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QUANTUM INVARIANTS CAN PROVIDE SHARP HEEGAARD GENUS BOUNDS

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Abstract

Using Seifert fibered three-manifold examples of Boileau and Zieschang, we demonstrate that the Reshetikhin–Turaev quantum invariants may be used to provide a sharp lower bound on the Heegaard genus which is strictly larger than the rank of the fundamental group.

1. Introduction

For a closed oriented, connected three-manifold M, the Heegaard genus $g(M)$ is defined to be the smallest integer so that M has a Heegaard splitting of that genus. Classically studied, the Heegaard genus is notoriously difficult to compute. In this paper, we investigate the effectiveness of a lower bound on $g(M)$ deriving from the Reshetikhin–Turaev quantum invariants, as was discovered in [6] and in [18].

The Reshetikhin–Turaev quantum invariants for three-manifolds were originally conceived by Witten in [20] as a generalization of the Jones polynomial for knots and links. As such, they allow an algorithmic and combinatorial definition, though the actual calculation is often computationally expensive. Of their known topological applications, the lower bound on $g(M)$ deriving from the quantum invariants may be one of the most powerful and useful.

Until the advent of the quantum invariants, the best known bounds on Heegaard genus came from algebraic topology. For a group G, let its rank $r(G)$ be the minimal number of elements required to generate G. The rank $r(\pi_1 M)$ of the fundamental group of a three-manifold is a lower bound on $g(M)$. By studying the Seifert fibered space examples of Boileau and Zieschang in [4], we show that quantum invariants may be used to provide a lower bound on $g(M)$ which is strictly larger than $r(\pi_1 M)$. Further, in this particular case, the calculation of the quantum invariant is significantly simpler and shorter than the determination of $r(\pi_1 M)$, as appears in [4].

It is shown in [5] that for a random Heegaard splitting of genus $g \geq 2$, the quantum invariants will not provide a sharp lower bound on $g(M)$ with probability approaching 1 as the complexity of the Heegaard splitting increases. Thus, such examples

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as presented here, where the quantum invariant is better than the rank of the fundamental group in determining Heegaard genus, are statistically rare.

We begin with a very brief description of the quantum invariant and the corresponding lower bound on Heegaard genus. We will use the version of the Reshetikhin–Turaev quantum invariant which corresponds to the gauge group $SO(3)$. In Section 3, we show that under appropriate assumptions and choices, the calculations for the $SO(3)$ quantum invariants may be simplified. We end in Section 4 by applying these results to the Boileau–Zieschang examples and determining the Heegaard genus.

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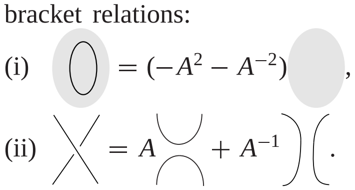
2. A lower bound on Heegaard genus

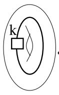
We define the $SO(3)$-quantum invariant for M, with the aim of describing a theorem due to Garoufalidis and Turaev relating the quantum invariants to the study of Heegaard genus. In this paper, we follow the exposition of Lickorish in [14] and of Turaev in [18], although the notation is changed slightly. Before proceeding, we remark that it is also possible to define the set of quantum invariants associated to the gauge group $SU(2)$ and to obtain lower bounds on Heegaard genus from them. The two versions are related by a factor depending on the first betti number.

DEFINITION 1. Let M be an oriented three-manifold and let A be a complex number. The skein space $\mathcal{S}(M)$ is the complex vector space generated by all possible framed links in M up to isotopy of framed links and subject to the following Kauffman bracket relations:

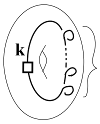


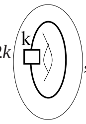
As described pictorially in the definition, relation (i) allows a small circle bounding a disk to be removed, at the cost of introducing a factor $(-A^2 - A^{-2})$. The diagrams in relation (ii) correspond to a small neighborhood of a single framed link, where it is understood that the suppressed remainder is identical in each diagram. For example, note that $\mathcal{S}(S^3) \cong \mathbb{C}$.

For a manifold M, choose a framed link L in S^3 so that surgery along L produces M. The framing on L determines the type of surgery performed, i.e. the curves to which disks are attached. Work of Dehn and Lickorish [13] guarantees such a link exists, and Kirby in [9] further shows that any such link is unique up to link isotopy and the two Kirby moves (stabilizations and handleslides). Following the combinatorial approach of [14], [18] and [3], the $SO(3)$ quantum invariant for M is obtained when each component of L is replaced by an element $\Omega \in \mathcal{S}(S^1 \times D^2)$ which we describe subsequently. Though the definition may seem complicated at first reading, it is because Ω is carefully crafted so that, under a suitable normalization, we obtain invariance under Kirby moves on L.

