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## A NOTE ON TODOROV SURFACES

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### Abstract

Let  $S$  be a *Todorov surface*, i.e., a minimal smooth surface of general type with  $q = 0$  and  $p_g = 1$  having an involution  $i$  such that  $S/i$  is birational to a  $K3$  surface and such that the bicanonical map of  $S$  is composed with  $i$ .

The main result of this paper is that, if  $P$  is the minimal smooth model of  $S/i$ , then  $P$  is the minimal desingularization of a double cover of  $\mathbb{P}^2$  ramified over two cubics. Furthermore it is also shown that, given a Todorov surface  $S$ , it is possible to construct Todorov surfaces  $S_j$  with  $K^2 = 1, \dots, K_S^2 - 1$  and such that  $P$  is also the smooth minimal model of  $S_j/i_j$ , where  $i_j$  is the involution of  $S_j$ . Some examples are also given, namely an example different from the examples presented by Todorov in [9].

### 1. Introduction

An *involution* of a surface  $S$  is an automorphism of  $S$  of order 2. We say that a map is *composed with an involution*  $i$  of  $S$  if it factors through the double cover  $S \rightarrow S/i$ . Involutions appear in many contexts in the study of algebraic surfaces. For instance in most cases the bicanonical map of a surface of general type is non-birational only if it is composed with an involution.

Assume that  $S$  is a smooth minimal surface of general type with  $q = 0$  and  $p_g \neq 0$  having bicanonical map  $\phi_2$  composed with an involution  $i$  of  $S$  such that  $S/i$  is non-ruled. Then, according to [10, Theorem 3],  $p_g(S) = 1$ ,  $K_S^2 \leq 8$  and  $S/i$  is birational to a  $K3$  surface (Theorem 3 of [10] contains the assumption  $\deg(\phi_2) = 2$ , but the result is still valid assuming only that  $\phi_2$  is composed with an involution).

Todorov ([9]) was the first to give examples of such surfaces. His construction is as follows. Consider a Kummer surface  $Q$  in  $\mathbb{P}^3$ , i.e., a quartic having as only singularities 16 nodes  $a_i$ . The double cover of  $Q$  ramified over the intersection of  $Q$  with a general quadric and over the 16 nodes of  $Q$  is a surface of general type with  $q = 0$ ,  $p_g = 1$  and  $K^2 = 8$ . Then, choose  $a_1, \dots, a_6$  in general position and let  $G$  be the intersection of  $Q$  with a general quadric through  $j$  of the nodes  $a_1, \dots, a_6$ . The double cover of  $Q$  ramified over  $Q \cap G$  and over the remaining  $16 - j$  nodes of  $Q$  is a surface of general type with  $q = 0$ ,  $p_g = 1$  and  $K^2 = 8 - j$ .

Imposing the passage of the branch curve by a 7-th node, one can obtain a surface with  $K^2 = p_g = 1$  and  $q = 0$ . This is the so-called *Kunev surface*. Todorov ([8]) has

shown that the Kunev surface is a bidouble cover of  $\mathbb{P}^2$  ramified over two cubics and a line.

I refer to [5] for an explicit description of the moduli spaces of Todorov surfaces.

We call *Todorov surfaces* smooth surfaces  $S$  of general type with  $p_g = 1$  and  $q = 0$  having bicanonical map composed with an involution  $i$  of  $S$  such that  $S/i$  is birational to a  $K3$  surface.

