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A NOTE ON KNOTS WITH H(2)-UNKNOTTING NUMBER ONE

YUANYUAN BAO

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Abstract

We give an obstruction to unknotting a knot by adding a twisted band, derived from Heegaard Floer homology.

1. Introduction

Many unknotting operations have been defined and studied in knot theory. For example, as well-known, (a), (b) (cf. [8, 10]) and (c) in Fig. 1 are three types of unknotting operations. Especially, (c) was introduced by Hoste, Nakanishi and Taniyama [4], which they called H(n)-move. Here n is the number of arcs inside the circle. Note that an H(n)-move is required to preserve the component number of the diagram. The H(n)-unknotting number of a knot is the minimal number of H(n)-moves needed to change the knot into the unknot. In this note, we focus on the special case when n equals two. Given two knots K and K', when K' is obtained from K by applying an H(2)-move, we also alternatively say that K' is obtained from K by adding a twisted band, as shown in Fig. 2. Following [4], we denote the H(2)-unknotting number of a knot K by u_2(K). In this note, we give a necessary condition for a knot K to have u_2(K) = 1, by using a method introduced by Ozsváth and Szabó [15].

The question whether a given knot has H(2)-unknotting number one should be traced back to Riley. He made the conjecture that the figure-eight knot could never be unknotted by adding a twisted band. Lickorish confirmed this conjecture in [7]. Here we give a brief review of his method. Given a knot K, let Σ(K) denote the double-branched cover of S^3 along K and let λ: H_1(Σ(K), Z) × H_1(Σ(K), Z) → Q/Z be the linking form of Σ(K). Lickorish proved that if the knot K can be unknotted by adding a twisted band, then H_1(Σ(K), Z) is cyclic and it has a generator g such that λ(g, g) = ±1/det(K), where det(K) is the determinant of K. For the figure-eight knot 4_1, the linking form has the form λ(g, g) = 2/5 for some generator g ∈ H_1(Σ(4_1)) ⊆ Z/5Z. If there is another generator g' = xg such that λ(g', g') = ±1/5, we have 2x^2 ≡ ±1 (mod 5), while there is no such an integer x satisfying the condition. Therefore Riley’s conjecture holds.

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Fig. 1. Some unknotting operations.

Fig. 2. Adding a twisted band to a knot diagram.
Fig. 3. The sign convention of a crossing.

Now we turn to the description of our result. Consider a negative-definite symmetric \( n \times n \) matrix \( Q \) over \( \mathbb{Z} \), and suppose \( |\det(Q)| = p \). Then define a group

\[
G_Q := \mathbb{Z}^n / \text{Im}(Q).
\]

A characteristic vector for \( Q \) is an element in

\[
\text{char}(Q) = \{ \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{Z}^n \mid \xi^t v \equiv v' Q v \text{ (mod 2)} \text{ for any } v \in \mathbb{Z}^n \}
\]

\[
= \{ \xi \in \mathbb{Z}^n \mid \xi_i \equiv Q_{ii} \text{ (mod 2)} \text{ for } 1 \leq i \leq n \}.
\]

Suppose \( p \) is odd, and consider the map (cf. [12, 15])

\[
M_Q : G_Q \to \mathbb{Q}
\]

defined by

\[
M_Q(\alpha) = \max \left\{ \frac{\xi^t Q^{-1} \xi + n}{4} \left| \xi \in \text{char}(Q), \ [\xi] = \alpha \in G_Q \right. \right\}.
\]

Now we recall the definition of Goeritz matrix. Given a knot diagram, color this diagram in checkerboard fashion such that the unbounded region has black color. Let \( f_0, f_1, \ldots, f_k \) denote the black regions and \( f_0 \) correspond to the unbounded one. Define the sign of a crossing as in Fig. 3. Then the Goeritz matrix \( A \) is the \( k \times k \) symmetric matrix defined as follows,

\[
q_{ij} = \begin{cases} 
\text{the signed count of crossings adjacent to } f_i & \text{if } i = j, \\
\text{minus the signed count of crossings joining } f_i \text{ and } f_j & \text{if } i \neq j
\end{cases}
\]

for \( i, j = 1, 2, \ldots, k \).

