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THE REDUCIBILITY OF THE BOUNDARY CONDITIONS IN THE ONE-PARAMETER FAMILY OF ELLIPTIC LINEAR BOUNDARY VALUE PROBLEMS II

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1. Introduction

Let $P_1(D)$ and $P_2(D)$ be linear partial differential operators with constant coefficients. Let the order of P_1 with respect to ξ_1 be m, that of P_2 be m', and $m>m'$. Let $b_{j_k}(D)$, $k=1, \cdots, \mu$ be normal boundary operators of order j_k and $\boldsymbol{R}^n_+=\{x_1>0\}$. We shall consider the following one-parameter family of unilateral boundary value problems:

$$(1.1)\qquad \begin{cases} (\varepsilon^{m-m'} P_1(D)+P_2(D))\, u(x) = 0 \text{ in } \boldsymbol{R}^n_+ \,; \\ b_{j_k}(D)\, u(x)\,|_{x_1\downarrow 0} = \phi_k(x'),\, k = 1, \cdots, \mu\,. \end{cases}$$

Here $\phi=(\phi_1, \cdots, \phi_\mu)$ belongs to $\mathrm{F}^{-1}(C_0^\infty(\boldsymbol{R}^{n-1}))^\mu$, where F^{-1} denotes the inverse Fourier transformation. We shall choose $b_{j_k}(D)$, $k=1, \cdots, \mu$ so that the bounded solutions are uniquely determined. We have introduced the notion of "reducibility" for the family of elliptic boundary value problems in [1] and that of "admissibility" for the family of Cauchy problems in [2]. In §3, by using the localization in the Fourier images of the solutions of (1.1), which we may call the local Fourier analysis, we shall introduce the notion of "micro-admissibility" and "micro-reducibility" of (1.1) and show the same kind of results as those on the reducibility of the family of elliptic boundary value problems in [1]. As a preliminary, we shall study in §2 asymptotic behaviour of the characteristic roots more deeply than [1]. In §4, we shall pacth up the localization in the Fourier images and study relation between the reducibility and the micro-reducibility on various examples. In §5, we shall show the normal reducibility of the following one-parameter family of non-characteristic Cauchy problems for kowalewskian operators:

$$(1.2)\qquad \begin{cases} (\varepsilon\cdot P_1(D)+P_2(D))\, u = 0, \text{ in } \boldsymbol{R}^n \,; \\ b_j(D)\, u\,|_{x_1=0} = \phi_j, j = 0, \cdots, m-1\,. \end{cases}$$

If the Cauchy problems (1.2) are uniquely solvable and the limit u_0 of the solutions u_ε of (1.2) exists in $C(\boldsymbol{R}_{x_1};\ \mathcal{D}'(\boldsymbol{R}^{n-1}_{x'}))$, which denotes the space of continuous functions of x_1 in $\boldsymbol{R}_{x_1}$ valued in $\mathcal{D}'(\boldsymbol{R}^{n-1}_{x'})$, then u_0 satisfies

$$(1.3)\qquad \begin{cases} P_2(D)\,u = 0, \text{ in } \boldsymbol{R}^n\ ; \\ b_j(D)\,u|_{x_1=0} = \phi_j, j = 0, \cdots, m'-1\,. \end{cases}$$

Here m=ord P_1 (the order of P_1), m'=ord P_2, and $m>m'$. In appendix, we shall give a brief survey of boundary values of solutions to a non-characteristic hyperplane according to [4].

2. Preliminaries

In this section, we shall study necessary properties of the characteristic roots and the asymptotic behaviour of determinants more deeply than [1].

Let $P_1(D)$ and $P_2(D)$ be linear partial differential operators with constant coefficients. Let the order of P_1 with respect to ξ_1 be m, that of P_2 be m', and $m>m'$. Let their symbols be

$$(2.1)\qquad P_1(\xi) = \xi_1^m + \textstyle\sum_{j=1}^m p_{1,j}(\xi')\,\xi_1^{m-j}\,,$$

$$(2.2)\qquad P_2(\xi) = p\cdot\xi_1^{m'} + \textstyle\sum_{j=1}^{m'} p_{2,j}(\xi')\,\xi_1^{m'-j}\,.$$

Here $p_{1,j}(\xi')$ and $p_{2,j}(\xi')$ are polynomials of ξ' without restrictions on orders, that is, $P_1(D)$ and $P_2(D)$ are non-kowlaewskian in general and p is a non-zero constant.

We shall deal with the following polynomial with a small positive parameter ε:

$$(2.3)\qquad \varepsilon^{m-m'}\cdot P_1(\xi) + P_2(\xi) = 0\,.$$

By replacing ε by $\varepsilon\cdot|p|^{1/(m-m')}$, we may assume that $|p|=1$. Denote the characteristic roots of (2.3) with respect to ξ_1 by $\tau_j(\varepsilon, \xi'), j=1, \cdots, m$ and those of

$$(2.4)\qquad P_2(\xi) = 0$$

with respect to ξ_1 by $\sigma_j(\xi'), j=1, \cdots, m'$, respectively.

Assumption 2.1. There exists a point ξ_0' in $\boldsymbol{R}^{n-1}$ such that for $1\leqq j<k\leqq m'$

$$\sigma_j(\xi_0') \neq \sigma_k(\xi_0')\,.$$

Remark. If Assumption 2.1 is satisfied, then there exists an open ball $B_0=B_0(r_0;\xi_0')$ of radius r_0 with the centre ξ_0' such that all $\sigma_j(\xi')$ are simple on the closure of B_0.

Under Assumption 2.1, we have essentially studied the asymtotic properties of the characteristic roots of (2.3) in [1]. We shall calculate the second and the third terms of the asymptotic expansions of the characteristic roots of (2.3). Denote by θ the argument of $-p$ satisfying $0\leqq\theta<2\pi$, that is, $-p=\exp\, i\theta$. Denote

$$\Theta = \exp\frac{i\theta}{m-m'},\ \zeta = \exp\frac{2\pi i}{m-m'},\quad \text{and}\quad \tau'_j = \zeta^{j-m'-1}, j = m'+1, \cdots, m\,.$$

Lemma 2.2. *Let Assumption* 2.1 *be satisfied and* B_0 *be the open ball in Remark to Assumption* 2.1. *If the suffixes* $\{j\}$ *of the characteristic roots* $\tau_j(\varepsilon, \xi')$, $j=1, \cdots, m$ *of* (2.3) *are properly chosen, then there exists a positive number* ε_0 *such that if* $0<\varepsilon<\varepsilon_0$, *then* $\tau_j(\varepsilon, \xi')$, $j=1, \cdots, m$ *satisfy the following asymptotic properties on the closure of* B_0:
For $j=1, \cdots, m'$

$$\tau_j(\varepsilon, \xi') = \sigma_j(\xi')+s_{j,2}(\xi')\,\varepsilon^{m-m'}+s_{j,3}(\xi')\,\varepsilon^{2(m-m')}+O(\varepsilon^{3(m-m')})\,, \tag{2.5}$$

where $\partial_1 = \dfrac{\partial}{\partial\xi_1}$,

$$s_{j,2} = -P_1(\sigma_j, \xi')\cdot\partial_1 P_2(\sigma_j, \xi')^{-1}\,, \tag{2.6}$$

$$\begin{aligned} s_{j,3} = &-\frac{1}{2!}\cdot\partial_1{}^2P_2(\sigma_j, \xi')\cdot\partial_1 p_2(\sigma_j, \xi')^{-1}\cdot s_{j,2}{}^2 \\ &-\partial_1 P_1(\sigma_j, \xi')\cdot\partial_1 P_2(\sigma_j, \xi')^{-1}\cdot s_{j,2}\,. \end{aligned} \tag{2.7}$$

For $j=m'+1, \cdots m$

$$\tau_j(\varepsilon, \xi') = \Theta\tau'_j\cdot\frac{1}{\varepsilon}+t_2(\xi')+(\Theta\tau'_j)^{-1}\cdot t_3(\xi')\cdot\varepsilon+O(\varepsilon^2)\,, \tag{2.8}$$

where

$$t_2 = \frac{-p_{1,1}+p_{2,1}\cdot p^{-1}}{m-m'}\,, \tag{2.9}$$

$$\begin{aligned} t_3 = (m-m')^{-1}\Big[&\frac{m'(m'-1)-m(m-1)}{2!}\cdot t_2{}^2 \\ &+((m'-1)\,p^{-1}p_{2,1}-(m-1)\,p_{1,1})\,t_2+p^{-1}p_{2,2}-p_{1,2}\Big]\,. \end{aligned} \tag{2.10}$$

Proof. In [1], we have calculated the first terms of the expansion of the characteristic roots. Put $\varepsilon'=\varepsilon^{m-m'}$. When Assumption 2.1 is satisfied, we know that m' characteristic roots of $\varepsilon'\cdot P_1(\xi)+P_2(\xi)=0$ are analytic for sufficiently small ε' and ξ' in a neighbourhood of the closure of B_0. As we need the first three terms of the expansion of the characteristic roots $\tau_j(\varepsilon, \xi')$, we may assume that

$$\tau_j(\varepsilon, \xi') = \sigma_j(\xi')+s_{j,2}(\xi')\cdot\varepsilon'+s_{j,3}(\xi')\cdot\varepsilon'^2, j = 1, \cdots, m'\,.$$

Expand the left-hand side of

$$\varepsilon'\cdot P_1(\tau_j, \xi')+P_2(\tau_j, \xi') = 0$$

as a power series of ε'. Differentiate the power series by ε' and put $\varepsilon'=0$. Then the coefficient of ε' is

$$P_1(\sigma_j, \xi')+\partial_1 P_2(\sigma_j, \xi')\cdot s_{j,2}$$

and this must be zero. Since $\sigma_j(\xi')$ are simple on the closure of B_0, it implies that $\partial_1 P_2(\sigma_j, \xi')\neq 0$. Hence we have (2.6). By differentiating two times the power series by ε' and putting $\varepsilon'=0$, we have (2.7). Thus we have (2.5).