Let A to be a $2r$-th or $4r$-th primitive root of unity for some odd integer $r \geq 3$. (We will usually choose A to be one of $\pm e^{\pm 2\pi i/4r}$ or $\pm i e^{\pm 2\pi i/4r}$.) In this case, the Temperley–Lieb algebra on a square with n marked points on top and n marked points on bottom may be generated by a set of idempotents called the Jones–Wenzl projectors. By identifying the top and bottom of the square, we obtain elements of $\mathcal{S}(S^1 \times D^2)$ which furthermore form a basis for $\mathcal{S}(S^1 \times D^2)$. We denote the k-th basis element obtained from a Jones–Wenzl projector by a open box labeled by k, drawn as follows: $S_k =$ 

. Due to the following identity regarding the addition of n positive twists,

(1) 

n times $= (-1)^k A^{k^2+2k}$ 

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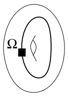
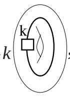
the Jones–Wenzl basis elements are often thought of as eigenvectors for the linear action on the skein space induced by a Dehn twist along a meridian of $S^1 \times D^2$.

The element $\Omega \in \mathcal{S}(S^1 \times D^2)$ is defined as a weighted average of Jones–Wenzl idempotents with *even* labels:

(2) 

$= \mu \sum_{\substack{k=0 \\ k \text{ is even}}}^{r-3} \Delta_k$ 

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where $\mu^2 = (A^2 - A^{-2})^2/(-r)$ and $\Delta_k = (-1)^k (A^{2(k+1)} - A^{-2(k+1)})/(A^2 - A^{-2})$. Δ_k is the evaluation of S_k when embedded into a neighborhood of the unknot with framing 0 in S^3.

Let $\langle \Omega, \ldots, \Omega \rangle_L$ denote the skein in $\mathcal{S}(S^3)$ obtained when Ω is embedded into a neighborhood of each component of the link $L \subset S^3$. Since $\mathcal{S}(S^3) \cong \mathbb{C}$, we can reduce $\langle \Omega, \ldots, \Omega \rangle_L$ to a complex number dependent on A.

Theorem 2 ([10], [2]). *Let $r \geq 3$ be an odd integer, and A be a $2r$-th or $4r$-th primitive root of unity. Let M be the closed three-manifold which results from surgery along a framed link L in S^3, and let $\sigma(L)$ denote the signature of the linking matrix for L. Also let U_- denote the unknot with framing -1 in S^3. Then*

$$I_A(M) = \langle \Omega, \ldots, \Omega \rangle_L (\langle \Omega \rangle_{U_-})^{\sigma(L)}$$

is an invariant of the three-manifold M.

Theorem 2 appeared in various papers, notably [10], [2], and also later in [14] and [3]. The version we state here is presented by Lickorish in [14] and is referred to there as the "invariant with zero spin structure". In Turaev's book [18], it is the "$SO(3)$ quantum invariant". However, we note that the normalization provided above differs by a factor of μ.

As an example, consider the manifold S^3. As it can be obtained from S^3 without any surgery, its corresponding framed link will be the empty link. So $I_A(S^3) = 1$ for us here. Since S^3 can also be obtained by surgery along the unknot with framing 1 in S^3 and $I_A(S^3) = 1$, it follows from the definition of I_A that $\langle\Omega\rangle_{U_-}\langle\Omega\rangle_{U_+} = 1$, where U_- and U_+ denote the unknot with framing -1 and $+1$ respectively. As all the crossings in U_- and U_+ are reversed, $\langle\Omega\rangle_{U_-}$ and $\langle\Omega\rangle_{U_+}$ are conjugates of each other. Thus, $|\langle\Omega\rangle_{U_-}| = 1$.

The $SO(3)$ quantum invariant enjoys many properties; for instance, it behaves well under reversal of orientation and under connect sum. That is,

$$I_A(\overline{M}) = \overline{I_A(M)}$$

and

$$I_A(M_1 \sharp M_2) = I_A(M_1) \cdot I_A(M_2)$$

for three-manifolds M, M_1, and M_2.

Further, for particular choices of A, it can be shown that $I_A(M)$ is related to the Heegaard genus $g(M)$. This may be thought of as a consequence of the $SO(3)$ topological quantum field theory. The proof for the $SU(2)$ version, which is nearly identical as that for the $SO(3)$, may be found in [6] and in [18].