Our result about \( H(2) \)-unknotting number is as follows:
Theorem 1.1. Let $K$ be an alternating knot with $|\det K| = p$, and let $A$ be the negative-definite Goeritz matrix corresponding to a reduced alternating diagram of $K$ or its mirror image. Since $K$ is a knot, we see that $p$ is an odd number. Suppose $G_A$ is the group presented by $A$. If $u_2(K) = 1$, then there is an isomorphism $\mathbb{Z}/p\mathbb{Z} \rightarrow G_A$ and a sign $\epsilon \in \{+1, -1\}$ with the properties that for all $i \in \mathbb{Z}/p\mathbb{Z}$:

$$I_{\phi, \epsilon}(i) := \epsilon \cdot M_A(\phi(i)) + \frac{1}{4} \left( \frac{1}{p} \left( \frac{p + (-1)^i p}{2} - i \right)^2 - 1 \right) \equiv 0 \pmod{2},$$

and

$$I_{\phi, \epsilon}(i) \geq 0.$$

Here we abuse $i$ to denote both the element in $\mathbb{Z}/p\mathbb{Z}$ and its representative in the set $\{0, 1, 2, \ldots, p-1\}$.

If one is familiar with the work in [15], the proof is immediate. We will give the proof in Section 2.

The $H(2)$-unknotting number of a knot is an interesting knot invariant. It is closely related to the 3-dimensional and 4-dimensional crosscap numbers of a knot. It can be defined in some different viewpoints, as indicated by Taniyama and Yasuhara [17]. Many researches concerning it can be found in [18, 6, 1] and other papers.

In order to check that Theorem 1.1 works better in some cases than the existing criteria, we post the knot $P(13, 4, 11)$ as an example. We determine that it has $H(2)$-unknotting number 2, which cannot seem to be detected by the other methods that the author knows.

Corollary 1.2. The pretzel knot $P(13, 4, 11)$ has $H(2)$-unknotting number 2.

2. Proofs

2.1. Preliminaries. Almost all the ingredients contained in this subsection can be found in [15], or an earlier paper [13]. But for intactness, we include them here. If $X$ is an oriented 3- or 4-manifold, the second cohomology $H^2(X, \mathbb{Z})$ acts on the set of spin$^c$-structures $\text{Spin}^c(X)$ freely and transitively. Each spin$^c$-structure $s \in \text{Spin}^c(X)$ has the first Chern class $c_1(s) \in H^2(X, \mathbb{Z})$, and the relation to the action is $c_1(s + h) = c_1(s) + 2h$ for any $h \in H^2(X, \mathbb{Z})$.

Let $Y$ be an oriented rational homology 3-sphere and $s$ be a spin$^c$-structure over $Y$. Then there is Heegaard Floer homology associated with the pair $(Y, s)$. In this note, we use Heegaard Floer homology with coefficients in the field $\mathbb{F} := \mathbb{Z}/2\mathbb{Z}$. There are several versions of this homology. One version is $HF^+(Y, s)$, which is a $\mathbb{Q}$-graded
module over the polynomial algebra $\mathbb{F}[U]$. That is

$$HF^+(Y, s) = \bigoplus_{i \in \mathbb{Q}} HF^+_i(Y, s),$$

where multiplication by $U$ lowers the grading by two. In each grading $i \in \mathbb{Q}$, $HF^+_i(Y, s)$ is a finite-dimensional $\mathbb{F}$-vector space. A simpler version is $HF^\infty(Y)$, and it satisfies $HF^\infty(Y, s) = \mathbb{F}[U, U^{-1}]$ for each $s \in \text{Spin}^c(Y)$ [14, Theorem 10.1]. It is also $\mathbb{Q}$-graded and multiplication by $U$ lowers its grading by two.

