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K-GROUPS OF SYMMETRIC SPACES I

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Introduction

In this paper we consider the unitary K-groups of compact homogeneous spaces of Lie groups and in particular, we lay emphasis on the compact symmetric spaces.

For a compact connected Lie group G with $\pi_1(G)$ torsion-free as a symmetric space it is known that the K-group $K^*(G)$ of G is an exterior algebra on the elements of $K^{-1}(G)$ induced by the basic representations of G [2], [3], [7]. Making use of this result Hodgkin constructed the Kunneth formula spectral sequence in equivariant K-theory [8], [9]. This spectral sequence is our main tool in the present study. Besides we find some examples of the K-groups of symmetric spaces in [5].

The main theorem of this paper is the following

Theorem A. Let G be a compact connected simply-connected Lie group together with the involutive automorphism σ and K the subgroup of G consisting of fixed points of σ . When we write M for the homogeneous space G/K, we have

(i) There are elements ρ_1, \dots, ρ_l of R(G) such that

$$\sigma^*(\rho_k) = \rho_k \ (r+1 \leq k \leq l) \quad \text{for some } r \text{ and}$$
$$R(G) = Z[\rho_1, \cdots, \rho_r, \sigma^*(\rho_1), \cdots, \sigma^*(\rho_r), \rho_{r+1}, \cdots, \rho_l]$$

(ii) The natural homomorphism $\alpha: Z \bigotimes_{R(G)} R(K) \to K^{\circ}(M)$ becomes a monomorphism (Section 1) and if we identify an element of $Z \bigotimes_{R(G)} R(K)$ with its image by α , then we can write

$$K^*(M) = \Lambda(\beta(\rho_1 - \sigma^*(\rho_1)), \cdots, \beta(\rho_r - \sigma^*(\rho_r))) \otimes (Z \bigotimes_{r \in I} R(K)).$$

where $\beta(\rho_k - \sigma^*(\rho_k))$ is the element of $K^{-1}(M)$ induced by the representations ρ_k and $\sigma^*(\rho_k)$ in (i) for $k=1, \dots, r$ (Section 1).

(iii) $K^*(M)$ is torsion-free.

The arrangement of this paper is as follows.

In section 1 we describe the definitions of the α - and β - elements of $K^*(M)$ in Theorem A and summarize some of the facts on the Künneth formula spectral sequence.

In sections 2-4 we give a remark (Theorem 2.1) on Snaith's collapsing theorem ([14], Theorem 5.5) for the Künneth formula spectral sequence in equivariant K-theory. Professor V.P. Snaith informed the author that Theorem 2.1 is known to him and the author agreed with him in an outline of a proof. For a proof of Theorem A we have need of Proposition 4.1 obtained as a corollary to the proof of Theorem 2.1.

Sections 5–8 are devoted to the proof of Theorem A.

1. The α and β constructions and the spectral sequence

Let G be a compact Lie group and H a closed subgroup of G. The K-group of the homogeneous space G/H has two kind of elements induced by the unitary representations of G and H.

Over G/H we have the canonical principal H-bundle η . Then, for an Hvector space V, the vector bundle with fibre V associated with η defines an element $\alpha(V)$ of $K^{\circ}(G/H)$. Thus η defines a homomorphism of rings $\alpha: R(H) \rightarrow$ $K^{\circ}(G/H)$ and we see that α is clearly factored through the natural projection $R(H) \rightarrow Z \bigotimes_{R(G)} R(H)$ where R(G) (resp. R(H)) is the complex representation ring of G (resp. H) ([1], [9] §9). We shall denote this factored homomorphism $Z \bigotimes_{R(G)} R(H) \rightarrow K^{\circ}(G/H)$ by the same letter α .

The other elements are defined in the following way. Consider a representation of G viewed as a homomorphism of G to the unitary group U(n). If $\rho_1, \rho_2: G \to U(n)$ are representations of G agreeing on H, then we can define a map $f: G/H \to U(n)$ by $f(gH) = \rho_1(g) \rho_2(g)^{-1}$ for $gH \in G/H$. Then the composition of f and the inclusion of U(n) to the stable unitary group forms an element of $K^{-1}(G/H)$. We denote this element by $\beta(\rho_1 - \rho_2)$.

