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Osaka University
COMPACT MINIMAL GENERIC SUBMANIFOLDS
WITH PARALLEL NORMAL SECTION
IN A COMPLEX PROJECTIVE SPACE

YEONG-WU CHOE*, U-HANG KI** and RYOICHI TAKAGI

(Received July 6, 1998)

Introduction

Generic submanifold have been investigated by many authors (e.g. [5], [7], [8], [9], [21]). Here a submanifold $M$ in a Kaehlerian manifold is called generic if each normal space of $M$ is mapped into the tangent space of $M$ by the complex structure of the ambient space (cf. [2], [4], [22]). Any real hypersurface in a Kaehlerian manifold is a typical example of the generic submanifold.

In particular, the model space of the so called $A_1$, $A_2$, $B$, $C$, $D$ and $E$-type are typical examples of a real hypersurface in a complex projective space $P(\mathbb{C})$. Recently, the third named author, B. H. Kim and I.-B. Kim [19] proved that those model spaces exhaust all intrinsic homogeneous real hypersurfaces in $P(\mathbb{C})$.

On the other hand, the model spaces of the type $A_1$ and $A_2$ was first introduced by Lawson [13], and he gave a characterization of them. Moreover, Choe and Oukumura [5] gave a generalization of Lawson's theorem in [13] from a viewpoint of the CR-submanifold (see §1 for the definition).

The purpose of the present paper is to give another generalization (Theorem A) of Lawson's theorem, from a viewpoint of the generic submanifold, and to give new examples of a generic submanifold in $P(\mathbb{C})$.

The authors would like to thank the referee for his suggestions, which resulted in many improvements of the present paper.

1. Preliminaries

Let $\tilde{M}$ be a Kaehlerian manifold of real dimension $n + r$ equipped with an almost complex structure $J$ and a Hermitian metric tensor $G$. Then for any vector fields $X$ and $Y$ on $M$, we have

$$J^2X = -X, \quad G(JX, JY) = G(X, Y), \quad \nabla J = 0,$$

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where $\tilde{V}$ denotes the Riemannian connection of $\tilde{M}$.

Let $M$ be an $n$-dimensional Riemannian manifold covered by a system of co-
ordinate neighborhoods $\{U; x^i\}$ and isometrically immersed in $\tilde{M}$ by the immersion $i : M \rightarrow \tilde{M}$. When the argument is local, $M$ need not distinguished from $i(M)$ itself. Throughout this paper the indices $i, j, k, \cdots$ run from 1 to $n$. We represent the immersion $i$ locally by

$$y^A = y^A(x^i), \quad (A = 1, \cdots, n, \cdots, n + r)$$

and put $B_j^A = \partial_j y^A$, ($\partial_j = \partial/\partial x^j$) then $B_j = (B_j^A)$ are $n$-linearly independent lo-
cal tangent vectors of $M$. We choose $r$-mutually orthogonal unit normals $C_x = (C_x^A)$ to $M$. Hereafter the indices $u, v, w, x, \cdots$ run from $n + 1$ to $n+r$ and the summa-
tion convention will be used. The immersion being isometric, the induced Riemannian metric tensor $g$ with components $g_{ij}$ and the metric tensor $\delta$ with components $\delta_{xy}$ of the normal bundle are respectively obtained

$$g_{ij} = G(B_j, B_i), \quad \delta_{xy} = G(C_y, C_x).$$

By denoting $\nabla_j$ the operator of van der Waerden-Bortolotti covariant differentia-
tion with respect to $g$ and $G$, the equations of Gauss and Weingarten for the subman-
ifold $M$ are respectively given by

$$\nabla_j B_i = A_{ij}^a C_a, \quad \nabla_j C_x = -A_{ji}^x B_i,$$

where $A_{ji}^x$ are components of the second fundamental tensor and the shape operator $A^x$ in the direction $C_x$ are related by

$$A^x = (A_J^x) = (A_{ji}^x g^{ih} \delta_{xy}), \quad (g^{ij}) = (g_{ij})^{-1}.$$

For $x \in M$ we denotes by $T_x(M)$ and $N_x(M)$ the tangent space and the normal space of $M$, respectively.