In this paper we prove the following:

**Theorem 1.** *Let  $S$  be a Todorov surface with involution  $i$  and  $P$  be the smooth minimal model of  $S/i$ . Then:*

- a) *there exists a generically finite degree 2 morphism  $P \rightarrow \mathbb{P}^2$  ramified over two cubics;*
- b) *for each  $j \in \{1, \dots, K_S^2 - 1\}$ , there is a Todorov surface  $S_j$ , with involution  $i_j$ , such that  $K_{S_j}^2 = j$  and  $P$  is the smooth minimal model of  $S_j/i_j$ .*

The idea of the proof is the following. First we verify that the evenness of the branch locus  $B' + \sum A_i \subset P$  implies that each nodal curve  $A_j$  can only be contained in a Dynkin graph  $G$  of type  $A_{2n+1}$  or  $D_n$ . Then we use a Saint-Donat result to show that  $A_j$  can be chosen such that the linear system  $|B' - G|$  is free. This implies b). Finally we conclude that there is a free linear system  $|B'_0|$  with  $B'_0{}^2 = 2$ , which gives a).

**Notation and conventions.** We work over the complex numbers; all varieties are assumed to be projective algebraic. For a projective smooth surface  $S$ , the *canonical class* is denoted by  $K$ , the *geometric genus* by  $p_g := h^0(S, \mathcal{O}_S(K))$ , the *irregularity* by  $q := h^1(S, \mathcal{O}_S(K))$  and the *Euler characteristic* by  $\chi = \chi(\mathcal{O}_S) = 1 + p_g - q$ .

A *(-2)-curve* or *nodal curve* on a surface is a curve isomorphic to  $\mathbb{P}^1$  such that  $C^2 = -2$ . We say that a curve singularity is *negligible* if it is either a double point or a triple point which resolves to at most a double point after one blow-up.

The rest of the notation is standard in algebraic geometry.

## 2. Preliminaries

The next result follows from [7, (4.1), Theorem 5.2, Propositions 5.6 and 5.7].

**Theorem 2** ([7]). *Let  $|D|$  be a complete linear system on a smooth  $K3$  surface  $F$ , without fixed components and such that  $D^2 \geq 4$ . Denote by  $\varphi_D$  the map given by  $|D|$ . If  $\varphi_D$  is non-birational and the surface  $\varphi_D(F)$  is singular then there exists an elliptic pencil  $|E|$  such that  $ED = 2$  and one of these cases occur:*

- (i)  *$D \equiv \mathcal{O}_F(4E + 2\Gamma)$  where  $\Gamma$  is a smooth rational irreducible curve such that  $\Gamma E = 1$ . In this case  $\varphi_D(F)$  is a cone over a rational normal twisted quartic in  $\mathbb{P}^4$ ;*
- (ii)  *$D \equiv \mathcal{O}_F(3E + 2\Gamma_0 + \Gamma_1)$ , where  $\Gamma_0$  and  $\Gamma_1$  are smooth rational irreducible curves such that  $\Gamma_0 E = 1$ ,  $\Gamma_1 E = 0$  and  $\Gamma_0 \Gamma_1 = 1$ . In this case  $\varphi_D(F)$  is a cone over a rational*

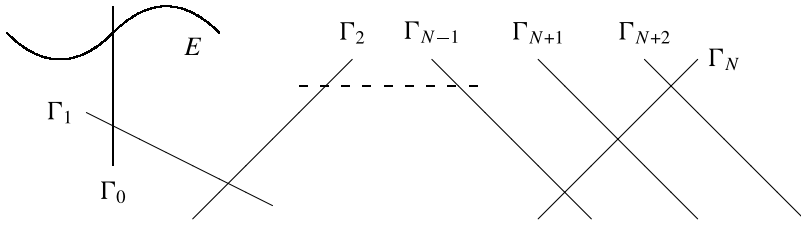


Fig. 1. Configuration (iii) b).

normal twisted cubic in  $\mathbb{P}^3$ ;

(iii) a)  $D \equiv \mathcal{O}_F(2E + \Gamma_0 + \Gamma_1)$ , where  $\Gamma_0$  and  $\Gamma_1$  are smooth rational irreducible curves such that  $\Gamma_0 E = \Gamma_1 E = 1$  and  $\Gamma_0 \Gamma_1 = 0$ ;

b)  $D \equiv \mathcal{O}_F(2E + \Delta)$ , with  $\Delta = 2\Gamma_0 + \dots + 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2}$  ( $N \geq 0$ ), where the curves  $\Gamma_i$  are irreducible rational curves as in Fig. 1.