Multiply (2.3) by $\varepsilon^{m'}$, and put $t=\varepsilon\cdot\xi_1$. Then

$$t^m+\sum_{j=1}^{m} p_{1,j}(\xi')\,\varepsilon^j\, t^{m-j}+p\cdot t^{m'}+\sum_{j=1}^{m'} p_{2,j}(\xi')\,\varepsilon^j\, t^{m'-j} = 0\,. \tag{2.11}$$

We know that $m-m'$ roots of (2.11) are analytic in a neighbourhood of the closure of B_0 for sufficiently small ε. Put $t_j=\varepsilon\cdot\tau_j(\varepsilon, \xi')$, $j=m'+1, \cdots, m$. Then t_j, $j=m'+1, \cdots, m$ are the roots of (2.11). As we need first three terms of the expansion of t_j, we may assume that

$$t_j = \Theta\tau_j'+t_{j,2}(\xi')\cdot\varepsilon+t_{j,3}(\xi')\cdot\varepsilon^2, j = m'+1, \cdots, m\,.$$

Substitute t_j for t in (2.11) and expand the left-hand side of (2.11) as a power series of ε. Then the coefficient of ε is

$$m(\Theta\tau_j')^{m-1}\, t_{j,2}+p_{1,1}(\Theta\tau_j')^{m-1}+pm'(\Theta\tau_j')^{m'-1}\, t_{j,2}+p_{2,1}(\Theta\tau_j')^{m'-1}$$

and this must be zero. As $(\Theta\tau_j')^{m-m'}=-p$, we have

$$t_{j,2} = \frac{-p_{1,1}+p_{2,1}\cdot p^{-1}}{m-m'}\,. \tag{2.12}$$

Since the right-hand side of (2.12) is independent of j, we may write $t_{j,2}=t_2$. The coefficient of ε^2 is

$$\begin{aligned} &mt_{j,3}(\Theta\tau_j')^{m-1}+\frac{m(m-1)}{2!}\cdot t_{j,2}^2(\Theta\tau_j')^{m-2}+(m-1)\,p_{1,1}\, t_{j,2}(\Theta\tau_j')^{m-2}+p_{1,2}(\Theta\tau_j')^{m-2} \\ &+m'pt_{j,3}(\Theta\tau_j')^{m'-1}+\frac{m'(m'-1)}{2!}\cdot pt_{j,2}^2(\Theta\tau_j')^{m'-2} \\ &\quad+(m'-1)\,p_{2,1}\, t_{j,2}(\Theta\tau_j')^{m'-2}+p_{2,2}(\Theta\tau_j')^{m'-2} \end{aligned}$$

and this must be zero. Hence

$$\begin{aligned} t_{j,3} = (m-m')^{-1}(\Theta\tau_j')^{-1}\Big[&\frac{m'(m'-1)-m(m-1)}{2!}\cdot t_2^2 \\ &+((m'-1)\,p^{-1}p_{2,1}-(m-1)\,p_{1,1})\, t_2+p^{-1}p_{2,2}-p_{1,2}\Big] = (\Theta\tau_j')^{-1}\cdot t_3\,. \end{aligned}$$

Thus we have (2.8). [Q.E.D].

Let ν and μ be integers such that $1\leqq\nu\leqq m'$ and $\nu+1\leqq\mu\leqq m$. Let $j_1, \cdots, j_\mu$ be a series of integers with

$$0\leqq j_1<\cdots<j_\mu\leqq m-1\,. \tag{2.13}$$

Let $b_j(\tau, \xi')$, $j=j_1, \cdots, j_\mu$ be polynomials of order j as

$$b_j(\tau, \xi') = \tau^j+\textstyle\sum_{k=1}^{j} b_{j,k}(\xi')\,\tau^{j-k},\ j=j_1, \cdots, j_\mu\,, \tag{2.14}$$

which are denoted by $b_j(\tau)$ when regarded as polynomials of τ with polynomial coefficients. We shall use the same notation as in [1] except T'_k and $\partial D'_k$ as follows.

NOTATION 2.3. For polynomials $b_j(\tau)$, $j=1, \cdots, \mu$ and for complex numbers or functions τ_j and ϕ_j, $j=1, \cdots, \mu$,

$$\text{Mat}\,D_0 = \text{Mat}\,D_0(\tau_1, \cdots, \tau_\mu; b_1, \cdots, b_\mu) = \begin{bmatrix} b_1(\tau_1) & \cdots & b_1(\tau_\mu) \\ \vdots & & \vdots \\ b_\mu(\tau_1) & \cdots & b_\mu(\tau_\mu) \end{bmatrix},$$

$$\text{Mat}\,D_k = \text{Mat}\,D_k(\tau_1, \cdots, \tau_\mu; b_1, \cdots, b_\mu; \phi_1, \cdots, \phi_\mu)$$

$$= \begin{bmatrix} b_1(\tau_1) & \cdots & b_1(\tau_{k-1}) & \phi_1 & b_1(\tau_{k+1}) & \cdots & b_1(\tau_\mu) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ b_\mu(\tau_1) & \cdots & b_\mu(\tau_{k-1}) & \phi_\mu & b_\mu(\tau_{k+1}) & \cdots & b_\mu(\tau_\mu) \end{bmatrix},$$

where $k=1, \cdots, \mu$.

$$\text{Mat}\,V_n(\zeta; j_1, \cdots, j_n) = \begin{bmatrix} 1 & \cdots & 1 \\ \zeta^{j_1} & & \zeta^{j_n} \\ \vdots & & \vdots \\ (\zeta^{j_1})^{n-1} & \cdots & (\zeta^{j_n})^{n-1} \end{bmatrix}$$

$$\text{Mat}\,V_{\mu-\nu-1,0} = \text{Mat}\,V_{\mu-\nu-1}(\zeta; j_{\nu+2}, \cdots, j_\mu)\,,$$

$$\text{Mat}\,V_{\mu-\nu,0} = \text{Mat}\,V_{\mu-\nu}(\zeta; j_{\nu+1}, \cdots, j_\mu)\,.$$

For $1\leqq k\leqq\mu-\nu$,

$$j'_k = j_{\nu+k}-1$$

$$\text{Mat}\,V_{\mu-\nu,k} = \text{Mat}\,V_{\mu-\nu}(\zeta; j_{\nu+1}, \cdots, j_{\nu+k-1}, j'_k, j_{\nu+k+1}, \cdots, j_\mu)$$

$$T'_k = (j_{\nu+1}\cdot(\zeta^{k-1})^{j'_1}, \cdots, j_\mu\cdot(\zeta^{k-1})^{j'_{\mu-\nu}})$$

$$\text{Mat}\,\partial D'_k = \text{Mat}\,D_k(1, \zeta, \cdots, \zeta^{\mu-\nu-1}; \tau^{j_{\nu+1}}, \cdots, \tau^{j_\mu}; T'_k)\,.$$

We shall abbreviate the determinant of Mat D as D, where Mat D is any of the matrices abbreviated as above. Denote $J=j_{\nu+1}+\cdots+j_\mu$ and $J'=J-j_{\nu+1}$.
For $\nu+1\leqq k\leqq\mu$,

$$D_{(k)} = \Theta^{J'} \cdot D_0(1, \cdots, \zeta^{k-\nu-2}, \zeta^{k-\nu}, \cdots, \zeta^{\mu-\nu-1}; \tau^{j_{\nu+2}}, \cdots, \tau^{j_\mu}) .$$
$$B_{\mu-\nu}(\xi') = \sum_{k=1}^{\mu-\nu} (b_{j_{\nu+k},1}(\xi') \cdot V_{\mu-\nu,k} + t_2(\xi') \cdot \partial D'_k) .$$

By the same method as in Lemma 2.4 in [1], we have the following:

Lemma 2.4. *Let Assumption* 2.1 *be satisfied and* B_0 *be the open ball in Remark to Assumption* 2.1. *Then*

$$\text{(2.15)} \quad \lim_{\varepsilon \downarrow 0} D_0(\tau_1, \cdots, \tau_\mu; b_{j_1}, \cdots, b_{j_\mu}) \cdot \varepsilon^J = D_0(\sigma_1, \cdots, \sigma_\nu; b_{j_1}, \cdots, b_{j_\nu}) \cdot \Theta^J \cdot V_{\mu-\nu,0} .$$

For $k=1, \cdots, \nu$

$$\text{(2.16)} \quad \lim_{\varepsilon \downarrow 0} D_k(\tau_1, \cdots, \tau_\mu; b_{j_1}, \cdots, b_{j_\mu}; \hat{\phi}_1, \cdots, \hat{\phi}_\mu) \cdot \cdot \varepsilon^J = D_k(\sigma_1, \cdots, \sigma_\nu; b_{j_1}, \cdots, b_{j_\nu}; \hat{\phi}_1, \cdots, \hat{\phi}_\nu) \cdot \Theta^J \cdot V_{\mu-\nu,0} ,$$

and for $k=\nu+1, \cdots, \mu$

$$\text{(2.17)} \quad \lim_{\varepsilon \downarrow 0} D_k(\tau_1, \cdots, \tau_\mu; b_{j_1}, \cdots, b_{j_\mu}; \hat{\phi}_1, \cdots, \hat{\phi}_\mu) \cdot \varepsilon^{J'} = D_{\nu+1}(\sigma_1, \cdots, \sigma_{\nu+1}; b_{j_1}, \cdots, b_{j_{\nu+1}}; \hat{\phi}_1, \cdots, \hat{\phi}_{\nu+1}) \cdot (-1)^{k-\nu-1} \cdot \Theta^{J'} \cdot V_{\mu-\nu-1,0} ,$$

where $\sigma_{\nu+1}$ *is a dummy variable, that is, the right-hand side of* (2.17) *is independent of* $\sigma_{\nu+1}$.

