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COMPACT MINIMAL GENERIC SUBMANIFOLDS WITH PARALLEL NORMAL SECTION IN A COMPLEX PROJECTIVE SPACE

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Introduction

Generic submanifold have been investigated by many authors (e.g. [5], [7], [8], [9], [21]). Here a submanifold M in a Kaehlerian manifold is called *generic* if each normal space of M is mapped into the tangent space of M by the complex structure of the ambient space (cf. [2], [4], [22]). Any real hypersurface in a Kaehlerian manifold is a typical example of the generic submanifold.

In particular, the model space of the so called A_1, A_2, B, C, D and E-type are typical examples of a real hypersurface in a complex projective space $P(\mathbb{C})$. Recently, the third named author, B. H. Kim and I.-B. Kim [19] proved that those model spaces exhaust all intrinsic homogeneous real hypersurfaces in $P(\mathbb{C})$.

On the other hand, the model spaces of the type A_1 and A_2 was frist introduced by Lawson [13], and he gave a characterization of them. Moreover, Choe and Okumura [5] gave a generalization of Lawson's theorem in [13] from a viewpoint of the CR-submanifold (see §1 for the definiton).

The purpose of the present paper is to give another generalization (Theorem A) of Lawson's theorem, from a viewpoint of the generic submanifold, and to give new examples of a generic submanifold in $P(\mathbb{C})$.

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1. Preliminaries

Let $\widetilde{M}$ be a Kaehlerian manifold of real dimension $n+r$ equipped with an almost complex structure J and a Hermitian metric tensor G. Then for any vector fields X and Y on M, we have

$$J^2X = -X,\ \ G(JX, JY) = G(X, Y),\ \ \widetilde{\nabla} J = 0,$$

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where $\widetilde{\nabla}$ denotes the Riemannian connection of $\widetilde{M}$.

Let M be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$ and isometrically immersed in $\widetilde{M}$ by the immersion $i : M \to \widetilde{M}$. When the argument is local, M need not distinguished from $i(M)$ itself. Throughout this paper the indices i, j, k, $\cdots$ run from 1 to n. We represent the immersion i locally by

$$y^A = y^A(x^h), \quad (A = 1, \cdots, n, \cdots, n+r)$$

and put $B_j{}^A = \partial_j y^A$, $(\partial_j = \partial/\partial x^j)$ then $B_j = (B_j{}^A)$ are n-linearly independent local tangent vectors of M. We choose r-mutually orthogonal unit normals $C_x = (C_x{}^A)$ to M. Hereafter the indices u, v, w, x, $\cdots$ run from $n+1$ to n+r and the summation convention will be used. The immersion being isometric, the induced Riemannian metric tensor g with components g_{ji} and the metric tensor δ with components δ_{yx} of the normal bundle are respectively obtained

$$g_{ji} = G(B_j, B_i), \ \ \delta_{yx} = G(C_y, C_x).$$

By denoting ∇_j the operator of van der Waerden-Bortolotti covariant differentiation with respect to g and G, the equations of Gauss and Weingarten for the submanifold M are respectively given by

$$\nabla_j B_i = A_{ji}{}^x C_x, \ \ \nabla_j C_x = -A_j{}^h{}_x B_h, \tag{1.1}$$

where $A_{ji}{}^x$ are components of the second fundamental tensor and the shape operator A^x in the direction C_x are related by

$$A^x = (A_j{}^{hx}) = (A_{jiy}\, g^{ih}\, \delta^{yx}), \ \ (g^{ji}) = (g_{ji})^{-1}.$$

For $x \in M$ we denotes by $T_x(M)$ and $N_x(M)$ the tangent space and the normal space of M, respectively.

A submanifold M of a Kaehlerian manifold $\widetilde{M}$ is called *CR submanifold* of $\widetilde{M}$ if there exists a differentiable distribution $D : x \to D_x \subset T_x(M)$ on M satisfying the following conditons (see [2], [4], [22]):

(1) D is invariant with respect to J, and

(2) the complementary orthogonal distribution $D^\perp : x \to D^\perp_x \subset T_x(M)$ is totally real with respect to J.

