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# *On Monomial Representations of Finite Groups*

By Noboru ITÔ*

In 1933 Shoda obtained remarkable results concerning monomial representations of finite groups [1]. Above all, he established a comprehensible criterion whether a transitive monomial representation of a finite group is irreducible or not, which is of general character; so that it is applicable to imprimitive representations of not necessarily finite groups. Further he proved the precise relation between the degree of a faithful irreducible representation of a metabelian group and the order of a maximal abelian normal subgroup containing the commutator subgroup. Giving alternative proofs to the above results of Shoda with some remarks, we shall show now the following

**Theorem.** *Every irreducible monomial representation of a finite group which is induced by its cyclic subgroup (which is different from the whole group) contains at least one not scalar diagonal matrix.*

## §1.

First of all, for the completeness of the description, we give a proof to a theorem due to Frobenius [2]:

**Proposition 1** (FROBENIUS). *Let $G$ be an irreducible matrix group of finite order and let $N$ be a normal subgroup of $G$. Let $N=r_1\Delta_1+\cdots+r_n\Delta_n$ be the irreducible decomposition of $N$. Then $r_1=\cdots=r_n$ and $\Delta_1,\cdots,\Delta_n$ are $G$-conjugate with each other.*

PROOF. We may assume, by the complete reducibility, that $G$ is transformed into the form in which $N$ is completely reduced: $N=\begin{pmatrix}\Delta^{(1)} & & \\ & \ddots & \\ & & \Delta^{(n)}\end{pmatrix}$, where $\Delta^{(1)}=r_1\Delta_1,\cdots,\Delta^{(n)}=r_n\Delta_n$. Let $X=\begin{pmatrix}X_{11} & \cdots & X_{1n}\\ \vdots & & \vdots \\ X_{n1} & \cdots & X_{nn}\end{pmatrix}$ be any matrix of $G$, where $X_{ij}$ is of type $(\deg\Delta^{(i)}, \deg\Delta^{(j)})$ $(i,j=1,\cdots,n)$. Then we have

$$\begin{pmatrix}X_{11} & \cdots & X_{1n}\\ \vdots & & \vdots \\ X_{n1} & \cdots & X_{nn}\end{pmatrix}\begin{pmatrix}\Delta^{(1)}(Y) & & \\ & \ddots & \\ & & \Delta^{(n)}(Y)\end{pmatrix}=\begin{pmatrix}\Delta^{(1)}(XYX^{-1}) & & \\ & \ddots & \\ & & \Delta^{(1)}(XYX^{-1})\end{pmatrix}\begin{pmatrix}X_{11} & \cdots & X_{1n}\\ \vdots & & \vdots \\ X_{n1} & \cdots & X_{nn}\end{pmatrix}$$

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where $Y$ runs all the matrices of $N$. If there exists an $X$ such that $X_{ik}$, $X_{il} \neq 0$ for some $i$ and for some two $k \neq l$, then we have

$$X_{ik}\Delta^{(k)}(Y) = \Delta^{(i)}(XYX^{-1})\, X_{ik} \text{ and } X_{il}\Delta^{(l)}(Y) = \Delta^{(i)}(XYX^{-1})\, X_{il}\,.$$

Since all the irreducible parts of $\Delta^{(i)}(XYX^{-1})$ are equivalent one another, we come, by the so-called Schur's lemma, to the fact that $\Delta^{(k)}(Y)$ and $\Delta^{l}(Y)$ have at least one equivalent irreducible part, which is clearly a contradiction. Further more finely we see that if $X_{ij} \neq 0$, then, since $X_{ik} = 0$, $k \neq j$ as above and det $X \neq 0$, deg $\Delta^{(i)} =$ deg $\Delta^{(j)}$ and det $X_{ij} \neq 0$. Therefore $\Delta^{(i)}$ and $\Delta^{(j)}$ are $G$-conjugate one another, whence follows $r_i = r_j$ and $\Delta_i$ and $\Delta_j$ are $G$-conjugate one another. Now, by a fundamental relation of Schur, there exists an $X$ such that $X_{ij} \neq 0$, when the pair $i, j$ is arbitrarily given. Thus all the $r_i$ are equal one another and all the $\Delta_i$ are $G$-conjugate one another $(i = 1, \cdots, n)$. This completes the proof.