The convergences are uniform on the closure of B_0.

Dente the asymptotic expansions of D_k by

$$D_k = d_{k,0}(\xi') \cdot \varepsilon^{-J} + d_{k,1}(\xi') \cdot \varepsilon^{-J+1} + O(\varepsilon^{-J-2}), \ k = 0, \cdots, \mu .$$

By the same method as in Lemma 2.6 in [1], we have the following:

Lemma 2.5. *Let Assumption* 2.1 *be satisfied and* B_0 *be the open ball in Remark to Assumption* 2.1. *Assume that* $V_{\mu-\nu,0}=0$.

When $j_{\nu+1}-j_\nu \geqq 2$,

$$\text{(2.18)} \quad d_{0,1} = D_0(\sigma_1, \cdots, \sigma_\nu; b_{j_1}, \cdots, b_{j_\nu}) \cdot \Theta^{J-1} \cdot B_{\mu-\nu} .$$

For $k=1, \cdots, \nu$

$$\text{(2.19)} \quad d_{k,1} = D_k(\sigma_1, \cdots, \sigma_\nu; b_{j_1}, \cdots, b_{j_\nu}; \hat{\phi}_1, \cdots, \hat{\phi}_\nu) \cdot \Theta^{J-1} \cdot B_{\mu-\nu} .$$

For $k=\nu+1, \cdots, \mu$

$$\text{(2.20)} \quad d_{k,1} = 0 .$$

When $j_{\nu+1}-j_\nu=1$,

(2.21)
$$d_{0,1} = D_0(\sigma_1, \cdots, \sigma_\nu; b_{j_1}, \cdots, b_{j_\nu})\cdot\Theta^{J-1}\cdot B_{\mu-\nu} + D_0(\sigma_1, \cdots, \sigma_\nu; b_{j_1}, \cdots, b_{j_{\nu-1}}, b_{j_{\nu+1}})\cdot\Theta^{J-1}\cdot V_{\mu-\nu,1}\,.$$

For $k=1, \cdots, \nu$

(2.22)
$$d_{k,1} = D_k(\sigma_1, \cdots, \sigma_\nu; b_{j_1}, \cdots, b_{j_\nu}; \hat{\phi}_1, \cdots, \hat{\phi}_\nu)\cdot\Theta^{J-1}\cdot B_{\mu-\nu} + D_0(\sigma_1, \cdots, \sigma_\nu; b_{j_1}, \cdots, b_{j_{\nu-1}}, b_{j_{\nu+1}}; \hat{\phi}_1, \cdots, \hat{\phi}_{\nu-1}, \hat{\phi}_{\nu+1})\cdot\Theta^{J-1}\cdot V_{\mu-\nu,1}\,.$$

For $k=\nu+1, \cdots, \mu$ *there are two cases as follows.*

When $\nu=1$ *and* $j_\nu=0$, *it may be assumed that* $b_{j_1}=b_0=1$ *and* $b_{j_2}=b_1=\xi_1+b_{1,1}(\xi')$. *Then*

(2.23)
$$d_{k,1} = (-1)^{k-2}(\hat{\phi}_2-(\sigma_1+b_{1,1}(\xi'))\,\hat{\phi}_1)\cdot D_{(k)}\,.$$

When $\nu\geqq 2$ *or* $j_\nu\geqq 1$,

(2.24)
$$d_{k,1} = 0\,.$$

3. The micro-reducibility

Let the symbol of P_1 be (2.1) and that of P_2 be (2.2). Let $b_{j_k}(D)$, $k=1, \cdots, \mu$ be normal and $\boldsymbol{R}^n_+=\{x_1>0\}$. We shall consider the following one-parameter family of unilateral boundary value problems:

(3.1)
$$\begin{cases} (\varepsilon^{m-m'}P_1(D)+P_2(D))\,u(x) = 0 \text{ in } \boldsymbol{R}^n_+\,; \\ b_{j_k}(D)\,u(x)\,|_{x_1\downarrow 0} = \phi_k(x'),\ k = 1, \cdots, \mu\,. \end{cases}$$

Here we shall choose $b_{j_k}(D)$, $k=1, \cdots, \mu$ so that the bounded solutions solved by the partial Fourier transformation with respect to x' are uniquely determined. In this paper, we hsall only deal with such solutions in order to determine a unique solution of (3.1) for every fixed ε. Denote by $B(\boldsymbol{R}^n_+)$ the space of bounded continuous functions in $\boldsymbol{R}^n_+$.

DEFINITION 3.1. A one-parameter family of the unilateral boundary value problems (3.1) is said to be *micro-admissible at* ξ'_0 if there exist an open ball B with centre ξ'_0 in $\boldsymbol{R}^{n-1}_{\xi'}$ and a positive number ε_0 such that the one-parameter family of (3.1) satisfies the following two conditions:

(1) For every ε with $0<\varepsilon<\varepsilon_0$ and for every $\Phi=(\phi_1, \cdots, \phi_\mu)$ in $\mathrm{F}^{-1}(C^\infty_0(B))^\mu$, the unilateral boundary value problem (3.1) has a unique solution $u_\varepsilon(x; \Phi)$ in $B(\boldsymbol{R}^n_+)$.

(2) For every Φ in $\mathrm{F}^{-1}(C^\infty_0(B))^\mu$, there exists a function $u_0(x; \Phi)$ such that

$$\lim_{\varepsilon\downarrow 0} u_\varepsilon(x; \Phi) = u_0(x; \Phi) \text{ in } C(\boldsymbol{R}^n_+)\,.$$

A one-parameter family of the unilateral boundary value problems (3.1) is said to be *micro-reducible at* ξ'_0 if the family (3.1) is micro-admissible at ξ'_0 and

satisfies the following two conditions:
(3) There exists a series $(k_1, \cdots, k_\nu)$ such that

$$1 \leqq k_1 < \cdots < k_\nu \leqq \mu \; ; \quad 0 \leqq j_{k_1} < \cdots < j_{k_\nu} \leqq m'-1 \; ;$$

and every $u_0(x; \Phi)$ satisfies the following unilateral bounadry value problem:

$$\begin{cases} P_2(D)\, u(x) = 0 \text{ in } \boldsymbol{R}^n_+ \; ; \\ b_{j_{k_l}}(D)\, u(x)|_{x_1 \downarrow 0} = \phi_{k_l}(x'),\, l = 1, \cdots, \nu \, . \end{cases} \tag{3.2}$$

(4) The reduced unilateral boundary value problem (3.2) is uniquely solvable.
In particular, when $k_l = l$, $l = 1, \cdots, \nu$, the family (3.1) is said to be *normally micro-reducible at* ξ'_0. The family (3.1) is said to be *abnormally micro-reducible at* ξ'_0 if the family (3.1) is micro-reducible at ξ'_0 but not normally micro-reducible at ξ'_0.

REMARK. When $\nu = m'$, the micro-reducibility is equivalent to the normal micro-reducibility. We can also define the micro-admissibility at $(x'_0; \xi'_0)$ and the micro-reducibility at $(x'_0; \xi'_0)$ by replacing $\boldsymbol{R}^{n-1}_{x'}$ with a neighbourhood U' of x'_0. Since we only treat solutions solved by the partial Fourier transformation, we do not need licaliaztion in x'-space.

Let us consider the partial Fourier transform with respect to x' of (3.1):

$$\begin{cases} (\varepsilon^{m-m'} P_1(D_1, \xi') + P_2(D_1, \xi'))\, \hat{u}(x_1, \xi') = 0 \; ; \\ b_{j_k}(D_1, \xi')\, \hat{u}(x_1, \xi')|_{x_1 \downarrow 0} = \hat{\phi}_k(\xi'),\, k = 1, \cdots, \nu \, . \end{cases} \tag{3.3}$$

Let Assumption 2.1 be satisfied, B_0 be the open ball in Remark to Assumption 2.1, and $\Phi = (\phi_1, \cdots, \phi_\mu)$ belong to $\mathrm{F}^{-1}(C_0^\infty(B_0))^\mu$. If the suffixes $\{j\}$ of the characteristic roots $\tau_j(\varepsilon, \xi')$, $j = 1, \cdots, m$ are properly chosen, which are simple in B_0 for sufficiently small ε, then the solutions of (3.3) are represented as

$$\hat{u}(x_1, \xi') = Y(x_1) \cdot \textstyle\sum_{k=1}^{\mu} C_k(\varepsilon, \xi'; \Phi)\, (\exp i\tau_k(\varepsilon, \xi')\, x_1) \, . \tag{3.4}$$

Here $Y(x_1)$ is the Heaviside function and for $k = 1, \cdots, \mu$,

$$C_k(\varepsilon, \xi'; \Phi) = \frac{D_k(\tau_1, \cdots, \tau_\mu; b_{j_1}, \cdots, b_{j_\mu}; \hat{\phi}_1, \cdots, \hat{\phi}_\mu)}{D_0(\tau_1, \cdots, \tau_\mu; b_{j_1}, \cdots, b_{j_\mu})} \, . \tag{3.5}$$

Next we shall study sufficient conditions for the unique solvability of (3.3). Assume that there exists an open ball B' with the centre ξ'_0 included in B_0 such that on the closure of B' and for sufficiently small ε,

$$\operatorname{Im} \tau_k(\varepsilon, \xi') > 0,\, k = 1, \cdots, \mu \tag{3.6}$$

and

(3.7) $$\operatorname{Im}\tau_k(\varepsilon, \xi')<0,\ k=\mu+1, \cdots, m\,,$$

where the suffixes $\{j\}$ of $\tau_j(\varepsilon, \xi')$, $j=1, \cdots, m$ are properly chosen. Then bounded solutions of (3.3) are uniquely determined for Φ in $F^{-1}(C_0^\infty(B'))^\mu$. We shall only deal with bounded solutions of (3.1) whose partial Fourier transforms are (3.4).