Theorem 3 ([6], [18]). *Let $r \geq 3$ be odd. If $A = e^{\pm 2\pi i/4r}$ or $ie^{\pm 2\pi i/4r}$, then $|I_A(M)| \leq \mu^{-g(M)}$.*

Recall that $\mu^2 = (A^2 - A^{-2})^2/(-r)$. When $A = e^{\pm 2\pi i/4r}$ or $ie^{\pm 2\pi i/4r}$, note that then $0 < \mu = (2/\sqrt{r})\sin(\pi/r) < 1$. Define

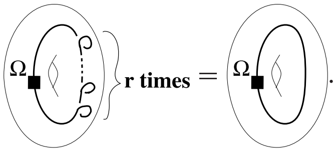
$$q_A(M) = \frac{\log(|I_A(M)|)}{\log(\mu)}$$

so that $q_A(M) \leq g(M)$. In other words, $q_A(M)$ is the lower bound on Heegaard genus provided by the $SO(3)$ quantum invariant.

3. Changing the framing number by r

We present some methods for simplifying the computation of the $SO(3)$ quantum invariants in some special cases. In particular, it is possible for two non-homeomorphic manifolds to have $SO(3)$ quantum invariants with the same value. Similar results appear in [7] and also in [11] for the $SU(2)$ case. All such results rely on a simple observation.

Proposition 4. *The skein element $\Omega \in \mathcal{S}(S^1 \times D^2)$ does not change when any multiple of r twists are added to it.*



Proof. This is an application of equation (1). Because $A^{4r} = 1$, then also $((-1)^k A^{k^2+2k})^r = 1$ whenever k is even. Since the definition of Ω involves only labels k which are even, the result immediately follows. □

Theorem 5 ([7], [11]). *Let $r \geq 5$ be odd and A be a $2r$-th or $4r$-th primitive root of unity. Let L be a link in S^3 with two distinct framings, f and f'. Surgery along L using the framings f and f' will result in two manifolds, which we call M and M' respectively. Suppose that the framings f and f' are congruent modulo r on each component of L. Then*

$$|I_A(M)| = |I_A(M')|.$$

Proof. A difference in framing numbers can be accounted for by introducing a corresponding number of twists into the diagram. In particular, in the blackboard framing, the framing f' can be obtained from f by inserting r twists, possibly more than once, to each link component. Although this changes the signature of the framed link, the absolute value of the quantum invariant is left unchanged because $\langle \Omega \rangle_{U_-}$ has unit norm and because of Proposition 4. □

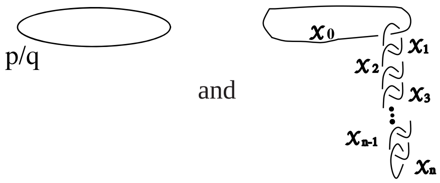
Notice that Theorem 5 is true only at the specified level r. Examples of pairs of manifolds with all values of the $SO(3)$ invariants identical for all choices of level r can be found in [15] and [8].

We next recall a fact from number theory: any rational number $p/q \in \mathbb{Q}$ has a *continued fraction decomposition*, where

$$\frac{p}{q} = x_0 - \cfrac{1}{x_1 - \cfrac{1}{\cdots - \cfrac{1}{x_n}}}.$$

We will denote this by $p/q = [x_0, x_1, \ldots, x_n]$. We will say that two fractions p/q and p'/q' have *entries in their continued fraction decompositions equal modulo r* if $p/q = [x_0, x_1, x_2, \ldots, x_n]$ and $p'/q' = [x_0 + ra_0, x_1 + ra_1, x_2 + ra_2, \ldots, x_n + ra_n]$ for $a_i \in \mathbf{Z}$.

Related by a series of Kirby moves, the following two surgery descriptions



yield the same manifold ([17]). This allows us to convert a rational (p/q)-surgery along a knot into the language of integral surgery along a framed link, thus facilitating computation of the $SO(3)$ quantum invariants.

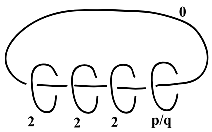
Let L be a link with l components in a three-manifold M. Denote the manifold obtained by (p_i/q_i)-surgery along the i-th component of L $(1 \le i \le l)$ by $M_{\{p_i/q_i\}}$. Theorem 5 has the following corollary.

