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SELF-BUMPINGS ON KLEINIAN ONCE-PUNCTURED TORUS GROUPS

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Abstract

We give a new proof of the characterization of self-bumping points on the space of Kleinian once-punctured torus groups of Ito [8], based on some recent work of Bromberg [4].

1. Introduction

Let F be a finite type hyperbolic surface, $AH(F)$ be the set of discrete faithful representations of $\pi_1(F)$ into $PSL_2(\mathbb{C})$ up to conjugacy, under the condition that each loop around a cusp of F is mapped to a parabolic element in $PSL_2(\mathbb{C})$. Then $AH(F)$ admits a natural topology, named the algebraic topology. The interior of $AH(F)$ is parameterized by the product of two Teichmüller spaces, i.e., the quasi-Fuchsian space $\mathcal{QF}(F) = \mathcal{T}(F) \times \mathcal{T}(F)$, due to works of Ahlfors and Bers, see [10]. But the boundary of $AH(F)$ is fairly complicated. For example, there are self-bumping points in the boundary of $AH(F)$ [2, 12, 5], and further more, there are non-locally-connected points in the boundary of $AH(F)$ [4, 9]. We refer the reader to Canary [6] for a survey on the pathological phenomena of the deformation spaces of hyperbolic structures on general hyperbolic 3-manifolds.

In this note, we consider the simplest case, $F = F_{1,1}$, the once-punctured torus. A point $\rho \in \partial AH(F_{1,1})$ is *self-bumping* if for any neighborhood U of ρ in $AH(F_{1,1})$, there is a small neighborhood V of ρ contained in U, such that $V \cap (\mathcal{T}(F_{1,1}) \times \mathcal{T}(F_{1,1}))$ is disconnected.

There is a one-to-one correspondence between the set of simple closed curves in $F_{1,1}$ and $\mathbb{Q} \cup \infty$. We fix two oriented curves α (corresponding to ∞) and β (corresponding to 0) in $F_{1,1}$ which intersect in one point, they give us a marking of $\pi_1(F_{1,1})$. Then $AH(F_{1,1})$ is also the deformation space of marked hyperbolic structures on $F_{1,1} \times (-1, 1)$, with a rank-one cusp corresponding to the cusp of $F_{1,1}$.

Take $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B_\mu = \begin{pmatrix} i\mu & i \\ i & 0 \end{pmatrix}$ in $PSL_2(\mathbb{C})$, then A and B_μ generate a free group $G_\mu = \langle A, B_\mu \rangle$ in $AH(F_{1,1})$ for suitable $\mu \in \mathbb{C}$, where $AB_\mu A^{-1} B_\mu^{-1}$ is a parabolic element which corresponds to the cusp of $F_{1,1}$.

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The set

$$\{\mu \in \mathbb{C} \mid \Im(\mu) > 0,\ \langle A, B_\mu \rangle \in AH(F_{1,1})\} \tag{1.1}$$

is called the *Maskit slice*, we denote it by $\mathscr{M}$, which is homeomorphic to $\overline{\mathbb{H}^2} - \infty$ (see [13]), but its boundary in $\mathbb{C}$ is highly complicated. In particular, the boundary is fractal-like. By [15], if $\Im(\mu) \geq 2$, then $\mu \in \mathscr{M}$.

The set

$$\{\mu \in \mathbb{C} \mid \mu = (p+1)x + py,\ x, y \in \mathscr{M}\} \tag{1.2}$$

is denoted by $\mathscr{M}(p)$ for $p \in \mathbb{N}$, then from a simple calculation, we have $\mathscr{M}(p) \subseteq \mathscr{M}(1) \subset \mathscr{M}$, see [8].