In particular if dim $D^\perp$ = codim$M$, then M is a *generic submanifold* of $\widetilde{M}$(see [8], [20]). If M is a CR submanifold, then the maximal J-invariant subspace $JT_x(M) \cap T_x(M)$ of $T_x(M)$ has constant dimension for $x \in M$ and this constant is called *CR dimension*.

If we assume that M is CR submanifold of CR dimension $n-1$, that is,

$$\dim(JT_x(M) \cap T_x(M)) = n - 1.$$

This implies that there exists a unit vector field C_* normal to M such that $JT(M) \subset T(M) \oplus$ span $\{C_*\}$. Then, we have the following theorem by the first named author and Okumura [5].

Theorem A. *Let M be an n-dimensional compact, minimal CR submanifold of CR dimension $n-1$ of $P^{(n+r)/2}(\mathbb{C})$. If the normal vector field C_* is parallel with respect to the normal connection and scalar curvature $\geq (n+2)(n-1)$, then M is an $M^C_{p,q}$ for some p, q satisfying $2(p+q)=n-1$.*

The model space $M^C_{p,q}$ in the above theorem is described in the following. Let $M_{p,q}$ be the hypersurface in S^{n+2} which is defined by

$$\sum_{j=0}^{p} |z_j|^2 = \cos^2\theta, \quad \sum_{j=p+1}^{p+q+1} |z_j|^2 = \sin^2\theta, \quad 0 < \theta < \frac{\pi}{2}.$$

$M_{p,q}$ is a standard product $S^{2p+1} \times S^{2q+1}$, $2(p+q) = n-1$. The Hopf fibration $\pi : S^{n+2} \to P^{(n+1)/2}(\mathbb{C})$ submerses $M_{p,q}$ onto a real hypersurface of $P^{(n+1)/2}(\mathbb{C})$ which we denote by $M^C_{p,q}$. Cecil-Ryan [3] proved that $M^C_{p,q}$ is a tube of radius θ over a totally geodesic $P^p(\mathbb{C})$, namely, $M^C_{p,q}$ is a homogeneous type A_1 or A_2 [18].

In the following, we assume that M is a generic submanifold of a Kaehlerian manifold. Then our hypothesis implies that the transformations of B_i and C_x by J are respectively represented in each coordinate neighborhood as follows:

$$JB_j = f_j{}^h B_h - J_j{}^x C_x, \quad JC_x = J_x{}^h B_h, \tag{1.2}$$

where we have put $f_{ji} = G(JB_j, B_i)$, $J_{jx} = -G(JB_j, C_x)$, $J_{xj} = G(JC_x, B_j)$, $f_j{}^h = f_{ji}g^{ih}$ and $J_j{}^x = J_{jy}\delta^{yx}$. From these definitions, it follows from (1.2) that

$$f_j{}^t f_t{}^h = -\delta_j{}^h + J_j{}^x J_x{}^h, \quad f_{jt}J_x{}^t = 0, \tag{1.3}$$

$$J_x{}^t J_t{}^z = \delta_x{}^z. \tag{1.4}$$

By differentiating (1.2) covariantly along M, using $\widetilde{\nabla} J=0$, and by comparing the tangential and normal parts, we obtain

$$\nabla_j f_i{}^h = A_{ji}{}^x J_x{}^h - A_j{}^{hx} J_{ix}, \tag{1.5}$$

$$\nabla_j J_{ix} = A_{jtx} f_i{}^t, \tag{1.6}$$

$$A_{jty} J^{tx} = A_{jt}{}^x J_y{}^t. \tag{1.7}$$

If the ambient space $\widetilde{M}$ is a Kaehlerian manifold of constant holomorphic sectional curvature 4, the equations of Gauss, Codazzi and Ricci of M are respectively given by

$$R_{kjih} = g_{kh}g_{ji} - g_{jh}g_{ki} + f_{kh}f_{ji} - f_{jh}f_{ki} - 2f_{kj}f_{ih} + A_{kh}{}^x A_{jix} - A_{jh}{}^x A_{kix}, \tag{1.8}$$

