is homotopy equivalent to  $S^{r+dn}$ . Moreover, G acts semifreely on X with  $X^{c}=F$ .

In Swan [8], he proved that G can act freely on some finite complex X which is homotopy equivalent to  $S^{dn-1}$ , where G, d and n are as in 1.3.

We also can define another obstruction theory by using the  $\tilde{G}$ -resolution  $\tilde{\phi}$  which is defined exactly as a *G*-resolution of *f* with  $X^{G} = F^{G}$  except that *G* is not necessarily acting freely on X - F. Set

 $\tilde{B}(G) = \{\tilde{\phi} : \tilde{\phi} \text{ is a } G \text{-resolution of } c : pt \rightarrow pt\}$ 

which is again a subgroup of  $\tilde{K}_0(Z(G))$ . Thus if f satisfies (EP) for  $\tilde{G}$ -resolutions,  $\tilde{\gamma}_G(f) = [\tilde{\gamma}_G(f, \tilde{\phi})] \in \tilde{K}_0(Z(G))/\tilde{B}(G)$  is well defined so that the obstruction

theory can be established as in  $\gamma_G$ -theory. The  $\tilde{\gamma}_G(f)$  was defined by Oliver in [6] for the case  $\chi(F)=1 \mod m(G)$ , G acts trivially on F and P a point.

2. Preliminaries. Let X and W be finite G complexes, G finite, and  $f: X \rightarrow W$  be an equivariant cellular map. There is an exact sequence of the map f:

$$\cdots \to \pi_{n+1}(X) \to \pi_{n+1}(W) \to \pi_{n+1}(f) \to \pi_n(X) \to \cdots$$

The element of  $\pi_{n+1}(f)$  is represented by a pair  $(\beta, \alpha)$  of cellular maps so that the diagram below is commutative:

$$\begin{array}{c} S^{n} \xrightarrow{\alpha} X \\ j \downarrow & \downarrow f \\ D^{n+1} \xrightarrow{\beta} W \end{array}$$

Define  $\overline{\alpha}$ :  $G \times S^n \to X$  by  $\overline{\alpha}(g, y) = g\alpha(y)$  for  $(g, y) \in G \times S^n$ , and let

$$Y = X \bigcup_{\overline{a}} G \times D^{n+1} = X \bigcup_{\overline{a}} (\bigcup_{g \in \mathcal{G}} g \times D^{n+1}) = X \cup (\bigcup_{g \notin g \in \mathcal{G}} D^{n+1}).$$

Then there is a naturally induced action of G on Y defined by

$$h(y) = \begin{cases} hx & \text{if } y = x \in X \\ (hg, x) & \text{if } y = (g, x) \in G \times D^{n+1} \end{cases}$$

where  $h \in G$ . Thus Y is a G-complex which is obtained from X be adding free orbits of (n+1)-cells. Define an equivariant map  $f: Y \to W$  by

$$\bar{f}(y) = \begin{cases} f(x) & \text{if } y = x \in X \\ g\beta(z) & \text{if } y = (g, z) \in G \times D^{n+1}. \end{cases}$$

By construction, f is an equivariant map, and

$$\pi_i(\bar{f}) = \pi_i(f) \quad \text{for } i \le n$$
  
$$\pi_{n+1}(\bar{f}) = \pi_{n+1}(f)/K,$$

where K is the normal subgroup containing the Z(G)-submodule generated by the class  $(\beta, \alpha)$  in  $\pi_{n+1}(f)$ .

**Lemma 2.1.** Let  $f: X \to Y$  be an (n-1)-connected equivariant map with  $H_n(f) = N \oplus M$ , where M and N are Z(G)-modules,  $n \ge 2$ . Then we can add n-cells of free orbits to kill off M to produce an (n-1)-connected equivariant map  $\hat{f}: \hat{X} \to Y$  such that (1)  $H_i(\hat{f}) = H_i(f), i \ge n+2$ , (2) 0→H<sub>n+1</sub>(f)→H<sub>n+1</sub>(f)→Ker ∂<sub>n+1</sub>→0 and 0→Ker ∂<sub>n+1</sub>→H<sub>n</sub>(X, X)→M→0 exact.
(3) H<sub>n</sub>(f)=N.
Moreover if M is Z(G)-projective (or Ker ∂<sub>n+1</sub> is Z(G)-projective) then
(4) H<sub>n+1</sub>(f)=H<sub>n+1</sub>(f)⊕Ker ∂<sub>n+1</sub>.
Furthermore if M is a free Z(G)-module, ∂<sub>n+1</sub>: H<sub>n</sub>(X, X)≈M and
(5) H<sub>i</sub>(f)=H<sub>i</sub>(f), i≥n+1.

Proof. Since f is (n-1)-connected,  $\pi_n(f) \to H_n(f)$  is surjective by the Hurewicz Theorem. Hence there exist pairs of cellular maps  $(\beta_i, \alpha_i)$ ,  $\alpha_i \colon S^{n-1} \to X$ ,  $1 \le i \le v$  such that  $\sum_{i=1}^{v} (\beta_i, \alpha_i)_* \{H_n(D^n, S^{n-1})\}$  generates M as Z(G)-module. Let

$$\hat{X} = X \bigcup_{\bigcup \bar{\alpha}_i} \{\bigcup G \times D_i^n\}.$$

By the construction above f extends to an equivariant map  $\hat{f}: \hat{X} \rightarrow Y$ . There is an exact sequence

$$0 \to H_{n+1}(f) \to H_{n+1}(\hat{f}) \to H_n(\hat{X}, X) \xrightarrow{\partial_{n+1}} H_n(f) \xrightarrow{j_*} H_n(\hat{f}) \to 0.$$

This can be easily obtained by looking at the algebraic mapping cones of  $f_*: C_*(X) \to C_*(Y)$  and  $\hat{f}_*: C_*(\hat{X}) \to C_*(Y)$ . Now  $\operatorname{Ker} j_* = M$  by construction, hence  $H_n(\hat{f}) \approx H_n(f)/M \approx N$ , and

$$0 \to H_{n+1}(f) \to H_{n+1}(\hat{f}) \to H_n(\hat{X}, X) \xrightarrow{\partial_{n+1}} M \to 0$$

is exact. The results follow easily.

