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DEFORMATIONS OF SINGULARITIES OF PLANE CURVES: TOPOLOGICAL APPROACH

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Abstract

In this paper we use a knot invariant, namely the Tristram–Levine signature, to study deformations of singular points of plane curves. We bound, in some cases, the difference between the M-number of the singularity of the central fiber and the sum of M-numbers of the generic fiber.

1. Introduction

A deformation of a plane curve singularity is, roughly speaking, a smooth family of plane algebraic curves $\{C_s\}_{s\in D}$ (we consider here only deformations over a disk D in $\mathbb{C}$) such that $C_s \subset \mathbb{C}^2$ and a distinguished member, say C_0, has a singular point at $z_0 \in \mathbb{C}^2$. The question we address is the following: how are related to each other singular points of C_0 and of C_s with s sufficiently small? This question, although already very difficult, becomes even more involved if we impose some topological constrains on the general members C_s. For example, we can require all of them to be rational, which means that each C_s is a union of immersed disks.

This rationality condition is justified for various reasons. For example, let us be given a flat family C_s of projective curves in some surface Z and this family specializes to a curve C_0 with the same geometric genus as C_s. Then, for each singular point $z \in C_0$, we can take a sufficiently small ball B around z and the family $C_s \cap B$ provides a deformation of a singular point such that all curves $C_s \cap B$ are rational.

To show a more specific example, we can take $C = C_{mn}$ to be a polynomial curve given in parametric form by $C = \{(t^n, t^m),\ t \in \mathbb{C}\}$ with n, m coprime, and assume C' is also parametric $C' = \{(\phi(t), \psi(t)),\ t \in \mathbb{C}\}$ with $\deg\phi = n$, $\deg\psi = m$. Then for $s \in \mathbb{C}\setminus\{0\}$, the mapping $(s^n\phi(t/s), s^m\psi(t/s))$ parametrizes a curve that is algebraically isomorphic to C' and, for sufficiently small s, is very close to C. In other words, every polynomial curve of bidegree (n,m) specializes to (t^n,t^m). In particular if a polynomial curve of bidegree (n,m) has some singularity, this singularity can be specialized to the quasi-homogeneous singularity (t^n, t^m). So, classification of parametric deformations encompasses the problem of finding possible singularities of a polynomial curve of a

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given bidegree. The characterization of possible singularities of polynomial curves is, in turn, a problem with applications beyond algebraic geometry itself, for example in determining the order of weak focus of some ODE systems (see [6] and [5, Section 5]).

In [11, 5] there was defined a new invariant of plane curve singularities, namely the codimension, also known as the $\bar{M}$-number (or the rough $\bar{M}$-number). It is, roughly speaking, the codimension of the (topological) equisingularity stratum in the appropriate space of parametric singularities. A naive parameter counting argument suggests that this invariant is upper-semicontinuous under parametric deformations. Yet proving this appears to be an extremely difficult task. On the one hand, the $\bar{M}$-number can be expressed by some intersection number of divisors in the resolution of singularity, but then the blow-up diagram changes after a deformation in a way that we are still far from understand. In an algebraic approach, the geometric genus of nearby fibers is quite difficult to control. On the other hand, the famous Hirano's example [9] can be used to show, that a natural generalization of this expected semicontinuity property fails if we allow the curves C_s to have higher genera.

A possible rescue comes from a very unexpected place, namely from knot theory. It turns out that the $\bar{M}$-number, or its more subtle brother, the M-number (also called the fine M-number), is very closely related to the integral of the Tristram–Levine signature of the knot of the singularity ([3]). We say a knot, instead of a link, to emphasize that this relationship has been proved only in the case of cuspidal singularities. On the other hand, we can apply methods from [2] to study the changes of the Tristram–Levine signature. Putting things together we obtain a bound for the difference between the sum of M-numbers of singular points of a generic fiber and the sum of M-numbers of singular points of the central fiber, provided that the curves have only cuspidal singularities or double points.