For any spin$^c$-structure $s$, there is a natural $\mathbb{F}[U]$-equivariant map

$$\pi : HF^\infty(Y, s) \to HF^+(Y, s),$$

which preserves the $\mathbb{Q}$-grading. We use $\pi_i$ to denote the restriction of $\pi$ on the grading $i$. Then $\pi_i$ is zero for all sufficiently negative gradings and an isomorphism in all sufficiently positive gradings. Ozsváth and Szabó defined an invariant $d(Y, s)$ from the map $\pi$, which is called the correction term of the pair $(Y, s)$. Precisely, we have

$$d(Y, s) := \min\{i \in \mathbb{Q} \mid \pi_i \text{ is non-zero}\}.$$

The correction terms for $Y$ and $-Y$, where “$-$” means the reversion of orientation, are related by the formula

$$d(-Y, s) = -d(Y, s)$$

under the natural identification $\text{Spin}^c(Y) \cong \text{Spin}^c(-Y)$.

The map $\pi$ behaves naturally under cobordisms. Let $Y_1$ and $Y_2$ be two oriented rational homology 3-spheres. We say a smooth connected oriented 4-manifold $X$ is a cobordism from $Y_1$ to $Y_2$ if the boundary of $X$ is given by $\partial X = -Y_1 \cup Y_2$. Suppose $X$ is a cobordism from $Y_1$ to $Y_2$ and $t$ is a spin$^c$-structure of $X$. Then there is a homomorphism

$$F^o_{X, t} : HF^o(Y_1, s_1) \to HF^o(Y_2, s_2),$$

where $HF^o$ denotes any version of Heegaard Floer homology and $s_i$ is the restriction of $t$ to $Y_i$ for $i = 1, 2$ (we simply express it as $s_i = t|_{Y_i}$). The map $\pi$ and the map $F^o_{X, t}$ fit into the following commutative diagram:

$$\begin{array}{ccc}
HF^\infty(Y_1, s_1) & \xrightarrow{F^\infty_{X, t}} & HF^\infty(Y_2, s_2) \\
\pi \downarrow & & \downarrow \pi \\
HF^+(Y_1, s_1) & \xrightarrow{F^+_{X, t}} & HF^+(Y_2, s_2).
\end{array}$$
If $X$ is a negative-definite cobordism, the proof of Theorem 9.1 in [13] (also mentioned in the proof of [13, Proposition 9.9]) tells us that $F_{X,t}^\infty$ is an isomorphism.

Suppose that $Y$ is an oriented rational homology 3-sphere, that $X$ is a negative-definite simply connected 4-manifold with $\partial X = Y$ and that $t \in \text{Spin}^c(X)$. Then it is shown in [13] that

\begin{align}
(1) \quad d(Y, t|_Y) &\geq \frac{c_1^2(t) + b_2(X)}{4}, \\
(2) \quad d(Y, t|_Y) &\equiv \frac{c_1^2(t) + b_2(X)}{4} \pmod{2}.
\end{align}

Here (1) follows directly from [13, Theorem 9.6], while (2) is not clearly written. For readers’ convenience, we explain it here. Consider $X$ minus a point as a cobordism $W$ from $S^3$ to $Y$. Then we have the following commutative diagram

\[
\begin{array}{ccc}
HF^\infty(S^3, t|_{S^3}) & \xrightarrow{F_{W,t}^\infty} & HF^\infty(Y, t|_Y) \\
\pi \downarrow & & \downarrow \pi \\
HF^+(S^3, t|_{S^3}) & \xrightarrow{F_{W,t}^+} & HF^+(Y, t|_Y),
\end{array}
\]

and $F_{W,t}^\infty$ is an isomorphism. There is an element $\xi \in HF^\infty(Y, t|_Y)$ with the property that its $\mathbb{Q}$-grading $gr(\xi)$ is $d(Y, t|_Y)$. Suppose the preimage of $\xi$ in $HF^\infty(S^3, t|_{S^3})$ is $\eta$. Then we have

\[
d(Y, t|_Y) - gr(\eta) = gr(\xi) - gr(\eta) = \frac{c_1^2(t) - 2\chi(W) - 3\sigma(W) + b_2(X)}{4} = \frac{c_1^2(t) + b_2(X)}{4}.
\]