Suppose that G is a compact connected Lie group such that $\pi_1(G)$ is torsion-free and let K_G^* denote the equivariant K-theory associated with G [13]. In [8], [9], Hodgkin constructed a strongly convergent spectral sequence

(1.1)
$$E_2^{*,*} = \operatorname{Tor}_{R(G)}^{*,*}(K_G^*(X), K_G^*(Y)) \Rightarrow F_G^*(X; Y),$$

and showed that there is a natural transformation λ of $F^*_G(X; Y)$ to $K^*_G(X \times Y)$ and if either X or Y is a free G-space then λ is an isomorphism ([8], Propositions 6.3 and 7.2).

In particular, when X=G and Y=G/H, a homogeneous space, in the spectral sequence (1.1), (1.1) becomes

(1.2)
$$E_2^{*,0} = \operatorname{Tor}_{R(G)}^{*,0}(Z, R(H)) \Rightarrow K^*(G/H)$$

because of $K^*_G(G/H) = R(H)$.

2. A collapsing theorem for (1.2)

We have the following

Theorem 2.1. Let G be a compact connected Lie group such that $\pi_1(G)$ is torsion-free and H a closed connected subgroup of G. Then the spectral sequence

$$E_2^{*,0} = \operatorname{Tor}_{R(G)}^{*,0}(Z, R(H) \Rightarrow K^*(G/H)$$

collapses.

From now we write $E_r^{*,*}(X; Y)_G$ for the *r*-th term of the spectral sequence (1.1) and also $\{E_r^{*,*}(X; Y)_G\}$ for this spectral sequence.

To prove Theorem 2.1 we reduce this theorem to Theorem 5.5 of [14] (which requires the conditions that $H^*(BG, Z)$ and $H^*(BH, Z)$ are polynomial algebras). For this purpose we prepare two lemmas.

Lemma 2.1. Let T be a maximal torus of H in Theorem 2.1. If the spectral sequence

$$E_2^{*,0} = \operatorname{Tor}_{R(G)}^{*,0}(Z, R(T)) \Rightarrow K^*(G/T)$$

collapses, then so does the spectral sequence

$$E_2^{*,0} = \operatorname{Tor}_{R(G)}^{*,0}(Z, R(H)) \Rightarrow K^*(G/H).$$

Proof. The natural projection $G/T \rightarrow G/H$ induces a morphism of the spectral sequences

$$\{\varphi_r\}: \{E_r^{*,*}(G, G/H)_G\} \to \{E_r^{*,*}(G, G/T)_G\}$$
.

For a proof of Lemma 2.1 it is sufficient to to prove that φ_2 is injective. However it follows easily from the facts that $\varphi_2 = \operatorname{Tor}_{R(G)}^{*,*}(1, i^*)$ where i^* is the restriction of R(H) to R(T) and R(H) is a direct summand of R(T) as an R(G)-module (via restriction). q.e.d.

By choosing unitary representations of G suitably, we can embed G into a finite product of unitary groups U such that if we denote this embedding by $i: G \rightarrow U$, then

(2.1)
$$i^* \colon R(U) \to R(G)$$
 is surjective.

Let X be a compact, locally contractible G-space of finite covering dimension. Then we have

Lemma 2.2. Suppose that the spectral sequence

$$E_2^{*,*} = \operatorname{Tor}_{R(U)}^{*,*}(Z, K_G^*(X)) \Rightarrow K^*(U \times X)$$

collapses, then so does

$$E_2^{*,*} = \operatorname{Tor}_{R(G)}^{*,*}(Z, K_G^*(X)) \Rightarrow K(X)$$

3. Proof of Lemma 2.2

Here we shall give a proof of Lemma 2.2.

Let L be a compact connected Lie group. We define I(L) to be the kernel of the augumentation $\mathcal{E}:R(L) \rightarrow Z$ of R(L) and J(L) the quotient $I(L)/(I(L))^2$. Then we know that if the fundamental group of L is torsion-free then J(L) is a free abelian group of rank l where l is the rank of L ([7], Lemma 4.2).

From (2.1) we see obviously that the homomorphism induced by i^*

(3.1)
$$i_1^*: J(U) \to J(G)$$
 is surjective.

Define $J(U, G) = \text{Ker } i_1^*$. We can choose a basis $\xi_1, \dots, \xi_l, \nu_1, \dots, \nu_s$ for J(U) such that $i_1^*(\xi_1), \dots, i_1^*(\xi_l)$ form a basis J(G) and ν_1, \dots, ν_s a basis for J(U, G). For brevity we denote the representatives of these elements in R(U) by the same notation and then we may assume that

(3.2)
$$i^*(\nu_k) = 0$$
 for $k = 1, \dots, s$.

