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<td>Nunokawa, Mamoru; Sokol, Janosz</td>
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Osaka University
ON SOME LENGTH PROBLEMS FOR ANALYTIC FUNCTIONS

MAMORU NUNOKAWA and JANUSZ SOKÓŁ

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Abstract

Let \( \mathcal{A} \) be the class of functions

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \]

which are analytic in the unit disk \( \mathbb{D} = \{ z : |z| < 1 \} \). Let \( C(r) \) be the closed curve which is the image of the circle \( |z| = r < 1 \) under the mapping \( w = f(z) \), \( L(r) \) the length of \( C(r) \), and let \( A(r) \) be the area enclosed by the curve \( C(r) \). It was shown in [13] that if \( f \in \mathcal{A} \), \( f \) is starlike with respect to the origin, and for \( 0 \leq r < 1 \), \( A(r) < A \), an absolute constant, then

\[ L(r) = O \left( \log \frac{1}{1-r} \right) \quad \text{as} \quad r \to 1 . \]

It is the purpose of this work to prove, using a modified method than that in [13], a strengthened form of (0.1) for Bazilevič functions, strongly starlike functions and for close-to-convex functions.

1. Introduction

Let \( \mathcal{A} \) be the class of functions

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \]

which are analytic in the unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \). Let \( \mathcal{S} \) denote the subclass of \( \mathcal{A} \) consisting of all univalent in \( \mathbb{D} \).

If \( f \in \mathcal{A} \) satisfies

\[ \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, \quad z \in \mathbb{D} \]

then \( f(z) \) is said to be convex in \( \mathbb{D} \) and denoted by \( f(z) \in \mathcal{K} \).

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If \( f \in \mathcal{A} \) satisfies
\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0, \quad z \in \mathbb{D}
\]
then \( f(z) \) is said to be starlike with respect to the origin in \( \mathbb{D} \) and denoted by \( f(z) \in S^* \).

Furthermore, if \( f \in \mathcal{A} \) satisfies
\[
(1.2) \quad \Re \left\{ \frac{zf'(z)}{e^{ia}g(z)} \right\} > 0, \quad z \in \mathbb{D}
\]
for some \( g(z) \in S^* \) and some \( \alpha \in (-\pi/2, \pi/2) \), then \( f(z) \) is said to be close-to-convex in \( \mathbb{D} \) and denoted by \( f(z) \in C \). An univalent function \( f \in \mathcal{S} \) belongs to \( C \) if and only if the complement \( E \) of the image-region \( F = \{ f(z) : |z| < 1 \} \) is the union of rays that are disjoint (except that the origin of one ray may lie on another one of the rays).

On the other hand, if \( f \in \mathcal{A} \) satisfies
\[
\Re \left\{ \frac{zf''(z)}{f(z)} \right\} > 0, \quad z \in \mathbb{D}
\]
for some \( g(z) \in S^* \) and some \( \beta \in (0, \infty) \), then \( f(z) \) is said to be a Bazilević function of type \( C \) and denoted by \( f(z) \in B(\beta) \).

Let \( SS^*(\alpha) \) denote the class of strongly starlike functions of order \( \alpha \), \( 0 < \alpha \leq 1 \),
\[
SS^*(\alpha) := \left\{ f \in \mathcal{A} : \left| \frac{zf'(z)}{f(z)} \right| < \frac{\alpha \pi}{2}, \quad z \in \mathbb{D} \right\},
\]
which was introduced in [12] and [1].

Let \( C(r) \) be the closed curve which is the image of \( |z| = r < 1 \) under the mapping \( w = f(z) \). Let \( L(r) \) denote the length of \( C(r) \) and let \( A(r) \) be the area enclosed by \( C(r) \).

Let us define \( M(r) \) by
\[
M(r) = \max_{|z|<r<1} |f(z)|.
\]
Then F.R. Keogh [4] has shown that

**Theorem 1.1.** Suppose that \( f(z) \in S^* \) and
\[
|f(z)| \leq M < \infty, \quad z \in \mathbb{D}.
\]

Then we have
\[
L(r) = \mathcal{O} \left( \log \frac{1}{1-r} \right) \quad \text{as} \quad r \to 1,
\]
where $O$ means Landau’s symbol.