**Lemma 2.2.** Let G be a finite group of order q, and  $f: M \rightarrow P \oplus T$  be a surjective map of Z(G)-modules. Suppose M is Z(G)-free, P Z(G)-projective and T torsion prime to q. Then Ker f is Z(G)-projective.

Proof. Let L be any Sylow subgroup of G. Then both M and P are Z(L)-projective. Since we have exact sequence of Z(L)-modules

$$0 \to \operatorname{Ker} f \to M \to P \oplus T \to 0,$$

we obtain an exact sequence

$$\hat{H}^{i}(L: M) = 0 \to \hat{H}^{i}(L: T) \approx \hat{H}^{i}(L: P \oplus T) \to \hat{H}^{i+1}(L: \operatorname{Ker} f)$$
$$\to \hat{H}^{i+1}(L: M) = 0,$$

where  $\hat{H}^{i}(L: M) = \hat{H}^{i+1}(L: M) = 0$  and  $\hat{H}^{i}(L: P \oplus T) \approx \hat{H}^{i}(L: T)$  because M and P are both Z(L)-projective. Since T is torsion prime to |L|, the map  $|L|: T \rightarrow T$  which is a multiplication by |L| is an isomorphism. Thus it

induces an isomorphism  $|L|: \hat{H}^{i}(L; T) \rightarrow \hat{H}^{i}(L; T)$ . But  $|L| \hat{H}^{i}(L; T) = 0$ , hence  $\hat{H}^{i}(L; T) = 0$ , and so

$$\hat{H}^{i+1}(L;\operatorname{Ker} f)=0,$$

for any Sylow subgroup L of G and any i. It follows from [7] that Ker f is cohomologically trivial. Therefore Ker f is Z(G)-projective by [7]. The proof extends the idea due to Oliver (cf. [3]).

**Lemma 2.3.** Let  $G=Z_q$ , (n, q)=1, M and N be free Z(G)-modules and  $P_1: N \oplus Z_n \to Z_n$  the projection. Suppose  $h: M \to N \oplus Z_n$  is an epimorphism such that  $p_1h: M \to Z_n$  is an augmentation map. Then Ker h is stably Z(G)-free.

Proof. Let  $p: N \oplus Z_n \to N$  be the projection. Since N is Z(G)-free, there is a Z(G)-homomorphism  $\phi: N \to M$  such that  $(ph)\phi=1$ . Thus  $M=K\oplus\phi(N)$ as Z(G)-modules and  $h|\phi(N):\phi(N)\approx N$ , where K=Ker(ph). It follows that h induces an epimorphism  $\bar{h}: K \to Z_n$  with Ker  $\bar{h}=\text{Ker } h$ . Note that we have exact sequence

$$0 \to \operatorname{Ker} h \oplus \phi(N) \to M = K \oplus \phi(N) \xrightarrow{p_1 h = (h, 0)} Z_n \to 0.$$

Hence Ker  $h \oplus \phi(N)$  is Z(G)-free by applying [4, Lemma 1.1] inductively on the rank of M. As  $\phi(N)$  is Z(G)-free, Ker h is stably Z(G)-free.

## 3. An obstruction theory for finite group actions

**Lemma 3.1.** Let  $\psi: Y \rightarrow P$  be an equivariant map and  $n = \dim Y \geq P$  such that  $\psi|F=f$  and  $\psi$  is 1-connected. Assume that  $H_i(\psi)$  is Z(G)-projective for  $i\geq 2$ . Then  $\psi$  can be embedded in a G-resolution  $\phi: X \rightarrow P$  of f such that

$$\gamma_{G}(f, \phi) = \sum_{i=2}^{n+1} (-1)^{i} [H_{i}(\psi)].$$

Proof. Let k be the smallest integer such that  $H_{k+1}(\psi) \neq 0$ . If k=n,  $\phi=\psi$  is a G-resolution of f.

Suppose k < n. Since the Hurewitz homomorphism  $h: \pi_{k+1}(\psi) \rightarrow H_{k+1}(\psi)$ is an epimorphism by the Hurewitz theorem, we can add free orbits of (k+1)cells to kill off  $H_{k+1}(\psi)$ . This creates an equivariant map  $f_k: X_k \rightarrow P$ . According to 2.1 we have

$$egin{aligned} & \hat{H}_i(f_k) = 0\,, \qquad i \leq k+1\,. \ & H_i(f_k) = H_i(\psi)\,, \qquad i \geq k+3\,. \ & H_{k+2}(f_k) = H_{k+2}(\psi) \oplus M\,, \qquad M = \operatorname{Ker} \partial_{k+2}\,, ext{ and } \ & H_{k+1}(X_k,\,Y) = M \oplus H_{k+1}(\psi)\,, \end{aligned}$$

where  $H_{k+1}(X_k, Y)$  is Z(G)-free. Thus  $[M] = -[H_{k+1}(\psi)] \in \tilde{K}_0[(Z(G))$ . It follows that

H.T. KU AND M.C. KU

$$\sum_{i=k+2}^{n+1} (-1)^{i} [H_{i}(f_{k})] = \sum_{i=2}^{n+1} (-1)^{i} [H_{i}(\psi)].$$

By repeating this process, eventually we will produce a G-resolutiont  $\phi$  of f with

$$\gamma_G(f, \phi) = (-1)^{n+1} [H_{n+1}(\phi)] = \sum_{i=2}^{n+1} (-1)^i [(H_i(\psi)]].$$

**Lemma 3.2.** Let  $B(G) = \{\gamma_G(c, \phi) : \phi \text{ is a } G \text{-resolution of } c\}$ , where  $c : pt \rightarrow pt$  is the constant map. Then B(G) is a subgroup of  $\tilde{K}_0(Z(G))$ .