(1.9) $$\nabla_k A_{jix} - \nabla_j A_{kix} = J_{jx} f_{ki} - J_{kx} f_{ji} + 2J_{ix} f_{kj},$$

(1.10) $$R_{jiyx} = J_{jx} J_{iy} - J_{ix} J_{jy} + A_{jtx} A_i{}^t{}_y - A_{itx} A_j{}^t{}_y,$$

where R_{kjih} and R_{jiyx} are components of the Riemannian curvature tensor and those with respect to the connection induced in the normal bundle respectively.

From (1.8) the Ricci tensor S of M is verified that

$$S_{ji} = (n+2)g_{ji} - 3J_j{}^x J_{ix} + h^x A_{jix} - A_{jt}{}^x A_i{}^t{}_x$$

because of (1.3), where h^x=trace A^x. Thus the scalar curvature ρ of M is given by

(1.11) $$\rho = n(n+2) - 3J_{ix} J^{ix} + h_x h^x - A_{jix} A^{jix}$$

since we have (1.4).

In what follows, to write our formula in convention forms $n+1$ denoted by the symbol $*$ and we put $h_{(2)} = A_{ji}{}^* A_*^{ji}$, $(A_{ji}{}^*)^2 = A_{jr}{}^* A_i{}^{r*}$ and $P_{xyz} = A_{jix} J_y{}^j J_z{}^i$. Then P_{xyz} is symmetric for all indices because of (1.7).

2. Parallel normal section

Here we consider the case of a complex projective space $\widetilde{M} = P^{(n+r)/2}(\mathbb{C})$ of constant holomorphic sectional curvature 4. A normal vector field ξ=(ξ^x) is called a *parallel section* in the normal bundle if it satisfies $\nabla_j \xi^x = 0$.

From now on we suppose that M is an n-dimensional compact generic submanifold of $P^{(n+r)/2}(\mathbb{C})$ with parallel unit normal vector field C_* with respect to the normal connection, that is, $\nabla_j^{\perp} C_* = 0$. Then (1.10) shows that R_{ji*x} vanishes identically for any index x and hence

(2.1) $$A_{jtx} A_i{}^t{}_* - A_{itx} A_j{}^t{}_* = J_{j*} J_{ix} - J_{i*} J_{jx},$$

which together with (1.4) and (1.7) implies that

(2.2) $$(J^{j*} A_j{}^{t*})(J^{ix} A_{itx}) = (A_j{}^{tx} J_*{}^j)(A_{itx} J^{i*}) + 1 - r.$$

From (1.5) and (1.6) we have

(2.3) $$\nabla_k \nabla_j J_i{}^* = (\nabla_k A_{jt}{}^*) f_i{}^t + A_{jt}{}^* (A_{ki}{}^x J_x{}^t - A_k{}^{tx} J_{ix}),$$

or, using (1.3), (1.4) and (2.2)

$$J^{i*} \Delta J_{i*} = (A_j{}^{tx} J_{t*})(A^{ji}{}_x J_{i*}) - h_{(2)},$$

where $\Delta = g^{ji} \nabla_j \nabla_i$.