Use the same suffixes $\{j\}$ of $\tau_j(\varepsilon, \xi')$, $j=1, \cdots, m$ as in Lemma 2.2. Denote

$$N^+(\theta)=\#\{j;\ \operatorname{Im}\Theta\tau'_j>0, j=m'+1, \cdots, m\}\,,$$

$$N^0(\theta)=\#\{j;\ \operatorname{Im}\Theta\tau'_j=0, j=m'+1, \cdots, m\}\,,$$

and

$$N^-(\theta)=\#\{j;\ \operatorname{Im}\Theta\tau'_j<0, j=m'+1, \cdots, m\}\,,$$

where θ is the argument of $-p$ and $\Theta=\exp\dfrac{i\theta}{m-m'}$. Then we have the following:

(1) The case when $m-m'=2l-1$, where l is a positive integer.

(1—a) If $\theta=0$ or π, then

$$N^+(\theta)=l-1, N^0(\theta)=1\,,\quad\text{and}\quad N^-(\theta)=l-1\,.$$

(1—b) If $0<\theta<\pi$, then

$$N^+(\theta)=l, N^0(\theta)=0\,,\quad\text{and}\quad N^-(\theta)=l-1\,.$$

(1—c) If $\pi<\theta<2\pi$, then

$$N^+(\theta)=l-1, N^0(\theta)=0\,,\quad\text{and}\quad N^-(\theta)=l\,.$$

(2) The case when $m-m'=2l$, where l is a positive integer.

(2—a) If $\theta=0$, then

$$N^+(0)=l-1, N^0(0)=2\,,\quad\text{and}\quad N^-(0)=l-1\,.$$

(2—b) If $0<\theta<2\pi$, then

$$N^+(\theta)=l, N^0(\theta)=0\,,\quad\text{and}\quad N^-(\theta)=l\,.$$

It must be remarked that

$$\begin{aligned}&\{\Theta\tau'_k;\ \operatorname{Im}\Theta\tau'_k\geq 0, k=m'+1, \cdots, m\}\\&\quad=\{\Theta\tau'_k;\ k=m'+1, \cdots, m'+N^+(\theta)+N^0(\theta)\}\,.\end{aligned}$$

In order to seek sufficient conditions for (3.6) and (3.7), we introduce

ASSUMPTION 3.2.

$$\operatorname{Im}\sigma_j(\xi'_0)>0, j=1, \cdots, \nu$$

$$\operatorname{Im} \sigma_j(\xi_0') < 0, j = \nu+1, \cdots, m' .$$

REMARK. Here the number ν may be changed by ξ_0'.

Assumption 3.2 implies that there exists an open ball B_1 with the centre ξ_0' included in B_0 such that on the closure of B_1 and for sufficiently small ε,

$$\operatorname{Im} \tau_j(\varepsilon, \xi') > 0, j = 1, \cdots, \nu \tag{3.8}$$

$$\operatorname{Im} \tau_j(\varepsilon, \xi') < 0, j = \nu+1, \cdots, m' . \tag{3.9}$$

Lemma 2.2 implies that if $\operatorname{Im} \Theta\tau_j' > 0$ (resp. $\operatorname{Im} \Theta\tau_j' < 0$), then there exists an open ball B_2 with the centre ξ_0' included in B_1 such that $\operatorname{Im} \tau_j(\varepsilon, \xi') > 0$ (resp. $\operatorname{Im} \tau_j(\varepsilon, \xi') < 0$) on the closure of B_2 and for sufficiently small ε. When $\operatorname{Im} \Theta\tau_j' = 0$, we need the following:

ASSUMPTION 3.3.

$$\operatorname{Im} (-p_{1,1}(\xi_0') + p_{2,1}(\xi_0') \cdot p^{-1}) \neq 0 .$$

Lemma 2.2 implies that if $\operatorname{Im} \Theta\tau_j' = 0$ and

$$\operatorname{Im} (-p_{1,1}(\xi_0') + p_{2,1}(\xi_0') \cdot p^{-1}) > 0 , \tag{3.10}$$

then $\operatorname{Im} \tau_j(\varepsilon, \xi_0') > 0$ and that if $\operatorname{Im} \Theta\tau_j' = 0$ and

$$\operatorname{Im} (-p_{1,1}(\xi_0') + p_{2,1}(\xi_0') \cdot p^{-1}) < 0 , \tag{3.11}$$

then $\operatorname{Im} \tau_j(\varepsilon, \xi_0') < 0$. Put

$$\mu = \nu + N^+(\theta) + N^0(\theta) , \qquad \text{(The case when (3.10).)} , \tag{3.12}$$

$$\mu = \nu + N^+(\theta) , \qquad \text{(The case when (3.11).)} . \tag{3.13}$$

Then there exists an open ball B' with the centre ξ_0' included in B_2 such that on the closure of B' and for sufficidntly small ε,

$$\operatorname{Im} \tau_j(\varepsilon, \xi') > 0, j = m'+1, \cdots, m'+\mu-\nu , \tag{3.14}$$

$$\operatorname{Im} \tau_j(\varepsilon, \xi') < 0, j = m'+\mu-\nu+1, \cdots, m . \tag{3.15}$$

When $m-m'=2l-1$ and ($0<\theta<\pi$ or $\pi<\theta<2\pi$) or when $m-m'=2l$ and $0<\theta<2\pi$, that is, when $N^0(\theta)=0$, Assumption 3.3 is not required, Thus, by permuting the suffixes $\{\nu+1, \cdots, m\}$ of the characteristic roots properly, we can find an open ball B' with the centre ξ_0' included in B_0 such that for sufficiently small ε, (3.6) and (3.7) are valid on the closure on B'.

NOTATION 3.4.

$$D_0(\sigma)(\xi') = D_0(\sigma_1, \cdots, \sigma_\nu; b_{j_1}, \cdots, b_{j_\nu})$$

$$D_0(\sigma;\nu)(\xi')=D_0(\sigma_1,\cdots,\sigma_\nu;b_{j_1},\cdots,b_{j_{\nu-1}},b_{j_{\nu+1}})$$

We shall need the following assumption of the "micro-ellipticity" of the boundary conditions.

ASSUMPTION 3.5.

(1) $D_0(\sigma)(\xi_0')\neq 0$.

(2) $D_0(\sigma;\nu)(\xi_0')\neq 0$.

(3) $D_0(\sigma)(\xi_0')\cdot B_{\mu-\nu}(\xi_0')+D_0(\sigma;\nu)(\xi_0')\cdot V_{\mu-\nu,1}\neq 0$.

REMARK. If Assumption 3.5 is satisfied, then there exists an open ball B included in B' such that (1), (2), and (3) are valid for all ξ_0' on the closure of B. If $\nu=m'$, then we have $D_0(\sigma)(\xi')\neq 0$ on the closure of B.

Recall that $B_{\mu-\nu}(\xi')$ is a polynomial of ξ' and that $V_{\mu-\nu,0}$ and $V_{\mu-\nu,1}$ are constants independent of ξ'. Then, by the same kind of method as in Theorem 4.4 in [1], we have the following:

Theorem 3.6. *Let Assumption* 2.1, 3.2, 3.3, *and* 3.5 *be satisfied and B be the open ball in Remark to Assumption* 3.5. *When* $m-m'=2l-1$ *and* $(0<\theta<\pi$ *or* $\pi<\theta<2\pi)$ *or when* $m-m'=2l$ *and* $0<\theta<2\pi$, *Assumption* 3.3 *is not required. Let* μ *be* (3.12) *or* (3.13) *and the boundary data space be* $\mathrm{F}^{-1}(C_0^\infty(B))^\mu$.

(1) *The case when* rank Mat $V_{\mu-\nu,0}=\mu-\nu$, *that is,* $V_{\mu-\nu,0}\neq 0$. *The family* (3.1) *is normally micro-reducible at* ξ_0'. *In particular, if the boundary conditions are Dirichlet's*

$$b_{j_k}(D)=D_1^{k-1},\ k=1,\cdots,\mu\,, \tag{3.16}$$

then the family (3.1) *is normally micro-reducible at* ξ_0'.

(2) *The case when* rank Mat $V_{\mu-\nu,0}=\mu-\nu-1$. *Then* $V_{\mu-\nu,0}=0$.

(2–1) *If* $j_{\nu+1}-j_\nu\geqq 2$ *and* $B_{\mu-\nu}(\xi_0')\neq 0$, *then the family* (3.1) *is normally micro-reducible at* ξ_0'.

(2–2) *If* $j_{\nu+1}-j_\nu=1$, *then there are three cases as follows.*

(2–2–a) *If* $B_{\mu-\nu}(\xi_0')\neq 0$ *and* $V_{\mu-\nu,1}=0$, *then the family* (3.1) *is normally micro-reducible at* ξ_0'.