Next, with a slight modification in the formulation, we give a proof to Shoda's theorem concerning metabelian groups.

**Proposition 2** (Shoda). *Let $G$ be a metabelian group of finite order with a faithful irreducible representation. Then all the maximal abelian normal subgroups $\{A\}$ containing the commutator subgroup possess the same order and all the faithful irreducible representations $\{\Gamma\}$ possess the same degree, and between these two numbers holds the equality:*

$$deg\ \Gamma\ ord\ A = ord\ G\,.$$

*Therefore $G$ can be induced from a suitable linear representation of an arbitrarily given $A$.*

Proof. We take any $\Gamma$. Further we take any $A$ as an $N$ in Proposition 1 and notice that $\Delta_i$ is of degree 1 $(i = 1, \cdots, n)$.
(We use the same notation as in the proof of Proposition 1).

Let $X = \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & & \vdots \\ X_{n1} & \cdots & X_{nn} \end{pmatrix}$ be any matrix of $G$ such that $X_{11} \neq 0$.

Then we have

$$X_{11}\Delta^{(1)}(Y) = \Delta^{(1)}(XYX^{-1})X_{11}\,. \quad (Y \text{ runs all the elements of } A).$$

Since $\Delta^{(1)}(Y)$ and $\Delta^{(1)}(XYX^{-1})$ are scalar matrices, we have

$$\Delta^{(1)}(Y) = \Delta^{(1)}(XYX^{-1}) \qquad (Y \text{ runs all the elements of } A).$$

But this implies that

$$\Delta^{(i)}(Y) = \Delta^{(i)}(XYX^{-1}) \qquad i = 1, \cdots, n\,.$$

In fact, by Proposition 1, there exists an element $Z$ (depending on $i$) of $G$ such that $\Delta^{(i)}(Y)=\Delta^{(1)}(ZYZ^{-1})$ and $\Delta^{(i)}(XYX^{-1})=\Delta^{(1)}(ZXYX^{-1}Z^{-1})$. Now since $A$ contains the commutator subgroup of $G$ and is abelian, we have $\Delta^{(1)}(ZXYX^{-1}Z^{-1})=\Delta^{(1)}(XZYZ^{-1}X^{-1})$. Thus we have the conclusion that $\Delta^{(i)}(Y)=\Delta^{(i)}(XYX^{-1})$. Since $\begin{pmatrix}\Delta^{(1)} & & \\ & \ddots & \\ & & \Delta^{(n)}\end{pmatrix}$ is faithful for $A$, this implies that $Y=XYX^{-1}$. Further $X$ belongs to $A$, because $A$ is a maximal abelian normal subgroup containing the commutator subgroup of $G$. Put $\Gamma(X)=(\gamma_{ij}(X))$. If $r=r_1=\cdots=r_n>1$, then we have, by a fundamental relation of Schur, a contradiction:

$$\sum\gamma_{11}(X)\,\gamma_{22}(X^{-1})=0=\sum\gamma_{11}(X)\,\gamma_{11}(X^{-1})=\operatorname{ord}G\,;\ \deg\Gamma.$$

Therefore $r=1$. Finally we have, by a fundamental relation of Schur,

$$\frac{\operatorname{ord}G}{\deg\Gamma}=\sum_G\gamma_{11}(X)\,\gamma_{11}(X^{-1})=\sum_A\gamma_{11}(Y)\,\gamma_{11}(Y^{-1})=\operatorname{ord}A\,.$$

This completes the proof.

Further we prove the following.