We also have from (2.3)

$$J^{j*}(\nabla_i \nabla_j J^{i*}) = h^x P_{x**} - (A_j{}^{tx} J_{t*})(A^{ji}{}_x J_{i*}) + n - 1,$$

where we have used (1.3), (1.4) and (1.9). From the last two equations, we obtain

$$J^{i*} \Delta J_{i*} + J^{j*}(\nabla_i \nabla_j J^{i*}) = -h_{(2)} + h^x P_{x**} + n - 1. \tag{2.4}$$

Let us put $U_j = J^{i*}\nabla_j J_i{}^* + J^{i*}\nabla_i J_j{}^*$. Then we have

$$\operatorname{div} U = (\nabla_j J_{i*})(\nabla^i J^{j*}) + (\nabla_j J^{i*})(\nabla^j J_{i*}) + J^{i*}\Delta J_{i*} + J^{j*}\nabla^i \nabla_j J_{i*},$$

which together with (1.6) and (2.4) yields

$$\operatorname{div} U = \frac{1}{2} \mid A^* f - f A^* \mid^2 - h_{(2)} + h^x P_{x**} + n - 1. \tag{2.5}$$

On the other hand, we have from (1.4)

$$J_{j*} J^{j*} = 1, \quad J_{jx} J^{jx} = r \tag{2.6}$$

because r is the codimension of M and consequently we obtain

$$J_{j(x)} J^{j(x)} = r - 1, (x) \geq n + 2. \tag{2.7}$$

Thus, (1.11) turns out to be

$$\rho = (n+3)(n-1) - 3J_{j(x)} J^{j(x)} + h_x h^x - h_{(2)} + A_{ji(x)} A^{ji(x)}. \tag{2.8}$$

Lemma 1. *Let M be an n-dimensional generic, minimal submanifold of $P^{(n+r)/2}(\mathbb{C})$ with parallel unit normal C_*. Then we have*

$$\operatorname{div} U = \frac{1}{2} \mid A^* f - f A^* \mid^2 + \rho - (n+2)(n-1) + 3J_{j(x)} J^{j(x)} + A_{ji(x)} A^{ji(x)}. \tag{2.9}$$

Proof. Since M is minimal, it follows, using (2.5), (2.6) and (2.8), that required equation is obtained. This completes the proof. $\square$

Further, suppose that M is compact and the scalar curvature ρ of M satisfies $\rho \geq (n+2)(n-1)$ in Lemma 1, Then we have

$$A^* f = f A^*,$$
$$A_{ji}{}^{(x)} = 0, \quad J_{j(x)} = 0 \quad \text{for all } (x) \geq n+2$$

and $\rho = (n+2)(n-1)$. Thus (2.7) means $r = 1$, that is, M is a real hypersurface of $P^{(n+1)/2}(\mathbb{C})$.

Thus we have

Lemma 2. *Let M be an n-dimensional compact generic, minimal submanifold in $P^{(n+r)/2}(\mathbb{C})$. Suppose that M admits a parallel unit normal vector field C_* and the scalar curvature $\rho \geq (n+2)(n-1)$ on M. Then M is a real hypersurface in $P^{(n+1)/2}(\mathbb{C})$ satisfying $A^* f = f A^*$ and $\rho = (n+2)(n-1)$.*

From Lemma 2 and Theorem 4.4 in [15] due to Okumura, we have

Theorem 3. *Let M be an n-dimensional compact generic, minimal submanifold in $P^{(n+r)/2}(\mathbb{C})$. Suppose that M admits a parallel unit normal vector field and the scalar curvature $\geq (n+2)(n-1)$. Then $r = 1$ and M is an ${M_{p,q}}^C$ for some p, q satisfying $2(p+q) = n-1$.*

3. Examples of generic submanifolds in $P^n(\mathbb{C})$

In this section we shall give two examples of a compact homogeneous generic submanifold in $P^n(\mathbb{C})$, and another example of a compact homogeneous minimal generic submanifold in $P^n(\mathbb{C})$ admitting a parallel normal vector field.