The first equality follows from our choice of $\xi$, the second one follows from Equation (4) in [13], and the last one holds because of the fact that $2\chi(W) + 3\sigma(W) + b_2(X) = 0$. Precisely we have

\[
2\chi(W) + 3\sigma(W) + b_2(X) \\
= 2(b_0(W) - b_1(W) + b_2(W) - b_3(W) + b_4(W)) - 3b_2(W) + b_2(W) \\
= 2(b_0(W) - b_1(W) - b_3(W) + b_4(W)) \\
= 2(b_0(W) - 2b_1(W) - 1 + b_4(W)) = 0.
\]

Here $b_i(W)$ denotes the $i$-th Betti number of $W$. The first equality comes from our assumption that $X$ is negative-definite. The third equality follows from the fact that $b_3(W) = b_1(W) + 1$, obtained from the relation $H_3(W) \cong H_3(W, S^3 \cup Y) \oplus \mathbb{Z}$, Poincaré duality and the universal coefficient theorem. The last equality comes from the facts that $b_0(W) = 1$ and $b_4(W) = 0$, and our assumption that $X$ is simply connected. For
the $3$-sphere $S^3$, as an $\mathbb{F}$-vector space, we know that ([14, Theorem 10.1])

$$HF^\infty(S^3, t|_{S^3}) = \bigoplus_{i=-\infty}^{\infty} \mathbb{F}(2i),$$

where $\mathbb{F}(j)$ denotes the summand supported on grading $j$. Therefore we see that $gr(\eta) = 0 \pmod{2}$. Now (2) follows.

Remember that $d(S^3, t|_{S^3}) = 0$ and that $HF^\infty(S^3, t|_{S^3}) = \mathbb{F}[U, U^{-1}]$, and therefore we obtain $gr(\eta) = 0 \pmod{2}$. Now (2) follows obviously.

Suppose further for simplicity that $X$ is simply-connected and that the order of $H^2(Y, \mathbb{Z})$ is odd. Then there exists a group structure on the space $\text{Spin}^c(Y)$ by identifying $s \in \text{Spin}^c(Y)$ with $c_1(s) \in H^2(Y, \mathbb{Z})$. In the following, we denote the correction term $d(Y, s)$ by $d(Y, c_1(s))$ if necessary. Let $r$ denote the second Betti number of $X$.

Then we have the following exact sequence:

$$0 \to H_2(X) = \mathbb{Z}^r \xrightarrow{j^*} H^2(X) = \mathbb{Z}^r \xrightarrow{j^*} H^2(Y) \to H_1(X) = 0.$$ 

Fix a basis for $H_2(X)$ and let $B$ be the matrix of the intersection form of $X$. Then $B$ is a symmetric negative-definite $r \times r$ integer matrix with $|\text{det } B| = |H^2(Y, \mathbb{Z})|$. A spin$^c$-structure $s \in \text{Spin}^c(Y)$ is the restriction of a spin$^c$-structure $t \in \text{Spin}^c(X)$ on $Y$ if and only if $j^*(c_1(t)) = c_1(s)$.

In fact, the map $\tau$ under the given basis of $H_2(X)$ is presented by the matrix $B$. We define $\varphi$ as the map $\text{Coker}(\tau) = G_B \xrightarrow{j^*} H^2(Y)$, where $j^*$ is the map induced from $j^*$ on the cokernel of $\tau$. It is obvious from the exact sequence that $\varphi$ is an isomorphism. Under $\varphi$ the set of characteristic vectors char($B$) is equal to the set \{ $c_1(t) \mid t \in \text{Spin}^c(X)$ \} $\subset H^2(X, \mathbb{Z})$. If we suppose the first Chern class $c_1(t)$ corresponds to the characteristic vector $\xi$, then $c_1(t)$ is equal to $\xi^t B^{-1} \xi$.