Then we have the Koszul complex given by

$$(3.3) C^* = \Lambda(x_1, \cdots, x_s, y_1, \cdots, y_l) \otimes R(U)$$

where $d(x_i) = \nu_i$ $(1 \le i \le s)$, $d(y_j) = \xi_j$ $(1 \le j \le l)$ and d is a derivation.

Proof of Lemma 2.2. The inclusion $X \to U \underset{\sigma}{\times} X$ induces a morphism $\{\phi_r\}$ of $\{E_r^{*,*}(U, U \underset{a}{\times} X)_U\}$ to $\{E_r^{*,*}(U, X)_G\}$.

Using (3.3) we have isomorphisms

(3.4)

$$E_{2}^{*,*}(U, U \underset{\sigma}{\times} X)_{U} = \operatorname{Tor}_{R(U)}^{*,*}(Z, K_{G}^{*}(X))$$

$$\cong H^{*}(C^{*} \underset{R(U)}{\otimes} K_{G}^{*}(X))$$

$$\cong \Lambda(x_{1}, \cdots, x_{s}) \otimes \operatorname{Tor}_{R(G)}^{*,*}(Z, K_{G}^{*}(X)) \qquad \text{by (3.2)}.$$

Next we consider $E_2^{*,*}(U, X)_G$. For this we need that

(3.5) $K^*(U/G)$ is torsion-free.

Suppose that (3.5) is true for the moment. Then we have an isomorphism

(3.6)
$$E_{2}^{*,*}(U, X)_{G} = \operatorname{Tor}_{R(G)}^{*,*}(K^{*}(U/G), K^{*}_{G}(X)) \\ \simeq K^{*}(U/G) \otimes \operatorname{Tor}_{R(G)}^{*,*}(Z, K^{*}_{G}(X))$$

from (2.1) and (3.5).

From (3.4) and (3.6) we see that ϕ_2 induces an epimorphism from the torsion-part of $E_2^{*,*}(U, U \underset{\sigma}{\times} X)_U$ to that of $E_2^{*,*}(U, X)_G$, and therefore we see by the assumption that $E_2^{*,*}(U, X)_G$ consists of permanent cycles. Moreover when we consider the morphism of the spectral sequences

$$\psi_r \colon \{E_r^{*,*}(U, X)_G\} \to \{E_r^{*,*}(G, X)_G\}$$

induced by the embedding of G to U, it is easy to see that $E_z^{*,*}(G, X)_G$ also consists of permanent cycles.

It remains to prove (3.5). Put X=G in the above. Then, from Lemma 7.3 of [8], it follows that $\{E_r^{*,*}(U, U)_U\}$ collapses, and (3.6) when X=G follows from the facts that $\operatorname{Tor}_{R(G)}^{*,*}(Z, Z)$ is torsion-free ([8], Lemma 7.2). Hence we see that $\{E_r^{*,*}(U, G)_G\}$ collapses by using the above argument and so $K^*(U/G)$ is isomorphic to a subgroup of $K^*(U)$. This shows (3.5). Therefore Lemma 2.2 is proved.

4. Proof of Theorem 2.1 and a corollary

Proof of Theorem 2.1. Putting X=G/T in Lemma 2.2 where T is a maximal torus of H, Lemmas 2.1 and 2.2 imply that $\{E_r^{*,*}(G, G/H)_G\}$ collapses because $\{E_r^{*,*}(U, U/T)_U\}$ does so by Theorem 5.5 of [14]. q.e.d.

Next we describe a result obtained from the proof of Lemma 2.2.

Proposition 4.1 (Cf. [5], Proposition 2.3).

Let G and H be as in Theorem 2.1. Suppose that $\pi_1(H)$ is torsion-free and the restriction i^* : $R(G) \rightarrow R(H)$ is surjective. Then we have

- (i) There exist elements ν_1, \dots, ν_s of R(G) such that $i^*(\nu_k)=0$ for $k=1, \dots, s$ and $\pi(\nu_1), \dots, \pi(\nu_s)$ form a basis for the free abelian group Ker $(J(G) \rightarrow J(H))$ where π is the composition of the natural projections $R(G) \rightarrow I(G) \rightarrow J(G)$.
- (ii) $K^*(G/H)$ is an exterior algebra on $\beta(\nu_1), \dots, \beta(\nu_s)$.