Let p, q $(p \leq q)$ be positive integers. We denote by $M_{p,q}(\mathbb{C})$ the space of $p \times q$ matrices over $\mathbb{C}$, which can be considered as a complex Euclidean space $\mathbb{C}^{pq}$ with the standard Hermitian inner product. Let $U(p)$ denote the unitary group of degree p. Then the Lie group $G := S(U(p) \times U(q))$ acts on $\mathbb{C}^{pq} \equiv M_{p,q}(\mathbb{C})$ as follows:

$$(\sigma, \tau)X = \sigma X \tau^{-1}, \quad (\sigma, \tau) \in G, \ X \in \mathbb{C}^{pq}.$$

Thus we can consider G as a unitary subgroup of $U(pq)$. Remark that this action is nothing but the linear isotropic representation of the compact Hermitian symmetric space $SU(p+q)/S(U(p) \times U(q))$ of type AIII.

Let π be the canonical projection of $\mathbb{C}^{pq} - \{0\}$ onto $P^{pq-1}(\mathbb{C})$, and $S^{2pq-1}(r)$ the hypersphere in $\mathbb{C}^{pq}$ of radius r centered at the origin. Then, for any element A of $\mathbb{C}^{pq} - \{0\}$, the orbit $G(A)$ of A under G is a compact homogeneous submanifold in $S^{2pq-1}(|A|)$, and the space $\pi(G(A))$ is a compact homogeneous submanifolds in $P^{pq-1}(\mathbb{C})$ (see e.g. [19]). Moreover, for any normal vector N of $G(A)$ in $S^{2pq-1}(|A|)$, the mean curvature of $G(A)$ in the direction N is equal to the one of $\pi(G(A))$ in the direction $\pi_* N$ in $P^{pq-1}(\mathbb{C})$. (see e.g. [16]). In particular, $G(A)$ is minimal in $S^{2pq-1}(|A|)$ if and only if $\pi(G(A))$ is minimal in $P^{pq-1}(\mathbb{C})$.

Here, for $i = 1, \cdots, p$ we put

$$e_i := \begin{array}{c} \begin{matrix} & & & i^{th} & & & & & \end{matrix} \\ \begin{bmatrix} 0 & & & \vdots & & & & \vdots & \\ & \ddots & & \vdots & & & & \vdots & \\ & & 0 & \vdots & & & & \vdots & \\ & & & 1 & & & & \vdots & O \\ & & & & 0 & & & \vdots & \\ & & & & & \ddots & & \vdots & \\ & & & & & & 0 & \vdots & \end{bmatrix} \end{array} \in \mathbb{C}^{pq} - \{0\},$$

and denote by $\mathfrak{a}$ the vector space spanned by $e_1, \cdots, e_p$ over $\mathbb{R}$. In the sequel we shall show

(3.1) If $A = a_1e_1 + \cdots + a_pe_p$ satisfies $a_i \neq 0$, $a_i{}^2 \neq a_j{}^2$ for $1 \leq i < j \leq p$, then $\pi(G(A))$ is a $(2pq - p - 1)$-dimensional generic submanifold in $P^{pq-1}(\mathbb{C})$.

(3.2) If $A = e_1+e_2+a_3e_3+\cdots+a_pe_p$ satisfies $a_i{}^2 \neq 0, 1$ and $a_i{}^2 \neq a_j{}^2$ for $3 \leq i < j \leq p$, then $\pi(G(A))$ is a $(2pq - p - 3)$-dimensional generic submanifold in $P^{pq-1}(\mathbb{C})$.

(3.3) Let $p = 3 \leq q$. Then there exists a vector A in $\mathfrak{a} \setminus \{0\}$ such that $\pi(G(A))$ is a $(6q - 4)$ dimensional minimal generic submanifold in $P^{3q-1}(\mathbb{C})$ admitting a parallel normal vector field.