Furthermore, D.K. Thomas in [13] extended this result for bounded close-to-convex functions. Ch. Pommerenke in [9] has shown that

**Theorem 1.2.** If $f(z) \in \mathcal{C}$, then

$$L(r) = O \left\{ M(r) \left( \log \frac{1}{1-r} \right)^{5/2} \right\} \text{ as } r \to 1.$$  

Later, D.K. Thomas in [14] has shown that

**Theorem 1.3.** If $f(z) \in \mathcal{S}^*$, then

$$L(r) = O \left\{ \sqrt{A(r)} \log \frac{1}{1-r} \right\} \text{ as } r \to 1.$$  

M. Nunokawa in [6, 7] has shown that

**Theorem 1.4.** If $f(z) \in \mathcal{K}$, then

$$L(r) = O \left\{ A(r) \log \frac{1}{1-r} \right\}^{1/2} \text{ as } r \to 1.$$  

Moreover, D.K. Thomas in [15] has shown the following two theorems

**Theorem 1.5.** If $f(z) \in \mathcal{B}(\beta)$ and $|f(z)| < 1$ in $\mathbb{D}$, then we

$$L(r) = O \left( \log \frac{1}{1-r} \right) \text{ as } r \to 1.$$  

**Theorem 1.6.** If $f(z) \in \mathcal{B}(\beta)$ and $0 < \beta \leq 1$, then we

$$L(r) = O \left( M(r) \log \frac{1}{1-r} \right) \text{ as } r \to 1.$$  

M. Nunokawa, S. Owa et al. in [8] have shown that

**Theorem 1.7.** If $f(z) \in \mathcal{B}(\beta)$ and $zf'(z) = f^{1-\beta}(z)g^{\beta}(z)h(z)$, then we

$$L(r) = O \left\{ \sqrt{A^{1-\beta}(r)G^{\beta}(r)} \left( \log \frac{1}{1-r} \right)^2 \right\} \text{ as } r \to 1,$$
where

\[ G(r) = \int_0^r \int_0^{2\pi} e|g'(re^{i\theta})|^2 d\theta d\phi \]

or \( G(r) \) is the area of the image domain of \(|z| \leq r\) under the starlike mapping \( g \).

Ch. Pommerenke in [9] has also shown that

**Theorem 1.8.** If \( f(z) \in \mathcal{S} \), then

\[ M(r) \leq 4 \sqrt{A(r)} \log \frac{3}{1 - r} \quad (|z| = r < 1). \]

Therefore, we have

\[ M(r) = \mathcal{O} \left( A(r) \log \frac{1}{1 - r} \right)^{1/2} \text{ as } r \to 1. \]

It is the purpose of this work to prove, using a modified method than that in [13], a strengthened form of (0.1) for Bazilevič functions, strongly starlike functions and for close-to-convex functions.

2. **Lemmas**

**Lemma 2.1.** If \( h(z) \) is analytic and \( \Re\{h(z)\} > 0 \) in \( \mathbb{D} \) with \( h(0) = 1 \), then

\[ \frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\theta})|^2 d\theta \leq \frac{1 + 3r^2}{1 - r^2} < \frac{4}{1 - r^2} \]

for \( 0 < r < 1 \).

Lemma 2.1 can be easily proved using \(|h^{(n)}(0)| \leq 2n!\) and the Gutzmer’s theorem, see for example [3, p.31].

**Lemma 2.2.** If \( f(z) \in \mathcal{S} \), then we have

\[ \left| \frac{zf''(z)}{f(z)} \right| \leq \frac{1 + |z|}{1 - |z|} < \frac{2}{1 - |z|} \quad \text{in } \mathbb{D}, \]

\[ |f''(z)| \leq \frac{1 + |z|}{(1 - |z|)^3} \quad \text{in } \mathbb{D}. \]

A proof can be found in [10, p.21].
Lemma 2.3 ([2, p. 337]). If \( h(z) \) is analytic and \( \Re\{h(z)\} > 0 \) in \( \mathbb{D} \) with \( h(0) = 1 \), then we have

\[
|h'(z)| \leq \frac{2 \Re\{h(z)\}}{1 - |z|^2} < \frac{2}{1 - |z|} \quad \text{in} \quad \mathbb{D}.
\]

A proof can be found also in [5].