Proof. Let  $\gamma_G(c, \phi_i) \in B(G)$ , where  $\phi_i: X_i \to pt$  are two G-resolutions of c, and dim  $X_i = n_i$ , i = 1, 2. Then

$$H_{j+1}(\phi_1 \lor \phi_2) = H_{j+1}(\phi_1) \oplus H_{j+1}(\phi_2)$$
, for all *j*,

where  $\phi_1 \lor \phi_2$ :  $X_1 \bigvee_{p_t} X_2 \to pt$ . Thus  $\hat{H}_*(\phi_1 \lor \phi_2)$  is Z(G)-projective. It follows from this and 3.1 that there is a G-resolution  $\phi$  of c such that

$$\begin{split} \gamma_G(c, \phi) &= (-1)^{n_1+1} [H_{n_1+1}(\phi_1 \vee \phi_2)] + (-1)^{n_2+1} [H_{n_2+1}(\phi_1 \vee \phi_2)] \\ &= (-1)^{n_1+1} [H_{n_1+1}(\phi_1)] + (-1)^{n_2+1} [H_{n_2+1}(\phi_2)] \,, \end{split}$$

that is,  $\gamma_G(c, \phi_1) + \gamma_G(c, \phi_2) \in B(G)$ .

Suppose now that  $\gamma_c(c, \phi) \in B(G)$ ,  $\phi: X \to pt$  is a G-resolution of  $c \pmod{X=n}$ . Then  $\tilde{\phi}: \Sigma X \to pt$  is also a G-resolution of c, where  $\Sigma X$  denotes the reduced suspension of X. This implies that

$$-\gamma_{G}(c, \phi) = (-1)^{n+2}[H_{n+1}(\phi)] = (-1)^{n+2}[H_{n+2}(\tilde{\phi})] = \gamma_{G}(c, \tilde{\phi}) \in B(G).$$

**Proposition 3.3.** Suppose the map  $f: F \to P$  satisfies (EP). Then for any two G-resolutions  $\phi_1$  and  $\phi_2$ ,  $\gamma_G(f, \phi_1) - \gamma_G(f, \phi_2) \in B(G)$ .

Proof. By hypothesis (EP),  $\phi_1 \cup \phi_2$ :  $X_1 \bigcup_F X_2 \to P$  extends to a *G*-resolution  $\phi: X \to P$  of *f*. Let dim X=m dim  $X_i=n_i$ , i=1, 2. We can assume that  $m \ge n_i+2$  and  $n_i > \dim P$ , i=1, 2.

By assumptions,  $\phi_*$ :  $H_j(X) \approx H_j(P)$ , j < m; and  $\phi_{i*}$ :  $H_j(X_i) \approx H_j(P)$  for  $j < n_i$ , i=1, 2, hence from the homology exact sequences of the pairs  $(X, X_i)$  we have  $H_m(X) \approx H_m(X|X_i)$ ,  $H_{n_i+1}(X|X_i) \approx H_{n_i}(X_i)$  and  $\tilde{H}_j(X|X_i) = 0$  otherwise. For instance if  $j < n_j$ ,

$$\begin{array}{c} H_{j}(X_{i}) \xrightarrow{\approx} H_{j}(X) \to H_{j}(X|X_{i}) \to H_{j-1}(X_{i}) \xrightarrow{\approx} H_{j-1}(X) \\ \approx \bigvee \swarrow \swarrow & \swarrow & \swarrow & \swarrow & \swarrow \\ H_{j}(P) & H_{j-1}(P) \end{array}$$

From the homology exact sequences of  $\phi$  and  $\phi_i$  we obtain  $H_{m+1}(\phi) \approx H_m(X)$ and  $H_{n_i+1}(\phi_i) \approx H_{n_i}(X_i)$ .

Now G acts on  $X/X_i$  with  $(X/X_i)^c = pt$ . Apply 3.1 to the constant maps

 $f_i: X/X_i \rightarrow pt$ , since  $H_{m+1}(f_i) = H_m(X/X_i) = H_{m+1}(\phi)$  and  $H_{n_i+2}(f_i) = H_{n_i+1}((X/X_i))$ = $H_{n_i+1}(\phi_i)$  are Z(G)-projective and  $\tilde{H}_j(f_i) = 0$  otherwise, there exist G-resolutions  $\psi_i: Y_i \rightarrow pt$  of c such that

$$egin{aligned} &\gamma_{G}(c,\,\psi_{i})=(-1)^{n_{i}+2}[H_{n_{i}+2}(f_{i})]\!+\!(-1)^{m+1}\![H_{m+1}(f_{i})]\ &=-(-1)^{n_{i}+1}\![H_{n_{i}+1}(\phi_{i})]\!+\!(-1)^{m+1}\![H_{m+1}(\phi)]\ &=-\gamma_{G}(f,\,\phi_{i})\!+\!\gamma_{G}(f,\,\phi), \quad i=1,\,2\,. \end{aligned}$$

Clearly this implies that  $\gamma_G(f, \phi_1) - \gamma_G(f, \phi_2) \in B(G)$ .

**Theorem 3.4.** If  $\gamma_G(f)=0$  and  $F^G \neq \phi$ , then G acts on some finite complex  $\hat{X}$  and there is an equivariant map  $\hat{\phi}: \tilde{X} \rightarrow P$  which is a homology equivalence,  $\hat{\phi}|F=f$  and G acting freely on  $\hat{X}-F$ . If F is simply connected, then  $\hat{\phi}$  is a homotopy equivalence.