To show these, we need some preparations. Let Δ denote the positive restricted root system associated with the symmetric space $SU(p+q)/S(U(p) \times U(q))$ and $\mathfrak{a}$(cf. [6]). Let $\{\omega_1, \cdots, \omega_p\}$ be the dual basis of $e_1, \cdots, e_p$. Then Δ is given by

(3.4) $$\Delta = \{\omega_i, 2\omega_i, \omega_i \pm \omega_j\,;\ 1 \leq i < j \leq p\}$$

with multiplicities $2(q - p)$, 1, 2, respectively (cf. Helgason [6, 349p] or Araki [1, table]). An element $A = a_1e_1 + \cdots + a_pe_p$ is called regular if $\omega(A) \neq 0$ for any $\omega \in \Delta$, or equivalently $a_i \neq 0$, $a_i{}^2 \neq a_j{}^2$ for $1 \leq i < j \leq p$.
Thus a vector A in (3.2) is regular, and one in (3.2) is not regular. As seen later, a vector in (3.3) is also regular. In the following, A always denotes an element of $\mathfrak{a}\setminus\{0\}$.

By the difinition, the tangent space $T_A(G(A))$ of the orbit of A under G is generated by the vectors

$$XA \quad \text{and} \quad AY,$$

where X (resp. Y) ranges over all skew-Hermitian matrices of degree p (resp. q). In particular, if A is regular, the normal space of $G(A)$ in $\mathbb{C}^{2pq}$ is just $\mathfrak{a}$, and the normal space of $G(A)$ in $S^{2pq-1}(|A|)$ consits of all vectors $x_1e_1 + \cdots + x_pe_p$ satisfying $a_1x_1 +$

$\cdots + a_p x_p = 0$.

It is proved in [19] that if A is regular, for a unit normal vector N of $G(A)$ in $S^{2pq-1}(|A|)$, the mean curvature of $G(A)$ in the direction N is given by

$$\frac{-1}{\dim G(A)} \sum_{\lambda \in \Delta} \frac{\lambda(N)}{\lambda(A)},$$

where the summation is taken according to the multiplicities of λ. In particular, if A is regular, the orbit $G(A)$ and space $\pi(G(A))$ are minimal in $S^{2pq-1}(|A|)$ if and only if

$$\sum_{\lambda \in \Delta} \frac{\lambda(N)}{\lambda(A)} = 0 \quad \text{for } N = a_i e_1 - a_1 e_i \ (i = 2, \cdots, p). \tag{3.5}$$

Now, by a theorem of Kitagawa and Ohnita [11] we see that the mean curvature vector field $\eta(A)$ of the orbit $G(A)$ in $\mathbb{C}^{pq}$ is parallel with respect to the normal connection. We denote by by $\eta_s(A)$ the $S^{2pq-1}(|A|)$-component of $\eta(A)$. Then we easily see that $\eta_s(A)$ is the mean curvature vector field of $G(A)$ in $S^{2pq-1}(|A|)$ and parallel in $S^{2pq-1}(|A|)$. Moreover, by a theorem of Shimizu [17], the mean curvature vector field of the submanifold $\pi(G(A))$ is given by $\pi_* \eta_s(A)$ and parallel in $P^{pq-1}(\mathbb{C})$.

Now we are in a position to show (3.1)~(3.3).

Proof of (3.1). This is a special case of the results in [17]. Remark that the word *generic* is not used there. □

Proof of (3.2). By a simple calculation we find that the normal space of $T_A(G(A))$ in $\mathbb{C}^{pq}$ is generated by $\mathfrak{a}$ and the following two vectors:

$$B = \left[\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & O \\ \hline \multicolumn{2}{c|}{O} & O \end{array}\right], \quad C = \left[\begin{array}{cc|c} 0 & \sqrt{-1} & \\ -\sqrt{-1} & 0 & O \\ \hline \multicolumn{2}{c|}{O} & O \end{array}\right]. \qquad \square$$

Thus the space $\sqrt{-1}\mathfrak{a}$ and two vectors $\sqrt{-1}B$ and $\sqrt{-1}C$ are tangent to $G(A)$ at A, which implies that the space $\pi(G(A))$ is generic in $P^{pq-1}(\mathbb{C})$.

REMARK. Since this A is not regular, the space is not treated in [17].