The structure of the paper is the following. First we precise, what is a deformation (Section 2). Then we recall definitions of codimension (Section 3). Section 4 is devoted to the application of the Tristram–Levine signature. We recall a definition of the Tristram–Levine signature and cite two results from [2] and [3]. This allows to provide the promised estimates in Section 5.

2. What is a deformation?

Under a notion of a *deformation* of a plane curve singularity over a base space $(D, 0)$, where $D \subset \mathbb{C}$ is an open disk, we understand a pair $(\mathcal{X}, B)$ where B is a ball in $\mathbb{C}^2$ and $\mathcal{X}$ is an algebraic surface (called the *total space*) in $B \times D$. The sets $X_s = \mathcal{X} \cap B \times \{s\}$ (treated simply as subsets of B) are called the *fibers* of the deformation. We impose the following conditions on the pair $(\mathcal{X}, B)$.

flatness: The natural projection on the second factor $\pi_2 \colon \mathcal{X} \to D$ is a flat morphism.

transversality: For each $s \in D$, the curve X_s is transverse to the boundary ∂B.

locality: The curve X_0 (called the *central fiber*) has precisely one singular point z_0 (we will assume that this is $0 \in \mathbb{C}^2$), and the intersection $X_0 \cap \partial B$ is the link of singularity

of z_0.
The flatness condition is a standard one in the deformation theory. The locality means that we are concerned with the deformation of a given singular point z_0 at a local scope: roughly speaking it says that B is a small ball around z_0. The transversality will be crucial in our approach, it roughly means that the disk D is small: if X_0 is transverse to ∂B, then the transversality holds for all s sufficiently close to 0.

DEFINITION 2.1. The *genus* g of the deformation is the geometric genus (i.e. the topological genus of the normalization) of a generic fiber X_s. The deformation is *rational* if $g = 0$, in which case all X_s are sums of immersed disks. The deformation is *unibranched* if X_0 is a disk. The deformation is *parametric* if it is both rational and unibranched.

The intersection of X_s with the ball B by the transversality condition above is a link, which we shall denote L_s. As this intersection is transverse for each $s \in D$, the isotopy type of L_s does not depend on s.

DEFINITION 2.2. The (isotopy class of the) link L_s is called the *link of the deformation*. It is denoted by L_X.

REMARK 2.3. The locality property ensures that L_X can be identified with the link of singularity of X_0.

Lemma 2.4. *Let* $(\mathcal{X}, B)$ *be a parametric deformation. Then, there exists such an* $\varepsilon > 0$ *and a family of holomorphic functions*

$$x_s(t) = a_0(s) + a_1(s)t + \cdots,$$
$$y_s(t) = b_0(s) + b_1(s)t + \cdots$$

with $|s| < \varepsilon$ *that* (x_s, y_s) *locally parametrizes* X_s *and both* x_s *and* y_s *depend analytically on* s.

Proof. The assumptions on the parametricity and transversality guarantee that the deformation is δ-constant, hence equinormalizable (see [8, Section 2.6]). By assumptions, the normalization of $\mathcal{X}$ is a product $D \times D'$, where D' is a small disk. Let ρ be a normalization map. Then we consider the composition of ρ with the projection $\pi_1 \colon \mathcal{X} \to B$ onto the first factor, and then with coordinate functions π_x, $\pi_y \colon B \to \mathbb{C}$. We have $x_s = \pi_x \circ \pi_1 \circ \rho$ and $y_s = \pi_y \circ \pi_1 \circ \rho$. □

3. Codimension

The codimension is a topological invariant of a plane curve singularity. We recall here a definition from [5].

