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A NEW DISTINGUISHED FORM FOR 3-BRAIDS

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Abstract

We show that every 3-strand braid has a representative word of a given form, and furthermore, this form allows us, in most cases, to deduce positivity (or negativity) in the $\sigma$-ordering of $B_3$. The $\sigma$-ordering of $B_n$ was introduced by Patrick Dehornoy in the late 1990’s, however, other (equivalent) orderings were discovered soon after by Fenn, Greene, Rolfsen, et al.

1. Introduction

The braid groups, denoted $B_n$, were introduced in 1925 by Emil Artin [1] and can be defined for each $n > 1$ as the group generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ with relations:

(1) $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| > 1$, and
(2) $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$ if $|i - j| = 1$.

In this paper, we use this generator and relation description of $B_n$, but we will also view $B_n$ as a mapping class group of the space $D_n = D^2 - \{p_1, p_2, \ldots, p_n\}$, the unit disk with the set of distinguished points (called punctures) removed. As a mapping class, the generator $\sigma_i$ exchanges punctures $p_i$ and $p_{i+1}$ by a counterclockwise half-twist. The $n$-strand braid

$$\Delta_n = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})(\sigma_1 \sigma_2 \cdots \sigma_{n-2}) \cdots (\sigma_1 \sigma_2)(\sigma_1)$$

is a half-twist of all $n$ strands, and as a mapping class $\Delta_n^2$ is a full Dehn twist about the boundary of $D_n$ which generates the infinite cyclic center of $B_n$. We will denote $\Delta_3 = \sigma_1 \sigma_2 \sigma_1$ as simply $\Delta$ throughout this paper. Finally, we note that, here, we compose braids on the left.

A solution to the word and conjugacy problems for $B_n$ was discovered in 1968 by Garside in [6] and expanded upon by many others over the years. However, it still remains an interesting endeavor to find distinguished representative words for elements of the braid groups and monoids. The goal of this paper is to give a new distinguished representative word for all 3-braids which can be obtained from a simple algorithm. The geometric nature of this approach and the use of the mapping class group point of view of $B_3$ makes this approach novel and is an improvement upon previous normal forms which use brute force methods. This new standard form is particularly useful in

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deciding for a 3-braid $\beta$, whether $\beta > 1$ is true in the $\sigma$-ordering of $B_3$. Until now, there has been no standard form for braids which claims to determine positivity in the $\sigma$-ordering. This is a new and practical feature of the standard form given here for the cases where the integer $m$ in the standard form is greater than or equal to 3.

We give a brief description of each section of this paper. In Section 2, we give a proof of the main theorem by using the mapping class group definition of $B_3$ and using a Euclidean-like algorithm. In Section 3, we recall the $\sigma$-ordering of the braid groups, first defined by Patrick Dehornoy, and show how the distinguished form for a 3-braid can in most cases identify it as positive or negative in this ordering. Finally, in Section 4, we outline future directions in the continuation of this work. We start with the statement of the main theorem.

Theorem 1.1. Every 3-braid $\beta$ admits a representative word of the form

$$\mu = (\sigma_2 \sigma_1)^m \omega \tau^r,$$

where $\tau = \sigma_1^{-1} \sigma_2 \sigma_1$ and $\omega$ is a word in only $\sigma_1$ and $\sigma_2^{-1}$.

2. Proof of the main result

As a matter of convenience, we arrange the punctures in a triangular fashion. Let $e$ be a straight edge between $p_1$ and $p_3$, and let $\alpha_i$ be a properly embedded arc which separates $D_3$ into two components: a component which contains only the puncture $p_i$ and its complement. We also require that $\alpha_1, \alpha_2, \alpha_3$ be pairwise disjoint (see Fig. 1).