Under these identifications, (1) and (2) can be written as follows. For any $s \in \text{Spin}^c(Y)$ and any $\xi \in \text{char}(B)$ with $c_1(s) = \varphi([\xi])$, there are

$$d(Y, c_1(s)) \geq \frac{\xi^t B^{-1} \xi + r}{4}$$

and

$$d(Y, c_1(s)) = \frac{\xi^t B^{-1} \xi + r}{4} \pmod{2}. $$

This is equivalent to say under the isomorphism $\varphi: G_B \to H^2(Y, \mathbb{Z})$ the following hold for any $\alpha \in G_B$:

$$d(Y, \varphi(\alpha)) \geq M_B(\alpha),$$

$$d(Y, \varphi(\alpha)) = M_B(\alpha) \pmod{2}. $$

(3)
2.2. Proof of Theorem 1.1. When $K$ is an alternating knot in $S^3$, the correction terms for $\Sigma(K)$ have an extremely easy combinatorial description as follows.

**Theorem 2.1** (Ozsváth–Szabó [15, 16]). If $K$ is an alternating knot and $A$ denotes a Goeritz matrix associated to a reduced alternating projection of $K$, and $G_A$ is the group presented by $A$, then there is an isomorphism $\psi : H^2(\Sigma(K), \mathbb{Z}) \to G_A$, with the property that

$$d(\Sigma(K), \beta) = M_A(\psi(\beta))$$

for all $\beta \in H^2(\Sigma(K), \mathbb{Z})$.

For knots with $H(2)$-unknotting number one, we have the following lemma.

**Lemma 2.2** (Montesinos’s trick [9]). If the $H(2)$-unknotting number of a knot $K$ is one, then $\Sigma(K) = \epsilon \cdot S^3_{-p}(C)$ for some knot $C \subset S^3$ and $\epsilon \in \{+1, -1\}$. Here $p = |\det(K)|$ and $S^3_{-p}(C)$ denotes the $-p$-surgery of $S^3$ along the knot $C$.

Proof of Theorem 1.1. If the $H(2)$-unknotting number of $K$ is one, then by Lemma 2.2 $\Sigma(K) = \epsilon \cdot S^3_{-p}(C)$ for some knot $C \subset S^3$ and $\epsilon \in \{+1, -1\}$ and $p = |\det(K)|$. Therefore $\epsilon \cdot \Sigma(K) = S^3_{-p}(C)$ bounds a 4-manifold $X$, which is obtained by attaching a 2-handle to the 4-ball along $C$ with framing $-p$. The intersection form of $X$ is $B = (-p)$. In this case $G_B = \mathbb{Z}/p\mathbb{Z}$, and $X$ is a simply-connected negative-definite 4-manifold.

By (3), there exists a group isomorphism $\varphi : G_B = \mathbb{Z}/p\mathbb{Z} \to H^2(S^3_{-p}(C), \mathbb{Z})$ with

$$d(S^3_{-p}(C), \varphi(i)) = d(\epsilon \cdot \Sigma(K), \varphi(i)) = \epsilon \cdot d(\Sigma(K), \varphi(i)) \geq M_B(i)$$

and

$$\epsilon \cdot d(\Sigma(K), \varphi(i)) \equiv M_B(i) \pmod{2}$$

Theorem 2.1 implies that for the map $\phi = \psi \circ \varphi : \mathbb{Z}/p\mathbb{Z} \to G_A$ (here we automatically identify $H^2(S^3_{-p}(C), \mathbb{Z})$ with $H^2(\Sigma(K), \mathbb{Z})$) we have

$$\epsilon \cdot M_A(\phi(i)) \geq M_B(i)$$

and

$$\epsilon \cdot M_A(\phi(i)) \equiv M_B(i) \pmod{2}$$

for all $i \in \mathbb{Z}/p\mathbb{Z}$. In the following calculation, we abuse $i$ to denote both the element in $\mathbb{Z}/p\mathbb{Z}$ and its representative in the set $\{0, 1, 2, \ldots, p-1\}$. By definition we see that
for any $i \in \mathbb{Z}/p\mathbb{Z}$,

$$M_B(i) = \max \left\{ \frac{u'B^{-1}u + 1}{4} \mid u \text{ is odd, } [u] = i \right\}$$

$$= \max \left\{ \frac{-u^2 + p}{4p} \mid u \text{ is odd, } [u] = i \right\}$$

$$= \begin{cases} 
\frac{-(p - i)^2 + p}{4p} & \text{if } i \text{ is even,} \\
\frac{-(i)^2 + p}{4p} & \text{if } i \text{ is odd.}
\end{cases}$$

Writing these two cases in one form we have $M_B(i) = -(1/4)((1/p)((p + (-1)^i p)/2 - i)^2 - 1)$. This completes the proof of Theorem 1.1.