A submanifold $M$ of a Kaehlerian manifold $\tilde{M}$ is called $CR$ submanifold of $\tilde{M}$ if there exists a differentiable distribution $D : x \rightarrow D_x \subset T_x(M)$ on $M$ satisfying the following conditions (see [2], [4], [22]):

1. $D$ is invariant with respect to $J$, and
2. the complementary orthogonal distribution $D^\perp : x \rightarrow D^\perp_x \subset T_x(M)$ is totally
   real with respect to $J$.

In particular if $\dim D^\perp = \text{codim} M$, then $M$ is a generic submanifold of $\tilde{M}$ (see [8], [20]). If $M$ is a $CR$ submanifold, then the maximal $J$-invariant subspace $JT_x(M) \cap T_x(M)$ of $T_x(M)$ has constant dimension for $x \in M$ and this constant is called $CR$ dimension.

If we assume that $M$ is $CR$ submanifold of $CR$ dimension $n - 1$, that is,

$$\dim(JT_x(M) \cap T_x(M)) = n - 1.$$
This implies that there exists a unit vector field $C_*$ normal to $M$ such that $JT(M) \subset T(M) \oplus \text{span} \{C_*\}$. Then, we have the following theorem by the first named author and Okumura [5].

**Theorem A.** Let $M$ be an $n$-dimensional compact, minimal CR submanifold of CR dimension $n-1$ of $P^{(n+1)/2}(C)$. If the normal vector field $C_*$ is parallel with respect to the normal connection and scalar curvature $\geq (n+2)(n-1)$, then $M$ is an $M^C_{p,q}$ for some $p, q$ satisfying $2(p+q)=n-1$.

The model space $M^C_{p,q}$ in the above theorem is described in the following. Let $M_{p,q}$ be the hypersurface in $S^{n+2}$ which is defined by

$$\sum_{j=0}^p |z_j|^2 = \cos^2 \theta, \quad \sum_{j=p+1}^{p+q+1} |z_j|^2 = \sin^2 \theta, \quad 0 < \theta < \frac{\pi}{2}.$$ 

$M_{p,q}$ is a standard product $S^{2p+1} \times S^{2q+1}, 2(p+q) = n-1$. The Hopf fibration $\pi : S^{n+2} \rightarrow P^{(n+1)/2}(C)$ submerses $M_{p,q}$ onto a real hypersurface of $P^{(n+1)/2}(C)$ which we denote by $M^C_{p,q}$. Cecil-Ryan [3] proved that $M^C_{p,q}$ is a tube of radius $\theta$ over a totally geodesic $P^p(C)$, namely, $M^C_{p,q}$ is a homogeneous type $A_1$ or $A_2$ [18].

In the following, we assume that $M$ is a generic submanifold of a Kaehlerian manifold. Then our hypothesis implies that the transformations of $B_i$ and $C_x$ by $J$ are respectively represented in each coordinate neighborhood as follows:

$$JB_j = f_j^h B_h - J_j^x C_x, \quad JC_x = J_x^h B_h,$$

where we have put $f_{ji} = G(JB_j, B_i), J_{jx} = -G(JB_j, C_x), J_{kj} = G(JC_x, B_j), f_j^h = f_{ji} \delta^{ih}$ and $J_j^x = J_{jy} \delta^{zx}$. From these definitions, it follows from (1.2) that

$$f_j^t f_t^h = -\delta_j^h + J_J^x J_j^h, \quad f_j^t J_j^t = 0,$$

$$J_x^t J_y^z = \delta_x^z.$$

By differentiating (1.2) covariantly along $M$, using $\nabla J=0$, and by comparing the tangential and normal parts, we obtain

$$\nabla_J f_i^h = A_{ji} x J_x^h - A_j^h x J_{ix},$$

$$\nabla_J J_{ix} = A_{jix} f_i^t,$$

$$A_{jty} J_{ix}^x = A_{jtx} J_{iy}^t.$$

If the ambient space $\tilde{M}$ is a Kaehlerian manifold of constant holomorphic sectional curvature $\delta$, the equations of Gauss, Codazzi and Ricci of $M$ are respectively given by