Let $\beta$ be an element of $B_3$, and isotop $\beta(e)$ so that it intersects the set $\{\alpha_1, \alpha_2, \alpha_3\}$ transversally and minimally. The intersection of $\beta(e)$ and each once-punctured disk region is some number (possibly zero) of parallel arcs enclosing the puncture and possibly an arc which ends at the puncture. The complement of the three once-punctured disk regions is a hexagonal region bounded by the arcs $\alpha_i$ and three subarcs of $\partial D_3$. The intersection of $\beta(e)$ with this region consists of a disjoint union of embedded edges, each of which connects two of the arcs $\alpha_i$. Thus, there are three types of these edges; however, all three could not occur in the same diagram, for this would give rise to a closed loop. The braid $\sigma_2 \sigma_1$ rotates the diagram by an angle of $2\pi/3$, therefore by applying $(\sigma_2 \sigma_1)^m$ for $m = 0, 1$ or 2 to $\beta(e)$, we can assume there is no edge connecting $\alpha_1$ and $\alpha_3$ (see Fig. 2).

Let $a_i$ be the geometric intersection number between the arcs $e' = (\sigma_1 \sigma_2)^m \beta(e)$ and $\alpha_i$. Note that since there is no edge between $\alpha_1$ and $\alpha_3$, we have $a_2 = a_1 + a_3$. First, there are a few special cases to consider. If $a_3 = 0$, then it must be the case that $a_1 = a_2 = 1$ to avoid a closed loop. Thus, in this case, $e'$ is a straight arc between $p_1$ and $p_2$. We apply $\sigma_2$, and the resulting braid $\sigma_2 (\sigma_1 \sigma_2)^m \beta$ fixes $e$. Similarly, if $a_1 = 0$, then $e'$ is a straight arc between $p_2$ and $p_3$. We apply $\sigma_1^{-1}$ and the resulting braid $\sigma_1^{-1}(\sigma_1 \sigma_2)^m \beta$ fixes $e$. If $a_1 = a_3$, then we must have $a_1 = a_3 = 1$ to avoid a closed
loop. In this case $e'$ is a “U”-shaped arc with endpoints $p_1$ and $p_3$. We may apply $\sigma_2^2$ or $\sigma_1^{-2}$, and in either case, the resulting braid $\sigma_2^2(\sigma_1\sigma_2)^m\beta$ or $\sigma_1^{-2}(\sigma_1\sigma_2)^m\beta$ fixes $e$.

Now suppose that $0 < a_1 < a_3$, and consider $\sigma_2(e')$. This arc has fewer intersections with the set \{\alpha_1, \alpha_2, \alpha_3\} than $e'$. In particular, if $a'_i$ denotes the geometric intersection numbers of $\sigma_2(e')$ and $\alpha_i$, then $a'_1 = a_1$, $a'_2 = a_3$, and $a'_3 = a_3 - a_1$. Similarly, if $0 < a_3 < a_1$, we apply $\sigma_1^{-1}$ and find that the intersection numbers of $\sigma_1^{-1}(e')$ and $\alpha_1, \alpha_2, \alpha_3$ are $a'_1 = a_1 - a_3$, $a'_2 = a_1$, and $a'_3 = a_3$, respectively. We repeat the algorithm of applying $\sigma_2$ whenever $a_1 < a_3$ and applying $\sigma_1^{-1}$ whenever $a_3 < a_1$ a finite number of times until we eventually unravel the arc $e'$ so that the resulting arc is isotopic to $e$.

Thus, in each case we have generated a word $\omega^{-1}$ only involving negative powers of $\sigma_1$ and positive powers of $\sigma_2$ for which the braid $\omega^{-1}(\sigma_2\sigma_1)^m\beta$ fixes $e$. However, any braid that fixes $e$ must be a power of $\tau = \sigma_1^{-1}\sigma_2\sigma_1$, the half Dehn twist about $e$, times a power of the central element $\Delta^2 = (\sigma_2\sigma_1)^3$. Thus,

$$\omega^{-1}(\sigma_2\sigma_1)^m\beta = \Delta^k\tau^r$$
yielding

\[ \beta = (\sigma_2\sigma_1)^{-m} \omega \Delta^{2k} \tau^r \]

\[ = (\sigma_2\sigma_1)^{3k-m} \omega \tau^r, \]

where \( \omega \) is a word involving only positive powers of \( \sigma_1 \) and negative powers of \( \sigma_2 \). This is the desired form, so the result is shown.