Corollary 6. *Let $r \ge 5$ be odd and A be a $2r$-th or $4r$-th primitive root of unity. Let L be a link in a three-manifold M. If p_i/q_i and p_i'/q_i' have entries in their continued fraction decomposition equal modulo r, then*

$$|I_A(M_{\{p_i/q_i\}})| = |I_A(M_{\{p_i'/q_i'\}})|.$$

4. Boileau–Zieschang examples

In this section, we focus on a particular set of three-manifolds. Let M be the manifold which corresponds to the surgery presentation given by L below:

$L =$ 

where the continued fraction decomposition of p/q has even length and every other entry is divisible by r, i.e. $p/q = [rx_0, x_1, rx_2, x_3, \dots, x_{2n-1}, rx_{2n}]$ for odd $r \ge 5$ and integers x_i.

Such three-manifolds M are Seifert-fibered. As first noted by Boileau and Zieschang in [4], they are of especial interest because they are examples of the relatively rare phenomenon that $r(\pi_1 M) = 2$ is strictly less than $g(M) = 3$. Recall from Section 2 that the quantum invariants also provide a lower bound on the Heegaard genus, denoted $q_A(M)$. In the remainder of this section, we apply the results of Section 3 to show that $2 < q_A(M)$, and thus also $2 < g(M)$ for the chosen values of p/q. Viewed another way, we show that the quantum lower bound for Heegaard genus can be strictly larger than that provided by the rank of the fundamental group.

Proposition 7. *Let $r \geq 5$ be odd and A be a $2r$-th or $4r$-th primitive root of unity. Let M be the manifold corresponding to surgery using the indicated coefficients along the link L in S^3 described at the beginning of this section. Suppose that $p/q = [rx_0, x_1, rx_2, x_3, \ldots, x_{2n-1}, rx_{2n}]$ for some integers $x_0, \ldots, x_{2n}$. Then*

$$|I_A(M)| = |I_A(\mathbb{R}P^3 \sharp \mathbb{R}P^3 \sharp \mathbb{R}P^3)|.$$

Proof. Because $p/q = [rx_0, x_1, rx_2, x_3, \ldots, x_{2n-1}, rx_{2n}]$, the $SO(3)$ invariant does not change absolute value when we replace the surgery label p/q by $0 = [0, x_1, 0, x_3, \ldots, x_{2n-1}, 0]$ according to Corollary 6. Recall that if in a framed link, one component is an unknot with framing zero which links only one other component geometrically once, then these two components may be deleted from the surgery picture without affecting any of the other remaining framed components. This is a consequence of the Kirby moves. On the other hand, surgery along an unknot with framing 2 produces $\mathbb{R}P^3$. Thus, $|I_A(M)| = |I_A(\mathbb{R}P^3) \sharp I_A(\mathbb{R}P^3) \sharp I_A(\mathbb{R}P^3)|$ by Corollary 6. □

Proposition 8. *When $r \geq 5$ is odd and $A = e^{\pm 2\pi i/4r}$ or $ie^{\pm 2\pi i/4r}$, then*

$$|I_A(\mathbb{R}P^3)| = \frac{\cos(\pi/2r)}{\sin(\pi/r)}.$$

Proof. Let U_{++} denote the unknot with framing number 2 in S^3. In the blackboard framing, this would be drawn as an unknot with two positive twists. Hence, from the basic definitions, we obtain

$$\langle \Omega \rangle_{U_{++}} = \mu \sum_{\substack{k=0 \\ k \text{ is even}}}^{r-3} ((-1)^{k^2+2k})^2 \Delta_k^2.$$

By rearranging the terms and noting that $A^{4r} = 1$, then

$$\langle \Omega \rangle_{U_{++}} = \mu \frac{(A^{-4} - A^{2(r-1)}) \sum_{k=0}^{r-1} A^{8k^2}}{(A^2 - A^{-2})^2}.$$

With $A = e^{\pm 2\pi i/4r}$ or $ie^{\pm 2\pi i/4r}$, both have $\mu^2 = (A^2 - A^{-2})^2/(-r) = 4\sin^2(\pi/r)/r$. A standard result about Gauss sums (see for example [1]) gives $\sum_{k=0}^{r-1} A^{8k^2} = \sum_{k=0}^{r-1} e^{4\pi i k^2/r} = i^{r(r-1)/2}\sqrt{r}$. It follows that $|\langle \Omega \rangle_{U_{++}}| = \cos(\pi/2r)/\sin(\pi/r)$. Since surgery along U_{++} gives $\mathbb{R}P^3$, we have $I_A(\mathbb{R}P^3) = \langle \Omega \rangle_{U_-}^2 \langle \Omega \rangle_{U_{++}}$. Finally, recall that $|\langle \Omega \rangle_{U_-}| = 1$, so $|I_A(\mathbb{R}P^3)| = |\langle \Omega \rangle_{U_{++}}|$. □