Proof. Let  $\gamma_{c}(f) = [\gamma_{c}(f, \phi)] = 0$ , where  $\phi: X \to P$  is G-resolution of f. Then  $\gamma_{c}(f, \phi) \in B(G)$ . Hence there is a G-resolution  $\psi: Y \to pt$  of  $c: pt \to pt$ such that  $\gamma_{c}(f, \phi) = -\gamma_{c}(c, \psi)$ . Consider

$$\phi \lor \psi \colon X \lor Y \rightarrow P \lor pt = P$$

where the  $pt \in Y$  is joined to any fixed point of X. Thus we have

$${ ilde H}_{m{*}}(\phiee\psi)={ ilde H}_{m{*}}(\phi){ otheta}{ ilde H}_{m{*}}(\psi)$$

which is Z(G)-projective. By 3.1,  $\phi \lor \psi$  can be embedded in a G-resolution  $\tilde{\phi} \colon \tilde{X} \to P$  of f, and

$$\gamma_{\mathcal{G}}(f,\,\widetilde{\phi}) = \sum_{i} (-1)^{i} [H_{i}(\phi \lor \psi)] = \gamma_{\mathcal{G}}(f,\,\phi) + \gamma_{\mathcal{G}}(c,\,\psi) = 0 \,.$$

Thus  $\gamma_G(f, \tilde{\phi})$  is stably Z(G)-free. It follows from 2.1 that  $\tilde{\phi}$  can be extended to an equivariant map  $\tilde{\phi}: \hat{X} \to P$  which is a homology equivalence.

**Proposition 3.5.** Suppose G acts semifreely on both F and P and  $\gamma_G(f)=0$ . Then

$$f^{\scriptscriptstyle G}_*\colon H_*(F^{\scriptscriptstyle G};Z_p) \xrightarrow{\approx} H_*(P^{\scriptscriptstyle G};Z_p),$$

for every prime p such that p||G|, where  $f^G: F^G \to P^G$  is the restriction of f. Equivalently,  $\tilde{H}_*(f^G)$  is torsion prime to |G|, |G| = order of G.

Proof. Let  $\phi: X \to P$  be an equivariant map which is a homology equivalence and  $\phi|F=f$ . For every prime p, p||G|, there is a subgroup  $Z_p$  of G of order p and

$$X^{Z_{p}} = F^{Z_{p}} = F^{G}, P^{Z_{p}} = P^{G}.$$

Since the group  $Z_p$  acts semifreely on the mapping cone  $C_{\phi}$  which is  $Z_p$ -acyclic with  $C_{\phi}^{Z_p} = C_f G$ , hence  $C_f G$  is  $Z_p$ -acyclic by the Smith fixed point theorem, that is,  $\hat{H}_*(f^c; Z_p) = 0$ , or

$$f_*^G: H_*(F^G; Z_p) \approx H_*(P^G; Z_p)$$
 for every  $p \mid |G|$ .

4. Existence of G-resolutions. The obstruction theory which we have defined depends on the existence of G-resolutions satisfying (EP). In this section we shall investigate the existence of the equivariant maps  $f: F \rightarrow P$  satisfying (EP).

**Lemma 4.1.** Let |G| = q and  $\phi: X \to P$  be an n-connected equivariant cellular map,  $n \ge 2$ . Suppose  $\tilde{H}_*(\phi^H)$  is torsion prime to q (resp.  $\tilde{H}_*(\phi^H)=0$ ) for every  $\phi^H = \phi | X^H: X^H \to P^H$ , where  $e \neq H \subset G$ , dim  $X^H < n = \dim X$  and dim  $P \le \dim X$ . Then  $H_{n+1}(\phi)$  is Z(G)-projective (resp. stably Z(G)-free).

Proof. Let  $\hat{X}=U\{X^{H}: e \neq H \subset G\}$  and  $\hat{P}=\cup\{P^{H}: e \neq H \subset G\}$ . By the Mayer-Vietoris sequence and induction, it is easy to verify that  $\hat{H}_{*}(\hat{\phi})$  is torsion prime to q, where  $\hat{\phi}=\phi \mid \hat{X}: \hat{X} \rightarrow \hat{P}$ . Since  $\phi$  is *n*-connected, it induces an equivariant map  $\bar{\phi}: X/\hat{X} \rightarrow P/\hat{P}$  with  $H_{i}(\bar{\phi}) \approx \tilde{H}_{i-1}(\hat{\phi})$  for  $i \leq n$ . Hence  $\tilde{H}_{i}(\bar{\phi})$  is torsion prime to q for every  $i \leq n$ . We may assume that both  $X/\hat{X}$  and  $P/\hat{P}$  are simply connected. Otherwise we simply consider the suspension map  $\Sigma\bar{\phi}: \Sigma(X/\hat{X}) \rightarrow \Sigma(P/\hat{P})$ . Add free orbits to  $X/\hat{X}$  inductively to kill off  $\tilde{H}_{i}(\bar{\phi}), i \leq n$ . By 2.1. this creates equivariant *i*-connected cellular maps  $\phi_{i}: X_{i} \rightarrow P/\hat{P}, 2 \leq i \leq n$ , with

(1) 
$$\begin{cases} H_{i+1}(\phi_i) = H_{i+1}(\phi_{i-1}) \oplus \text{Ker } \partial_{i+1} \\ H_j(\phi_i) = H_j(\phi_{i-1}), \quad j \ge i+2, \end{cases}$$

where Ker  $\partial_{i+1}$  is Z(G)-projective for every *i* by 2.2.