3. An application to the \( \sigma \)-ordering of \( B_3 \)

The connection between the braid groups and orderable groups, though long overdue, was not widely recognized when announced by Patrick Dehornoy in 1992 [3] because the methods used were largely unfamiliar to topologists. Several years later, a topological proof of the orderability of \( B_n \) was discovered by Fenn, Rolfsen, Wiest, et al. [5]. This ordering used the description of \( B_n \) as a mapping class group of an \( n \)-punctured disk, yet surprisingly it leads to the same ordering given by Dehornoy.

In the years since these discoveries, several other approaches, including ideas from hyperbolic geometry and lamination theory, have been used to show orderability of \( B_n \). Here we describe Dehornoy’s ordering of \( B_n \) known as the \( \sigma \)-ordering starting with a few definitions.

A braid word \( \sigma \) is said to be \( \sigma \)-positive (resp. \( \sigma \)-negative) if, among the letters \( \sigma_i^{\pm 1} \) that occur in \( \sigma \), the one with lowest index occurs positively only (resp. negatively only). For example, the word \( \sigma = \sigma_2\sigma_3^{-1}\sigma_2\sigma_3 \) is \( \sigma \)-positive since \( \sigma_2 \) appears in \( \sigma \), but \( \sigma_3^{-1} \) does not. By contrast the word \( \sigma_1\sigma_2\sigma_1^{-1} \) is neither \( \sigma \)-positive nor \( \sigma \)-negative since both \( \sigma_1 \) and \( \sigma_1^{-1} \) appear in \( \sigma \). For \( \beta, \beta' \) in \( B_n \), we say that \( \beta <_n \beta' \) is true if \( \beta^{-1}\beta' \) admits an \( n \)-strand representative word that is \( \sigma \)-positive.

For instance, if \( \beta = \sigma_1 \) and \( \beta' = \sigma_2\sigma_1 \) in \( B_3 \), then \( \beta^{-1}\beta' \) admits the word \( \sigma_1^{-1}\sigma_2\sigma_1 \) which is neither \( \sigma \)-positive nor \( \sigma \)-negative. However, using the relation \( \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \) in \( B_3 \) we see that this braid also admits the word \( \sigma_2\sigma_1\sigma_2^{-1} \) which is \( \sigma \)-positive. Hence, \( \beta <_3 \beta' \). The relation \( <_n \) is a total left-invariant ordering of \( B_n \) for \( 2 \leq n \leq \infty \), and we shall refer to this ordering as the \( \sigma \)-ordering of \( B_n \). The proof is not included here but can be found in [4], among other sources. The essential properties of the relation \( <_n \) needed for the proof are summed up in the following two statements.

Property A (Acyclicity). A braid that admits at least one \( \sigma \)-positive representative word is nontrivial.

Property C (Comparison). Every braid in \( B_n \) admits an \( n \)-strand representative word that is \( \sigma \)-positive, \( \sigma \)-negative, or empty.

From the definition above, it can be seen that \( 1 <_n \beta \) if and only if \( \beta \) admits an \( n \)-strand representative word that is \( \sigma \)-positive. For a braid \( \beta \) in \( B_3 \), we use a representative
word in the form given by Theorem 1.1 to deduce $\sigma$-positivity. Since we will be referring to 3-strand braids for the remainder of this paper, we simply write $<$ for $<_3$.

First, we make a simple observation about $\sigma$-positivity in $B_3$. Since there are only the letters $\sigma_i^\pm 1$ and $\sigma_2^\pm 1$ to consider, a 3-strand braid word $w$ is $\sigma$-positive if and only if $\sigma_1$ appears in $w$, but $\sigma_1^{-1}$ does not; or if $w$ is a positive power of $\sigma_2$. We can immediately conclude the following.

**Proposition 3.1.** Let $\beta \in B_3$ be nontrivial, and choose a representative word for $\beta$ of the form given in Equation (1.1). If $m \geq 0$ and $r \geq 0$, then $1 < \beta$.