Proof. By Theorem 2.1 the spectral sequence

$$E_2^{*,0} = \operatorname{Tor}_{R(G)}^{*,0}(Z, R(H)) \Rightarrow K^*(G/H)$$

collapses. Here we consider the E_2 -term of this spectral sequence. In section 3 we can substitute the pair (G, H) for the pair (U, G) by the assumption and put X=a point. Then we have an isomorphism

$$E_2^{*,0}(G, G/H)_G \cong \Lambda(x_1, \cdots, x_s)$$

by using the notation of (3.4) and we see that the edge homomorphism of this

spectral sequence sends x_j to $\beta(\nu_j)$ for $j=1, \dots, s$ (See [9], §10). This completes the proof.

5. The classification of symmetric spaces

Let G, K, σ and M be as in Theorem A and let $j: K \rightarrow G$ be the inclusion of K throughout the remainder of this paper.

We know that K is connected ([11, I], Theorem 3.4 in Chapter IV). Now if K is of maximal rank, then we can easily check Theorem A as follows: Since the restriction $i^*: R(G) \rightarrow R(K)$ is injective, all elements of R(G) are fixed by σ^* , and since R(K) is a stably-free module as an R(G)-module [12], we see that

(5.1)
$$\alpha: Z \bigotimes_{R(G)} R(K) \to K^*(G/K)$$
 is an isomorphism

from the spectral sequence (1.2) ([9], §9) and $Z \bigotimes_{R(G)} R(K)$ is tronsion-free.

Therefore it suffices to prove Theorem A when rank $G > \operatorname{rank} K$. When M is a simply-connected Lie group as a symmetric space, we refer the reader to [2] and [3] or [7]. M is simply-connected and so it is a direct product of irreducible symmetric spaces. Hence we consider only the irreducible symmetric spaces such that rank $G > \operatorname{rank} K$. According to the classification of irreducible symmetric spaces [6], such irreducible symmetric spaces are the following six types:

 $\begin{array}{rcl} AI & SU(n)/SO(n) \\ AII & SU(2n)/Sp(n) \\ BDI(a) & Spin(p+q)/Spin(p) \underset{Z_2}{\times} Spin(q) & (where p and q are odd and \\ (5.2) & Z_2 = \{(1,1), (-1, -1))\} \\ BDII(a) & Spin(n)/Spin(n-1) & (where n is even) \\ EI & E_{\mathfrak{s}}/PSp(4) \\ EIV & E_{\mathfrak{s}}/F_4 \end{array}$

6. Proofs for AII, BDII(a) and EIV

The symmetric spaces of types AII, BDII(a) and EIV have the properties such that $\pi_1(K)$ is torsion-free and $i^* \colon R(G) \to R(K)$ is surjective. Hence we can apply Proposition 4.1 to this case.

Here we describe Proposition 4.1 for the above three symmetric spaces explicitly.

Type AII (M=SU(2n)/Sp(n)). Let I_n denote the $n \times n$ unit matrix and put $J=\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Then σ is given by

 $\sigma(g) = J\bar{g}J^{-1}$ for $g \in SU(2n)$

where \overline{g} is the complex conjugate of g.

Using the notation in [10] we have

(6.1)
$$R(SU(2n)) = Z[\lambda_1, \dots, \lambda_{2n-1}] \text{ and } R(Sp(n)) = Z[\lambda_1, \dots, \lambda_n]$$

([10], Theorems 3.1 and 6.1 in $\S13$). Then it is clear that

(6.2)
$$i^*(\lambda_k) = i^*(\lambda_{2n-k}) = \lambda_k$$
 and $\sigma^*(\lambda_k) = \lambda_{2n-k}$ for $k = 1, \dots, n$.

From (6.2) we see easily that i^* is surjective and $\pi(\lambda_k - \lambda_{2n-k})$ $(1 \le k \le n-1)$ form a basis for the free abelian group $\operatorname{Ker}(J(SU(2n)) \to J(Sp(n)))$. Therefore we get from Proposition 4.1 that

Proposition 6.1. The notation being as in (6.1),

$$K^*(SU(2n)/Sp(n)) = \Lambda(\beta(\lambda_1 - \lambda_{2n-1}), \dots, \beta(\lambda_{n-1} - \lambda_{n+1})).$$

Type BDII(a) (M=Spin(2n)/Spin(2n-1)). σ is given by
 $\sigma(g) = -e_{2n}ge_{2n}$ for any $g \in Spin(2n)$

where e_{2n} is the generator of the Clifford algebra C_{2n} in the 2*n*-th position ([10], §11).