DEFINITION 3.1. Let $\mathcal{T}$ be a topological type of a plane curve cuspidal singularity with multiplicity m. Let $\mathcal{H}$ be the space of polynomials in one variable. Consider the stratum $\Sigma \subset \mathcal{H}$ consisting of such polynomials y that a singularity parametrized by

$$t \to (t^m, y(t))$$

defines a singularity at 0 of type $\mathcal{T}$. Then the *external codimension* of the singularity $\mathcal{T}$ is

$$ext\nu = \operatorname{codim}_{\mathcal{H}} \Sigma + m - 2.$$

(Here $\operatorname{codim}_B A$ means the codimension of A in B.) The interpretation of the definition is the following. If we consider the space of pairs of polynomials $(x(t), y(t))$ of sufficiently high degree, then the subset of those parametrizing a curve with a singularity of type $\mathcal{T}$ forms a subspace of codimension $ext\nu(\mathcal{T})$. In fact, there are $m-1$ condition for the derivatives of x to vanish at some point, $\operatorname{codim}_{\mathcal{H}} \Sigma$ conditions for the polynomial y (the degree of y is assumed to be high enough so that these conditions are independent). The missing -1 comes from the fact that we do not require the singularity to be at $t = 0$, but we have here sort of freedom.

REMARK 3.2. In [5] the assumption that m is the multiplicity is not required. If m is not the multiplicity, then (3.1) below, does no longer hold.

The above definition can be generalized to multibranched singularities. We refer to [5] for detailed definitions.

There exists also a construction of the $ext\nu$ in a coordinate-free way. It can be done as follows. Let $(C, 0)$ be a germ of a plane curve singularity at 0, not necessarily unibranched. Let $\pi : (U, E) \to (\mathbb{C}^2, 0)$ be the minimal embedded resolution of this singularity, where $E = \sum E_i$ is the exceptional divisor with a reduced structure. Let K be a (local) canonical divisor on U, which means that $K = \sum \alpha_i E_i$ and $(K + E_i) \cdot E_i = -2$ for exceptional curves E_i. Let C' be the class of the strict transform of C, and $D = C' + E$.

DEFINITION 3.3. A *rough $\bar{M}$-number* of $(C, 0)$ is the quantity

$$K \cdot (K + D).$$

We have the following fact (see [5, Proposition 4.1])

$$\bar{M} = ext\nu. \tag{3.1}$$

REMARK 3.4. In [11], $\bar{M}$ is defined as $\bar{M} = \mu + (K+D)^2$. This definition agrees with Definition 3.3 for unibranched singularities, because $\mu = -D(K+D)$. For multi-branched, the Orevkov's version is bigger and the difference is the number of branches -1. See also [5, Remark 4.2].

Orevkov [11] defines, besides a rough $\bar{M}$-number, a fine M-number of a singularity. We should take the Zariski–Fujita decomposition

$$K + D = H + N,$$

where H is the nef part and N negative (i.e. the intersection form on the support of N is negative definite). We have the following definition (see [5, Definition 4.1]).

DEFINITION 3.5. The M-number of the singularity is equal to $\bar{M} - N^2$.

N^2 is always non-positive, so $\bar{M} \leq M$. For cuspidal singularities we have $N^2 < -1/2$, while for an ordinary d-tuple point $N = 0$ and $N^2 = 0$. For cuspidal singularities we have the formula $\bar{M} = \mu + (K+D)^2$, hence we recover Orevkov's original definition [11, Section 1]

$$M = \mu + H^2.$$

Both $\bar{M}$ and M-numbers can be very effectively calculated from the Eisenbud–Neumann diagram. An algorithm can be found for example in [5, Section 4.2]. We provide a simple, but important example.

EXAMPLE 3.6. Let p, q be coprime positive integers and consider the singularity $\{x^p - y^q = 0\}$. Its $\bar{M}$-number is equal to $p + q - \lceil p/q \rceil - \lceil q/p \rceil - 1$, while

$$M = p + q - \frac{p}{q} - \frac{q}{p} - 1. \tag{3.2}$$

EXAMPLE 3.7. Both $\bar{M}$ and M-numbers of an ordinary double point are zero.

We expect the $\bar{M}$-number to be upper-semicontinuous in parametric deformation. To be more specific we state a following conjecture.