Proof. Let $\beta = (\sigma_2 \sigma_1)^m \omega \tau^r$, where $\omega$ is a word in only $\sigma_1$ and $\sigma_2^{-1}$. We simply observe that $\tau = \sigma_1^{-1} \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2^{-1}$. Therefore, if $m \geq 0$ and $r \geq 0$, only positive powers of $\sigma_1$ appear in the word. This proves the result.

We can also conclude that if $m \geq 3$, the word is $\sigma$-positive no matter the value of $r$ due to the fact that the infinite cyclic center of $B_3$ is generated by $(\sigma_2 \sigma_1)^3$.

**Proposition 3.2.** Let $\beta \in B_3$ be nontrivial, and choose a representative word for $\beta$ of the form given in Equation (1.1). If $m \geq 3$, then $1 < \beta$.

Proof. If we have the word $\mu = (\sigma_2 \sigma_1)^m \omega \sigma_1^{-1} \sigma_2 \sigma_1$, then there is only one occurrence of the letter $\sigma_1^{-1}$. We can freely reduce this letter by commuting $\omega$ and $(\sigma_2 \sigma_1)^3$. This proves the result.

For further cases, we use a special case of a combinatorial method for comparing braid words called handle reduction introduced by Dehornoy in [2]. Handle reduction provides an algorithm for finding a $\sigma$-positive or $\sigma$-negative representative word for a nonempty braid word (which always exists by Property C). By definition, if a nonempty braid word $w$ is neither $\sigma$-positive nor $\sigma$-negative, then the letter $\sigma_i$ of lowest index must appear both positively and negatively in $w$. Therefore, $w$ contains a subword either of the form $\sigma_i v \sigma_i^{-1}$ or $\sigma_i^{-1} v \sigma_i$ where all letters of $v$ are $\sigma_k^{\pm 1}$ for $k > i$. A subword of this type is called a $\sigma_i$-handle, and handle reduction is a process of replacing a $\sigma_i$-handle in $w$ with an equivalent word in which the first and last letters $\sigma_i^{\pm 1}$ have been deleted. This process is iterated until no $\sigma_i$-handle is left in the word, resulting in an equivalent word that is either $\sigma$-positive, $\sigma$-negative, or empty.

So, specifically, here is how handle reduction works. A $\sigma_i$-handle $\sigma_i^c v \sigma_i^{-c}$ is said to be permitted if the word $v$ includes no $\sigma_i+1$-handle. If $v$ is a permitted $\sigma_i$-handle, we define the reduce of $v$ to be the word obtained from $v$ by replacing each letter $\sigma_i^{\pm 1}$ with $\sigma_i^-c \sigma_i^{\pm 1} \sigma_i^c$. We say that $w'$ is obtained from $w$ by handle reduction (or $w$ reduces to $w'$) if $w'$ is obtained by replacing a subword of $w$ that is a permitted handle with its reduce. It should be noted that handle reduction extends free reduction.
since reducing the $\sigma_i$-handle $\sigma_i^2 \sigma_1^{-e}$ amounts to deleting it. By a result of Dehornoy, handle reduction converges.

Sparing further exposition on the general case, we move directly to the $n = 3$ case. Here, the only nontrivial $\sigma_i$-handles are $\sigma_1$-handles $\sigma_1^d \sigma_1^{-e}$ which we simply refer to as “handles”. It is also immediately apparent that all handles are permitted. Therefore, if each $\sigma_{2 \pm 1}$ in a handle is replaced by $\sigma_{2 -e} \sigma_{2 \pm 1} \sigma_2^e$, after free reductions are performed, we see that handle reduction in this case simply replaces a subword of the form $\sigma_1^d \sigma_1^e$ with its reduct $\sigma_1^{2e} \sigma_1^e$.