$$R_{kijh} = g_{kh} g_{ji} - g_{jh} g_{ki} + f_{kh} f_{ji} - f_{jh} f_{ki} - 2 f_{kj} f_{ih} + A_{kh}^x A_{jix} - A_{jh}^x A_{kix},$$
\[
\n(1.9) \quad \nabla_k A_{jix} - \nabla_j A_{kix} = J_{jx} f_{ki} - J_{kx} f_{ji} + 2J_{ix} f_{kj},
\]
\[
(1.10) \quad R_{jix} = J_{jx} J_{iy} - J_{ix} J_{jy} + A_{jtx} A_{i}^{t} y - A_{itx} A_{i}^{t} y,
\]

where \( R_{kijh} \) and \( R_{jixy} \) are components of the Riemannian curvature tensor and those with respect to the connection induced in the normal bundle respectively.

From (1.8) the Ricci tensor \( S \) of \( M \) is verified that

\[
S_{ji} = (n+2)g_{ji} - 3J_{j}^{x} J_{ix} + h^{x} A_{jix} - A_{jit} A_{i}^{t} x,
\]

because of (1.3), where \( h^{x} = \text{trace} A^{x} \). Thus the scalar curvature \( \rho \) of \( M \) is given by

\[
(1.11) \quad \rho = n(n+2) - 3J_{ix} J_{ix} + h_{ix} h^{x} - A_{jix} A_{jix}
\]

since we have (1.4).

In what follows, to write our formula in convention forms \( n+1 \) denoted by the symbol * and we put \( h_{(2)} = A_{jix} A_{jix}^{*} \), \( (A_{jix}^{*})^{2} = A_{jx}^{*} A_{ix}^{*} \) and \( P_{xyz} = A_{jix} J_{y} J_{z}^{x} \). Then \( P_{xyz} \) is symmetric for all indices because of (1.7).

2. Parallel normal section

Here we consider the case of a complex projective space \( \tilde{M} = P^{(n+r)/2}(\mathbb{C}) \) of constant holomorphic sectional curvature 4. A normal vector field \( \xi = (\xi^{x}) \) is called a parallel section in the normal bundle if it satisfies \( \nabla_j \xi^{x} = 0 \).

From now on we suppose that \( M \) is an \( n \)-dimensional compact generic submanifold of \( P^{(n+r)/2}(\mathbb{C}) \) with parallel unit normal vector field \( C_{*} \) with respect to the normal connection, that is, \( \nabla_{j} C_{*} = 0 \). Then (1.10) shows that \( R_{jix} \) vanishes identically for any index \( x \) and hence

\[
(2.1) \quad A_{jix} A_{jix}^{*} - A_{itx} A_{itx}^{*} = J_{ix} J_{ix} - J_{ix} J_{ix},
\]

which together with (1.4) and (1.7) implies that

\[
(2.2) \quad (J^{*} A_{j}^{*})(J^{*} A_{itx}) = (A_{j}^{*} J_{x}^{*})(A_{itx} J_{ix}) + 1 - r.
\]

From (1.5) and (1.6) we have

\[
(2.3) \quad \nabla_{k} \nabla_{j} J_{i}^{*} = (\nabla_{k} A_{j}^{*}) f_{l}^{i} A_{j}^{*} (A_{kl}^{*} J_{x}^{*} - A_{l}^{*} J_{ix}),
\]

or, using (1.3), (1.4) and (2.2)

\[
J^{*} \Delta J_{ix} = (A_{j}^{*} J_{ix}^{*})(A_{j}^{*} J_{ix}) - h_{(2)},
\]