In both cases  $\varphi_D(F)$  is a quadric cone in  $\mathbb{P}^3$ .

Moreover in all the cases above the pencil  $|E|$  corresponds under the map  $\varphi_D$  to the system of generatrices of  $\varphi_D(F)$ .

### 3. Proof of Theorem 1

We say that a curve  $D$  is *nef* and *big* if  $DC \geq 0$  for every curve  $C$  and  $D^2 > 0$ . In order to prove Theorem 1, we show the following:

**Proposition 3.** Let  $P$  be a smooth K3 surface with a reduced curve  $B$  satisfying:  
 (\*)  $B = B' + \sum_1^t A_i$ ,  $t \in \{9, \dots, 16\}$ , where  $B'$  is a nef and big curve with at most negligible singularities, the curves  $A_i$  are disjoint  $(-2)$ -curves also disjoint from  $B'$  and  $B \equiv 2L$ ,  $L^2 = -4$ , for some  $L \in \text{Pic}(P)$ .

Then:

a) Let  $\pi: V \rightarrow P$  be a double cover with branch locus  $B$  and  $S$  be the smooth minimal model of  $V$ . Then  $q(S) = 0$ ,  $p_g(S) = 1$ ,  $K_S^2 = t - 8$  and the bicanonical map of  $S$  is composed with the involution  $i$  of  $S$  induced by  $\pi$ ;

b) If  $t \geq 10$ , then  $P$  contains a smooth curve  $B'_0$  and  $(-2)$ -curves  $A'_1, \dots, A'_{t-1}$  such that  $B_0'^2 = B'^2 - 2$  and  $B_0 := B'_0 + \sum_1^{t-1} A'_i$  also satisfies condition (\*).

Proof. a) Let  $L \equiv (1/2)B$  be the line bundle which determines  $\pi$ . From the double cover formulas (see e.g. [1]) and the Riemann-Roch theorem,

$$q(S) = h^1(P, \mathcal{O}_P(L)),$$

$$p_g(S) = 1 + h^0(P, \mathcal{O}_P(L)),$$

$$h^0(P, \mathcal{O}_P(L)) + h^0(P, \mathcal{O}_P(-L)) = h^1(P, \mathcal{O}_P(L)).$$

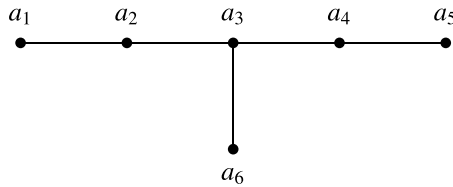


Fig. 2.  $E_6$ .

Since  $2L - \sum A_i$  is nef and big, the Kawamata-Viehweg’s vanishing theorem (see e.g. [3, Corollary 5.12, c)]) implies  $h^1(P, \mathcal{O}_P(-L)) = 0$ . Hence

$$h^1(P, \mathcal{O}_P(L)) = h^1(P, \mathcal{O}_P(K_P - L)) = h^1(P, \mathcal{O}_P(-L)) = 0$$

and then  $q(S) = 0$  and  $p_g(S) = 1$ . As

$$h^0(P, \mathcal{O}_P(2K_P + L)) = h^0(P, \mathcal{O}_P(L)) = 0,$$

the bicanonical map of  $S$  is composed with  $i$  (see [2, Proposition 6.1]).