**Proposition 3.** *Let $G$ be an irreducible matric group such that $G$ contains an abelian subgroup $A$ for which the inequality: $\operatorname{ord}A\ \deg G\geqq\operatorname{ord}G$ holds. Then it holds the equality: $\operatorname{ord}A\ \deg G=\operatorname{ord}G$ and $G$ is equivalent with a monomial matric group, which is induced by a suitable linear representation of $A$.*

PROOF. This proof we owe to Prof. T. Nakayama, our original proof was somewhat longer. Considering $G$ itself as a representation of $G$, we transform into a form in which $G(A)$ takes the completely reduced form: $G(A)=r\Delta+\cdots$, where $\deg\Delta=1$. Let $\Delta^*(G)$ be the induced representation of $G$ by $\Delta$. Then, by Frobenius' reciprocity theorem [2], we have $\Delta^*(G)=rG+\cdots$. Since $\deg\Delta^*(G)=\operatorname{ord}G:\ \operatorname{ord}A$, we have $\Delta^*(G)=G$. This completes the proof.

Next we give a proof to Shoda's criterion on the irreduciblity of transitive monomial representations.

**Proposition 4.** (SHODA). *Let $G$ be a transitive monomial matric group of finite order. Considering $G$ itself as a representation of $G$, we put $G(X)=(\gamma_{ij}(X))=X$ for every matrix $X$ of $G$. Then $G$ is irreducible if and only if, for every pair of $i,j$, $i\neq j$, $G$ contains a matrix $X$ (depending on $j,j$) such that $\gamma_{ii}(X)\,\gamma_{jj}(X)\neq0$, $\gamma_{ii}(X)\neq\gamma_{jj}(X)$.*

PROOF. "Only if"-part holds for general irreducible matric groups. In fact, otherwise, by a fundamental relation of Schur, we have $0=\sum_G \gamma_{ii}(X)\gamma_{jj}(X^{-1})=\sum_G \gamma_{ii}(X)\gamma_{ii}(X^{-1})=\operatorname{ord} G:\deg G$. Now we prove "If" part. To do this, let $A=(a_{ij})$ be any commutor of $G$. We take an $X$ such that $\gamma_{ii}(X)\,\gamma_{jj}(X)\neq 0$ and $\gamma_{ii}(X)\neq\gamma_{jj}(X)$. Then we have $XA=AX$. Now equating the $(i,j)$-components of both sides, we have $\gamma_{ii}(X)a_{ij}=a_{ij}\gamma_{jj}(X)$. Thus we come to the fact that $A$ is diagonal. Now since $G$ is transitive, $G$ contains, for every $i$, a matrix $X$ (depending on $i$) such that $\gamma_{ii}(X)\neq 0$. Then we have $XA=AX$. Again equating the $(1,i)$-components of both sides, we have $\gamma_{1i}(X)a_{ii}=a_{11}\gamma_{1i}(X)$. Thus we come to the fact that $A$ is scalar. This completes the proof.

Now in the extremal case: $\operatorname{ord} G=\deg G.\ \operatorname{ord} A$, we can say nothing on the structure of the factor group $G/A$. In fact,

**Example 1.** Let $H$ be the regular representation of any group $H$. Now let $k$ be a natural number $>1$ and let $\rho$ be a primitive $k$-th root of 1. Let $A$ be the totality of the matrices such that $\begin{pmatrix}\rho^{e_1} & & \\ & \ddots & \\ & & \rho^{e_{\operatorname{ord} H}}\end{pmatrix}$, where $0\leqq e_1,\cdots,e_{\operatorname{ord} H}<k$. Then clearly the product $AH=G$ constitutes a group in which $A$ is abelian normal, and holds the equality $\operatorname{ord} G=\deg G\ \operatorname{ord} A$. By Proposition 4, actually $G$ is irreducible.

§2.

Now we treat the question to what extent the orders of two maximal abelian normal subgroups are correlated. First we give the following.

**Example 2.** Let $G$ be a $p$-group of order $p^{2p+2}$ which is defined by the following generators and relations: $A_i^{p^2}=[A_i,A_j]=B^p=C^p=1$, $BA_iB^{-1}=A_i^{1+p}$ $(i=1,\cdots,p)$, $CA_1C^{-1}=A_2,\cdots,CA_pC^{-1}=A_1$. Then $G$ is metabelian and the centre of $G$ is cyclic. Now $\{A_1,\cdots,A_p\}$ and $\{A_1^p,\cdots,A_p^p,B\}$ are two maximal abelian normal subgroups and are of order $p^{2p}$ and $p^{p+1}$ respectively. Thus in Proposition 2 the adjective "containing the commutator subgroup" cannot be omitted.