Conjecture 3.8. *Let $(\mathcal{X}, B)$ be a parametric deformation with a central fiber X_0 having a singular point z_0 with $\bar{M}$-number $\bar{M}_0$. Then, for all $s \in D^*$ we have*

$$\sum_{k=1}^{n} \bar{M}_k \leq \bar{M}_0, \tag{3.3}$$

where we sum the $\bar{M}$-numbers of all singular points of the fiber X_s.

Without assumption for the deformation to be parametric, one could naturally expect that the left hand side of (3.3) should be replaced by $\bar{M}_0 + g$, where g is the geometric genus of the fiber X_s. But then we can give a counterexample to this extended conjecture. Namely, Hirano [9] constructs a series of curves $H_d \subset \mathbb{C}P^2$ (for infinitely many d's) such that each H_d is of degree d and has approximately $\frac{9}{32}d^2$ ordinary cusps. Now, it is well known that any algebraic curve C of degree d in $\mathbb{C}P^2$ specializes to a curve C_d given by $x^d - y^d = 0$. So let us take a deformation (satisfying only the flatness condition) $\mathcal{Z} \subset \mathbb{C}P^2 \times D$, with $Z_s = \mathcal{Z} \cap \mathbb{C}P^2 \times \{s\}$, such that $Z_0 = C_d$, and, for $s \in D^*$, Z_s is isomorphic to H_d. For s sufficiently small and non-zero, all singularities of Z_s are close to $(0, 0) \in \mathbb{C} \subset \mathbb{C}P^2$, so we can restrict our deformation to a small ball B around $(0, 0)$. Shrinking D if necessary we can guarantee that

$$\mathcal{X} = \mathcal{Z} \cap B \times D \subset \mathbb{C}^2 \times D$$

is a deformation satisfying flatness, transversality and locality conditions. Now we compare codimensions. As the codimension of the ordinary d-tuple point is $d - 2$ by (3.1), and the codimension of an ordinary cusp is one, we get that the geometric genus of X_s for $s \neq 0$ should be at least $(9/32)d^2$ (we neglect terms of lower order in d). Thus the geometric genus of H_d must be at least $(9/32)d^2$. But this contradicts the classical genus formula, because a degree d curve with $(9/32)d^2$ cusps can have geometric genus at most $(7/32)d^2$.

4. Tristram–Levine signatures

Let L be a link in S^3. Let V be a Seifert matrix of L. Finally, let $\zeta \in \mathbb{C}$, $|\zeta| = 1$.

DEFINITION 4.1. The *Tristram–Levine signature* of L is the signature $\sigma_L(\zeta)$ of the Hermitian form given by the matrix

$$(1 - \zeta)V + (1 - \bar{\zeta})V^T.$$

It is well-known that σ_L is a link invariant. It is also easily computable for algebraic links.

EXAMPLE 4.2. Let us consider the singularity $\{x^p - y^q = 0\}$ as in Example 3.6 and let $T_{p,q}$ be its link (note, that this is exactly the (p, q)-torus knot). Its Tristram–Levine signature can be computed as follows: consider a set

$$\Sigma = \left\{ \frac{i}{p} + \frac{j}{q} \colon 1 \leq i \leq p - 1,\ 1 \leq j \leq q - 1 \right\} \subset (0, 2).$$

Let $\zeta = e^{2\pi i x}$ with $x \in (0, 1)$ and $x \notin \Sigma$. Then

$$\sigma(\zeta) = -\#\Sigma \cap (x, x + 1) + \#\Sigma \setminus (x, x + 1).$$

Here # denote the cardinality of a finite set.

In general, $\sigma(\zeta)$ is a piecewise constant function with jumps only at the roots of the Alexander polynomial. Its values are computable, yet they can not always be expressed by a nice, compact formula. However, the main feature we shall use is that Tristram–Levine signatures behave well under knot cobordism. This behavior was studied in [2] in the context of the plane algebraic curves. We use one result from this paper, that in our setting can be formulated as follows.