(2–2–b) *If* $V_{\mu-\nu,1}\neq 0$ *and* $D_{\xi'}^\alpha B_{\mu-\nu}(\xi_0')=0$ *for all multi-indexes* α, *that is,* $B_{\mu-\nu}(\xi')\equiv 0$, *then the limit* u_0 *of the solutions of* (3.1) *satisfies the following boundary conditions:*

$$b_{j_k}(D)\,u(x)|_{x_1\downarrow 0}=\phi_k(x'),\ k=1,\cdots,\nu-1,\nu+1. \tag{3.17}$$

In particular, when $\nu\leqq m'-1$, *the family* (3.1) *is abnormally micro-reducible at* ξ_0'. *When* $\nu=m'$, *the family* (3.1) *is micro-admissible at* ξ_0' *but not micro-reducible at* ξ_0'.

(2–2–c) *If* $V_{\mu-\nu,1}\neq 0$ *and there exists a multi-index* α *such that* $D_{\xi'}^\alpha B_{\mu-\nu}(\xi_0')\neq 0$, *that is,* $B_{\mu-\nu}(\xi')\not\equiv 0$, *then the family* (3.1) *is micro-admissible at* ξ'_0 *but not micro-reducible at* ξ'_0.

4. Various examples

We shall patch up the localization in ξ'-space and study the reducibility in various examples. We shall require the following "global-ellipticity" of the boundary conditions:

Assumption 4.1. There exist positive numbers I, C, and M independent of $0<\varepsilon<1$ and ξ' in $\boldsymbol{R}^{n-1}$ such that

$$|(D_0\cdot\varepsilon^I)^{-1}|\leqq C\langle\xi'\rangle^M ,$$

and every cofactor $D_{0,k,l}$ of D_0 k, $l=1, \cdots, \mu$ satisfies

$$|D_{0,k,l}\cdot\varepsilon^I|\leqq C\langle\xi'\rangle^M .$$

Here $\langle\xi'\rangle=(1+|\xi'|)^{1/2}$.

Remark. In some cases, instead of Assumption 4.1, it might be better to assume

$$|(D_0\cdot\varepsilon^I)^{-1}|\leqq C\langle\xi'\rangle^M/|\xi'|^{n-1-\delta} ,$$

where $\delta>0$ and to deal with L^2-solutions instead of $\mathcal{S}'$-solutions. Then we can admit some algebraic singularities of D_0 at $\xi'=0$.

Example 4.2. Let $P_1(\xi)$ be an elliptic polynomial of order 2μ with real coefficients such that $P_1(\xi)>0$ for ξ in $\boldsymbol{R}^n$ and

$$P_1(\xi) = \xi_1^{2\mu}+\textstyle\sum_{j=1}^{2\mu} p_{1,j}(\xi')\,\xi_1^{2\mu-j} . \tag{4.1}$$

Let $P_2(\xi')$ be an elliptic polynomial of ξ' with real coefficients such that $P_2(\xi')>0$ for ξ' in $\boldsymbol{R}^{n-1}$ and $\operatorname{ord} P_2<\operatorname{ord} P_1$. Then, for ξ in $\boldsymbol{R}^n$ and for $0<\varepsilon<1$,

$$\varepsilon^{2\mu-1}\cdot P_1(\xi)+i\xi_1+P_2(\xi') \neq 0 . \tag{4.2}$$

This implies that the characteristic roots $\tau_j(\varepsilon, \xi')$, $j=1, \cdots, 2\mu$ satisfy $\operatorname{Im}\tau_j(\varepsilon, \xi')>0$ or $\operatorname{Im}\tau_j(\varepsilon, \xi')<0$ alternatively in $\boldsymbol{R}^{n-1}_{\xi'}$ for $0<\varepsilon<1$.

Let us consider the following one-parameter family of unilateral boundary value problems:

$$\begin{cases} (\varepsilon^{2\mu-1}\cdot P_1(D)+iD_1+P_2(D'))\,u(x) = 0 \text{ in } \boldsymbol{R}^n_+ \ ; \\ b_{j_k}(D)\,u(x)|_{x_1\downarrow 0} = \phi_k(x'),\ k = 1, \cdots, \mu , \end{cases} \tag{4.3}$$

with $0<\varepsilon<1$. Let $b_{j_1}=b_0=1$ and b_{j_k}, $k=2, \cdots, \mu$ be normal and satisfy Assumption 3.5 for $B=\boldsymbol{R}^{n-1}$. Then the family (4.3) is normally micro-reducible at every point ξ' in $\boldsymbol{R}^{n-1}$, which will be shown under.

Let Φ in $\mathrm{F}^{-1}(C_0^\infty(B(0;R)))^\mu$, where $B(0;R)=\{|\xi'|<R\}$. Since $j_\mu \leqq 2\mu-1$ and $j_2\geqq 1$, it follows that for every pair (l_1, l_2), $l_1, l_2 \in A=\{j_2, \cdots, j_\mu\}$ with $l_1<l_2$, we have $0<l_2-l_1\leqq j_\mu-j_2\leqq 2\mu-2$. Hence $l_1 \not\equiv l_2 \pmod{2\mu-1}$, and we have rank Mat $V_{\mu-1,0}=\mu-1$. Since the characteristic roots are simple for $|\xi'|<R$ and $\varepsilon<\varepsilon_R$, the partial Fourier transforms of the solutions u_ε of (4.3) can be represented as (3.4). By Lemma 2.2, we have

$$\tau_1(\varepsilon, \xi') = iP_2(\xi')+O(\varepsilon^{2\mu-1}), \tag{4.4}$$

and

$$\tau_j(\varepsilon, \xi')\cdot\varepsilon = \Theta\zeta^{j-2}+O(\varepsilon), j=2, \cdots, 2\mu\,. \tag{4.5}$$

Here $\Theta=\exp\dfrac{3\pi i}{2(2\mu-1)}$ and $\zeta=\exp\dfrac{2\pi i}{2\mu-1}$. The imaginary parts of $\tau_j(\varepsilon, \xi')$, $j=1, \cdots, \mu$ are positive. Hence Lemma 2.4 implies

$$\lim_{\varepsilon\downarrow 0} C_1(\varepsilon, \xi'; \Phi) = \lim_{\varepsilon\downarrow 0}(D_1\cdot\varepsilon^J)/(D_0\cdot\varepsilon^J) = \hat{\phi}_1\,,$$

and for $k=2, \cdots, \mu$

$$\lim_{\varepsilon\downarrow 0} C_k(\varepsilon, \xi'; \Phi) = \lim_{\varepsilon\downarrow 0}(D_k\cdot\varepsilon^J)/(D_0\cdot\varepsilon^J) = 0\,.$$

Therefore

$$\lim_{\varepsilon\downarrow 0} u_\varepsilon = u_0 = Y(x_1)\cdot\mathrm{F}_{\xi'}^{-1}(\exp -P_2(\xi')\,x_1\cdot\hat{\phi}_1(\xi'))\,.$$

This implies that u_0 satisfies

$$\begin{cases} (iD_1+P_2(D'))\,u(x) = 0 \text{ in } \boldsymbol{R}_+^n\,; \\ u(x)\,|_{x_1\downarrow 0} = \phi_1(x')\,. \end{cases} \tag{4.6}$$

Thus (4.3) is normally micro-reducible at every point in $B(0;R)$, where R is an arbitrary positive number.

When Assumption 4.1 is satisfied, the family (4.3) is normally reducible. In fact, Assumption 4.1 assures the commutation of the limit $\varepsilon\downarrow 0$ and the inverse Fourier transformation. Then we have only to calculate the pointwise limit of (3.4) in ξ'-space, but this is the micro-reducible version.

The above example can be generlized as follows:

EXAMPLE 4.3. Assume the same assumptions as in Example 4.2. Let $P_3(\xi)$ be an elliptic polynomial of order 2κ such that $P_3(\xi)\neq 0$ for ξ in $\boldsymbol{R}^n$ and the characteristic roots of $P_3(\xi)=0$ with respect to ξ_1 are simple in $\boldsymbol{R}_{\xi'}^{n-1}$. Consider the following equation:

$$(4.7)\qquad (\varepsilon^{2\mu-1}\cdot P_1(\xi)+i\xi_1+P_2(\xi'))\,P_3(\xi)=0\,.$$

Renumber $\tau_1(\varepsilon, \xi')$ of (4.4) as $\tau_{\kappa+1}(\varepsilon, \xi')$ and $\tau_j(\varepsilon, \xi')$ of (4.5) as $\tau_{\kappa+j}(\varepsilon, \xi'), j=2, \cdots, \mu$, respectively. We denote by $\sigma_j(\xi'), j=1, \cdots, \kappa$ the characteristic roots of $P_3(\xi)=0$, which have positive imaginary parts. Put $\sigma_{\kappa+1}(\xi')=i\cdot P_2(\xi')$ and $\tau_j=\sigma_j, j=1, \cdots, \kappa+1$. Let us consider the following one-parameter family:

$$(4.8)\qquad \begin{cases}(\varepsilon^{2\mu-1}\cdot P_1(D)+iD_1+P_2(D'))\,P_3(D)\,u(x)=0 \text{ in } \boldsymbol{R}^n_+\,;\\ b_{j_k}(D)\,u(x)|_{x_1\downarrow 0}=\phi_k(x'),\ k=1, \cdots, \mu+\kappa\,.\end{cases}$$

Here we assume that $j_k\leqq 2\mu+2\kappa-1$ and that b_{j_k} satisfy Assumption 3.5 for $B=\boldsymbol{R}^{n-1}$. Then we can apply Theorem 3.6 to this example for $B=\boldsymbol{R}^{n-1}$. When Assumption 4.1 is satisfied, we can have the same result as in Theorem 4.4 in [1].