Now G acts semifreely on both  $X_n$  and  $P/\hat{P}$  with exactly one fixed point, say  $x_0$  and  $p_0$  respectively. Let  $h=\phi_n|x_0$ . Then  $C_*=C_*(C_{\phi_n}, C_h)$  is a free Z(G)-module with  $H_i(C_*)=\tilde{H}_i(\phi_n)$  for all  $i\geq 0$ . As  $\phi_n$  is *n*-connected, there is an exact sequence

$$(2) 0 \to H_{n+1}(\phi_n) \to C_{n+1} \to \cdots \to C_0 \to 0.$$

This implies that if we let  $N=C_n\oplus C_{n-2}\oplus \cdots$ , then  $H_{n+1}(\phi_n)\oplus N$  is Z(G)-free. Let  $M=\text{Ker }\partial_{n+1}$ . According to (1)

$$H_{n+1}(\phi_n) = H_{n+1}(\phi_{n-1}) \oplus M = \cdots = H_{n+1}(\bar{\phi}) \oplus M.$$

Since the sequence

$$(3) \qquad 0 \to H_{n+1}(\phi) \to H_{n+1}(\bar{\phi}) \to H_n(\phi) \to 0$$

is exact, we obtain an exact sequence

$$0 \to H_{n+1}(\phi) \oplus M \oplus N \to H_{n+1}(\bar{\phi}) \oplus M \oplus N = H_{n+1}(\phi) \oplus N \to H_n(\hat{\phi}) \to 0.$$

But  $H_n(\hat{\phi})$  is torsion prime to q, and  $H_{n+1}(\phi_n) \oplus N$  is Z(G)-free, hence  $H_{n+1}(\phi) \oplus M \oplus N$  is Z(G)-projective by 2.2. It follows that  $H_{n+1}(\phi)$  is Z(G)-projective.

Suppose now that  $\hat{H}_{*}(\phi^{H})=0$  for every  $H \neq e$ . Then  $\hat{H}_{i}(\bar{\phi})=0$  for  $i \leq n$ . Hence  $\bar{\phi}$  is *n*-connected and (2) holds for  $\bar{\phi}$ . Thus  $H_{n+1}(\bar{\phi})$  is stably Z(G)-free. But  $H_{n}(\hat{\phi})=0$  and so  $H_{n+1}(\phi)=H_{n+1}(\bar{\phi})$  by (3). This proves that  $H_{n+1}(\phi)$  is stably Z(G)-free.

**Theorem 4.2** (Existence theorem I). Let  $f: F \rightarrow P$  be an 1-connected equivariant cellular map.

(1) Suppose  $\hat{H}_*(f^H)$  is torsion prime to q = |G| for every  $f^H = f|F^H : F^H \to P^H$ , where  $e \neq H \subset G$ . Then  $\gamma_G(f)$  is well defined, i.e., f satisfies (EP).

(2) If  $\tilde{H}_*(f^H) = 0$  for every  $f^H$ ,  $e \neq H \subset G$ , then  $\gamma_G(f) = 0$ .

Proof. Add free orbits of cells of dimensions  $\leq n$  inductively to get an *n*-connected cellular map  $\phi: X \rightarrow P$  with  $n = \dim X \geq \dim P$ . Now  $\phi^H = f^H$  for every  $e \neq H \subset G$  by construction, hence by 4.1  $H_{n+1}(\phi)$  is Z(G)-projective, i.e.,  $\phi$  is a G-resolution of f.

If  $\phi_i: X_i \to P$  are two *G*-resolutions of *f*, i=1, 2, add free orbits of cells to  $X_1 \bigcup_F X_2$  extending  $\phi_1 \cup \phi_2: X_1 \bigcup_F X_2 \to P$  to an *n*-connected equivariant cellular map  $\phi: X \to P$ . Again  $\phi$  is a *G*-resolution of *f* which extends both  $\phi_1$  and  $\phi_2$ , i.e., *f* satisfies (EP). Hence  $\gamma_G(f)$  is well defined.

If  $\tilde{H}_*(f^H)=0$  for every  $G \supset H \neq e$ , then  $H_{n+1}(\phi)$  is stably Z(G)-free. Hence  $\gamma_G(f)=0$ .

**Theorem 4.3 (Existence theorem II).** Suppose f is 1-connected and  $H_i(f) = M_i \oplus T_i$  for all  $i \ge 2$ , where  $M_i$ 's are Z(G)-projective and  $T_i$ 's torsion prime to q = |G|. Then the G-resolutions of f always exist and f satisfying (EP). Moreover if  $H_i(f)$  is Z(G)-free for every  $i \ge 2$ , then  $\gamma_G(f) = 0$ .

Proof. We shall construct inductively s-connected equivariant cellular maps  $f_s: X_s \rightarrow P$  satisfying the following:

(1)  $H_{s+i}(f_s) = H_{s+i}(f) \oplus a Z(G)$ -projective module, i=1, 2.

(2)  $H_i(f_s) = H_i(f), i \ge s+3.$ 

Assume that  $f_s$  has been constructed. By 2.1. add free orbits of (s+1)-cells to kill off  $H_{s+1}(f_s)$  to obtain a (s+1)-connected cellular map  $g_{s+1}$ :  $Y_s \rightarrow P$  such that

(3) 
$$H_i(g_{s+1}) = H_i(f_s)$$
, hence  $H_i(g_{s+1}) = H_i(f)$ ,  $i \ge s+3$  by (2).  
(4)  $0 \rightarrow H_{s+2}(f_s) \rightarrow H_{s+2}(g_{s+1}) \rightarrow \text{Ker } \partial_{s+2} \rightarrow 0$  and  
 $0 \rightarrow \text{Ker } \partial_{s+2} \rightarrow H_{s+1}(Y_s, X_s) \xrightarrow{\partial_{s+2}} H_{s+1}(f_s) \rightarrow 0$  exact,

where Ker  $\partial_{s+2}$  is Z(G)-projective 2.2. Thus

$$H_{s+2}(g_{s+1}) = H_{s+2}(f_s) \oplus \operatorname{Ker} \partial_{s+2}.$$

Again by 2.1. add (s+2)-cells to  $Y_s$  to kill off ker  $\partial_{s+2}$  to produce a (s+1)connected equivariant cellular map  $f_{s+1}$ :  $X_{s+1} \rightarrow P$  such that