Assume that $(\mathcal{X}, B)$ is a deformation. Let $Y = X_s$ be a non-central fiber (i.e. $s \neq 0$). Assume that $z_1, \ldots, z_N$ are the singular points of Y and $L_1, \ldots, L_N$ the corresponding links of singularities. Let, finally, $b_1(Y)$ denotes the first Betti number of Y. Recall that L_0 is the link of the singularity X_0.

Proposition 4.3. *For almost all* $\zeta \in S^1$

$$\left| \sigma_{L_0}(\zeta) - \sum_{k=1}^{N} \sigma_{L_k}(\zeta) \right| \leq b_1(Y). \tag{4.1}$$

Proof. Let x, y be the coordinates in $\mathbb{C}^2$. If the function $|x|^2 + |y|^2$ is Morse on Y, then the statement follows from [2, Proposition 6.8] (L_0 in the present paper corresponds to L_r in [2], with r being the radius of our ball B). If the above function is not Morse, we can still find its subharmonic perturbation which is sufficiently close to the original one in B and finish the proof in the way like above. □

Proposition 4.3 gives a strong obstruction for the singularities occurring in the perturbations. Yet the Tristram–Levine signature function is difficult to handle as we have already seen in Example 4.2. Fortunately, there is a result of [3, 4] that allows to draw some consequences from Proposition 4.3 in a ready-to-use form.

Proposition 4.4 (see [3, 4, Proposition 4.6])**.** *Let C be a germ of a curve singular at z_0. Let K be the corresponding link of the singularity, μ and M the Milnor and M-numbers of C. If K is a knot then*

$$0 < -3 \int_0^1 \sigma(e^{2\pi i x})\, dx - (M + \mu) < \frac{2}{9}. \tag{4.2}$$

REMARK 4.5. There is a mistake in the formulation of [3, Lemma 4.4] and [3, Lemma 4.5]. The updated version [4] on the arxiv has this error corrected. The quantities on the left hand sides of formulae (4.2) and (4.3) in [3] should be read $2\mu + (K + D)^2$ and $2\mu + H^2$, respectively. Indeed, the correct version of [3, (4.2)] is explicitly written e.g. in [12, formula 29]. It can be also deduced from the formula [5, (4.7)] and [5, Corollary 4.7]. The correct formula, as stated in [4, (4.3)] follows as well, because

$2\mu + H^2 = 2\mu + (K + D)^2 - N^2$ and $-N^2$ is computed e.g. in [5, Proposition 4.9]. Thus, the correct estimate in [3, Proposition 4.6], should be $0 < -3\rho_0 - (2\mu + H^2) < 2/9$, exactly as we wrote in [4] above. The essential part of the proof of [3, Proposition 4.6] is not changed. We are grateful to the referee of the present article for having spotted that mistake.

EXAMPLE 4.6. If C is a germ of a quasi-homogeneous singularity $\{x^p - y^q = 0\}$, $p, q > 1$, $\gcd(p, q) = 1$, then its link is the torus knot $T_{p,q}$. It is known (see e.g. [3, Corollary 2.10], or [10, Remark 3.9]), that for the torus knot

$$-3\int_0^1 \sigma(e^{2\pi ix})\,dx = \left(p - \frac{1}{p}\right)\left(q - \frac{1}{q}\right) = pq - \frac{p}{q} - \frac{q}{p} + \frac{1}{pq}.$$

As $\mu = (p-1)(q-1)$, by Example 3.6 we have $M + \mu = pq - p/q - q/p$. Hence

$$-3\int_0^1 \sigma(e^{2\pi ix})\,dx - (M + \mu) = \frac{1}{pq} \in \left(0, \frac{1}{6}\right).$$

Now we have all pieces to prove the main result.