For any $\mu \in \operatorname{int}\mathscr{M}$, the manifold $M_{G_\mu} = (\mathbb{H}^3 \cup \Omega_{G_\mu})/G_\mu$ has two conformal boundaries, where Ω_{G_μ} is the set of discontinuity of G_μ on $\partial\mathbb{H}^3 = \hat{\mathbb{C}}$. The thrice-punctured sphere boundary of M_{G_μ} is quasi-conformally rigid, but the once-punctured torus boundary is quasi-conformally flexible, and every marked conformal structure on $F_{1,1}$ is realizable by a unique $\mu \in \operatorname{int}\mathscr{M}$. M_{G_μ} is a hyperbolic manifolds in $AH(F_{1,1})$ which has one more rank-one cusp than manifolds in $\operatorname{int}AH(F_{1,1})$, the new cusp corresponds to the parabolic element A. In other words, we can view $\mathscr{M} \subset \partial AH(F_{1,1})$ if we choose the marking on $F_{1,1}$ suitably, i.e., when $\rho(\alpha) = A$ and $\rho(\beta) = B_\mu$.

Our aim here is to reprove the following theorem of Ito [8]:

Theorem 1.1. *$\mu \in \mathscr{M}$ corresponds to a self-bumping point on $AH(F_{1,1})$ if and only if $\mu \in \mathscr{M}(1)$.*

Given Theorem 1.1, it is easy to give a complete characterization of self-bumping points on the space of $AH(F_{1,1})$ as what have done by Ito [8], see Theorem 2.4 in next section.

Comparing to the proof in [8], where Ito emphasizes on exotic convergence sequences of [2], our proof here depends on the result in [4], i.e., the local homeomorphism between the deformation space and the (nearly) product of a sub-manifold of the Maskit slice and a sub-space of $\hat{\mathbb{C}}$. See also [14] for the generalization of Ito's result to high genus Kleinian surface groups.

2. Proof of the theorem

For a point $x \in \mathscr{M}$, we denote by $\mathscr{S}^\circ_{x,k}$ the solution space of

$$\{z \in \mathbb{C} \mid x - kz \in \operatorname{int}\mathscr{M}, x - (k+1)z \in \operatorname{int}\mathscr{M}_-\} \tag{2.1}$$

for $k \in \mathbb{Z}_{\geq 0}$, and we denote by $\mathscr{S}_{x,k}$ the solution space of

$$\{z \in \mathbb{C} \mid x - kz \in \mathscr{M}, x - (k+1)z \in \mathscr{M}_-\} \tag{2.2}$$

for $k \in \mathbb{Z}_{\geq 0}$, where $\mathscr{M}_- = -\mathscr{M}$ as a subset of $\mathbb{C}$. Note that $\mathscr{S}_{x,i} \cap \mathscr{S}_{x,j} = \emptyset$ for any $0 \leq i < j$. If $\mathscr{S}^\circ_{x,k}$ is non-empty, then the closure of it is contained in $\mathscr{S}_{x,k}$. A priori, it is possible that $\mathscr{S}^\circ_{x,k} = \emptyset$ but $\mathscr{S}_{x,k} \neq \emptyset$, for example, if $\mathscr{S}_{x,k}$ is a sequence of discrete points in $\mathbb{C}$.

Proposition 2.1. *A point $x \in \mathscr{M}$ corresponds to a self-bumping point of $AH(F_{1,1})$ if and only if $\mathscr{S}_{x,k}$ is non-empty for some $k \in \mathbb{N}$.*

Proof. We first assume that $\mathscr{S}^\circ_{x,k}$ is non-empty for some $k \in \mathbb{N}$. Since for any $y \in \partial\mathscr{M}$, we have $1 \leq \Im(y) \leq 2$ (see [15]). Then a simple calculation shows that $\Im(x) \geq 3$ provided $\mathscr{S}^\circ_{x,k}$ is non-empty for some $k \in \mathbb{N}$, thus we have $x \in \operatorname{int}\mathscr{M}$. We note that $\mathscr{S}^\circ_{x,0}$ is non-empty, which is a translation of $\operatorname{int}\mathscr{M}$, so it is homeomorphic to an open half-space of $\mathbb{C}$. It is easy to see that for any $y \in \mathbb{C}$, we have $y \in \mathscr{M}$ if and only if $(y+2) \in \mathscr{M}$.