Now we prove following

**Proposition 5.** *Let $A$ and $B$ be two maximal abelian normal subgroups of a finite group $G$. If the commutator subgroup of $AB$ is cyclic, then the orders of $A$ and $B$ are coincident.*

PROOF. First we remark that if $A$ is maximal abelian normal and if $A_p$ is the $p$-Sylow subgroup of $A$, then $A_p$ is a maximal abelian normal $p$-subgroup. Therefore, to prove the Proposition 5, it is sufficient to prove the following: Let $A$ and $B$ be two maximal abelian normal $p$-subgroups of a finite group. If the commutator subgroup of $AB$ is cyclic, then the orders of $A$ and $B$ are coincident. Now put $H=AB$. We show that $A$ and $B$ are maximal abelian subgroups of $H$. In fact, otherwise, we have, say, the inequality: $K(A)\frown H \gneqq A$, where $K(A)$ is the centralizer of $A$ in $G$. Now since $K(A)\frown H$ is normal in $G$, this contradicts the maximality of $A$. Thus the problem is reduced to that of $p$-groups. So we assume that $G$ is a $p$-group such that $G=AB$, where $A$ and $B$ are maximal abelian (normal) subgroups of $G$. Let $Z(G)$ be the centre of $G$. Then $Z(G)=A\frown B$. In fact, otherwise, $A$ say, is not maximal abelian. Let $D(G)$ be the commutator subgroup of $G$. Then obviously $D(G)\leq A\frown B$ $=Z(G)$. Therefore $G$ is of class 2. Therefore if $A\frown B=Z(G)$ is cyclic, then $A$ and $B$ are of the same order by Proposition 2. So we assume that $A\frown B=Z(G)$ is not cyclic. Let $C$ be a central subgroup of order $p$ such that $C \not\subseteq D(G)$. Let us consider the factor group $\frac{G}{C}=\frac{A}{C}\cdot\frac{B}{C}$. Then $\frac{A}{C}$ and $\frac{B}{C}$ are also maximal abelian (normal) subgroups of $\frac{G}{C}$. In fact, otherwise, say, $\frac{A^*}{C}$ be a maximal abelian normal subgroup of $\frac{G}{C}$ containing $\frac{A}{C}$ properly. Let $a_1^*$ and $a_2^*$ be any two elements of $A^*$. Then $a_1^*a_2^*a_1^{*-1}a_2^{*-1}$ belongs to $C\frown D(G)=1$. This contradicts the maximality of $A$ in $G$. Therefore we have the assertion by virtue of an induction argument.

## § 3.

Now we prove the theorem stated at the beginning, which is, we think, the main result of the present paper. First we give a lemma which is an immediate consequence of Proposition 4.

**Lemma.** *Let $G$ be a transitive monomial matric group of finite order, which is induced by its subgroup $M$. Let $Z(G)$ be the centre of $G$. If $G$ contains an element $X$ such that $M\frown XMX^{-1}\leq Z(G)$, then $G$ is reducible.*

Proof of the Theorem. Let $M$ be a cyclic subgroup of a finite group $G$, which is distinct from $G$. Let $\Gamma$ be a transitive monomial representation of $G$ induced by $M$. Let $\underline{M}$ be the largest normal subgroup of $G$ contained in $M$. Let $Z(G)$ be the centre of $G$. If $\Gamma$ is irreducible, then $\underline{M}$ is not contained in $Z(G)$.

Let $p$ be any prime divisor of the order of $G$. Let $P_p$ be a $p$-Sylow subgroup of $G$. We may assume that $P_p$ is not contained in

$Z(G)$. In fact, otherwise, by Schur's theorem, there exists the $p$-Sylow complement $C_p$ of $G$. Further, by the so-called Schur's lemma $\Gamma(P_p)$ is scalar and therefore $\Gamma(C_p)$ is irreducible. Since $\Gamma(C_p)$ is a transitive monomial representation of $C_p$ induced by $M \frown C_p$, we obtain the assertion by virtue of an induction argument with respect to the order of groups.