From Theorem 10.3 in §13 of [10],

(6.3)
$$\begin{array}{l} R(Spin(2n)) = Z[\lambda^{1}(\rho_{2n}), \cdots, \lambda^{n-2}(\rho_{2n}), \Delta_{2n}^{+}, \Delta_{2n}^{-}], \\ R(Spin(2n-1)) = Z[\lambda^{1}(\rho_{2n-1}), \cdots, \lambda^{n-2}(\rho_{2n-1}), \Delta_{2n-1}]. \end{array}$$

using the same notation. Then we can easily verify that

(6.4)
$$i^{*}(\lambda^{k}(\rho_{2n})) = \lambda^{k}(\rho_{2n-1}) + \lambda^{k-1}(\rho_{2n-1}) \quad (1 \leq k \leq n-2),$$
$$i^{*}(\Delta^{\pm}_{2n}) = \Delta_{2n-1},$$
$$\sigma^{*}(\lambda^{k}(\rho_{2n})) = \lambda^{k}(\rho_{2n}) \quad (1 \leq k \leq n-2) \text{ and}$$
$$\sigma^{*}(\Delta^{\pm}_{2n}) = \Delta^{\pm}_{2n}.$$

From (6.4) we see that i^* is surjective and the element $\Delta_{2n}^+ - \Lambda_{2n}^-$ holds the conditions required in (i) of Proposition 4.1 and therefore we have by Proposition 4.1

Propositiom 6.2. The notation being as in (6.3),

$$K^*(Spin(2n)/Spin(2n-1)) = \Lambda(\beta(\Delta_{2n}^+ - \Delta_{2n}^-))$$

Type $EIV(M=E_6/F_4)$. We look at the Dynkin diagrams of E_6 and F_4 with the irreducible representations corresponding to the vertexes and their dimensions written next to the vertexes:



where $\overline{\lambda^{k}\rho'}$ (k=1 or 2) is the greatest component of $\lambda^{k}\rho'$ (See Supplement and Table 30 of [4]).

The involutive automorphism σ of E_6 for *EIV* is the normal extension of a symmetry of the Dynkin diagram E_6 indicated by the arrow in the diagram (6.5) (See [11, II], *p*. 130). Hence it follows immediately that

(6.6)
$$\sigma^*(\lambda^k \rho_1) = \lambda^k \rho_2 \ (1 \le k \le 3) \quad \text{and} \quad \sigma^*(Ad_{E_6}) = Ad_{E_6}.$$

Consider the highest weights of ρ_1 and ρ' and their dimensions, then we get

(6.7)
$$i^*(\rho_1) = \rho' + 1$$

and moreover we obtain

(6.8)
$$i^*(Ad_{E_6}) = Ad_{F_4} + \rho'$$

by enumerating the all weights of the adjoint representations Ad_{E_6} and Ad_{F_4} .

From (6.7) and (6.8) we see that $i^*: R(E_6) \rightarrow R(F_4)$ is surjective because of $R(F_4) = Z[\rho', \lambda^2 \rho', \lambda^3 \rho', Ad_{F_4}]$ [15], and therefore from (6.6) and Proposition 4.1 follows

Proposition 6.3 (Cf. [5]). The notation being as in the diagram (6.5),

$$K^*(E_6/F_4) = \Lambda(\beta(\rho_1 - \rho_2), \beta(\lambda^2 \rho_1 - \lambda^2 \rho_2)).$$

7. Proofs for BDI(a) and EI

Let L be a compact connected Lie group, H be a closed connected subgroup of maximal rank of L and $j: H \rightarrow L$ the inclusion of H. Then,

Proposition 7.1. For a compact L-space X, there is a natural homomorphism of $K_L^*(X)$ -modules $j_*: K_H^*(X) \to K_L^*(X)$ such that $j_*(1)=1$, and therefore j_*j^* is an identity isomorphism where j^* is the restriction $K_L^*(X) \to K_H^*(X)$.

Proof. The proof is immediate from Proposition (3.8) of [13].