Theorem 9. *Let M be the manifold corresponding to surgery using the indicated coefficients along the link L pictured above in* Theorem 7. *Let $r \geq 5$ be odd, and let*

$A = e^{\pm 2\pi i/4r}$ or $ie^{\pm 2\pi i/4r}$. Suppose also that $p/q = [rx_0, x_1, rx_2, x_3, \ldots, x_{2n-1}, rx_{2n}]$ for some integers $x_0, \ldots, x_{2n}$. Then

$$2 < q_A(M) \leq g(M).$$

Proof. First note that since the quantum invariants are multiplicative under connect sum, we have $I_A(\mathbb{R}P^3 \sharp \mathbb{R}P^3 \sharp \mathbb{R}P^3) = I_A(\mathbb{R}P^3)^3$. Recall from Proposition 8 that $|I_A(\mathbb{R}P^3)| = \cos(\pi/2r)/\sin(\pi/r)$. When $r \geq 5$, a quick calculus argument shows that $\mu^{-2} = r/(4\sin^2(\pi/r)) < (\cos(\pi/2r)/\sin(\pi/r))^3 = |I_A(M)|$, and so $2 < q_A(M)$. $\square$

References

[1] B.C. Berndt, R.J. Evans and K.S. Williams: Gauss and Jacobi Sums, Canadian Mathematical Society Series of Monographs and Advanced Texts, Wiley, New York, 1998.

[2] C. Blanchet: *Invariants on three-manifolds with spin structure*, Comment. Math. Helv. **67** (1992), 406–427.

[3] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel: *Topological quantum field theories derived from the Kauffman bracket*, Topology **34** (1995), 883–927.

[4] M. Boileau and H. Zieschang: *Heegaard genus of closed orientable Seifert* 3*-manifolds*, Invent. Math. **76** (1984), 455–468.

[5] N. Dunfield and H. Wong: *Quantum invariants of random Heegaard splittings*, in preparation.

[6] S. Garoufalidis: *Applications of quantum invariants in low-dimensional topology*, Topology **37** (1998), 219–224.

[7] P.M. Gilmer: *Congruence and quantum invariants of* 3*-manifolds*, Algebr. Geom. Topol. **7** (2007), 1767–1790.

[8] J. Kania-Bartoszynska, *Examples of different* 3*-manifolds with the same invariants of Witten and Reshetikhin–Turaev*, Topology **32** (1993), 47–54.

[9] R. Kirby: *A calculus for framed links in* S^3, Invent. Math. **45** (1978), 35–56.

[10] R. Kirby and P. Melvin: *The* 3*-manifold invariants of Witten and Reshetikhin–Turaev for* sl(2, **C**), Invent. Math. 105 (1991), 473–545.

[11] M. Lackenby: *Fox's congruence classes and the quantum*-SU(2) *invariants of links in* 3*-manifolds*, Comment. Math. Helv. **71** (1996), 664–677.

[12] W.B.R. Lickorish: *Quantum invariants of* 3*-manifolds*; in Handbook of Geometric Topology, North-Holland, Amsterdam, 707–734, 2002.

[13] W.B.R. Lickorish: *A representation of orientable combinatorial* 3*-manifolds*, Ann. of Math. (2) **76** (1962), 531–540.

[14] W.B.R. Lickorish: *The skein method for three-manifold invariants*, J. Knot Theory Ramifications **2** (1993), 171–194.

[15] W.B.R. Lickorish: *Distinct* 3*-manifolds with all* $\mathrm{SU}(2)_q$ *invariants the same*, Proc. Amer. Math. Soc. **117** (1993), 285–292.

[16] N. Reshetikhin and V.G. Turaev: *Invariants of* 3*-manifolds via link polynomials and quantum groups*, Invent. Math. **103** (1991), 547–597.

[17] D. Rolfsen: Knots and Links, Mathematics Lecture Series **7**, Publish or Perish, Berkeley, 1976.

[18] V.G. Turaev: Quantum Invariants of Knots and 3-Manifolds, de Gruyter Studies in Mathematics **18**, de Gruyter, Berlin, 1994.

[19] F. Waldhausen: *Heegaard–Zerlegungen der* 3*-Sphäre*, Topology **7** (1968), 195–203.

[20] E. Witten: *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989), 351–399.

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