The  $(-2)$ -curves  $A_1, \dots, A_t$  give rise to  $(-1)$ -curves in  $V$ , therefore

$$K_S^2 = K_V^2 + t = 2(K_P + L)^2 + t = 2L^2 + t = t - 8.$$

b) Denote by  $\xi \subset P$  the set of irreducible curves which do not intersect  $B'$  and denote by  $\xi_i, i \geq 1$ , the connected components of  $\xi$ . Since  $B'^2 \geq 2$ , the Hodge index theorem implies that the intersection matrix of the components of  $\xi_i$  is negative definite. Therefore, following [1, Lemma I.2.12], the  $\xi_i$ ’s have one of the five configurations: the support of  $A_n, D_n, E_6, E_7$  or  $E_8$  (see e.g. [1, III.3] for the description of these graphs).

**Claim 1.** *Each nodal curve  $A_i$  can only be contained in a graph of type  $A_{2n+1}$  or  $D_n$ .*

Proof. Suppose that there exists an  $A_i$  which is contained in a graph of type  $E_6$ . Denote the components of  $E_6$  as in Fig. 2. If  $A_i = a_3$  or  $A_i = a_6$ , then  $a_6B = a_6a_3 = 1$  or  $a_3B = 1$ , contradicting  $B \equiv 2L$ . If  $A_i = a_1$  or  $A_i = a_2$ , then  $a_2B = 1$  or  $a_1B = 1$ , the same contradiction. By the same reason,  $A_i \neq a_4$  and  $A_i \neq a_5$ .

Analogously one can verify that each  $A_i$  can not be in a graph of type  $A_{2n}, E_7$  or  $E_8$ . □

The possible configurations for the curves  $A_i$  in the graphs are shown in Fig. 3. Fix one of the curves  $A_i$  and denote by  $G$  the graph containing it.

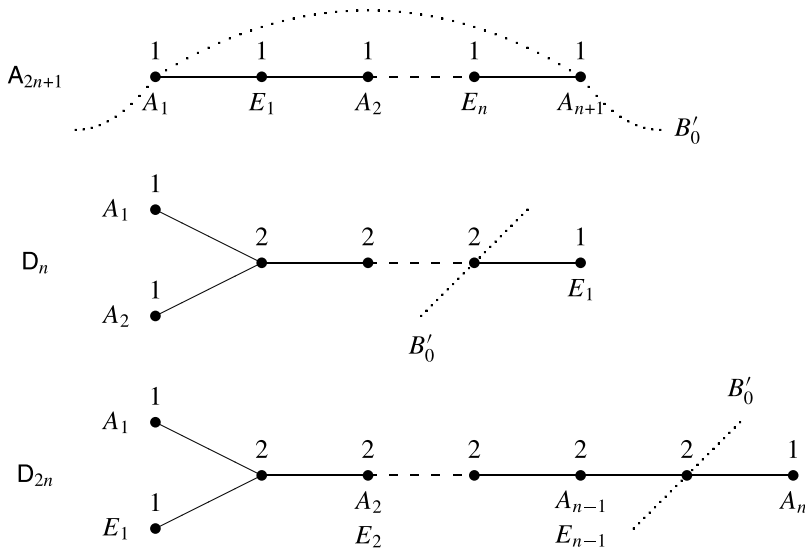


Fig. 3. The numbers represent the multiplicity and the dotted curve represent a general element  $B'_0$  in  $|B' - G|$ .

**Claim 2.** *We can choose  $A_i$  such that the linear system  $|B' - G|$  has no fixed components (and thus no base points, from [7, Theorem 3.1]).*

Proof. Denote by  $\varphi_{|B'|}$  the map given by the linear system  $|B'|$ . We know that  $\varphi_{|B'|}$  is birational or it is of degree 2 (see [7, Section 4]). If  $\varphi_{|B'|}$  is birational or the point  $\varphi_{|B'|}(G)$  is a smooth point of  $\varphi_{|B'|}(P)$ , the result is clear, since  $|B' - G|$  is the pullback of the linear system of the hyperplanes containing  $\varphi_{|B'|}(G)$  and  $\varphi_{|B'|}^*(\varphi_{|B'|}(G)) = G$  (see [1, Theorems III 7.1 and 7.3]).