Let us give an example of the micro-admissible family, which is not micro-reducible. This is a special case of Example 4.3.

EXAMPLE 4.4. Put $P_1=\xi_1^6+\langle\xi'\rangle^6$, $P_2=\langle\xi'\rangle^2$, and $P_3=\xi_1^2+\frac{1}{4}\langle\xi'\rangle^2$ in (4.7). Denote $\Theta=\exp\frac{3\pi i}{10}$ and $\zeta=\exp\frac{2\pi i}{5}$. Lemma 2.2 implies that $\sigma_1(\xi')=\frac{i}{2}\langle\xi'\rangle$, $\tau_2=i\langle\xi'\rangle^2+O(\varepsilon^4)$, $\tau_3\cdot\varepsilon=\Theta-\frac{i}{5}\langle\xi'\rangle^2\cdot\varepsilon+O(\varepsilon^2)$, and $\tau_4\cdot\varepsilon=\Theta\zeta-\frac{i}{5}\langle\xi'\rangle^2\cdot\varepsilon+O(\varepsilon^2)$. We set the following boundary conditions:

$$u|_{x_1\downarrow 0}=\phi_1,\ D_1u|_{x_1\downarrow 0}=\phi_2,\ D_1^2u|_{x_1\downarrow 0}=\phi_3,\quad\text{and}\quad D_1^7u|_{x_1\downarrow 0}=\phi_8\,.$$

Here ϕ_1, ϕ_2, ϕ_3, and ϕ_8 belong to $\mathrm{F}^{-1}(C_0^\infty(\boldsymbol{R}^{n-1}))$. Then we have

$$D_0\Big(\frac{i}{2}\langle\xi'\rangle, i\langle\xi'\rangle^2;\,1, \tau\Big)=i\langle\xi'\rangle\Big(\langle\xi'\rangle-\frac{1}{2}\Big)\neq 0\,,$$

$$D_0\Big(\frac{i}{2}\langle\xi'\rangle, i\langle\xi'\rangle^2;\,1, \tau^2\Big)=-\langle\xi'\rangle^2\Big(\langle\xi'\rangle^2-\frac{1}{4}\Big)\neq 0\,,$$

$$B_{4-2}=i\zeta(\zeta-1)\langle\xi'\rangle^2,\quad\text{and}\quad V_{4-2,1}=\zeta(\zeta-1)\,.$$

Thus

$$d_{0,1}/\Theta^8=-\zeta(\zeta-1)\langle\xi'\rangle^2\Big(\langle\xi'\rangle-\frac{1}{2}\Big)\Big(2\langle\xi'\rangle+\frac{1}{2}\Big),$$

$$d_{1,1}/\Theta^8=(i\langle\xi'\rangle^2\hat{\phi}_1-\hat{\phi}_2)\cdot i\zeta(\zeta-1)\langle\xi'\rangle^2-(\langle\xi'\rangle^4\hat{\phi}_1+\hat{\phi}_3)\cdot\zeta(\zeta-1)\,,$$

and

$$d_{2,1}/\Theta^8=\Big(\hat{\phi}_2-\frac{i}{2}\langle\xi'\rangle\,\hat{\phi}_1\Big)\cdot i\zeta(\zeta-1)\langle\xi'\rangle^2+\Big(\hat{\phi}_3+\frac{1}{4}\langle\xi'\rangle^2\hat{\phi}_1\Big)\cdot\zeta(\zeta-1)\,.$$

Obviously, this family is micro-admissible at every point in $\boldsymbol{R}^{n-1}_{\xi'}$. Denote by u_0 the limit of u_ε when $\varepsilon\downarrow 0$. Since

$$(\lim_{\varepsilon\downarrow 0} u_\varepsilon)|_{x_1\downarrow 0} = u_0|_{x_1\downarrow 0} = \mathrm{F}^{-1}((d_{1,1}+d_{2,1})/d_{0,1}) = \mathrm{F}^{-1}(\hat{\phi}_1),$$

u_0 satisfies the boundary condition $u|_{x_1\downarrow 0}=\phi_1$. We have

$$D_1 u_0|_{x_1\downarrow 0} = \mathrm{F}^{-1}\Big(\Big(\frac{i}{2}\langle\xi'\rangle\, d_{1,1}+i\langle\xi'\rangle^2\, d_{2,1}\Big)/d_{0,1}\Big)$$
$$= \mathrm{F}^{-1}(C_1\hat{\phi}_1+C_2\hat{\phi}_2+C_3\hat{\phi}_3),$$

where $C_1=\dfrac{i\langle\xi'\rangle^2}{4\langle\xi'\rangle+1}$, $C_2=\dfrac{2\langle\xi'\rangle}{4\langle\xi'\rangle+1}$, and $C_3=\dfrac{-2i}{\langle\xi'\rangle(4\langle\xi'\rangle+1)}$. We also have

$$D_1^2 u_0|_{x_1\downarrow 0} = \mathrm{F}^{-1}\Big(\Big(-\frac{1}{4}\langle\xi'\rangle^2\, d_{1,1}-\langle\xi'\rangle^4\, d_{2,1}\Big)/d_{0,1}\Big)$$
$$= \mathrm{F}^{-1}(C_4\hat{\phi}_1+C_5\hat{\phi}_2+C_6\hat{\phi}_3),$$

where $C_4=\dfrac{\langle\xi'\rangle^4}{4\langle\xi'\rangle+1}$, $C_5=\dfrac{i\langle\xi'\rangle^2(2\langle\xi'\rangle+1)}{4\langle\xi'\rangle+1}$, and $C_6=\dfrac{2\langle\xi'\rangle+1}{4\langle\xi'\rangle+1}$. Hence u_0 does not satisfy the boundary conditions $D_1u|_{x_1\downarrow 0}=\phi_2$ and $D_1^2u|_{x_1\downarrow 0}=\phi_3$. Thus this family is not micro-reducible at every point in $\boldsymbol{R}^{n-1}_{\xi'}$.

Since ν depends on ξ'_0, μ the number of the boundary conditions may be changed by ξ'_0. When μ is changed by ξ'_0, we can not set the problem of the reducibility. The following example, to which Theorem 3.6 can not be applied, will show us such a situation.

EXAMPLE 4.5. Let $P_1(\xi)=\langle\xi\rangle^4$ and $P_2(\xi)=-\langle\xi\rangle^2$. Then the characteristic roots of $\varepsilon^4\cdot P_1(\xi)+P_2(\xi)=0$ are $\pm i\langle\xi'\rangle$ and $\pm\dfrac{1}{\varepsilon}\cdot(1-\langle\xi'\rangle^2\cdot\varepsilon^2)^{1/2}$. For fixed ε, two characteristic roots have positive imaginary parts for sufficienty large ξ'. Therefore, let us consider the following one-parameter family of unilateral boundary value problems:

$$(4.9)\qquad \left[\begin{array}{l}(\varepsilon^4\cdot P_1(D)+P_2(D))\,u = 0 \text{ in } \boldsymbol{R}^n_+\,;\\ u|_{x_1\downarrow 0} = \phi_1,\ D_1u|_{x_1\downarrow 0} = \phi_2\,,\end{array}\right.$$

where ϕ_1 and ϕ_2 belong to $\mathcal{S}(\boldsymbol{R}^n)$. But the $\mathcal{S}'$-solutions of (4.9) are not unique. In fact, if ϕ_1, ϕ_2, and ϕ_3 belong to $\mathrm{F}^{-1}(C_0^\infty(B(0;R)))$ and $\varepsilon<R^{-1}<1$, then the following family

$$(4.10)\qquad \left[\begin{array}{l}(\varepsilon^4\cdot P_1(D)+P_2(D))\,u = 0 \text{ in } \boldsymbol{R}^n_+\,;\\ u|_{x_1\downarrow 0} = \phi_1,\ D_1u|_{x_1\downarrow 0} = \phi_2,\ D_1^2u|_{x_1\downarrow 0} = \phi_3\,,\end{array}\right.$$

is micro-reducible at every point in $B(0;R)$.

5. The convergence of canonical extensions

We shall deduce some results from the convergence of the canonical extensions, referring to Appendix. Let P_1 and P_2 be kowalewskian with their symbols:

$$P_1(\xi) = \xi_1^m + \sum_{j=0}^{m-1} p_{1,m-j}(\xi')\, \xi_1^j\,,$$
$$P_2(\xi) = p_{2,0}\, \xi_1^{m'} + \sum_{j=0}^{m'-1} p_{2,m'-j}(\xi')\, \xi_1^j\,.$$

Denote $P_\varepsilon(\xi) = \varepsilon \cdot P_1(\xi) + P_2(\xi)$ and

$$p_{\varepsilon,j} = \varepsilon \cdot p_{1,m-j} + p_{2,m'-j},\ j = 0, \cdots, m-1\,,$$

where $p_{2,k}=0$ for $k<0$. Let us consider a sequence of prolongable solutions u_ε of

$$(5.1) \qquad \begin{cases} P_\varepsilon(D)\, u = 0, \text{ in } \boldsymbol{R}^n_+ \\ b_j(D)\, u|_{x_1 \downarrow 0} = \phi_j,\ j = 0, \cdots, m-1\,. \end{cases}$$

Here every $b_j(D)$ is a normal boundary operator of order j and every ϕ_j belongs to $\mathcal{D}'(\boldsymbol{R}^{n-1})$. Then we have the following lemma.