An analytic function \( f \) is said to be subordinate to an analytic function \( F \), or \( F \) is said to be superordinate to \( f \), if there exists a function an analytic function \( w \) such that

\[
w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{D}),
\]
and

\[
f(z) = F(w(z)) \quad (z \in \mathbb{D}).
\]

In this case, we write \( f \prec F \) (\( z \in \mathbb{D} \)) or \( f(z) \prec F(z) \) (\( z \in \mathbb{D} \)). If the function \( F \) is univalent in \( \mathbb{D} \), then we have

\[
[f \prec F \ (z \in \mathbb{D})] \Leftrightarrow [f(0) = F(0) \text{ and } f(\mathbb{D}) \subset F(\mathbb{D})].
\]

Lemma 2.4. If \( f(z) \) is subordinate to \( g(z) \) in \( \mathbb{D} \) and if \( 0 < p \), then

\[
\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \leq \int_0^{2\pi} |g(re^{i\theta})|^p \, d\theta
\]

for all \( r, 0 < r < 1 \).

W. Rogosinski has shown Lemma 2.4 in [11].

3. Main results

Theorem 3.1. If \( f(z) \in \mathcal{S} \) satisfies the condition

\[
\Re\left\{ 1 + \frac{zf''(z)}{f(z)} \right\} \geq -\Re\left\{ \frac{1 + z}{1 - z} \right\} \quad \text{in} \quad \mathbb{D},
\]

then we have

\[
L(r) = O \left( A(r) \log \frac{1}{1 - r} \right)^{1/2} \quad \text{as} \quad r \to 1.
\]

Proof. For the case \( 0 < r \leq 1/2 \), from Lemma 2.2 we have

\[
L(r) = \int_0^{2\pi} |zf'(z)| \, d\theta
\]
\[
\leq \int_0^{2\pi} \frac{|z|(1 + |z|)}{(1 - |z|)^3} \, d\theta
\]
\[
< 12\pi.
\]
For the case $1/2 < r < 1$, we have

$$L(r) = \int_0^{2\pi} |zf'(z)| \, d\theta$$

$$= \int_0^{2\pi} \int_0^r \left| f'(z) + zf''(z) \right| \, d\theta \, dQ$$

$$= \int_0^{2\pi} \int_0^r \left| f'(z) \left( 1 + \frac{zf''(z)}{f(z)} \right) \right| \, d\theta \, dQ$$

$$\leq \left( \int_0^{2\pi} \int_0^r |f'(z)|^2 \, d\theta \, dQ \right)^{1/2} \left( \int_0^{2\pi} \int_0^r \left| 1 + \frac{zf''(z)}{f(z)} \right|^2 \, d\theta \, dQ \right)^{1/2}$$

$$< \left( 2 \int_0^{2\pi} \int_0^r \theta |f'(z)|^2 \, d\theta \, dQ \right)^{1/2} \left( \int_0^{2\pi} \int_0^r \left| 1 + \frac{zf''(z)}{f(z)} \right|^2 \, d\theta \, dQ \right)^{1/2}$$

$$= \sqrt{2A(r)} \left( \int_0^{2\pi} \int_0^r \left| 1 + \frac{zf''(z)}{f(z)} \right|^2 \, d\theta \, dQ \right)^{1/2}.$$

From the hypothesis (3.1), we have

$$\Re \left\{ 1 + \frac{zf''(z)}{f(z)} + \frac{1 + z}{1 - z} \right\} > 0 \quad \text{in} \quad \mathbb{D}$$

or

$$\frac{1 + zf''(z)/f(z) + (1 + z)/(1 - z)}{2} < \frac{1 + z}{1 - z} \quad \text{in} \quad \mathbb{D}, \quad (3.3)$$