where \( \Delta = g^{ij} \nabla_{j} \nabla_{i} \).
We also have from (2.3)
\[ J^i*(\nabla_i \nabla_j J^i*) = h^x P_{x**} - (A^i J^i J^i*) + n - 1, \]
where we have used (1.3), (1.4) and (1.9). From the last two equations, we obtain
\[ J^i* \Delta J^i* + J^i*(\nabla_i \nabla_j J^i*) = -h_{(2)} + h^x P_{x**} + n - 1. \]  
Let us put \( U_j = J^i* \nabla_i J^i* + J^i* \nabla_i J^i*. \) Then we have
\[ \text{div} \ U = (\nabla i J^i*) (\nabla i J^i*) + (\nabla J^i*) (\nabla J^i*) + J^i* \Delta J^i* + J^i* \nabla J^i* \]
which together with (1.6) and (2.4) yields
\[ \text{div} \ U = \frac{1}{2} \left| A^* f - f A^* \right|^2 - h_{(2)} + h^x P_{x**} + n - 1. \]
On the other hand, we have from (1.4)
\[ J_{j*} J^i* = 1, \quad J_{j*} J^j* = r \]
because \( r \) is the codimension of \( M \) and consequently we obtain
\[ J_{j(*)} J^{j(*)} = r - 1, \quad (x) \geq n + 2. \]
Thus, (1.11) turns out to be
\[ \rho = (n + 3)(n - 1) - 3 J_{j(*)} J^{j(*)} + h^x - h_{(2)} + A_{ji(*)} A^{ji(*)}. \]

**Lemma 1.** Let \( M \) be an \( n \)-dimensional generic, minimal submanifold of \( P^{(n+1)/2}(\mathbb{C}) \) with parallel unit normal \( C_\ast \). Then we have
\[ \text{div} \ U = \frac{1}{2} \left| A^* f - f A^* \right|^2 + \rho - (n + 2)(n - 1) + 3 J_{j(*)} J^{j(*)} + A_{ji(*)} A^{ji(*)}. \]

Proof. Since \( M \) is minimal, it follows, using (2.5), (2.6) and (2.8), that required equation is obtained. This completes the proof.

Further, suppose that \( M \) is compact and the scalar curvature \( \rho \) of \( M \) satisfies \( \rho \geq (n + 2)(n - 1) \) in Lemma 1, then we have
\[ A^* f = f A^*, \]
\[ A_{ji(*)} = 0, \quad J_{j(*)} = 0 \quad \text{for all} \quad (x) \geq n + 2 \]
and \( \rho = (n + 2)(n - 1) \). Thus (2.7) means \( r = 1 \), that is, \( M \) is a real hypersurface of \( P^{(n+1)/2}(\mathbb{C}) \).
Thus we have

**Lemma 2.** Let $M$ be an $n$-dimensional compact generic, minimal submanifold in $P^{(n+r)/2}(\mathbb{C})$. Suppose that $M$ admits a parallel unit normal vector field $C_*$ and the scalar curvature $\rho \geq (n+2)(n-1)$ on $M$. Then $M$ is a real hypersurface in $P^{(n+1)/2}(\mathbb{C})$ satisfying $A^*f = fA^*$ and $\rho = (n+2)(n-1)$.

From Lemma 2 and Theorem 4.4 in [15] due to Okumura, we have

**Theorem 3.** Let $M$ be an $n$-dimensional compact generic, minimal submanifold in $P^{(n+r)/2}(\mathbb{C})$. Suppose that $M$ admits a parallel unit normal vector field and the scalar curvature $\geq (n+2)(n-1)$. Then $r = 1$ and $M$ is an $M_{p,q}^C$ for some $p, q$ satisfying $2(p+q) = n - 1$.

**3. Examples of generic submanifolds in $P^n(\mathbb{C})$**

In this section we shall give two examples of a compact homogeneous generic submanifold in $P^n(\mathbb{C})$, and another example of a compact homogeneous minimal generic submanifold in $P^n(\mathbb{C})$ admitting a parallel normal vector field.

