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CONVERGENCE OF LOGISTIC PARAMETERS IN BAYESIAN APPROACH

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1. Introduction

In this paper, we study the statistical model of the independent, 0-1-valued observations with the following distributions:

$$P(Y_i = 1) = \frac{e^{\alpha+\beta x_i}}{1 + e^{\alpha+\beta x_i}}, \quad P(Y_i = 0) = \frac{1}{1 + e^{\alpha+\beta x_i}} \\ (i = 1, 2, \dots, n),$$

where x_i 's are known real numbers called the *observation points*. It is sometimes more natural to consider the parameters α and β in the above logistic way than something like $e^\alpha/(1 + e^\alpha)$'s. For example, let us consider a random variable Y on $\{0, 1\}$ with small $P[Y = 1]$, where the value 1 stands for a serious accident which we must avoid definitely. Since we are sensitive on the value $P[Y = 1]$, we take the measurement $\log P[Y = 1]$ instead of the value itself. In this case, the logistic parametrization is suitable. In the same reason, it is natural to assume that the prior distribution α and β is uniform, that is, the joint prior density for (α, β) is given by $p(\alpha, \beta) \equiv 1$ on \mathbf{R}^2 . Then we discuss the posterior distribution on (α, β) under a set of observations $Y_i = y_i$ ($i = 1, \dots, n$).

By the Bayes formula, the posterior probability density, is given by

$$p(\alpha, \beta | y_1, \dots, y_n) = c^{-1} \prod_{i=1}^n \left(\frac{e^{\alpha+\beta x_i}}{1 + e^{\alpha+\beta x_i}} \right)^{y_i} \left(\frac{1}{1 + e^{\alpha+\beta x_i}} \right)^{1-y_i}$$

if and only if the normalizing constant exists, that is

$$c := \iint \prod_{i=1}^n \left(\frac{e^{\alpha+\beta x_i}}{1 + e^{\alpha+\beta x_i}} \right)^{y_i} \left(\frac{1}{1 + e^{\alpha+\beta x_i}} \right)^{1-y_i} d\alpha d\beta < \infty.$$

We obtain in Theorem 1 a necessary and sufficient condition for the existence of the posterior probability distribution, or equivalently, for $c < \infty$.

Theorem 1. *A necessary and sufficient condition for $c < \infty$ is that $1 \leq m \leq n - 1$ and*

$$\min \left\{ \sum_{i \in S} x_i : \#S = m \right\} < \sum_{i=1}^n x_i y_i < \max \left\{ \sum_{i \in S} x_i : \#S = m \right\},$$

where we put $m = \sum_{i=1}^n y_i$ and $S \subset \{1, 2, \dots, n\}$.

Under this condition, we consider α and β to be random variables and use the notations A and B for α and β in this sense to avoid a confusion with their sample values α and β .

We are interested in the convergence of the random variables A, B under the observations y_1, y_2, \dots, y_{kn} satisfying that k number of the observation points are fixed where the same number n of observations are allocated and the ratio of 1 among them converges as $n \rightarrow \infty$ to a value in $(0, 1)$. That is, we assume the following set of observation points:

$$x_{i,j} \quad (i = 1, \dots, k; j = 1, \dots, n)$$

with

$$x_{i,1} = \dots = x_{i,n} := x_i \quad (i = 1, \dots, k)$$

and $x_i < x_{i+1}$ for $i = 1 \dots k - 1$ with a fixed integer k not less than 2. Let $y_{i,j}$ be the set of corresponding observations, for which we assume that

$$p_i := \lim_{n \rightarrow \infty} \frac{t_i}{n} \quad (i = 1, \dots, k)$$

exist for

$$t_i := \sum_{j=1}^n y_{i,j}$$

and it holds that $0 < p_i < 1$ ($i = 1, \dots, k$).

Then, the posterior density $p(\alpha, \beta | t_1, \dots, t_k)$ for (A, B) under these observations satisfies that

$$\begin{aligned} p(\alpha, \beta | t_1, \dots, t_k) &= c_n^{-1} \prod_{i=1}^k \prod_{j=1}^n \left(\frac{e^{\alpha+\beta x_{i,j}}}{1 + e^{\alpha+\beta x_{i,j}}} \right)^{y_{i,j}} \left(\frac{1}{1 + e^{\alpha+\beta x_{i,j}}} \right)^{1-y_{i,j}} \\ (1) \quad &= c_n^{-1} \frac{\exp\{\sum_{i=1}^k t_i(\alpha + \beta x_i)\}}{\prod_{i=1}^k \{1 + \exp(\alpha + \beta x_i)\}^n} \end{aligned}$$

$$= c_n^{-1} \exp \left[n \sum_{i=1}^k \left\{ \frac{t_i}{n} (\alpha + \beta x_i) - \log(1 + \exp(\alpha + \beta x_i)) \right\} \right]$$

where c_n is the normalizing constant. We put

$$(2) \quad \begin{aligned} f(\alpha, \beta) &:= \sum_{i=1}^k [p_i(\alpha + \beta x_i) - \log \{1 + \exp(\alpha + \beta x_i)\}] \\ G_n(\alpha, \beta) &:= \sum_{i=1}^k \left[\frac{t_i}{n} (\alpha + \beta x_i) - \log \{1 + \exp(\alpha + \beta x_i)\} \right]. \end{aligned}$$

The *maximal likelihood estimator* $(\hat{\alpha}_n, \hat{\beta}_n)$ is, by definition, a point (α, β) which maximize $G_n(\alpha, \beta)$. Similary, $(\hat{\alpha}, \hat{\beta})$ is defined to be (α, β) which maximize $f(\alpha, \beta)$.