If the solution space $\mathscr{S}^\circ_{x,k}$ is non-empty for some $k \in \mathbb{N}$, by [15] and (2.1), for any $z \in \mathscr{S}^\circ_{x,k}$, we have

$$\frac{\Im(x+1)}{k+1} \leq \Im(z) \leq \frac{\Im(x-1)}{k}. \tag{2.3}$$

So $\mathscr{S}^\circ_{x,k}$ is contained in an infinite length horizontal strip in $\mathbb{C}$ of bounded width, $\mathscr{S}^\circ_{x,k}$ is invariant under the translation $z \to z+2$, and which is a 2-dimensional manifold (may be disconnected) in $\mathbb{C}$. In particular, $\mathscr{S}^\circ_{x,0}$ and $\mathscr{S}^\circ_{x,k}$ approach to $\infty \in \hat{\mathbb{C}}$ simultaneously, this gives us the 2-dimensional bumping of all solution spaces corresponding to x.

Now we can use a result of Bromberg (Theorem 4.13 of [4]), which said that there is a small neighborhood U of x in $\mathscr{M}$, such that there is a local homeomorphism Φ between

$$\{(y, w) \mid y \in U,\ w \in \overline{\sqcup_{k\in\mathbb{Z}_{\geq 0}}\mathscr{S}_{y,k}}\} \tag{2.4}$$

and $AH(F_{1,1})$ at (x,∞), where $\overline{\sqcup_{k\in\mathbb{Z}_{\geq 0}}\mathscr{S}_{y,k}}$ is the one-point compactification of $\sqcup_{k\in\mathbb{Z}_{\geq 0}}\mathscr{S}_{y,k}$ by the point $\infty = \infty_y$ for each y. The local homeomorphism Φ maps each (y, ∞_y) to the manifold M_{G_y}, maps each (y, w) to a manifold $\Phi((y, w))$ in $AH(F_{1,1})$, which is obtained from a preferred Dehn filling of a manifold $M_{y,w}$. The holonomy group of $M_{y,w}$ is generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} iy & i \\ i & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$. $M_{y,w}$ is a manifold with a rank-two cusp, the Dehn filling can be performed if $|w|/\sqrt{2\Im w}$ is large enough. $\Phi((y, w))$ lies in $\operatorname{int} AH(F_{1,1})$ if $w \in \mathscr{S}^\circ_{y,k}$, and $\Phi((y, w))$ lies in $\partial AH(F_{1,1})$ if $w \in \mathscr{S}_{y,k} - \mathscr{S}^\circ_{y,k}$.

The solution space $\mathscr{S}^\circ_{y,k}$ is continuous on y, so if $\mathscr{S}^\circ_{x,k}$ is non-empty for some $k \in \mathbb{N}$, then $\mathscr{S}^\circ_{y,k}$ is non-empty for any y which is near to x enough. Then from the 2-dimensional bumping of all solution spaces corresponding to y, it is easy to see the self-bumping of $\Phi((x, \infty)) = M_{G_x}$ on $AH(F_{1,1})$.

If $\mathscr{S}^{\circ}_{x,k}$ is empty for any $k \in \mathbb{N}$, but $\mathscr{S}_{x,k}$ is non-empty for some $k \in \mathbb{N}$, we claim that for any small neighborhood U of x in $\mathscr{M}$, there is a 2-manifold V in U, such that x lies in the boundary of V, and for any y in $\operatorname{int} V$, $\mathscr{S}^{\circ}_{y,k}$ is non-empty. This is true from the Bers–Sullivan–Thurston density theorem in our case: $AH(F_{1,1})$ is the closure of $\operatorname{int} AH(F_{1,1})$, which is from Minsky [13]. Now, if $\mathscr{S}^{\circ}_{y,k}$ is empty for any y in U which is near to x, then $\Phi((x, w))$ is a manifold in $\partial AH(F_{1,1})$ which can not be approached by a sequence of quasi-Fuchsian manifolds for $w \in \mathscr{S}_{x,k}$ when $\Re(w)$ is large enough (due to the fact that $w \in \mathscr{S}_{x,k}$ if and only if $(w+2) \in \mathscr{S}_{x,k}$, we can assume $\Re(w)$ is large enough, and then $|w|/\sqrt{2\Im w}$ is large enough). Since $\operatorname{int} AH(F_{1,1})$ is 4-dimensional, $\operatorname{int} \mathscr{M}$ is 2-dimensional, and $\mathbb{C}$ is 2-dimensional, our claim holds. Then, similar to the case that $\mathscr{S}^{\circ}_{x,k}$ is non-empty, since $\mathscr{S}^{\circ}_{y,k}$ is non-empty for each $y \in \operatorname{int} V$, we have x corresponds to a self-bumping point of $\Phi((x, \infty)) = M_{G_x}$ on $AH(F_{1,1})$.