**2.3. An example: proof of Corollary 1.2.** The pretzel knot $K = P(13, 4, 11)$ is an alternating knot as shown in Fig. 4. A negative-definite Goeritz matrix associated with the mirror image of this diagram is

$$A = \begin{pmatrix} -17 & 4 \\ 4 & -15 \end{pmatrix},$$

and the determinant is $\det(A) = \det(K) = 239$. Suppose $G_A$ is the group presented by $A$. In fact, the group $G_A$ is isomorphic to $\mathbb{Z}/239\mathbb{Z}$. In the following calculation, we take the vector $(0, 1)^t$ as a generator of $G_A$. The inverse of the matrix $A$ is

$$A^{-1} = \frac{1}{239} \begin{pmatrix} -15 & -4 \\ -4 & -17 \end{pmatrix}.$$
Then by definition for any $0 \leq r \leq 238$ it holds that

$$M_A((0, r)') = \max \left\{ \frac{(u, v)'A^{-1}(u, v) + 2}{4} \left| (u, v)' \in \text{char}(A), [(u, v)'] = (0, r)' \in G_A \right. \right\}$$

$$= \max \left\{ \frac{478 - (15u^2 + 8uv + 17v^2)}{956} \left| u \text{ and } v \text{ are odd}, [(u, v)'] = (0, r)' \in G_A \right. \right\}.$$  

From this expression, we see that in order to obtain the maximum we only need to focus on those representatives $(u, v)'$ satisfying $|u| \leq 17$ and $|v| \leq 15$.

By calculation, it is easy to see that for any isomorphism $\phi : \mathbb{Z}/239\mathbb{Z} \rightarrow \mathbb{Z}/239\mathbb{Z}$ there is

$$I_{\phi, \epsilon}(0) = \epsilon \cdot M_A(\phi(0)) + \frac{119}{2} = \epsilon \cdot M_A((0, 0)') + \frac{119}{2} = \frac{\epsilon \cdot (-11) + 119}{2}.$$  

The vector which realizes the value of $M_A((0, 0)')$ is $(u, v)' = (13, 11)'$ or $(-13, -11)'$.

We assume that $u_2(K) = 1$. Then by Theorem 1.1 the value $I_{\phi, \epsilon}(0)$ has to be an even number, and therefore $\epsilon = 1$. Next by calculation we have $I_{\phi, 1}(1) = M_A(\phi(1)) - 119/478$. Since 239 is a prime number, any $\phi_j = \text{"multiplication by } j"$ is an automorphism of $\mathbb{Z}/239\mathbb{Z}$. To guarantee that $I_{\phi_j, 1}(1)$ is an even number, the isomorphism $\phi_j$ has to be either $\phi_{15}$ or $\phi_{224}$. By calculation, we see that

$$I_{\phi_{15}, 1}(1) = M_A((0, 15)') - \frac{119}{478} = -4.$$  

The vector which realizes the value of $M_A((0, 15)')$ is $(u, v)' = (-9, -11)'$. Same calculation tells us that $I_{\phi_{224}, 1}(1) = -4$ as well, which is realized by the vector $(u, v)' = (9, 11)'$. Now we see $-4$ is a negative number, which conflicts with the necessary condition stated in Theorem 1.1. Therefore the $H(2)$-unknotting number of $P(13, 4, 11)$ has to be at least two. On the other hand, the knot $P(13, 4, 11)$ can be changed into the unknot by adding two twisted bands as shown in Fig. 4. Hence the $H(2)$-unknotting number of $P(13, 4, 11)$ is two. This completes the proof of Corollary 1.2.

### 2.4. Comparisons with other criterions.

There have been many criterions and properties which can be used to bound the $H(2)$-unknotting number of a knot. We want to apply them to the knot $P(13, 4, 11)$ and compare the results with Corollary 1.2.