Let $p, q$ ($p \leq q$) be positive integers. We denote by $M_{p,q}(\mathbb{C})$ the space of $p \times q$ matrices over $\mathbb{C}$, which can be considered as a complex Euclidean space $\mathbb{C}^{pq}$ with the standard Hermitian inner product. Let $U(p)$ denote the unitary group of degree $p$. Then the Lie group $G := S(U(p) \times U(q))$ acts on $\mathbb{C}^{pq} = M_{p,q}(\mathbb{C})$ as follows:

$$(\sigma, \tau)X = \sigma X \tau^{-1}, \quad (\sigma, \tau) \in G, \quad X \in \mathbb{C}^{pq}.$$ 

Thus we can consider $G$ as a unitary subgroup of $U(pq)$. Remark that this action is nothing but the linear isotropic representation of the compact Hermitian symmetric space $SU(p+q)/S(U(p) \times U(q))$ of type AIII. Let $\pi$ be the canonical projection of $\mathbb{C}^{pq} - \{0\}$ onto $P^{pq-1}(\mathbb{C})$, and $S^{2pq-1}(r)$ the hypersphere in $\mathbb{C}^{pq}$ of radius $r$ centered at the origin. Then, for any element $A$ of $\mathbb{C}^{pq} - \{0\}$, the orbit $G(A)$ of $A$ under $G$ is a compact homogeneous submanifold in $S^{2pq-1}(\{A\})$, and the space $\pi(G(A))$ is a compact homogeneous submanifolds in $P^{pq-1}(\mathbb{C})$ (see e.g. [19]). Moreover, for any normal vector $N$ of $G(A)$ in $S^{2pq-1}(\{A\})$, the mean curvature of $G(A)$ in the direction $N$ is equal to the one of $\pi(G(A))$ in the direction $\pi* N$ in $P^{pq-1}(\mathbb{C})$. (see e.g. [16]). In particular, $G(A)$ is minimal in $S^{2pq-1}(\{A\})$ if and only if $\pi(G(A))$ is minimal in $P^{pq-1}(\mathbb{C})$.

Here, for $i = 1, \cdots, p$ we put
and denote by $\alpha$ the vector space spanned by $e_1, \ldots, e_p$ over $\mathbb{R}$. In the sequel we shall show

(3.1) If $A = a_1 e_1 + \cdots + a_p e_p$ satisfies $a_i \neq 0$, $a_i^2 \neq a_j^2$ for $1 \leq i < j \leq p$, then $\pi(G(A))$ is a $(2pq - p - 1)$-dimensional generic submanifold in $P^{pq-1}(\mathbb{C})$.

(3.2) If $A = e_1 + a_2 e_3 + \cdots + a_p e_p$ satisfies $a_i^2 \neq 0$, $1$ and $a_i^2 \neq a_j^2$ for $3 \leq i < j \leq p$, then $\pi(G(A))$ is a $(2pq - p - 3)$-dimensional generic submanifold in $P^{pq-1}(\mathbb{C})$.

(3.3) Let $p = 3 \leq q$. Then there exists a vector $A$ in $\alpha \setminus \{0\}$ such that $\pi(G(A))$ is a $(6q - 4)$-dimensional minimal generic submanifold in $P^{2q-1}(\mathbb{C})$ admitting a parallel normal vector field.

To show these, we need some preparations. Let $\Delta$ denote the positive restricted root system associated with the symmetric space $SU(p+q)/S(U(p) \times U(q))$ and $\omega$ (cf. [6]). Let $\{\omega_1, \ldots, \omega_p\}$ be the dual basis of $e_1, \ldots, e_p$. Then $\Delta$ is given by

$$\Delta = \{\omega_i, 2\omega_i, \omega_i \pm \omega_j ; 1 \leq i < j \leq p\}$$

with multiplicities $2(q - p)$, $1$, $2$, respectively (cf. Helgason [6, 349p] or Araki [1, table]). An element $A = a_1 e_1 + \cdots + a_p e_p$ is called regular if $\omega(A) \neq 0$ for any $\omega \in \Delta$, or equivalently $a_i \neq 0$, $a_i^2 \neq a_j^2$ for $1 \leq i < j \leq p$.

Thus a vector $A$ in (3.2) is regular, and one in (3.2) is not regular. As seen later, a vector in (3.3) is also regular. In the following, $A$ always denotes an element of $\alpha \setminus \{0\}$.