Suppose now that  $\varphi_{|B'|}$  is non-birational and that  $\varphi_{|B'|}(G)$  is a singular point of  $\varphi_{|B'|}(P)$ . Then  $B'$  is linearly equivalent to a curve with one of the configurations described in Theorem 2. Except for the last configuration,  $G$  contains at most two  $(-2)$ -curves. But  $t \geq 9$ , thus in these cases there exists other graph  $G'$  containing a curve  $A_j$  such that  $\varphi_{|B'|}(G')$  is a non-singular point of  $\varphi_{|B'|}(P)$  (notice that Theorem 2 implies that  $\varphi_{|B'|}(P)$  contains only one singular point).

So we can suppose that  $B'$  is equivalent to a curve with a configuration as in Theorem 2, (iii), b). None of the curves  $\Gamma_0, \dots, \Gamma_N$  can be one of the curves  $A_j$ . For this note that: if  $\Gamma_0 = A_j$ , then  $EB = E(B' + \sum A_i) = 2 + E\Gamma_0 = 3 \not\equiv 0 \pmod{2}$ ; if  $\Gamma_1 = A_j$ , then  $\Gamma_0 B = \Gamma_0 \Gamma_1 = 1 \not\equiv 0 \pmod{2}$ ; etc. Again this configuration can contain at most two curves  $A_j$ , the components  $\Gamma_{N+1}, \Gamma_{N+2}$ . □

Let  $B'_0$  be a smooth curve in  $|B' - G|$ . If  $G$  is an  $A_{2n+1}$  graph, then, using the notation of Fig. 3,

$$\begin{aligned} \left( B'_0 + \sum_1^n E_i \right) + \sum_{n+2}^t A_i &\equiv \left( B' - \sum_1^{n+1} A_i \right) + \sum_{n+2}^t A_i \\ &\equiv B' + \sum_1^t A_i - 2 \sum_1^{n+1} A_i \equiv 0 \pmod{2}. \end{aligned}$$

Therefore the curve

$$B_0 := B'_0 + \sum_1^n E_i + \sum_{n+2}^t A_i$$

satisfies condition (\*).

The case where  $G$  is a  $D_m$  graph is analogous. □

Proof of Theorem 1. Let  $V \rightarrow S$  be the blow-up at the isolated fixed points of the involution  $i$  and  $W$  be the minimal resolution of  $S/i$ . We have a commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & S \\ \pi \downarrow & & \downarrow \\ W & \longrightarrow & S/i. \end{array}$$

The branch locus of  $\pi$  is a smooth curve  $B = B' + \sum_1^t A_i$ , where the curves  $A_i$  are  $(-2)$ -curves which contract to the nodes of  $S/i$ . Let  $P$  be the minimal model of  $W$  and  $\overline{B} \subset P$  be the projection of  $B$ . Let  $L \equiv (1/2)B$  be the line bundle which determines  $\pi$ .

First we verify that  $\overline{B}$  satisfies condition (\*) of Proposition 3: from [2, Proposition 6.1],  $\chi(\mathcal{O}_W) - \chi(\mathcal{O}_S) = K_W(K_W + L)$ , hence  $K_W(K_W + L) = 0$ , which implies that  $\overline{B}$  has at most negligible singularities; now from [5, Theorem 5.2] we get  $K_S^2 = (1/2)\overline{B}^2$  and  $1 = p_g(S) = (1/4)(K_S^2 - t) + 3$ , thus  $t = K_S^2 + 8$  and  $\overline{B}^2 = \overline{B}'^2 - 2t = 2K_S^2 - 2t = -16$ , which gives  $(\overline{B}/2)^2 = -4$  and  $\overline{B}'^2 \geq 2$ ; finally  $\overline{B}'$  is nef because, on a  $K3$  surface, an irreducible curve with negative self intersection must be a  $(-2)$ -curve.