Lemma 5.1. *If there exists a sequence of prolongable solutions u_ε of* (5.1) *and a distribution v such that*

$$(5.2) \qquad [u_\varepsilon]^+ \to v \text{ in } \mathcal{D}'(\boldsymbol{R}^n)\,,$$

then

$$(5.3) \qquad P_2(D)\, v = 0 \quad in \quad \boldsymbol{R}^n_+\,;\ v = [v]^+\,;$$

$$(5.4) \qquad b_j(D)\, v|_{x_1 \downarrow 0} = b_j(D)\, u_\varepsilon|_{x_1 \downarrow 0} = \phi_j,\ j = 0, \cdots, m'-1\,.$$

Proof. First we shall prove the assertion when $b_j = D_1^j$, $j=0, \cdots, m-1$. Denote by $\{q_{l,j}\}$, $l=1, 2, \varepsilon$ the dual boundary systems of $\{D_1^j\}$ with respect to $P_l(D)$, $l=1, 2, \varepsilon$, respectively. Then by (A.5), $q_{l,j}(\xi) = {}^t\!\left(\frac{1}{i} \cdot \sum_{k=0}^{j} p_{l,k}(\xi')\, \xi_1^{j-k}\right)$, $l=1, 2, \varepsilon$. For every $u(x)$ in $C^\infty(\boldsymbol{R}^n)$, we have

$$\begin{aligned} P_\varepsilon(D)\,(Y(x_1) \cdot u) &= \varepsilon \cdot P_1(D)\,(Y(x_1) \cdot u) + P_2(D)\,(Y(x_1) \cdot u) \\ &= Y(x_1) \cdot \varepsilon \cdot P_1(D)\, u + Y(x_1) \cdot P_2(D)\, u + \varepsilon \cdot \textstyle\sum_{j=0}^{m-1} {}^t q_{1,m-j-1}(D) \{\delta(x_1)\, \phi_j\} \\ &\quad + \textstyle\sum_{j=0}^{m'-1} {}^t q_{2,m'-j-1}(D) \{\delta(x_1)\, \phi_j\} \\ &= Y(x_1) \cdot P_\varepsilon(D)\, u + \textstyle\sum_{j=0}^{m-1} {}^t q_{\varepsilon,m-j-1}(D) \{\delta(x_1)\, \phi_j\}\,, \end{aligned}$$

where $\phi_j(x') = D_1^j u|_{x_1=0}$, $j=0, \cdots, m-1$. Since the dual boundary system is uniquely determined in the case of constant coefficients, it implies that

$${}^t q_{\varepsilon,k}(D) = \varepsilon \cdot {}^t q_{1,k}(D) + {}^t q_{2,k}(D),\ k = 0, \cdots, m'-1$$

and

$${}^t q_{\varepsilon,k}(D) = \varepsilon \cdot {}^t q_{1,k}(D),\ k = m', \cdots, m\,.$$

Thus we can write

$$(5.5)\quad P_\varepsilon(D)[u_\varepsilon]^+ = \sum_{j=0}^{m-1} {}^t q_{\varepsilon,m-j-1}(D)\{\delta(x_1)\phi_j\}$$
$$= \varepsilon\cdot\sum_{j=0}^{m-1} {}^t q_{1,m-j-1}(D)\{\delta(x_1)\phi_j\} + \sum_{j=0}^{m'-1} {}^t q_{2,m'-j-1}(D)\{\delta(x_1)\phi_j\} .$$

Letting $\varepsilon \downarrow 0$ in (5.5), we have

$$(5.6)\qquad P_2(D)\, v = \sum_{j=0}^{m'-1} {}^t q_{2,m'-j-1}(D)\{\delta(x_1)\phi_j\} .$$

Since the support of the right-hand side of (5.6) is included in $x_1=0$, it follows that $P_2(D)\,v=0$ in $\boldsymbol{R}_+^n$. The expression (5.6) and the definition of the boundary values of v imply (5.4). The uniqueness of the expression (5.6) of $[v]^+$ implies $v=[v]^+$.

Denote by $\{c_{\varepsilon,j}\}$ the dual boundary system of $\{b_j\}$ with respect to $P_\varepsilon(D)$. Then by (A.6),

$$c_{\varepsilon,j} = \varepsilon\cdot\sum_{k=0}^{j} {}^t a_{m-1-j+k,k}\, q_{1,j-k} + \sum_{k=0}^{j} {}^t a_{m'-1-j+k,k}\, q_{2,j-k} .$$

Here $a_{j,k}$ satisfy (A.4) and $a_{j,k}=0$, for $j<0$. Since $a_{j,k}$ are independent of ε, we have

$$\lim_{\varepsilon\downarrow 0} c_{\varepsilon,j} = \sum_{k=0}^{j} {}^t a_{m'-1-j+k,k}\, q_{2,j-k} .$$

Thus we can reduce this general case into the first. [Q.E.D.]

In [2], we have already studied the necessary conditions for the convergence of solutions of the one-parameter family of Cauchy problems. The following theorem shows that an admissible one-parameter family of Cauchy problems is normally reducible.

Theorem 5.2. *Assume that there exists a sequence of solutions u_ε of the following Cauchy problems*:

$$(5.7)\qquad \begin{cases} P_\varepsilon(D)\,u = 0, \textit{ in } \boldsymbol{R}^n ; \\ b_j(D)\,u|_{x_1=0} = \phi_j, j = 0, \cdots, m-1 , \end{cases}$$

and a distirbution v such that

$$(5.8)\qquad \lim_{\varepsilon\downarrow 0} u_\varepsilon(x) = v(x) \text{ in } C(\boldsymbol{R}_{x_1}; \mathcal{D}'(\boldsymbol{R}_{x'}^{n-1})) .$$

Then v satisfies the following reduced Cauchy problem:

$$(5.9)\qquad \begin{cases} P_2(D)\,u = 0, \textit{ in } \boldsymbol{R}^n ; \\ b_j(D)\,u|_{x_1=0} = \phi_j, j = 0, \cdots, m'-1 . \end{cases}$$

Proof. By (5.8), we have

$$\lim_{\varepsilon\downarrow 0} Y(x_1)\, u_\varepsilon(x) = Y(x_1)\, v(x) \text{ in } \mathcal{D}'(\boldsymbol{R}^n)\,.$$

Since $[u_\varepsilon]^+ = Y(x_1)\, u_\varepsilon(x)$, it follows that

$$\lim_{\varepsilon\downarrow 0} [u_\varepsilon]^+ = Y(x_1)\, v(x) \text{ in } \mathcal{D}'(\boldsymbol{R}^n)\,.$$

We know that

$$b_j(D)\, u_\varepsilon|_{x_1\downarrow 0} = b_j(D)\, u_\varepsilon|_{x_1=0} = \phi_j,\, j=0, \cdots, m-1\,.$$

Hence u_ε satisfies the boundary value problem (5.1). By applying Lemma 5.1, we have $P_2(D)\, v=0$ in $\mathcal{D}'(\boldsymbol{R}^n_+)$, $v=[v]^+$, and

$$b_j(D)\, v|_{x_1\downarrow 0} = b_j(D)\, v|_{x_1=0} = \phi_j,\, j=0, \cdots, m'-1\,. \qquad \text{[Q.E.D]}$$

Remark. For example, when P_1 is strongly hyperbolic and the data belong to $C_0^\infty(\boldsymbol{R}^{n-1})$, then the Cauchy problem (5.7) is uniquely solvable for every $\varepsilon<1$. See Theorem 4.7 and 4.10 in [6]. Hence when P_1 is strongly hyperbolic and P_2 is hyperbolic, the admissibility implies the normal reducibility.

The following example shows that as for the one-parameter family of boundary value problems, it is not appropriate to require the convergence of canonical extensions.

Example 5.3. Let us consider the one-parameter family of boundary value problems of ordinary differential operators:

$$(5.10)\qquad \left[\begin{array}{l} \left(\left(\varepsilon\cdot\dfrac{d}{dx}\right)^2 + 2\left(\varepsilon\cdot\dfrac{d}{dx}\right)+1\right) u = 0\,, \\ u(0) = u(\infty) = 0\,. \end{array}\right.$$

Put $u_\varepsilon = \dfrac{x}{\varepsilon^3}\cdot\exp\left(-\dfrac{x}{\varepsilon}\right)$. Then u_ε is the global solution in $\mathcal{D}'(\boldsymbol{R})$, especially in $\mathcal{D}'(\boldsymbol{R}^+)$. The canonical extension of u_ε is $Y(x)\, u_\varepsilon(x)$ as we refer in Remark to Lemma *A*.2. Since

$$\langle Yu_\varepsilon, \phi\rangle = \frac{1}{\varepsilon}\int_0^\infty te^{-t}\, \phi(\varepsilon t)\, dt,\ \phi\in C_0^\infty(\boldsymbol{R})\,,$$

$\langle Yu_\varepsilon, \phi\rangle$ does not converge in $\mathcal{D}'(\boldsymbol{R})$ when $\varepsilon\downarrow 0$. Put

$$v_\varepsilon(x) = \left[\begin{array}{ll} u_\varepsilon(x), & \text{for } x\geqq 0\,; \\ -u_\varepsilon(-x), & \text{for } x<0\,. \end{array}\right.$$

Then

$$\langle v_\varepsilon, \phi\rangle = \int_0^\infty t^2\, e^{-t}\cdot\frac{1}{\varepsilon t}\,(\phi(\varepsilon t) - \phi(-\varepsilon t))\, \mathrm{dt}$$

and $\langle v_\varepsilon, \phi\rangle \to 2\phi'(0)\,\Gamma(3)$, when $\varepsilon \downarrow 0$. Here $\Gamma(z)$ is the gamma function. Therefore v_ε is a solution not in $\boldsymbol{R}$ but in $\boldsymbol{R}^+$ and converges in $\mathcal{D}'(\boldsymbol{R})$.