- (5)  $H_{s+2}(f_{s+1}) = H_{s+2}(f_s)$ , hence
- $H_{s+2}(f_{s+1}) = H_{s+2}(f) \oplus a Z(G)$ -projective module by (1).
- (6)  $H_i(f_{s+1}) = H_i(g_{s+1}) = H_i(f), i \ge s + 4$  (by (3))
- (7)  $0 \rightarrow H_{s+3}(g_{s+1}) \rightarrow H_{s+3}(f_{s+1}) \rightarrow \text{Ker } \tilde{\partial}_{s+2} \rightarrow 0 \text{ and}$

$$0 \to \operatorname{Ker} \tilde{\partial}_{s+2} \to H_{s+2}(X_{s+1}, Y_s) \xrightarrow{\partial_{s+2}} \operatorname{Ker} \partial_{s+2} \to 0$$

exact, where Ker  $\hat{\partial}_{s+2}$  is Z(G)-projective. Thus

$$H_{s+3}(f_{s+1}) = H_{s+3}(g_{s+1}) \oplus \operatorname{Ker} \partial_{s+2}$$
  
=  $H_{s+3}(f) \oplus \operatorname{Ker} \tilde{\partial}_{s+2}$  (by (3)).

This completes the inductive construction.

Now if  $s > \max\{\dim F, \dim P\}$ , then  $H_i(f) = 0$  for  $i \ge s+1$  and  $\dim X_s = s+1$ . It follows from (1) and (2) that we have an s-connected equivariant cellular map  $f_s: X_s \to P$  with  $H_i(f_s) Z(G)$ -projective for i=s+1 and s+2. By 2.1, we can add (s+1)-cells to X to kill off  $H_{s+1}(f_s)$  to obtain a G-resolution  $\phi: X \to P$  of f with

$$H_{s+2}(\phi) = H_{s+2}(f_s) \oplus \operatorname{Ker} \partial_{s+2}$$

which is Z(G)-projective.

Next, let  $\phi_i: X_i \to P$  be any two G-resolutions of f, i=1, 2. We may assume that dim  $X_i = n_i > \max \{\dim F, \dim P\}$ . Clearly we have an exact sequence

$$\cdots \to H_{i+1}(f) \to H_{i+1}(\phi_1) \oplus H_{i+1}(\phi_2) \to H_{i+1}(\phi_1 \cup \phi_2) \to H_i(f) \to \cdots$$

But  $H_{i+1}(\phi_j)=0$  for  $i \neq n_j$ , j=1,2. Thus

$$H_{i+1}(\phi_1 \cup \phi_2) \approx H_i(f) \quad \text{for } i \neq n_1, n_2 ,$$

$$H_{i+1}(\phi_1 \cup \phi_2) = \begin{cases} H_{n_1+1}(\phi_1) \oplus H_{n_2+1}(\phi_2) , & \text{if } i = n_1 = n_2 \\ H_{n_1+1}(\phi_1) & \text{if } i = n_1, n_1 \neq n_2 \\ H_{n_2+1}(\phi_2) & \text{if } i = n_2, n_1 \neq n_2 . \end{cases}$$

Hence  $\phi_1 \cup \phi_2$  can be embedded in a G-resolution of f. This proves that f satisfies (EP).

If  $H_i(f)$  is Z(G)-free for every  $i \ge 2$ , it is not difficult to see from the proof that  $H_{s+2}(\phi)$  is stably Z(G)-free, hence  $\gamma_G(f)=0$ .

**Proposition 4.4.** (1) If  $\gamma_G(f)$  is well defined for  $f: F \rightarrow P$ , then  $\gamma_G(\Sigma f) = -\gamma_G(f)$ .

(2) Suppose that  $f_1: F \rightarrow P_1$  and  $f_2: P_1 \rightarrow P_2$  both satisfy 4.2  $\gamma_G(1)$  or 4.3 so that  $\gamma_G(f_i)$  are well defined, i=1, 2. Then

$${\gamma}_{\scriptscriptstyle G}(f_2f_1) = {\gamma}_{\scriptscriptstyle G}(f_1) + {\gamma}_{\scriptscriptstyle G}(f_2) \ .$$

Proof. (1) Obvious.

(2) Let  $\phi_1: X_1 \rightarrow P_1$  be a *G*-resolution of  $f_1$  with  $n_1 = \dim X_1 > \max{\dim F, \dim P_1, \dim P_2}$ . We can see that

$$\begin{split} H_{i+1}(f_2\phi_1) &= H_{i+1}(f_2), \quad i < n_1, \\ H_{n_1+1}(f_2\phi_1) &= H_{n_1}(X_1) = H_{n_1+1}(\phi), \text{ a } Z(G) \text{-projective module,} \\ H_{i+1}(f_2\phi_1) &= 0, \quad i \ge n_1 + 1. \end{split}$$

For example, if  $i < n_1$ , we have

$$\cdots \to H_{i+1}(f_2\phi_1) = H_{i+1}(f_2) \to H_i(X_1) \xrightarrow{(f_2\phi_1)_*} H_i(P_2) \to \cdots$$
$$\phi_{1^*} \swarrow f_{2^*}$$
$$H_i(P_1)$$

Now let  $\psi_2: M_2 \rightarrow H_2(f_2\phi_1) = H_2(f_2)$  be a surjective map with  $M_2$  a Z(G)-free module. Adding cells to both  $X_1$  and  $P_1$  to kill off both  $H_2(f_2\phi_1)$  and  $H_2(f_2)$  realizing  $\psi_2$ . This creates 2-connected equivariant cellular maps  $g_2: Y_2 \rightarrow P_2$  and  $h_2: W_2 \rightarrow P_2$  extending  $f_2\phi_1$  and  $f_2$  respectively. By 2.1 we can see that  $H_3(g_2) = H_3(h_2)$ . Continuing this construction, eventually we will get a G-resolution  $\phi_2 = h_{n_2}: W_{n_2} \rightarrow P_2$  of  $f_2$ , dim  $W_{n_2} = n_2$  and an  $n_2$ -connected equivariant map  $g_{n_2}: Y_{n_2} \rightarrow P_1$  extending  $f_2\phi_1$  such that  $H_{n_2+1}(h_{n_2}) = H_{n_2+1}(g_{n_2})$  and  $H_i(g_{n_2}) = H_i(f_2\phi_1)$  for  $i \ge n_2+2$ . We may assume that  $n_1 > n_2$ . By construction, it is easy to see that

$$H_{i+1}(g_{n_2}) = \begin{cases} H_{n_j+1}(\phi_j), & \text{if } i = n_j, j = 1, 2 \text{ (hence } Z(G)\text{-projective)} \\ 0, & \text{otherwise.} \end{cases}$$