5. The main result

The setup in this section is the following. $(\mathcal{X}, B)$ is a deformation, X_0 the central fiber and $Y = X_s$ $(s \neq 0)$ some other fiber (not necessarily a generic one). We introduce the following notation:

- μ_0 is the Milnor number of the singularity of X_0 and M_0 its M-number;
- g is the geometric genus of Y;
- $z_1, \dots, z_N$ are singular points of Y, $L_1, \dots, L_N$ are corresponding links of singularities. Then $\mu_1, \dots, \mu_N$ (respectively $M_1, \dots, M_N$) are Milnor numbers (resp. M-numbers) of the singular points;
- b_1 is the first Betti number of Y.

We shall put a following additional assumption. It is dictated by the fact that we do not have the formula for the integral of the Tristram–Levine signature for general algebraic links.

ASSUMPTION 5.1. There is $n \leq N$ that $z_1, \dots, z_n$ are cuspidal and $z_{n+1}, \dots, z_N$ are ordinary double points. Furthermore, the singularity of X_0 is cuspidal.

Let

$$R = N - n$$

be the number of the double points of Y. We have the following important result.

Theorem 5.2. *In the above notation.*

$$\sum_{k=1}^{n} M_k - M_0 < 8g + 2R + \frac{2}{9}. \tag{5.1}$$

Proof. Let us observe that

$$b_1(Y) = 2g + R, \tag{5.2}$$

$$\mu_0 = 2g + R + \sum_{k=1}^{N} \mu_k = 2g + 2R + \sum_{k=1}^{n} \mu_k. \tag{5.3}$$

The equality (5.3) is exactly the genus formula. It can be proved by comparing the Euler characteristics of smoothings of X_0 and Y (they must agree). Since the signature of a link of a double point is exactly -1 we deduce from Proposition 4.3 that for almost all ζ

$$\sum_{k=1}^{n}(-\sigma_{L_k}(\zeta)) - (-\sigma_{L_0}(\zeta)) \leq 2g. \tag{5.4}$$

The signs in (5.4) are written in this way on purpose. Now we integrate the inequality (5.4). Using (4.2) we get

$$\sum_{k=1}^{n}(\mu_k + M_k) - \mu_0 - M_0 < 6g + \frac{2}{9}.$$

Applying (5.3) finishes the proof. □

We see that in this approach, the control of the genus is vital. In particular we can have the following result.

Proposition 5.3 (BMY like estimate)**.** *Let C be a curve in $\mathbb{C}^2$ given in parametric form by*

$$C = \{(x, y) \in \mathbb{C}^2 \colon x = \phi(t),\ y = \psi(t),\ t \in \mathbb{C}\},$$

where ϕ and ψ are polynomials of degree p and q respectively. Assume that p and q are coprime and C has cuspidal singularities $z_1, \ldots, z_n$ with M-numbers $M_1, \ldots, M_n$ and, besides, C has precisely R ordinary double points. Then

$$\sum_{k=1}^{n} M_k < p + q - \frac{p}{q} - \frac{q}{p} - \frac{7}{9} + 2R.$$

Proof. Consider a family of curves

$$C_s = \{(x, y) \in \mathbb{C}^2 \colon x = s^p \phi(s^{-1}t),\ y = s^q \psi(s^{-1}t),\ t \in \mathbb{C}\},$$

where s is in the unit disk in $\mathbb{C}$. For $s \neq 0$ all these curves are isomorphic, while for $s = 0$ we have a homogeneous curve (t^p, t^q). Let B be a sufficiently large ball such that for each s with $|s| < 1$, C_s is transverse to the boundary ∂B. Then, $B \cap C_s$ gives raise to a deformation in the sense of Section 2. The central fiber is C_0, a homogeneous curve, while a non-central is isomorphic to the intersection of C with a large ball. We can apply Theorem 5.2 in this context, noting that the M-number of the singularity (t^p, t^q) is equal to $p + q - p/q - q/p - 1$ (see (3.2)). □

We remark that the estimate in Proposition 5.3 is very similar to Theorem 4.2 in [5]. That result, however, relies on very difficult BMY inequality.

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