It follows that

$$1 + \frac{zf''(z)}{f(z)} + \frac{1 + z}{1 - z} < 2 \frac{1 + z}{1 - z} \quad \text{in} \quad \mathbb{D},$$

where the symbol $<$ means the subordination. Then we have

$$\int_0^r \int_0^{2\pi} \left| 1 + \frac{zf''(z)}{f(z)} \right|^2 \, d\theta \, dQ$$

$$= \int_0^r \int_0^{2\pi} \left| 1 + \frac{zf''(z)}{f(z)} + \frac{1 + z}{1 - z} \right| \, d\theta \, dQ$$

$$\leq \int_0^r \int_0^{2\pi} \left| 1 + \frac{zf''(z)}{f(z)} + \frac{1 + z}{1 - z} \right|^2 \, d\theta \, dQ$$

$$+ 2 \int_0^r \int_0^{2\pi} \frac{\left| 1 + \frac{zf''(z)}{f(z)} + \frac{1 + z}{1 - z} \right|^2}{|1 - z|^2} \, d\theta \, dQ$$

$$= I_1 + 2I_2 + I_3.$$
From Lemma 2.4, (3.3) and Lemma 2.1, we have

\[ I_1 = \int_0^r \int_0^{2\pi} \left| 1 + \frac{zf''(z)}{f(z)} + \frac{1 + z}{1 - z} \right|^2 d\theta d\varphi \]
\[ \leq \int_0^r \int_0^{2\pi} 4 \left| \frac{1 + z}{1 - z} \right|^2 d\theta d\varphi \]
\[ < 32\pi \int_0^r \frac{1}{1 - \varphi^2} d\varphi \]
\[ = 16\pi \log \frac{1 + r}{1 - r}. \]

By Lemma 2.1, we have

\[ 2I_2 = \left( \int_0^r \int_0^{2\pi} \left| 1 + \frac{zf''(z)}{f(z)} + \frac{1 + z}{1 - z} \right|^2 d\theta d\varphi \right)^{1/2} \left( \int_0^r \int_0^{2\pi} \left| \frac{1 + z}{1 - z} \right|^2 d\theta d\varphi \right)^{1/2} \]
\[ \leq \left( 16\pi \log \frac{1 + r}{1 - r} \right)^{1/2} \left( 8\pi \int_0^r \frac{1}{1 - \varphi^2} d\varphi \right)^{1/2} \]
\[ = \left( 16\pi \log \frac{1 + r}{1 - r} \right)^{1/2} \left( 4\pi \log \frac{1 + r}{1 - r} \right)^{1/2} \]
\[ = O\left( \log \frac{1}{1 - r} \right) \quad \text{as} \quad r \to 1. \]

By Lemma 2.1, we have

\[ I_3 = \int_0^r \int_0^{2\pi} \left| \frac{1 + z}{1 - z} \right|^2 d\theta d\varphi \]
\[ = 4\pi \log \frac{1 + r}{1 - r} \]
\[ = O\left( \log \frac{1}{1 - r} \right) \quad \text{as} \quad r \to 1. \]

This shows (3.2) which completes the proof of Theorem 3.1.

**Theorem 3.2.** If \( f(z) \in B(\beta) \) is a Bazilevič function of type \( \beta, \) \( 0 < \beta \leq 1 \), then we have

\[ L(r) = O\left\{ A(r) \left( \log \frac{1}{1 - r} \right)^{3/2} \right\} \quad \text{as} \quad r \to 1. \]

**Proof.** Because \( f(z) \in B(\beta) \), there exists \( g(z) \in S^* \) and there exists an analytic function \( h(z) \), \( h(0) = 1 \), \( \Re\{h(z)\} > 0 \) in \( \mathbb{D} \), such that

\[ zf'(z) = f^{1-\beta}(z)g^\beta(z)h(z). \]
Therefore we have
\[
L(r) = \int_0^{2\pi} |zf'(z)| \, d\theta \\
= \int_0^{2\pi} |f^{1-\beta}(z)g^\beta(z)h(z)| \, d\theta \\
\leq M^{1-\beta}(r) \int_0^{2\pi} |g^\beta(z)h(z)| \, d\theta \\
\leq M^{1-\beta}(r) \{ \int_0^r \int_0^{2\pi} \beta |g^{\beta-1}(z)g'(z)h(z)| \, d\theta \, dQ + \int_r^1 \int_0^{2\pi} |g^\beta(z)h'(z)| \, d\theta \, dQ \} \\
\leq M^{1-\beta}(r)(I_1(r) + I_2(r)).
\]