Let $M_p$ be the $p$-Sylow subgroup of $M$. If $M_p$ is not contained in $Z(G)$, then we call $p$ an essential prime divisor of the order of $G$. Now we assume that $p$ is essential. Let $Z_p$ be the $p$-Sylow subgroup of $Z(G)$. We denote by $T_p$ a minimal subgroup over $Z_p$ of $M_p$. By Lemma every element of $G$ is contained in the normalizer $N(T_p)$ of $T_p$ for some essential $p$. Thus $G$ admits the set-theoretical decomposition:

$$G = \sum_p N(T_p)$$

where $p$ runs all the essential prime order divisor of the order of $G$. If $N(T_p) = G$ for some essential $p$, then $T_p$ is normal in $G$. This proves the theorem. Therefore we may assume that $N(T_p) \neq G$ for every essential $p$. In particular, we may assume that $M$ is not a $p$-subgroup. Now there exists at least one $p$ for which the index of $N(T_p)$ in $G$ is smaller than the number $R$ of all the essential prime divisors of the order of $G$.

Let $p_1$ be the largest prime divisor of the order of G. Then we have that $G : N(T_p) < R < p_1$. Let $\bar{P}_{p_1}$ be the least normal subgroup of $G$ containing $P_{p_1}$. Representing $G$ as a permutation group of the residue class of $G$ by $N(T_p)$ we immediately see that $N(T_p) \supseteqq \bar{P}_{p_1}$. First we assume that $p \neq p_1$. Since the automorphism group of $T_p$ is cyclic and is of order $(p-1)p^*$, we have $K(T_p) \supseteqq \bar{P}_{p_1}$, where $K(T_p)$ is the centralizer of $T_p$. We call an element $X$ of $N(T_p)$ $p$-essential, if $X$ is not contained in any $N(T_q)$, $q \neq p$. Let $P_{p_1}$ contain a $p$-essential element $A$. Naturally $A$ is not an element of $M$. Let $B$ be any element of $M$ such that $ABA^{-1}$ is contained in $M$, too. Then, as can be easily seen, we have $ABA^{-1} = B$. By virtue of Proposition 4 $\Gamma$ is reducible, which is a contradiction. Therefore $P_{p_1}$ does not contain a $p$-essential element. Thus $P_{p_1}$ admits the set-theoretical decomposition:

$$P_{p_1} = \sum_{q \neq p} N(T_q) \frown P_{p_1}$$

where $q$ runs all the essential prime divisors except $p$ of the order of $G$. Now, as above, there exists at least one $q$ such that $P_{p_1} : P_{p_1} \frown N(T_q) < R-1 < p_1$. This implies that $P_{p_1}$ is contained in $N(T_q)$.

Again we assume that $p_1 \neq q$. Since the automorphism group of $T_q$ is of order $(q-1)q^*$, we have $K(T_q) \supseteqq P_{p_1}$. We call an element $X$ of $N(T_q)$ $\{p, q\}$-essential, if $X$ is not contained in any $N(T_r)$, $r \neq p, q$. Let $P_{p_1}$ contain a $\{p, q\}$-essential element $A$. Naturally $A$ is not an element of $M$. Let $B$ be any element of $M$ such that $ABA^{-1}$ is contained in $M$, too. Then, as can be easily seen, we have $ABA^{-1} = B$. By virtue of Proposition 4 $\Gamma$ is reducible, which is a contradiction. Therefore $P_{p_1}$ does not contain a $\{p, q\}$-essential element. Thus $P_{p_1}$ admits the set-theoretical decomposition:

$$P_{p_1} = \sum_{r \neq p,\, q} N(T_r) \frown P_{p_1}$$

where $r$ runs all the essential prime divisor except $p, q$ of the order of $G$. ············ . Repeating this procedure, we come to the conclusion that either $p_1$ is an essential prime divisor of the order of $G$ and $N(T_{p_1})$ contains $P_{p_1}$ or $p_1$ is not an essential prime divisor of the order of $G$ and $P_{p_1}$ is contained in $K(T_p)$ for every essential $p$. Now we assume that the latter case actually occurs. Let $A$ be an element of $P_{p_1}$ not belonging to $Z_{p_1}$. Let $B$ be an element of $M$ such that $ABA^{-1}$ is contained in $M$, too. Then, as can be easily seen, we have $ABA^{-1} = B$. By virtue of Proposition 4 $\Gamma$ is reducible, which is a contradiction. Thus the latter case does not occur and $p_1$ is an essential prime divisor of the order of $G$ and $N(T_{p_1})$ contains $P_{p_1}$. Then since $P_{p_1}$ may be an arbitrary $p_1$-Sylow subgroup of $G$, we have $\bar{P}_{p_1} \subseteqq N(T_{p_1})$.

Now let us assume that $Z_{p_1} = 1$. Then the order of $T_{p_1}$ is $p_1$. Since $N(T_{p_1}) \supseteqq \bar{P}_{p_1}$ and the automorphism group of $T_{p_1}$ is of order $p_1 - 1$, we have $K(T_{p_1}) \supseteqq \bar{P}_{p_1}$. Let $X$ be an element of $G$ not belonging to $N(T_{p_1})$. Then we have $T_{p_1} \neq XT_{p_1}X^{-1}$ and $K(XT_{p_1}X^{-1}) \supseteqq \bar{P}_{p_1}$. Therefore, in particular, it holds that $[XT_{p_1}X^{-1}, M_{p_1}] = 1$. Put $XT_{p_1}X^{-1} = \{A\}$. Then $A$ is a $p_1$-element not belonging to $M$. Now let $B$ be an element of $M$ such that $ABA^{-1}$ is contained in $M$, too. Then, as can be easily seen, we have $ABA^{-1} = B$. This contradicts the irreducibility of $\Gamma$ in virtue of Proposition 4. Thus $Z_{p_1}$ can not be the identity subgroup. Then since any element of $N(T_{p_1})$ with order prime to $p_1$ must be commutative with any element of $Z_{p_1}$, and since, as can be seen from the just above argument, $K(T_{p_1})$ does not contain $\bar{P}_{p_1}$, we have $N(T_{p_1}) : K(T_{p_1}) = p_1$. In other words, any element of $N(T_{p_1})$ with order prime to $p_1$ belongs to $K(T_{p_1})$.

Let $p_2$ be the largest prime divisor except $p_1$ of the order of $G$. Let $P_{p_2}$ contain a $p_1$-essential element $A$. Let $B$ be an element of $M$

such that $ABA^{-1}$ is contained in $M$, too. Then, as can be easily seen, we have $ABA^{-1}=B$. By virtue of Proposition 4 $\Gamma$ is reducible, which is a contradiction. Thus $P_{p_2}$ does not contain a $p_1$-essential element. Therefore $P_{p_2}$ admits the set-theoretical decomposition:

$$P_{p_2} = \sum_{p \neq p_1} N(T_p) \frown P_{p_2}$$

where $p$ runs all the essential prime divisors except $p_2$ of the order of $G$. Now, as before, there exists at least one $p \neq p_1$ such that $P_{p_2} : P_{p_2} \frown N(T_p) < R-1 \leqq p_2$. This implies that $P_{p_2}$ is contained in $N(T_p)$, etc. Repeating this procedure, we come to the conclusion that every prime divisor $p$ of the order of $G$ is an essential prime divisor of the order of $G$ and that $N(T_p) \supseteqq \bar{P}_p$ and that $Z_p \neq 1$ and that $N(T_p) : K(T_p) = p$.