We establish an algorithm for performing handle reductions. First, at each step we choose to reduce the leftmost handle in $w$. Here is what is meant by leftmost. If $w$ is a word of length $l$, then a $(p, q)$-subword of $w$ is the word obtained by deleting all letters before position $p$ and after position $q$, for $1 \leq p \leq q \leq l$. A handle $v$ is said to be leftmost in $w$ if there exist $p, q$ such that $v$ is the $(p, q)$-subword of $w$, and there is no $p', q'$ with $q' < q$ such that the $(p', q')$-subword of $w$ is a handle. For example, in the 3-strand braid word $w = \sigma_2 \sigma_1 \sigma_2^3 \sigma_1^{-2} \sigma_2^2 \sigma_1$, the $(2, 6)$-subword $v = \sigma_1 \sigma_2^3 \sigma_1^{-1}$ is the leftmost handle in $w$. Secondly, we perform all possible free reductions rather than waiting for trivial handle $\sigma_1^d \sigma_1^{-e}$ or $\sigma_2^e \sigma_2^{-e}$ to be leftmost. This algorithm is referred to as left handle reduction, and we use it in proving the following.

**Proposition 3.3.** Let $\beta \in B_3$ be nontrivial, and choose a representative word for $\beta$ of the form given in Equation (1.1). If $m \leq -4$, then $\beta < 1$.

Proof. For simplicity’s sake, we replace $\sigma_1$ with the letter $a$ and $\sigma_2$ with the letter $b$. The inverse of each generator will be the corresponding capital letter. Using this notation, $\mu = (AB)^m \omega Ab a$, where $m \geq 4$ and $\omega$ is a word in letters $a$ and $B$ only. Thus, we show $\mu$ is equivalent to a word in which $A$ appears, but $a$ does not appear.

First, we consider the case that $a$ does not appear in the subword $\omega$. Then, $\omega$ is either empty or a power of $B$. In this case, only appearance of the letter $a$ is as the last letter of $\mu$. Since $(AB)^3$ is in the center of $B_3$, we can annihilate the letter $a$ by commuting the prefix $(AB)^3$ with the rest of the word. The resulting word has the letter $A$ but no $a$, hence it is $\sigma$-negative.

Now, suppose the letter $a$ does appear in $\omega$. So, $\omega = a^{s_1} B^{s_2-1} \cdots a^{s_k} B^{s_k} a^{s_1}$, where $s_i \geq 0$ and some power of $a$ does appear. Consider the leftmost $a$ in $\omega$; call it $a_1$. The letter $a_1$ is to the right of the prefix $(AB)^m$, so it is clear that the leftmost handle in $\mu$ is of the form $AB^{t_1} a_1$, where $t_1 \geq 1$. We replace $AB^{t_1} a_1$ with its reduct $bA^{t_1} B$. Notice that this replacement deletes $a_1$ and adds no more $a$’s. Furthermore, we have shaved one $AB$ from our prefix, so the reduced word now has at least three $AB$’s in its prefix.

If there are any remaining $a$’s in $\omega$, consider the leftmost, and call it $a_2$. Note that $a_2$ must be to the right of the reduct $bA^{t_1} B$ from the previous step. Therefore, again our leftmost handle is of the form $AB^{t_2} a_2$, where $t_2 \geq 1$. We replace this handle with its reduct $bA^{t_2} B$, again, deleting $a_2$ while adding no more $a$’s. We keep this repeating this process until there are no more occurrences of the letter $a$ in $\omega$. 
Now, the only appearance of $a$ is as the last letter of $\mu'$ (the reduction of $\mu$). But since we have $(AB)^3$ in our prefix, we can commute this subword with the rest of the word to annihilate this $a$. The result is a word in which $A$ appears but $a$ does not appear. Thus, in each case $\beta < 1$.

4. Conclusions

In this paper, we show that every 3-braid admits a distinguished word of the form $(\sigma_2\sigma_1)^m\omega\sigma_1^{-1}\sigma_2^{-1}\zeta_1$ where $\omega$ is a word in $\sigma_1$ and $\sigma_2^{-1}$ only. For example, it can be verified that $\sigma_2$ is equivalent to the word $(\sigma_2\sigma_1)^2\sigma_2^{-2}\sigma_1^{-1}\sigma_2^{-1}\sigma_1$. Even though it may be less practical to use this form for braids like $\sigma_2$, the value for $m$ sheds light on the positivity or negativity of the braid in the $\sigma$-ordering of $B_3$. Our results leave remaining cases $-3 \leq m < 3$ open, but initial investigations into these cases suggests that the integer $r$ plays a greater role when $|m|$ is small.

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References


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