By the definition, the tangent space $T_A(G(A))$ of the orbit of $A$ under $G$ is generated by the vectors

$$XA \quad \text{and} \quad AY,$$

where $X$ (resp. $Y$) ranges over all skew-Hermitian matrices of degree $p$ (resp. $q$). In particular, if $A$ is regular, the normal space of $G(A)$ in $\mathbb{C}^{2pq}$ is just $\alpha$, and the normal space of $G(A)$ in $S^{2pq-1}(\vert A \vert)$ consists of all vectors $x_1 e_1 + \cdots + x_p e_p$ satisfying $a_1 x_1 + \cdots + a_p x_p = 0$. 


\[ \cdots + a_p x_p = 0. \]

It is proved in [19] that if \( A \) is regular, for a unit normal vector \( N \) of \( G(A) \) in \( S^{2pq-1}(\|A\|) \), the mean curvature of \( G(A) \) in the direction \( N \) is given by

\[ \frac{-1}{\dim G(A)} \sum_{\lambda \in \Delta} \frac{\lambda(N)}{\lambda(A)}, \]

where the summation is taken according to the multiplicities of \( \lambda \). In particular, if \( A \) is regular, the orbit \( G(A) \) and space \( \pi(G(A)) \) are minimal in \( S^{2pq-1}(\|A\|) \) if and only if

\[ \sum_{\lambda \in \Delta} \frac{\lambda(N)}{\lambda(A)} = 0 \quad \text{for} \quad N = a_i e_1 - a_1 e_i \quad (i = 2, \ldots, p). \quad (3.5) \]

Now, by a theorem of Kitagawa and Ohnita [11] we see that the mean curvature vector field \( \eta(A) \) of the orbit \( G(A) \) in \( \mathbb{C}^{pq} \) is parallel with respect to the normal connection. We denote by \( \eta_s(A) \) the \( S^{2pq-1}(\|A\|) \)-component of \( \eta(A) \). Then we easily see that \( \eta_s(A) \) is the mean curvature vector field of \( G(A) \) in \( S^{2pq-1}(\|A\|) \) and parallel in \( S^{2pq-1}(\|A\|) \). Moreover, by a theorem of Shimizu [17], the mean curvature vector field of the submanifold \( \pi(G(A)) \) is given by \( \pi_s \eta_s(A) \) and parallel in \( P^{pq-1}(\mathbb{C}) \).

Now we are in a position to show (3.1)\( \sim \)(3.3).

**Proof of (3.1).** This is a special case of the results in [17]. Remark that the word *generic* is not used there.

**Proof of (3.2).** By a simple calculation we find that the normal space of \( T_A(G(A)) \) in \( \mathbb{C}^{pq} \) is generated by \( a \) and the following two vectors:

\[
B = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
O & O
\end{bmatrix}, \quad C = \begin{bmatrix}
0 & \sqrt{-1} & O \\
-\sqrt{-1} & 0 & O \\
O & O
\end{bmatrix}.
\]

Thus the space \( \sqrt{-1} a \) and two vectors \( \sqrt{-1} B \) and \( \sqrt{-1} C \) are tangent to \( G(A) \) at \( A \), which implies that the space \( \pi(G(A)) \) is generic in \( P^{pq-1}(\mathbb{C}) \).

**Remark.** Since this \( A \) is not regular, the space is not treated in [17].

**Proof of (3.3).** Put \( A = e_1 + ae_2 + be_3 \), where \( 0 < b < a < 1 \). Then \( A \) is regular. Thus as a basis for the normal space of \( G(A) \) at \( A \) in \( S^{3q-1}(\|A\|) \) we can take

\[ \{ae_1 - e_2, \ be_1 - e_3\}. \]