By 3.1  $g_{n_2}$  can be embedded in a G-resolution  $\phi$  of  $f_2\phi_1$  which is also a G-resolution of  $f_2f_1$  such that

$$\begin{split} \gamma_G(f_2f_1,\,\phi) &= \gamma_G(f_2\phi_1,\,\phi) = (-1)^{n_1+1}[H_{n_1+1}(\phi_1)] + (-1)^{n_2+1}[H_{n_2+1}(\phi_2)] \\ &= \gamma_G(f_1,\,\phi_1) + \gamma_G(f_2,\,\phi_2) \,. \end{split}$$

Hence  $\gamma_G(f_2f_1) = \gamma_G(f_1) + \gamma_G(f_2)$  as required.

5. Converse to the Smith fixed point theorem. The converse to the Smith fixed point theorem 1.1 is an immediate consequence of the following.

**Theorem 5.1.** Let  $f: F \rightarrow P$  be an equivariant 1-connected cellular map

such that G acts semifreely on both F and P.

(1) Suppose  $\tilde{H}_*(f^c; Z_p) = 0$  for all p, where p ||G| and p is prime. Then  $\gamma_G(f)$  is well defined, i.e., f satisfies (EP).

(2) If  $\tilde{H}_*(f^c)=0$ , then  $\gamma_c(f)=0$ .

Proof. Since G acts semifreely on F, P and X,  $f^c = \phi^c$  for all  $e \neq H \subset G$ . Hence the result follows from 4.2.

**Corollary 5.2.** (1) Let F be a Z-acyclic finite complex and G a finite group. Then there is a finite contractible complex X such that  $X^G = F$  and G acts semifreely on X.

(2) Let F be an integral r-homology sphere (r>0) and G acts semifreely on  $S^k$  with fixed point set  $S'(r\geq 2)$ . Suppose there is a cellular map  $f: F \rightarrow S^r$  such that  $f_*: H_*(F) \approx H_*(S^r)$ . Then there is a finite G-complex X which is homology equivalent to  $S^k$ , and G acts semifreely on X with  $X^G = F$ . Moreover X is homotopy equivalent to  $S^k$  if F is simply connected.

**Theorem 5.3.** Let  $G = Z_q$  and  $f: F \to P$  be a 1-connected equivariant cellular map. Suppose that  $H_i(f) = M_i \oplus T_i$  for all  $i \ge 2$ , where  $M_i$ 's are Z(G)-free and  $T_i$ 's torsion prime to q. Moreover assume that

(\*)  $T_i = torsion \ submodule \ of \ H_i(f^G) \ for \ all \ i \geq 2$ .

Then  $\gamma_G(f) = 0$ .

Proof. According to the proof of 4.3, we have the following

(1) 
$$H_{s+i}(f_s) = H_{s+i}(f) \oplus \operatorname{Ker} \partial_{s+i}$$
$$= T_{s+i} \oplus M_{s+i} \oplus \operatorname{Ker} \partial_{s+i}, \quad i = 1, 2.$$

By using the notation in the proof of 4.3, the following composition of maps is an augmentation map by (\*)

$$H_{s+1}(Y_s, X_s) \xrightarrow{\partial_{(s+2)}} H_{s+1}(f_s) \xrightarrow{\text{proj.}} T_{s+1}$$

Thus Ker  $\partial_{s+2}$  is a stably free Z(G)-module by 2.3. Hence (1) becomes (1)'  $H_{s+i}(f_s) = T_{s+i} \oplus a$  stably free Z(G)-module, i=1, 2. If follows that  $H_i(f_s)$  is stably Z(G)-free for i=s+1 and s+2 if  $s > \max\{\dim F, \dim P\}$ . Therefore  $\gamma_G(f)=0$ .

Now Theorem 1.2 is a simple corollary of the following.

**Theorem 5.4.** Let  $G=Z_q$  and F be a finite G-complex such that  $\hat{H}_*(F; Z_q) = 0$  and  $i_*: H_*(F^c) \xrightarrow{\approx} H_*(F)$ , where  $i: F^c \rightarrow F$  is an inclusion. Then  $\gamma_G(f)=0$ , where  $f: F \rightarrow pt$  is the constant map. Thus there exists a contractible finite G-complex X which contains F as a G-subcomplex and acts freely outside F.

Proof. First, add free orbits of 2-cells to F to get a simply connected G-complex Y. According to 2.3 and 2.1 we have  $Y^{c}=F^{c}$  and

$$H_2(Y) = H_2(F) \oplus M$$
,  $M$  a free  $Z(G)$ -module,  
 $H_i = H_i(F)$  for  $i \ge 3$ .

Let  $\psi: Y \to pt$  be the constant map. Then  $H_{i+1}(\psi^c) = H_{i+1}(f^c) = H_i(F^c) \approx H_i(F)$  and

$$egin{aligned} H_i(\psi) &= \hat{H}_{i-1}(Y) = 0\,, & i = 1,\,2\,. \ H_3(\psi) &= H_2(F) \oplus M\,, \ H_i(\psi) &= H_{i-1}(F)\,, & i \geq 4\,, \end{aligned}$$

where  $H_i(F)$  are torsion prime to q for  $i \ge 2$ . Thus the conclusion follows from 5.3 and 3.4.