Applying Ch. Pommerenke’s result (1.3), we have
\[
L(r) \leq \left( \frac{16}{\pi} A(r) \log \frac{3}{1-r} \right)^{(1-\beta)/2} (I_1(r) + I_2(r)).
\]

D.K. Thomas in [15] has shown that if \( f(z) \) is a Bazilević function of type \( \beta, 0 < \beta \), then
\[
I_1(r) \leq 2 \sqrt{2\pi} \beta K(\beta) \left( \frac{1}{r} \log \frac{1 + r}{1 - r} \right)^{1/2}
\]

(3.6)
\[
= \mathcal{O} \left\{ \left( \log \frac{1}{1-r} \right)^{1/2} \right\} \quad \text{as} \quad r \to 1,
\]
where
\[
K(\beta) = \max\{1,(4/r)^{1-\beta}\}
\]
is a bounded constant not necessarily the same each time. On the other hand
\[
I_2(r) = \int_0^r \int_0^{2\pi} |g^\beta(z)h'(z)| \, d\theta \, dQ.
\]
Using (2.1) we obtain
\[
I_2(r) \leq \int_0^r \int_0^{2\pi} |g(z)|^\beta \Re\{h(z)\} \frac{2}{1 - \varrho^2} \, d\theta \, dQ \\
\leq 2 \Re \left\{ \int_0^r \int_0^{2\pi} \frac{|g^\beta(z)|}{g^\beta(z)} g^\beta(z)h(z) \frac{1}{1 - \varrho^2} \, d\theta \, dQ \right\}.
\]
Using (3.5) we can write
\[
I_2(r) \leq 2 \Re \left\{ \int_0^r \int_0^{2\pi} \frac{zf'(z)f^{\beta-1}(z)e^{-i \arg g(z)}}{1 - \varrho^2} \, d\theta \, dQ \right\}.
\]
Because \( g(z) \) is a starlike function, then \( \arg g(\rho e^{i\theta}) \) is an increasing function of \( \theta \) and maps the interval \([0, 2\pi]\) onto itself. Applying D. K. Thomas method [15, p. 357], after a suitable substitution and integrating by parts, we obtain

\[
I_2(r) \leq \frac{2}{\beta} \Re \left\{ \int_0^r \int_{|l|=\varrho} \frac{z}{i} \left( \frac{df^\beta(z)}{dz} \right) e^{-i\beta \arg g(z)} dz \frac{d\varrho}{1-\varrho^2} d\varrho \right\} 
\]

\[
= 2 \Re \left\{ \int_0^r \int_{|l|=\varrho} \frac{1}{i\beta(1-\varrho^2)} \int_{|l|=\varrho} e^{-i\beta \arg g(z)} \left( \frac{df^\beta(z)}{d\varrho} \arg g(z) \right) d\varrho \, d\varrho g(z) d\varrho \right\} 
\]

\[
= 2 \Re \left\{ \int_0^r \int_{|l|=\varrho} \frac{d\varrho}{i\beta(1-\varrho^2)} \left\{ f^\beta(z)e^{-i\beta \arg g(z) \arg g(z)=2\pi} \right\} \arg g(z) d\varrho g(z) d\varrho \right\} 
\]

\[
= 2 \Re \left\{ \int_0^r \int_{|l|=\varrho} f^\beta(z)e^{-i\beta \arg g(z)} \frac{1}{1-\varrho^2} d\varrho \, d\varrho g(z) d\varrho \right\} 
\]

\[
\leq 4\pi \int_0^r M^\beta(\varrho)/(1-\varrho^2) d\varrho. 
\]