Let $p_R$ be the least prime divisor of the order of $G$. First we assume that $p_R > 2$. Let $X$ be an element of $G$ not belonging to $N(T_{p_R})$. Since $N(T_{p_R}) \supseteqq \bar{P}_{p_R}$, in other words, since $T_{p_R}$ is normal in $P_{p_R}$, we see that $XT_{p_R}X^{-1}$ is also normal in $\bar{P}_{p_R}$. Now let us consider the product $M_{p_R} \prod_X XT_{p_R}X^{-1}$, where $X$ runs all the elements of $G$ not belonging to $N(T_{p_R})$. Then it is immediately seen that $M_{p_R}$ is normal in this product. Then since the automorphism group of $M_{p_R}$ is cyclic, we have that $\prod_X XT_{p_R}X^{-1} : Z_{p_R} = p_R$. Thus this product is a $p_R$-group containing a cyclic subgroup of index $p_R$. Then, as is well known, the number of subgroups such as $XT_{p_R}X^{-1}$ in this product is at most equal to $p_R$. On the other hand, $N(T_{p_R}) \supseteqq \bar{P}_{p_R}$. Since $p_R$ is the least prime divisor of the order of $G$, this means that $G = N(T_{p_R})$. Because of our assumption that $N(T_{p_R}) \neq G$, this is a contradiction. Thus we have $p_R = 2$. As above, we see that $M_2$. $\overline{T}_2$ is a 2-group containing a cyclic subgroup of index 2, where $\overline{T}_2$ is the least normal subgroup of $G$ containing $T_2$. Then since $Z_2 \neq 1$, as is well known, if $\overline{T}_2$ is not a quaternion group, then, as above, we have $G = N(T_2)$, which is a contradiction. Therefore $\overline{T}_2$ is a quaternion group. Since $N(T_2) \neq G$ and since $N(T_2) \supseteqq P_2$ and further since the automorphism group of a quaternion group is the symmetric group of degree 4, we have $p_{R-1} = 3$, where $p_{R-1}$ is the least prime divisor except $p_R$ of the order of $G$, and moreover $G : N(T_2) = 3$. Now let us assume that $M_2 \neq T_2$. Then let $X$ be an element of $G$ not belonging to $N(T_2)$. Since $N(T_2) \supseteqq \bar{P}_2 \supseteqq XM_2X^{-1}$, we have $[T_2, XT_2X^{-1}] = 1$, which is a contradiction. Therefore we must have $M_2 = T_2$. Further let us assume that $P_2 \neq \bar{M}_2$. Then $P_2$ contains an element $A$ not belonging to $M_2$

such that $A$ is an element of $K(M_2)$, where $K(M_2)$ is the centralizer of $M_2$. Let $B$ be any element of $M$ such that $ABA^{-1}$ is contained in $M$, too. Then, as can be easily seen, we have $ABA^{-1}=B$. By virtue of Proposition 4 $\Gamma$ is reducible, which is a contradiction. Thus we must have $P_2=\bar{M}_2$, in other words, the 2-Sylow subgroup $P_2$ of $G$ is normal in $G$ and is a quaternion group. On the other hand, let us consider $N(T_3)$. Then, as above, we have $G:N(T_3)=2$. More exactly, considering $G$ as an automorphisms group of $\bar{T}_3$, we see $G/K(\bar{T}_3)$, where $K(\bar{T}_3)$ is the centralizer of $\bar{T}_3$, is of even order. Because of the normality of $P_2$, this shows the contradiction. This completes the proof.

Naturally for an arbitrary inducing subgroup the conclusion of this theorem is not always valid. We refer, for instance, to the icosahedral group $A_5$, that is, the alternating group of degree 5; since it seems to us that the example shows us the utility of Shoda's criterion (Proposition 4). We take a tetrahedral subgroup $A_4=\{1,2,3,4\}$ as an inducing subgroup and a four subgroup $V_4=\{1,2,3,4\}$ as its kernel. Then the transitive monomial representation of $A$ thus induced of degree 5 is irreducible. In fact,

$$\begin{aligned}
(15432)\ (123)\ (12345) &= (234) = (132)\ (13)\ (24)\\
(14253)\ (124)\ (13524) &= (134) = (142)\ (12)\ (34)\\
(13523)\ (134)\ (14253) &= (124) = (143)\ (13)\ (24)\\
(12345)\ (234)\ (15432) &= (123) = (243)\ (12)\ (34)
\end{aligned}$$

By virtue of Shoda's criterion, this shows the irreducibility. Since $A_5$ is simple and not abelian, this representation evidently can not contain a not scalar, diagonal matrix.

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