## 6. Semifree actions of groups with periodic cohomology on homology spheres

In [8], Swan has proved that if a finite group G acts freely on a compact integral cohomology *n*-sphere, then G has periodic cohomology with period n+1. This result can be generalized for semifree actions. More precisely, we have

**Theorem 6.1.** Let G be a finite group acting semifreely on a locally compact space X with dim,  $X < \infty$  and  $X \sim S^n$ . Suppose  $F \sim S_{\perp}(n-r \geq 1)$ . Then G has periodic cohomology with period n-r. (Here we use the Alexander Spanier cohomology with compact supports).

Proof. The cohomology exact sequence of the pair (X, F),  $F = X^c$ , gives

$$H^{n}(X-F; Z) = H^{r+1}(X-F; Z) = Z$$
 and  
 $H^{i}(X-F; Z) = 0$  for  $i \neq n, r+1$ .

From the spectral sequence of the fibration  $(X-F) \rightarrow (X-F)_{c} \xrightarrow{\pi_{2}} B_{c}$  (cf [1]), we have the following Gysin type exact sequence (cf. [2])

The map  $\pi_1: (X-F)_G \rightarrow (X-F)/G$  induces isomorphism

$$\pi_1^*: H^i((X-F)/G; Z) \approx H^i((X-F)_G, Z)$$

for i>0 by the Vietoris-Begle mapping theorem. But  $H^i((X-F)/G; Z)=0$ i>n by [1]. It follows that

$$H^{i-n}(B_G; Z) \approx H^{i-r}(B_G; Z)$$
 for  $i > n$ .

Now we shall establish the converse of this result, i.e. Theorem 1.3 which is a special case of the following:

**Theorem 6.2.** Let G be a finite group of order q with periodic cohomology of period n,  $d=(q, \phi(q))$ , and f:  $F \rightarrow P$  be an equivariant cellular map with F and P both simply connected. Assume

$$H_{r+1}(f) = H_k(f) = Z, \quad k = r+dn \quad and$$
  
 $\tilde{H}_i(f) = 0, \quad i \neq r+1, k.$ 

Then  $\gamma_{G}(f)$  is well defined and  $\gamma_{G}(f)=0$ .

Proof. According to Swan [9], there is a periodic free resolution over Z of period dn, i.e., an exact sequence

(\*) 
$$0 \to Z \to F_k \xrightarrow{\partial} F_{k-1} \xrightarrow{\partial} \cdots \to F_{r+1} \xrightarrow{\psi} Z = H_{r+1}(f) \to 0$$
,

with all  $F_i$ s Z(G)-free.

Now add free orbits of (r+1)-cells to kill off  $H_{r+1}(f)$  realizing  $\psi$ . This creates an (r+1)-connected equivariant cellular map  $f_{r+1}: X_{r+1} \rightarrow P$  such that

$$0 \to H_{r+2}(f_{r+1}) \to H_{r+1}(X, F) = F_{r+1} \xrightarrow{\Psi} Z \to 0$$

is exact. Hence Im  $\{\partial: F_{r+2} \rightarrow F_{r+1}\} = \text{Ker } \psi = H_{r+2}(f_{r+1})$ . Again, adding free orbits of cells to kill off  $H_{r+2}(f_{r+1})$  and realizing  $\partial: F_{r+2} \rightarrow H_{r+2}(f_{r+1})$ . This produces an (r+2)-connected equivariant cellular map  $f_{r+2}: X_{r+2} \rightarrow P$  such that Im  $\{\partial: F_{r+3} \rightarrow F_{r+2}\} = \text{Ker } \{\partial: F_{r+2} \rightarrow H_{r+2}(f_{r+1})\} = H_{r+3}(f_{r+2})$ . Repeating this procedure eventually we will get an (k-1)-connected equivariant cellular map  $f_{k-1}: X_{k-1} \rightarrow P$  such that

$$0 \to H_k(f_{k-2}) = Z \to H_k(f_{k-1}) \to \operatorname{Ker} \partial \to 0$$

is exact, where  $\partial: F_{k-1} \to H_{k-1}(f_{k-2})$ . It follows from this and (\*) that  $H_k(f_{k-1}) = F_k$ , a free Z(G)-module. Since both F and P are simply connected, we can add k-cells to  $X_{k-1}$  to get equivariant cellular map  $\phi: X \to P$  which is a homotopy equivalence by 2.1.

To verify that f satisfies (EP), let  $\phi_i: X_i \to P$  be any two G-resolutions of f, i=1,2. We may assume that dim  $X_i=n_i > k$ . Then

Thus

We can use the periodic free resolution

**Obstruction Theory for Finite Group Actions** 

(\*\*) 
$$0 \to H_{k+1}(\phi_1 \cup \phi_2) = Z \to \widetilde{F}_{k+1} \to \widetilde{F}_k \to \cdots \to \widetilde{F}_{r+2} \to Z$$
$$= H_{r+2}(\phi_1 \cup \phi_2) \to 0$$

where  $\tilde{F}_{i+1} = F_i$ ,  $r+1 \le i \le k$ , as above to kill off all homology groups of dimensions  $\le k+1$  to get a map  $\tilde{\phi}: \tilde{X} \to P$  such that  $\tilde{H}_*(\tilde{\phi})$  is Z(G)-projective. By 3.1 the G-resolutions of  $\tilde{\phi}$  exists. This proves that f satisfies (EP).

REMARK. We can combine (\*) to get new periodic free resolutions

$$\begin{array}{l} 0 \rightarrow Z \rightarrow F_k \rightarrow \cdots \rightarrow F_{r+1} \rightarrow F_k \rightarrow \cdots \rightarrow F_{r+1} \rightarrow \\ F_k \rightarrow \cdots \rightarrow F_{r+2} \rightarrow F_{r+1} \rightarrow Z \rightarrow 0 \; . \end{array}$$

Thus 1.3 and 6.2 also hold for k=r+sdn, s positive integers.

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