Applying Ch. Pommerenke’s result (1.3), we have

\[
I_2(r) \leq 16\sqrt{\pi} \int_0^r \left( A(\varrho) \log \frac{3}{1-\varrho} \right)^{\beta/2} (1-\varrho^2) d\varrho 
\]

\[
\leq 16\sqrt{\pi} A^{\beta/2}(r) \int_0^r \left( \log \frac{3}{1-\varrho} \right)^{\beta/2} \frac{1}{1-\varrho} d\varrho 
\]

\[
= 16\sqrt{\pi} A^{\beta/2}(r) \frac{2}{\beta} + 2 \int_0^r \left( \log \frac{3}{1-\varrho} \right)^{(\beta+2)/2} d\varrho 
\]

\[
= O \left\{ A^{\beta/2}(r) \left( \log \frac{1}{1-r} \right)^{(\beta+2)/2} \right\} \text{ as } r \to 1. 
\]

Applying it together with (3.6) we obtain (3.4).

\[\square\]

**Theorem 3.3.** If \( f(z) \in \mathcal{B}(\beta) \) is a Bazilevič function of type \( \beta \), \( 1 < \beta \), then we have

\[
L(r) = O \left\{ A^\beta(r) \left( \log \frac{1}{1-r} \right)^{\beta+2} \right\}^{1/2} \text{ as } r \to 1. 
\]
Proof. For the case \(0 < r \leq 1/2\), because \(B(\beta) \subset S\), by Lemma 2.2 we have

\[
L(r) = \int_0^{2\pi} |zf'(z)| \, d\theta \\
\leq \int_0^{2\pi} \frac{r(1 + r)}{(1 - r)^3} \, d\theta \\
< 12\pi,
\]

where \(r = |z|\). Assume that

\[
h(z) = \frac{zf'(z)}{f^{1-\beta}(z)g^\beta(z)}, \quad \Re\{h(z)\} > 0, \ z \in D, \ g \in S^*.
\]

For the case \(1/2 < r < 1\), we have

\[
L(r) = \int_0^{2\pi} |zf'(z)| \, d\theta \\
= \int_0^{2\pi} |f^{1-\beta}(z)g^\beta(z)h(z)| \, d\theta \\
\leq \int_0^{2\pi} \left( \frac{1 + r}{r} \right)^{\beta-1} |g^\beta(z)h(z)| \, d\theta \\
\leq \left( \frac{9}{2} \right)^{\beta-1} \int_0^{2\pi} |g^\beta(z)h(z)| \, d\theta \\
\leq \left( \frac{9}{2} \right)^{\beta-1} \left\{ \int_0^{2\pi} \int_0^r \beta |g'(z)g^{\beta-1}(z)h(z)| \, dq \, d\theta + \int_0^{2\pi} \int_0^r |g^\beta(z)h'(z)| \, dq \, d\theta \right\} \\
= \left( \frac{9}{2} \right)^{\beta-1} (I_1(r) + I_2(r)).
\]

Using the result (3.7) for \(1/2 < r < 1\), we have

\[
I_1(r) \leq 2\sqrt{2\pi} \beta K_1(\beta) \left( 2 \log \frac{1}{1 - r} \right)^{1/2},
\]

where \(K_1(\beta) \leq \max\{1, 8^{1-\beta}\}\). Furthermore, in the same way as in the previous proof, we obtain

\[
I_2(r) = \int_0^{2\pi} \int_0^r |g^\beta(z)h'(z)| \, dq \, d\theta \\
= \mathcal{O} \left\{ (A(r))^{\beta/2} \left( \log \frac{1}{1 - r} \right)^{(\beta+2)/2} \right\} \quad \text{as} \quad r \to 1,
\]

where \(K_2(r)\) is a bounded function of \(\beta\). This completes the proof. \(\Box\)
Remark 3.4. D.K. Thomas in [15] has shown that if \( f(z) \) is a Bazilević function of type \( \beta, 0 < \beta \leq 1 \), then

\[
L(r) \leq K(\beta)M(r) \log \frac{1}{1 - r},
\]

where \( K(\beta) \) is a bounded function of \( \beta \). On the other hand, from Ch. Pommerenke’s result [9], we have

\[
L(r) \leq K(\beta) \sqrt{A(r) \left( \log \frac{1}{1 - r} \right)^{3/2}}.
\]

From Theorems 3.2 and 3.3 we have that if \( f(z) \) is a Bazilević function of type \( \beta, 0 < \beta \leq 1 \), then

\[
L(r) = \begin{cases} 
\mathcal{O} \left( A^{\beta/2}(r) \left( \log \frac{1}{1 - r} \right)^{\beta + 2/2} \right) & \text{for } 1 < \beta, \\
\mathcal{O} \left( A^{1/2}(r) \left( \log \frac{1}{1 - r} \right)^{3/2} \right) & \text{for } 0 < \beta \leq 1,
\end{cases}
\]

as \( r \to 1 \).