It follows from (3.4) and (3.5) that the space \( \pi(G(A)) \) is minimal in \( P^{3q-1}(\mathbb{C}) \) if and
only if

\[
\begin{align*}
(3.6) \quad & \left( \frac{q - 5}{2} \right) \left( a - \frac{1}{a} \right) + \frac{a - 1}{a + b} + \frac{a}{1 + a} - \frac{1}{a - b} + \frac{a + 1}{a + b} + \frac{a}{1 - a} - \frac{1}{a - b} = 0, \\
& \left( \frac{q - 5}{2} \right) \left( b - \frac{1}{b} \right) + \frac{b - 1}{b + a} + \frac{b}{1 + a} - \frac{1}{b - a} + \frac{b + 1}{b + a} + \frac{b}{1 - b} - \frac{1}{b - a} = 0.
\end{align*}
\]

For simplicity we put

\[
m := \frac{(2q - 5)}{4}, \quad x := a^2, \quad y := b^2,
\]

\[
X(x, y) := m \left( \frac{1}{x} - 1 \right) - \frac{2}{1 - x} - \frac{1}{1 - y} + \frac{1}{x - y},
\]

\[
U := \{(x, y) \in \mathbb{R}^2; 0 < y < x < 1\}.
\]

Then (3.6) can be rewritten as

\[
(3.7) \quad X(x, y) = 0, \quad X(y, x) = 0, \quad (x, y) \in U.
\]

Now we define a differential mapping \( f \) of \( U \) into \( \mathbb{R}^2 \) by

\[
f(x, y) = (X(x, y), X(y, x)), \quad (x, y) \in U.
\]

It is sufficient to show that \( f(U) \) contains 0. We can easily check the following.

(3.8) The Jacobian matrix of \( f \) is non-singular everywhere. Hence \( f \) is locally diffeomorphic everywhere.

(3.9) For every sequence \( \{p_n\} \) in \( U \) converging to a point of the boundary \( \partial U \) of \( U \),

\[
\lim_{n \to \infty} |f(p_n)| = \infty.
\]

Assume that \( W := \mathbb{R}^2 - f(U) \neq \emptyset \). Then, choose any point \( r \) in \( \partial W \). Let \( \{p_n\} \) be a sequence in \( U \) such that \( f(p_n) \to r \) as \( n \to \infty \). Then there exists a subsequence \( \{p_{n_k}\} \) of \( \{p_n\} \) such that \( \{p_{n_k}\} \) converges to some point of \( U \), say \( p_0 \). If \( p_0 \in U \), then it contradicts (3.8). If \( p_0 \in \partial U \), then it contradicts (3.9). Thus we have shown that there are a point \( (a_0, b_0) \) in \( U \) and a neighbourhood \( V \) of \( (a_0, b_0) \) in \( U \) such that the space \( \pi(G(A)) \) where \( A = a_1 + a_0e_2 + b_0e_3 \) is minimal but for any \( (a, b) \in V - \{(a_0, b_0)\} \) the space \( \pi(G(A)) \) where \( A = a_1 + ae_2 + be_3 \) is not minimal. For an element \( (a, b) \) in \( V \), we denote by \( M(a, b) \) the space \( \pi(G(A)) \) where \( A = a_1 + ae_2 + be_3 \), and by \( \eta(a, b) \) the mean curvature vector field of \( M(a, b) \).

Finally we shall show that \( M(a_0, b_0) \) admits a parallel normal vector field. Since every \( M(a, b) \) is an equivariant homogeneous submanifold in \( \mathbb{P}^{3q-1}(\mathbb{C}) \), the length of its mean curvature vector field is constant. Thus for every \( (a, b) \) in \( V - \{(a_0, b_0)\} \) we
obtain a parallel unit vector field \( \xi(a, b) := \frac{\eta(a, b)}{|\eta(a, b)|} \) on \( M(a, b) \). Since this \( \xi \) is a differentiable vector field on the open subset
\[ \{ p \in M(a, b) \mid (a, b) \in V - \{(a_0, b_0)\} \} \]

of \( P^{3q-1}(\mathbb{C}) \), we obtain a unit vector field on \( M(a_0, b_0) \) as a limit of \( \xi \), say \( \xi_0 \). Since the normal connection \( M(a, b) \) differentiably depends on \( (a, b) \) in \( V \), the vector field \( \xi_0 \) on \( M(a_0, b_0) \) is also parallel.

\[ \square \]

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