Proof of (3.3). Put $A = e_1 + ae_2 + be_3$, where $0 < b < a < 1$. Then A is regular. Thus as a basis for the normal space of $G(A)$ at A in $S^{3q-1}(|A|)$ we can take

$$\{ae_1 - e_2, \ be_1 - e_3\}.$$

It follows from (3.4) and (3.5) that the space $\pi(G(A))$ is minimal in $P^{3q-1}(\mathbb{C})$ if and only if

$$(3.6)\quad \begin{cases} \left(q-\frac{5}{2}\right)\left(a-\frac{1}{a}\right)+\frac{a-1}{1+a}+\frac{a}{1+b}-\frac{1}{a+b}+\frac{a+1}{1-a}+\frac{a}{1-b}-\frac{1}{a-b}=0, \\ \left(q-\frac{5}{2}\right)\left(b-\frac{1}{b}\right)+\frac{b-1}{1+b}+\frac{b}{1+a}-\frac{1}{b+a}+\frac{b+1}{1-b}+\frac{b}{1-a}-\frac{1}{b-a}=0. \end{cases}$$

For simplicity we put

$$\begin{aligned} &m := (2q-5)/4, \quad x := a^2, y := b^2, \\ &X(x,y) := m\left(\frac{1}{x}-1\right)-\frac{2}{1-x}-\frac{1}{1-y}+\frac{1}{x-y}, \\ &U := \{(x,y)\in\mathbb{R}^2\,;\, 0<y<x<1\}. \end{aligned}$$

Then (3.6) can be rewritten as

$$(3.7)\qquad X(x,y)=0,\ \ X(y,x)=0,\quad (x,y)\in U.$$

Now we define a differential mapping f of U into $\mathbb{R}^2$ by

$$f(x,y)=(X(x,y),\ X(y,x)),\quad (x,y)\in U.$$

It is sufficient to show that $f(U)$ contains 0. We can easily check the following.

(3.8) The Jacobian matrix of f is non-singular everywhere. Hence f is locally diffeomorphic everywhere.

(3.9) For every sequence $\{p_n\}$ in U converging to a point of the boundary ∂U of U,

$$\lim_{n\to\infty} |\, f(p_n)\,| = \infty.$$

Assume that $W := \mathbb{R}^2 - f(U) \neq \phi$. Then, choose any point r in ∂W. Let $\{p_n\}$ be a sequence in U such that $f(p_n)\to r$ as $n\to\infty$. Then there exists a subsequence $\{p_{n_i}\}$ of $\{p_n\}$ such that $\{p_{n_i}\}$ converges to some point of $\overline{U}$, say p_0. If $p_0\in U$, then it contradicts (3.8). If $p_0\in\partial U$, then it contradicts (3.9). Thus we have shown that there are a point (a_0,b_0) in U and a neighbourhood V of (a_0,b_0) in U such that the space $\pi(G(A))$ where $A=e_1+a_0e_2+b_0e_3$ is minimal but for any $(a,b)\in V-\{(a_0,b_0)\}$ the space $\pi(G(A))$ where $A=e_1+ae_2+be_3$ is not minimal. For an element (a,b) in V, we denote by $M(a,b)$ the space $\pi(G(A))$ where $A=e_1+ae_2+be_3$, and by $\eta(a,b)$ the mean curvature vector field of $M(a,b)$.

Finally we shall show that $M(a_0,b_0)$ admits a parallel normal vector field. Since every $M(a,b)$ is an equivariant homogeneous submanifold in $P^{3q-1}(\mathbb{C})$, the length of its mean curvature vector field is constant. Thus for every (a,b) in $V-\{(a_0,b_0)\}$ we obtain a parallel unit vector field $\xi(a,b) := \eta(a,b)/|\eta(a,b)|$ on $M(a,b)$. Since this ξ is a differentiable vector field on the open subset

$$\{p \in M(a,b) \mid (a,b) \in V - \{(a_0,b_0)\}\}$$

of $P^{3q-1}(\mathbb{C})$, we obtain a unit vector field on $M(a_0,b_0)$ as a limit of ξ, say ξ_0. Since the normal connection $M(a,b)$ differentiably depends on (a,b) in V, the vecctor field ξ_0 on $M(a_0,b_0)$ is also parallel. $\square$

References

[1] S. Araki: *On root systems and an infinitisimal classification of irreducible symmetric spaces*, J. Math. Osaka City Univ. **13** (1962), 1–34.