Theorem 3.5. Let \( f \in SS^*(\alpha) \) be strongly starlike function of order \( \alpha, 0 < \alpha < 1 \). Then we have

(3.9) \[
L(r) = \mathcal{O} \left( A(r) \left( \log \frac{1}{1 - r} \right)^{1/2} \right) \quad \text{as} \quad r \to 1.
\]

Proof. From the hypothesis of the Theorem and applying Ch. Pommerenke’s [9] and Rogosinski’s [11] results, we have

\[
L(r) = \int_0^{2\pi} |zf'(z)| \, d\theta \\
= \int_0^{2\pi} |f(z)| \left| \frac{zf'(z)}{f(z)} \right| \, d\theta \\
\leq M(r) \int_0^{2\pi} \left| \frac{zf'(z)}{f(z)} \right| \, d\theta \\
\leq \sqrt{K}A(r) \log(1 - r) \int_0^{2\pi} \left| \frac{1 + z}{1 - z} \right|^\alpha \, d\theta \\
\leq \sqrt{K}A(r) \log(1 - r) \int_0^{2\pi} \frac{2}{|1 - z|^\alpha} \, d\theta \\
= \mathcal{O} \left( A(r) \left( \log \frac{1}{1 - r} \right)^{1/2} \right) \quad \text{as} \quad r \to 1,
\]
where $K$ is a bounded constant and because we have
\[
\int_0^{2\pi} \frac{2}{|1 - e^{i\theta}|^2} \, d\theta < \infty \quad \text{for} \quad 0 < \alpha < 1. \quad \Box
\]

**Corollary 3.6.** Let $f \in \mathcal{C}$ be close-to-convex function, satisfy (1.2) with $\alpha = 0$ in $\mathbb{D}$ and map $\mathbb{D}$ onto a domain of finite area $A$. Then by Theorem 3.2, $\beta = 1$, we have
\[
L(r) = O \left( \log \frac{1}{1-r} \right)^{3/2} \quad \text{as} \quad r \to 1.
\]

Notice that D.K. Thomas in Theorem 2 [13, p.431]. has shown that
\[
L(r) = O \left( \log \frac{1}{1-r} \right) \quad \text{as} \quad r \to 1.
\]
when $f \in \mathcal{C}$, satisfies (1.2) with $\alpha = 0$ and $f$ is bounded in $\mathbb{D}$.

**Corollary 3.7.** Let $f \in \mathcal{C}$ be close-to-convex function, satisfy (1.2) with $\alpha = 0$ in $\mathbb{D}$. Then by Theorem 3.2, $\beta = 1$, we have
\[
L(r) = O \left\{ A(r) \left( \log \frac{1}{1-r} \right)^{3/2} \right\} \quad \text{as} \quad r \to 1.
\]

In [13] it was shown that
\[
L(r) = O \left\{ M(r) \left( \log \frac{1}{1-r} \right) \right\} \quad \text{as} \quad r \to 1,
\]
when $f \in \mathcal{C}$, satisfies (1.2) with $\alpha = 0$. Compare also Theorems 1.1–1.8 in the introduction.

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**References**


Mamoru Nunokawa  
University of Gunma  
Hoshikuki-cho 798-8  
Chuo-Ward, Chiba, 260-0808  
Japan  
e-mail: mamoru_nuno@doctor.nifty.jp

Janusz Sokół  
Department of Mathematics  
Rzeszów University of Technology  
Al. Powstańców Warszawy 12  
35-959 Rzeszów  
Poland  
e-mail: jsokol@prz.edu.pl