Now using Proposition 3, b) and a) we obtain statement b). In particular we get also that  $P$  contains a curve  $B'_0$  and  $(-2)$ -curves  $A'_i$ ,  $i = 1, \dots, 9$ , such that  $B_0 := B'_0 + \sum_1^9 A'_i$  is smooth and divisible by 2 in the Picard group. Moreover, the complete linear system  $|B'_0|$  has no fixed component nor base points and  $B_0'^2 = 2$ . Therefore, from [7],  $|B'_0|$  defines a generically finite degree 2 morphism

$$\varphi := \varphi_{|B'_0|} : P \rightarrow \mathbb{P}^2.$$

Since  $g(B'_0) = 2$ , this map is ramified over a sextic curve  $\beta$ . The singularities of  $\beta$  are negligible because  $P$  is a  $K3$  surface.

We claim that  $\beta$  is the union of two cubics. Let  $p_i \in \beta$  be the singular point corresponding to  $A'_i$ ,  $i = 1, \dots, 9$ . Notice that the  $p_i$ 's are possibly infinitely near. Let  $C \subset \mathbb{P}^2$  be a cubic curve passing through  $p_i$ ,  $i = 1, \dots, 9$ . As  $C + \varphi_*(B'_0)$  is a plane quartic, we have

$$\left( \varphi^*(C) - \sum_1^9 A'_i \right) + B'_0 + \sum_1^9 A'_i \equiv \varphi^*(C + \varphi_*(B'_0)) \equiv 0 \pmod{2},$$

hence also  $\varphi^*(C) - \sum_1^9 A'_i \equiv 0 \pmod{2}$ , i.e. there exists a divisor  $J$  such that

$$2J \equiv \varphi^*(C) - \sum_1^9 A'_i.$$

Since  $P$  is a  $K_3$  surface, the Riemann-Roch theorem implies that  $J$  is effective. From  $JA'_i = 1$ ,  $i = 1, \dots, 9$ , we obtain that the plane curve  $\varphi_*(J)$  passes with multiplicity 1 through the nine singular points  $p_i$  of  $\beta$ . This immediately implies that  $\varphi_*(J)$  is not a line nor a conic, because  $\beta$  is a reduced sextic. Therefore  $\varphi_*(J)$  is a reduced cubic. So  $\varphi_*(J) \equiv C$  and then

$$\varphi^*(\varphi_*(J)) \equiv 2J + \sum_1^9 A'_i.$$

This implies that  $\varphi_*(J)$  is contained in the branch locus  $\beta$ , which finishes the proof of a). □

#### 4. Examples

Todorov gave examples of surfaces  $S$  with bicanonical image  $\phi_2(S)$  birational to a Kummer surface having only ordinary double points as singularities. The next sections contain an example with  $\phi_2(S)$  non-birational to a Kummer surface and an example with  $\phi_2(S)$  having an  $A_{17}$  double point.

**4.1.  $S/i$  non-birational to a Kummer surface.** Here we construct smooth surfaces  $S$  of general type with  $K^2 = 2, 3$ ,  $p_g = 1$  and  $q = 0$  having bicanonical map of degree 2 onto a  $K3$  surface which is not birational to a Kummer surface.

It is known since [4] that there exist special sets of 6 nodes, called Weber hexads, in the Kummer surface  $Q \in \mathbb{P}^3$  such that the surface which is the blow-up of  $Q$  at these nodes can be embedded in  $\mathbb{P}^3$  as a quartic with 10 nodes. This quartic is the Hessian of a smooth cubic surface.



The space of all smooth cubic surfaces has dimension 4 while the space of Kummer surfaces has dimension 3. Thus it is natural to ask if there exist Hessian “non-Kummer” surfaces, i.e. which are not the embedding of a Kummer surface blown-up at 6 points. This is studied in [6], where the existence of “non-Kummer” quartic Hessians  $H$  in  $\mathbb{P}^3$  is shown. These are surfaces with 10 nodes  $a_i$  such that the projection from one node  $a_1$  to  $\mathbb{P}^2$  is a generically  $2 : 1$  cover of  $\mathbb{P}^2$  with branch locus  $\alpha_1 + \alpha_2$  satisfying:  $\alpha_1, \alpha_2$  are smooth cubics tangent to a nondegenerate conic  $C$  at 3 distinct points. We use this in the following construction.