Type BDI(a) $(M = Spin(2m+2n+2)/Spin(2m+1) \underset{z_2}{\times} Spin(2n+1))$. σ is given by

$$\sigma(g) = -(e_1 \cdots, e_{2m+1})g(e_{2m+1} \cdots e_1)$$

for any $g \in Spin(2m+2n+2)$ where e_k is the generator of the Clifford algebra C_{2m+1} in the k-th position for $k=1, \dots, 2m+1$ ([10], §11). Then we have

(7.1)
$$\sigma^{*}(\lambda^{k}(\rho_{2m+2n+2})) = \lambda^{k}(\rho_{2m+2n+2}) \quad (1 \leq k \leq m+n-1), \\ \sigma^{*}(\Delta^{\pm}_{2m+2n+2}) = \Delta^{\pm}_{2m+2n+2}$$

using the notation in (6.3).

Put G=Spin(2m+2n+2), $K=Spin(2m+1) \underset{Z_2}{\times} Spin(2n+1)$, $G_1=Spin(2m+2n+1)$ and $K_1=Spin(2m+1) \underset{Z_2}{\times} Spin(2n)$, then we have an isomorphism induced by the external product homomorphism

(7.2)
$$K^*(G/G_1) \bigotimes_{R(G_1)} R(K_1) \simeq K^*(G/K_1)$$

by use of the Künneth formula spectral sequence in $K_{G_1}^*$ ([8], [9] and [12]). Furthermore, since the restriction $R(G) \rightarrow R(G_1)$ is surjective, we have isomorphisms

(7.3)
$$K^{*}(G/K_{1}) \simeq K^{*}(G/G_{1}) \otimes (Z \bigotimes_{R(G_{1})} R(K_{1})) \quad \text{by (7.2)}$$
$$\simeq \Lambda(\beta(\Delta_{2m+2n+2}^{+} - \Delta_{2m+2n+2}^{-})) \otimes (Z \bigotimes_{R(G_{1})} R(K_{1})) \quad \text{by Prop. 6.2}$$
$$\simeq \Lambda(\beta(\Delta_{2m+2n+2}^{+} - \Delta_{2m+2n+2}^{-})) \otimes (Z \bigotimes_{R(G)} R(K_{1}))$$

Let $j^*: K^*(G/K) \to K^*(G/K_1)$ be the homomorphism induced by the projection $G/K_1 \to G/K$ and $j_*: K^*(G/K_1) \to K^*(G/K)$ the homomorphism of $K^*(G/K)$ -modules mentioned in Proposition 7.1. $K^*(G/K_1)$ is torsion-free by (7.3) and j^* is injective by the property of j_* . Therefore,

(7.4) $K^*(G/K)$ is torsion-free.

Here we have a natural homomorphism of rings

$$\varphi \colon \Lambda(\beta(\Delta_{2m+2n+2}^+ - \Delta_{2m+2n+2}^-)) \otimes (Z \bigotimes_{R(G)}^{\infty} R(K)) \to K^*(G/K)$$

which is well-defined by (7.4). Then φ is injective because $j^*\varphi$ is so, and also it is easy to see that φ is surjective by the fact that j_*j^* =identity. Hence we conclude that

Proposition 7.2. The notation being as in (6.3), $K^*(Spin(2m+2n+2)/Spin(2m+1) \underset{Z_2}{\times} Spin(2n+1))$ is torsion-free and equals the ring

$$\Lambda(\beta(\Delta_{2m+2n+2}^+-\Delta_{2m+2n+2}^-))\otimes(Z\bigotimes_{R(\operatorname{Spin}(2m+2n+2))}R(\operatorname{Spin}(2m+1)\bigotimes_{Z_2}\operatorname{Spin}(2n+1))).$$

Type $EI(M=E_6/PS(4))$. σ is the composition of the involutive automorphism of E_6 for EIV and the inner automorphism (See [11, II], p. 131). So we have from (6.6)

(7.5)
$$\sigma^*(\lambda^k \rho_1) = \lambda^k \rho_2 \ (1 \le k \le 3) \quad \text{and} \quad \sigma^*(\Lambda d_{E_6}) = Ad_{E_6}$$

using the notation the diagram (6.5).

From the argument in p. 131 of [11, II] we know that E_6 has $Sp(3) \underset{Z_2}{\times} SU(2)$ as a subgroup which is contained in PSp(4) and F_4 where Z_2 is the subgroup of $Sp(3) \times SU(2)$ consisting of (1,1) and (-1, -1).