[2] A. Bejancu: *CR-submanifolds of a Kaehler manifold I*, Proc. Amer. Math. Soc. **69** (1979), 135–142.

[3] T.E. Cecil and P.J. Ryan: *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc. **269** (1982), 481–499.

[4] B.Y. Chen: *CR-submanifolds of a Kaehlerian manifold I, II*, J. Differential Geom. **16** (1981), 305–322, 493–509.

[5] Y.-W. Choe and M. Okumura: *Scalar curvature of a certain CR-submanifold of complex projective space*, Arch. Math. **68** (1997), 340–346.

[6] S. Helgason: Differential Geometry, Lie groups and Symmetric Space, Academic Press, 1978.

[7] U-H. Ki and Y.H. Kim: *Totally real submanifolds of a complex space form*, Int'l. J. Math. and Math. Sci. **19** (1996), 38–44.

[8] U-H. Ki, Y.H. Kim and M. Kon: *Generic submanifolds of a complex space form*, Kyungpook Math. J. **34** (1994), 25–33.

[9] U-H. Ki and M. Kon: *CR submanifolds with parallel mean curvature vector of a complex space form*, Accademia Peloritana dei Pericolanti, **LXX** (1992), 173–192.

[10] U-H. Ki, R. Takagi and T. Takahashi: *On some CR submanifolds with parallel mean curvature vector field in a complex space form*, Nihonkai Math. J. **6** (1995), 215–226.

[11] Y. Kitagawa and Y. Ohnita: *On the mean curvature of R-spaces*, Math. Ann. **262** (1983), 219–243.

[12] M. Kon: *Real minimal hypersurfaces in a complex projective space*, Proc. Amer. Math. Soc. **79** (1980), 285–288.

[13] H.B. Lawson, Jr.: *Rigidity theorems in rank-1 symmetric space*, J. Differential Geom. **4** (1970), 349–357.

[14] Y. Ohnita: *On differential geometrical properties of the standard imbeddings of R-spaces*, report at the Conference of Math. Soc. Japan, (April, 1982), in Japanese.

[15] M. Okumura: *On some real hypersurfaces of a complex projective space*, Trans. Amer. Math. Soc. **212** (1975), 355–364.

[16] M. Okumura: *Totally umbilical submanifolds of a Kaehlerian manifold*, J. Japan Math. Soc. **19** (1967), 317–327.

[17] Y. Shimizu: *On a construction of homogenous CR-submanifolds in a complex projective spaces*, Comm. Math. Univ. Sancti Pauli, **32** (1983), 203–207.

[18] R. Takagi: *Real hypersurfaces in a complex projective space with constant principal curvatures I, II*, J. Math. Soc. Japan, **27** (1975), 43–53, 507–516.

[19] B.H. Kim, I.-B. Kim and R. Takagi: *The rigidity for real hypersurfaces in a complex projective space*, preprint.

[20] R. Takagi and T. Takahashi: *On the principal curvatures of homogeneous hypersurfaces in a sphere*, Differential Geometry; in honor of K. Yano, Kinokuniya, Tokyo (1972), 469–481.

[21] K. Yano and M. Kon: *Generic minimal submanifolds with flat normal connection*, E. B. Christoffel, Aachen Birkäuser Verlag. Basel (1981), 592–599.

[22] K. Yano and M. Kon n: *Structures on submanifolds*, World Scientific Pub. (1984).

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