The first one is Lickorish’s obstruction that we recalled in the beginning. It does not work for the pretzel knot $K = P(13, 4, 11)$ because of the following reason. It is known that the Goeritz matrix $A$ is a presentation matrix of $H_1(\Sigma(K), \mathbb{Z})$, and $A^{-1}$ represents the linking form $\lambda$. It is not hard to see that $H_1(\Sigma(K))$ is cyclic of order 239, and that the generator $g = (0, 1)'$ satisfies $\lambda(g, g) = -17/239$. Then we see $\lambda(15g, 15g) = (225 \times (-17))/239 = -3825/239 = -1/239$ over $\mathbb{Q}/\mathbb{Z}$. Since 239 is a prime number, the vector $g' = (0, 15)'$ can work as a generator of $H_1(\Sigma(K), \mathbb{Z})$. 
There are two invariants of knots which are closely related to H(2)-unknotting number. Given a knot \( K \subset S^3 \), the crosscap number of \( K \) [2] is defined as follows:

\[
\gamma(K) = \min\{\beta_1(F) \mid F \text{ is a non-orientable connected surface in } S^3 \text{ and } \partial F = K\},
\]

where \( \beta_1(F) \) denotes the rank of the first homology group of \( F \). The 4-dimensional crosscap number of \( K \) [11], which we denote \( \gamma^*(K) \) here, is by name defined as follows:

\[
\gamma^*(K) = \min\{\beta_1(F) \mid F \text{ is a non-orientable connected smooth surface in } B^4 \text{ and } \partial F = K \subset \partial B^4 = S^3\}.
\]

Their relation with H(2)-unknotting number is as follows.

**Lemma 2.3.** Given a knot \( K \subset S^3 \), we have \( \gamma^*(K) \leq u_2(K) \leq \gamma(K) \).

Proof. The knot \( K \) can be reconstructed from the unknot by adding \( u_2(K) \) twisted bands successively. Let \( D \) be a disk bounded by the unknot and \( b_1, b_2, \ldots, b_{u_2(K)} \) be the bands added to the boundary of \( D \). Then \( F := D \cup \bigcup_{i=1}^{u_2(K)} b_i \) is a non-orientable surface in \( B^4 \) with \( \partial F = K \). We have \( \gamma^*(K) \leq \beta_1(F) = u_2(K) \). The second inequality is proved as follows. Suppose \( S \) is a non-orientable surface in \( S^3 \) which realizes the crosscap number of \( K \). Namely we have \( \beta_1(S) = \gamma(K) \) and \( \partial S = K \). Then there are \( \gamma(K) \) disjoint essential arcs in \( S \), say \( \tau_1, \tau_2, \ldots, \tau_{\gamma(K)} \), such that \( S - \tau_i \) has one boundary component for \( i = 1, 2, \ldots, \gamma(K) \) and \( S - \bigcup_{i=1}^{\gamma(K)} \tau_i \) is a disk. If we add twisted bands to \( K \) along \( \tau_i \) for \( i = 1, 2, \ldots, \gamma(K) \), the resulting knot is the unknot. Therefore we have \( u_2(K) \leq \gamma(K) \).

Ichihara and Mizushima [5] calculated the crosscap numbers of pretzel knots. According to their calculation, the crosscap number of \( P(13, 4, 11) \) is two. Gilmer and Livingston [3] studied the 4-dimensional crosscap number of a knot by using Heegaard Floer homology. Their method and our result in this note are both in spirit derived from Theorem 9.6 in [13]. The author does not know whether their method can verify that the 4-dimensional crosscap number of \( P(13, 4, 11) \) is 2 or not.

Yasuhara [18], and Kanenobu and Miyazawa [6] introduced some methods for detecting the H(2)-unknotting number of a knot, but simple calculation shows that their methods cannot be applied to the knot \( P(13, 4, 11) \). Taniyama and Yasuhara[17] established the equivalence between H(2)-unknotting number and other two invariants of knots, but there seems no obvious way to apply their relation to the calculation of H(2)-unknotting number.
References