Setting $G=E_6$, K=PSp(4), $G_1=F_4$ and $K_1=Sp(3) \underset{Z_2}{\times} SU(2)$ the similar argument to BDI(a) shows that

Proposition 7.3. The notation being as in the diagram (6.5), $K^*(E_{\epsilon}/PSp(4))$ is torsion-free and equals the ring

$$\Lambda(\beta(\rho_1-\rho_2), \beta(\lambda^2\rho_1-\lambda^2\rho_2))\otimes(Z\bigotimes_{R\in E_{\delta}}R(PSp(4))).$$

8. Proof for AI

In the case of AI-type, σ is given by $\sigma(g) = \overline{g}$ for any $g \in SU(n)$ where \overline{g} is the complex conjugate of g.

From Theorems 3.1 and 10.3 in §13 of [10], we have

(8.1)

$$R(SU(n)) = Z[\lambda_1, \dots, \lambda_{n-1}],$$

$$R(SO(2m+1)) = Z[\lambda_1, \dots, \lambda_m] \text{ where } \lambda_k = \lambda^k(\rho_{2m+1}) (1 \le k \le m),$$

$$R(SO(2m)) = Z[\lambda_1, \dots, \lambda_{m-1}, \lambda_m^+, \lambda_m^-]/\sim \text{ where } \lambda_k = \lambda^k(\rho_{2m})$$

$$(1 \le k \le m-1) \text{ and } \lambda_m^+ = \lambda_{\pm}^m(\rho_{2m})$$

using the same notation.

First we consider the case when n is odd. Put n=2m+1. Then,

(8.2) $i^*(\lambda_k) = i^*(\lambda_{2m+1-k}) = \lambda_k$ and $\sigma^*(\lambda_k) = \lambda_{2m+1-k}$ for $k=1, \dots, m$ clearly.

Using the Koszul complex

$$C^* = \Lambda(x_1, \cdots, x_{2m}) \otimes R(SU(2m+1))$$

where $d(x_k) = \tilde{\lambda}_{2m+1-k} (= \lambda_{2m+1-k} - \mathcal{E}(\lambda_{2m+1-k}))$ $(1 \leq k \leq 2m)$ and d is a derivation, we shall show

(8.3)
$$\operatorname{Tor}_{R(SU(2m+1))}^{*,0}(Z, SO(2m+1)) = \Lambda(x_1 - x_{2m}, \cdots, x_m - x_{m+1}).$$

For a proof we define E_i $(1 \le l \le 2m)$ to be the subcomplex $\Lambda_{R(SO(2m+1))}(x_1, \dots, x_l)$ of $C^* \bigotimes_{R(SU(2m+1))} R(SO(2m+1))$. Then there exist a natural short exact sequence of complexes

$$0 \rightarrow E_{l} \rightarrow E_{l+1} \rightarrow E_{l+1}/E_{l} \rightarrow 0$$

and an isomorphism of complexes

$$E_{I} \simeq E_{I+1}/E_{I}$$

defined by the correspondence $z \rightarrow zx_{l+1}$, $z \in E_l$ for $=1, \dots, 2m-1$. This permits us to apply the induction on l and then we obtain

$$H^{*}(E_{l}) = R(SO(2m+1))/(\lambda_{1}, \dots, \lambda_{l}),$$

$$H^{*}(E_{m+l}) = \Lambda(x_{m-l+1} - x_{m+l}, \dots, x_{m} - x_{m+1})$$

for $l=1, \dots, m$. Thus (8.3) is proved.

From [9], §10 it follows that the element $x_k - x_{2m+1-k}$ converges to $\beta(\lambda_k - \lambda_{2m+1-k})$ in the spectral sequence (1.2) for $k=1, \dots, m$. Hence we have

Proposition 8.1. The notation being as in (8.1),

$$K^*(SU(2m+1)/SO(2m+1)) = \Lambda(\beta(\lambda_1 - \lambda_{2m}), \cdots, \beta(\lambda_m - \lambda_{m+1})).$$

In a similar way when n is even, we can prove the following

Proposition 8.2. The notation being as in (8.1),

$$K^*(SU(2m)/SO(2m)) = \Lambda(\beta(\lambda_1 - \lambda_{2m-1}), \dots, \beta(\lambda_{m-1} - \lambda_{m+1})) \otimes \Lambda(\tilde{\lambda}_m^+)$$

where $\tilde{\lambda}_m^+ = \lambda_m^+ - \frac{1}{2} \left(\frac{2m}{m}\right)$.

This completes the proof of Theorem A.

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