REMARK. In case of initial value problems with variable coefficients, *A.* Yoshikawa, [8] studied the same kind of equations as in Example 5.3 in a smart treatment.

Appendix

The boundary values of solutions to a non-characteristic hyperplane

We shall give a brief survey of a general boundary value theory for solutions of linear partial differential equations with constant coefficients according to [4] based on hyperfunction theory.

Let $P(D)$ be a differential operator of order m with constant coefficeints and its symbol be

$$P(\xi) = \xi_1^{\,m} + \textstyle\sum_{j=1}^{m} p_j(\xi')\,\xi_1^{\,m-j}\,. \tag{A.1}$$

Here every $p_j(\xi')$ is a polynomial of ξ' with order $p_j \leqq j$. Let every $b_j(D)$, $j=0, \cdots, m-1$ be a differential operator of order j with constant coefficients and its symbol be

$$b_j(\xi) = \xi_1^{\,j} + \textstyle\sum_{k=1}^{j} b_{j,k}(\xi')\,\xi_1^{\,j-k}\,. \tag{A.2}$$

Here every $b_{j,k}(\xi')$ is a polynomial of ξ' with order $b_{j,k} \leqq k$. Such a differential operator as $b_j(D)$ is said to be *normal*, and

$$b_j(D)\,u(x)\,|_{x_1 \downarrow 0} = \phi_j(x'),\ j = 0, \cdots, m-1 \tag{A.3}$$

is called the *normal boundary condition.* A system $\{b_j\}_{j=0}^{m-1}$ is said to be *normal* if every b_j is normal. When $\{b_j\}_{j=0}^{m-1}$ is normal, there exist normal differential operators $a_{j,k}(D')$ such that for $j=0, \cdots, m-1$

$$D_1^{\,j} = \textstyle\sum_{k=0}^{j} a_{j,k}(D')\cdot b_{j-k}(D)\,. \tag{A.4}$$

A system $\{c_j(D)\}_{j=0}^{m-1}$ is called the *dual boundary system* of $\{b_j(D)\}_{j=0}^{m-1}$ with respect to $P(D)$ if for every C^m function $u(x)$ it satisfies

$$P(D)\,(Y(x_1)\,u(x)) = Y(x_1)\,P(D)\,u(x) + \textstyle\sum_{j=0}^{m-1} {}^t c_{m-j-1}(D)\,(\delta(x_1)\,b_j(D)\,u(x))\,,$$

in a neighbourhood of $x_1=0$. Here $Y(x_1)$ is the Heaviside function and $\delta(x_1)$ is the Driac measure. The symbols of the dual boundary system of $\{D_1^{\,j}\}_{j=0}^{m-1}$ with respect to $P(D)$ are

$$q_j(\xi) = {}^t\!\left(\frac{1}{i}\cdot\textstyle\sum_{k=0}^{j} p_k(\xi')\,\xi_1^{\,j-k}\right), \tag{A.5}$$

and those of $\{b_j\}_{j=0}^{m-1}$ are

$$c_j(\xi) = \sum_{k=0}^{j} {}^t a_{m-1-j+k,k}(\xi')\cdot q_{j-k}(\xi)\,. \tag{A.6}$$

Let U be a domain containing the origin. Put $U^+=U\cap\{x_1>0\}$, $U^0=U\cap\{x_1=0\}$, $U^-=U\cap\{x_1<0\}$, $\bar{U}^+=U^+\cup U^0$, and $\bar{U}^-=U^-\cup U^0$. When U^0 is regarded as an open set in $\boldsymbol{R}^{n-1}$, U^0 is denoted by U', that is, $U^0=\{0\}\times U'$.

A distribution u in $\mathcal{D}'(U^+)$ is said to be *prolongable* into $x_1\leqq 0$ if there exist an open set V and a distribution v in $\mathcal{D}'(V)$, which is called an extension of u, such that

$$V\cap\{x_1>0\} = U^+ \quad \text{and} \quad v|_{U^+} = u\,.$$

Lemma A.1. *Let $u(x)$ be a prolongable solution of $P(D)\,u(x)=0$ in U^+. Then there exist a unique extension $[u]^+$ in $\mathcal{D}'(U)$ of u and unique data $\phi_j(x')$ in $\mathcal{D}'(U')$, $j=0,\cdots,m-1$ satisfying* supp $[u]^+\subset\bar{U}^+$ *and*

$$P(D)\,[u]^+(x) = \sum_{j=0}^{m-1} {}^t c_{m-j-1}(D)\{\delta(x_1)\,\phi_j(x')\}\,. \tag{A.7}$$

Here the extension $[u]^+$ is said to be canonical and is independent of the choice of the boundary system. The data $\phi_j(x')$ are called the boundary values to $x_1=0$ with respect to $\{b_j(D)\}_{j=0}^{m-1}$. We write $b_j(D)\,u|_{x_1\downarrow 0}=\phi_j$.

Proof. Let $\{\rho_j\}$ be a partition of unity on U and χ be the difining function of the set U^+. We can write $\rho_j u=\sum_\alpha D^\alpha f_{j,\alpha}$, where $f_{j,\alpha}$ are continuous functions with supp $f_{j,\alpha}\subset$ supp ρ_j. Put $v=\sum_j\sum_\alpha D^\alpha(\chi f_{j,\alpha})$. Then $v|_{U^+}=u|_{U^+}$ and supp $v\subset\bar{U}^+$. Hence $P(D)\,v=0$ in U^+ and supp $P(D)\,v\subset U^0$. By the local structure theorem of a distribution whose support is included in $x_1=0$ (See Théorèm XXXVI in [7]), we can write locally

$$P(D)\,v = \sum_{k=0}^{M} D_j{}^k\delta(x_1)\,f_k(x')\,. \tag{A.8}$$

Here $f_k(x')$ are distributions. If $M\geqq m$, then

$$D_1{}^M\delta(x_1)\,f_M(x') = P(D)\,D_1{}^{M-m}(\delta(x_1)\,f_M(x'))+\sum_{k=0}^{M-1} D_1{}^k\delta(x_1)\,g_k(x')\,.$$

By replacing v by $v-D_1{}^{M-m}(\delta(x_1)\,f_M(x'))$, we can diminish M by one. Repeating this operation, we can finally let $M=m-1$. We denote this extension by $[u]^+$ and the coefficients in the right-hand side by $v_j(x')$, then we have a local representation of (A.7) when $c_j(D)={}^tD_1{}^j$ as

$$P(D)\,[u]^+ = \sum_{j=0}^{m-1} D_1{}^{m-j-1}\,\delta(x_1)\,v_j(x')\,. \tag{A.9}$$

Let $[u]'^+$ be another extension and

$$P(D)\,[u]'^+ = \sum_{j=0}^{m-1} D_1{}^{m-j-1}\,\delta(x_1)\,w_j(x')\,.$$

If $[u]^+-[u]'^+$ is not identically zero, then

$$[u]^+ - [u]'^+ = \sum_{j=0}^{M} D_1^{\,j}\delta(x_1)\, h_j(x')\,,$$

where $h_M(x')$ is not identiaclly zero. But

$$\begin{aligned} P(D)\,([u]^+ - [u]'^+) &= D_1^{\,M+m}\delta(x_1)\, p_m h_M(x') + \cdots \\ &= \sum_{j=0}^{m-1} D_1^{\,m-j-1}\delta(x_1)\,(v_j(x') - w_j(x'))\,. \end{aligned}$$

This contradicts the uniqueness of the coefficients in the structure theorem. Thus $[u]^+$ and $v_j(x')$ are uniquely determined locally. The sheaf property of distributions implies that (A.9) holds globally. In the case of general $\{c_j(D)\}$, put

(A.10)
$$c_j(D) = \sum_{k=0}^{j} c_{j,k}(D')\, {}^tD_1^{\,j-k}\,,$$

then

(A.11)
$${}^tD_1^{\,j} = \sum_{k=0}^{j} d_{j,l}(D')\, c_{j-k}\,.$$

Hence

$$\begin{aligned} P(D)\,[u]^+ &= \sum_{j=0}^{m-1} \sum_{k=0}^{m-j-1} {}^tc_{m-j-1-l}(D)\, {}^td_{m-j-1,k}(D')\,\{\delta(x_1)\, v_j(x')\} \\ &= \sum_{l=0}^{m-1} {}^tc_{m-l-1}(D)\,\{\delta(x_1) \sum_{k=0}^{l} {}^td_{m-l-1+k,k}(D')\, v_{l-k}(x')\}\,, \end{aligned}$$

that is,

$$\phi_j(x') = \sum_{k=0}^{j} {}^td_{m-j-1+k,k}(D')\, v_{j-k}(x')\,.$$

Since this equation can be solved with respect to $v_j(x')$, it follows that $\phi_j(x')$ are uniquely determined by u. [Q.E.D.]

REMARK. If u can be extended as a solution, then x_1 is a C^∞-parameter, that is, $u(x)$ is microlocally C^∞ at $(x; 1, 0, \cdots, 0)$ for every x. Hence the product $Y(x_1)\, u(x)$ can be defined and we have $[u]^+ = Y(x_1)\, u$.

The following lemma will clarify the meaning of the limits of boundary values. The proof will be omitted.

LEMMA A.2. *Let* $U = \{|x_1| < \delta\} \times U'$ *and* $u(x)$ *be a prolongable solution of* $P(D)\,u = 0$ *in* U^+. *Then*

(A.12)
$$b_j(D)\, u(x)\,|_{x_1=\delta} \to b_j(D)\, u(x)\,|_{x_1\downarrow 0}\,,$$

in $\mathcal{D}'(U')$ *when* $\delta \downarrow 0$.

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