Let  $\alpha_1, \alpha_2$  and  $C$  be as above. Take the morphism  $\pi : W \rightarrow \mathbb{P}^2$  given by the canonical resolution of the double cover of  $\mathbb{P}^2$  with branch locus  $\alpha_1 + \alpha_2$ . The strict transform of  $C$  gives rise to the union of two disjoint  $(-2)$ -curves  $A_1, A_2 \subset W$  (one of these correspond to the node  $a_1$  from which we have projected).

Let  $T \in \mathbb{P}^2$  be a general line. Let  $A_3, \dots, A_{11} \subset W$  be the disjoint  $(-2)$ -curves contained in  $\pi^*(\alpha_1 + \alpha_2)$ . We have  $\pi^*(T + \alpha_1) \equiv 0 \pmod{2}$ , hence, since  $\alpha_1$  is in the branch locus, also

$$\pi^*(T) + \sum_3^{11} A_i \equiv 0 \pmod{2}.$$

The linear systems  $|\pi^*(T) + A_2|$  and  $|\pi^*(T) + A_1 + A_2|$  have no fixed components nor base points (see [7, (2.7.3) and Corollary 3.2]). The surface  $S$  is the minimal model of the double cover of  $W$  ramified over a general element in

$$|\pi^*(T) + A_2| + \sum_2^{11} A_i$$

or

$$|\pi^*(T) + A_1 + A_2| + \sum_1^{11} A_i.$$

**4.2.  $\phi_2(S)$  with  $A_{17}$  and  $A_1$  singularities.** This section contains a brief description of a construction of a surface  $S$  of general type having bicanonical image  $\phi_2(S) \subset \mathbb{P}^3$  a quartic  $K3$  surface with  $A_{17}$  and  $A_1$  singularities. I omit the details, which were verified using the *Computational Algebra System Magma*.

Let  $C_1$  be a nodal cubic,  $p$  an inflection point of  $C_1$  and  $T$  the tangent line to  $C_1$  at  $p$ . The pencil generated by  $C_1$  and  $3T$  contains another nodal cubic  $C_2$ , smooth at  $p$ . The curves  $C_1$  and  $C_2$  intersect at  $p$  with multiplicity 9.

Let  $\rho : X \rightarrow \mathbb{P}^2$  be the resolution of  $C_1 + C_2$  and  $\pi : W \rightarrow X$  be the double cover with branch locus the strict transform of  $C_1 + C_2$ . Denote by  $\bar{l}$  the line containing the nodes of  $C_1$  and  $C_2$  and by  $l \subset W$  the pullback of the strict transform of  $\bar{l}$ . The map given by  $|(\rho \circ \pi)^*(\bar{l}) + l|$  is birational onto a quartic  $Q$  in  $\mathbb{P}^3$  with an  $A_1$  and  $A_{17}$  singularities (notice that  $l$  is a  $(-2)$ -curve and  $((\rho \circ \pi)^*(\bar{l}) + l) = 0$ ).

Let  $B' \in |(\rho \circ \pi)^*(\bar{l}) + l|$  be a smooth element and  $A_1, \dots, A_9$  be the disjoint  $(-2)$ -curves contained in  $(\rho \circ \pi)^*(p)$ . Let  $S$  be the minimal model of the double cover of  $W$  with branch locus  $B' + \sum_1^9 A_i + l$ . The surface  $Q$  is the image of the bicanonical map of  $S$  and  $p_g(S) = 1$ ,  $q(S) = 0$ ,  $K_S^2 = 2$ .

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