Conversely, if $\mathscr{S}_{x,k}$ is empty for all $k \in \mathbb{N}$, then $\mathscr{S}_{y,k}$ is empty for all $k \in \mathbb{N}$ and y near enough to x, then again by the theorem of Bromberg, we have x does not correspond to a self-bumping point of $AH(F_{1,1})$. □

Proof of Theorem 1.1. If $x \in \mathscr{M}(1)$, then by definition $x = 2u + v$ for some $u, v \in \mathscr{M}$. Let $z = u + v$, then $x - z \in \mathscr{M}$ and $x - 2z \in \mathscr{M}_-$. In other words, $\mathscr{S}_{x,1} \neq \emptyset$, so x corresponds to a self-bumping point by Proposition 2.1.

Conversely, if x corresponds to a self-bumping point, then $\mathscr{S}_{x,k}$ is non-empty for some $k \in \mathbb{N}$, i.e., there is a $z \in \mathbb{C}$ such that $a + bi = x - kz \in \mathscr{M}$ and $c + di = x - (k+1)z \in \mathscr{M}_-$. Now let $u = a + bi \in \mathscr{M}$ and $v = -c - di \in \mathscr{M}$, then $(k+1)u + kv = x$. So $x \in \mathscr{M}(k) \subseteq \mathscr{M}(1)$. □

Proposition 2.2. *If $\rho \in \partial AH(F_{1,1})$ is a self-bumping point, then ρ is geometrically finite and has exactly one accidental parabolic class.*

Proof. Recall that a representation $\rho \in AH(F_{1,1})$ is *non-wrapping* provided that for any sequence of representations ρ_n converging algebraically to ρ and geometrically to a Kleinian group Γ, then there is a compact core for M_ρ which embeds in M_Γ under the covering map.

If ρ is not geometrically finite with exactly one accidental parabolic class, i.e., if ρ is geometrically infinite or ρ is geometrically finite but ∂M_ρ is quasi-conformally rigid, then there is no accidental parabolic class since ρ is an once-punctured torus group. From [1], Corollary B, we have ρ is a non-wrapping group. Then by Theorem 3 of [7], non-wrapping implies non-self-bumping on once-punctured torus Kleinian groups. □

REMARK 2.3. Proposition 2.2 can also be obtained from the proof of Theorem 1.1 and Theorem 1.3 of [3], even in the statement of their theorems the hyperbolic 3-manifold M should be compact with incompressible boundary or an trivial I-bundle over a closed

surface S, but the proofs can extend word-by-word to the deformation space of once-punctured torus Kleinian groups. The proofs use some results on hierarchy path of Masur–Minsky [11], which holds in the once-punctured torus case more simply than the general case. Actually, this approach is overkilled.

From Proposition 2.2, if ρ is a self-bumping point, then $\operatorname{int} M_\rho$ is homeomorphic to $F_{1,1} \times (-1, 1)$ and ∂M_ρ has two components, one of them is homeomorphic to $F_{1,1}$ and the other one is a thrice-punctured sphere, which corresponds to cutting $F_{1,1}$ along a curve, we denote it by $\alpha = \infty$. So M_ρ is isometric to M_{G_μ} for some $\mu \in \mathscr{M}$ by the Ending Lamination Theorem of once-punctured torus groups [13] (re-choosing a marking on M_ρ, and the isometry may be orientation-reversing), then from Theorem 1.1 and Proposition 2.2, all self-bumping points in $AH(F_{1,1})$ can be characterized, see also Section 7 of [8] for a more precise formulation:

Theorem 2.4. *$\rho \in \partial AH(F_{1,1})$ is a self-bumping point in $AH(F_{1,1})$ if and only if M_ρ is isometric to M_{G_μ} for some $\mu \in \mathscr{M}(1)$.*

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