Theorem 2. *The maximal likelihood estimator $(\hat{\alpha}_n, \hat{\beta}_n)$ exists uniquely.*

Theorem 3. *It holds that $(\hat{\alpha}, \hat{\beta})$ exists uniquely, $(\hat{\alpha}_n, \hat{\beta}_n)$ converges to $(\hat{\alpha}, \hat{\beta})$ as $n \rightarrow \infty$.*

Theorem 4. *The random variable (A, B) converges to $(\hat{\alpha}, \hat{\beta})$ in law.*

Corollary 1 (Lehmann [5], A. Ibragimov and R.Z. Khas'Minskii [10]). *Assume that $t_i/n = p_i + o(n^{-1})$ ($i = 1, \dots, k$) as $n \rightarrow \infty$. Then the distribution of the random variable $((A - \hat{\alpha})/\sqrt{n}, (B - \hat{\beta})/\sqrt{n})$ converges to the 2-dimensional centered normal distribution with the covariance matrix M^{-1} , where*

$$M = \begin{pmatrix} u & v \\ v & w \end{pmatrix}$$

with

$$\begin{aligned} u &= \sum_{i=1}^k \frac{\exp(\hat{\alpha} + \hat{\beta} x_i)}{(1 + \exp(\hat{\alpha} + \hat{\beta} x_i))^2} \\ v &= \sum_{i=1}^k \frac{x_i \exp(\hat{\alpha} + \hat{\beta} x_i)}{(1 + \exp(\hat{\alpha} + \hat{\beta} x_i))^2} \\ w &= \sum_{i=1}^k \frac{x_i^2 \exp(\hat{\alpha} + \hat{\beta} x_i)}{(1 + \exp(\hat{\alpha} + \hat{\beta} x_i))^2}. \end{aligned}$$

The aim of this paper is to justify the Bayesian approach for the logistic parameters by proving the consistency in Theorem 4 and the approximate normality in Corollary 1. The consistency for the natural parameters $e^{\alpha+\beta x_i}/(1 + e^{\alpha+\beta x_i})$ ($i = 1, \dots, k$)

with the uniform distribution on $[0, 1]^k$ as their joint prior distribution is just the law of large number. One of the difficulties in our case is that the prior distribution is not a finite measure, so that we have to start with a condition for the posterior distribution to be a probability measure. As we already remarked, the logistic parameters are sometimes more natural than the natural parameters. This fact is also discussed in [1]. We refer to [2], [3], [4] for the meanings of Bayesian approach. Heberman [6] discussed the logit model with continuum observations, but did not discuss the binary data case which we discuss in this paper. Johan W. Pratt [7] discussed log likelihood for his model, but the model did not contain our case. Cox [9] gave a way to get maximum likelihood estimates, but he did not discuss the existence and uniqueness. Our results contains some of V.T. Farewell [11].

2. Proof of Theorem 1

For a given set of observation points x_i ($i = 1, \dots, n$) and a set of corresponding observations $y_i \in \{0, 1\}$ with $m := \sum_{i=1}^n y_i$ and $M := \sum_{i=1}^n x_i y_i$, we define a subset Ω of \mathbf{R}^2 as the closed convex set generated by the set

$$\left\{ \left(\#\mathcal{S}, \sum_{i \in \mathcal{S}} x_i \right); \mathcal{S} \subset \{1, \dots, n\} \right\}.$$

Let $\partial\Omega$ be the boundary of Ω . Then, the claimed condition in Theorem 1 is equivalent to $P := (m, M) \in \Omega \setminus \partial\Omega$, so that it is sufficient to prove that $c < \infty$ if and only if $P \in \Omega \setminus \partial\Omega$.

We put

$$Q_j = (\alpha_j, \beta_j) := \left(j, \min \left\{ \sum_{i \in \mathcal{S}} x_i ; \#\mathcal{S} = j \right\} \right)$$

for $j = 0, 1, \dots, n$, and

$$Q_j = (\alpha_j, \beta_j) := \left(2n - j, \max \left\{ \sum_{i \in \mathcal{S}} x_i ; \#\mathcal{S} = 2n - j \right\} \right)$$

for $j = n, n+1, \dots, 2n$. Then, it is easy to see that $\partial\Omega$ is the polygon $Q_0 Q_1 \cdots Q_{2n-1} Q_{2n}$ with $Q_{2n} = Q_0$. Let

$$\overrightarrow{PQ_j} = (r_j \cos \theta_j, r_j \sin \theta_j) \quad (j = 0, 1, \dots, 2n-1)$$

with $r_i \geq 0$ and $\theta_0 < \theta_1 < \dots < \theta_{2n-1} < \theta_0 + 2\pi =: \theta_{2n}$.

Now we prove the “if” part. Assume that $P \in \Omega \setminus \partial\Omega$. Since P is in the interior

of the convex set Ω , we have

$$\begin{aligned}\tau &:= \max_{0 \leq j \leq 2n-1} \frac{\theta_{j+1} - \theta_j}{2} < \frac{\pi}{2} \\ c_0 &:= \min_{0 \leq j \leq 2n-1} r_j > 0.\end{aligned}$$

Define

$$\Omega_j := \left\{ (r \cos \phi, r \sin \phi); \frac{\theta_{j-1} + \theta_j}{2} \leq \phi < \frac{\theta_j + \theta_{j+1}}{2}, r > 0 \right\}$$

for $j = 0, 1, \dots, 2n-1$, where $\theta_{-1} := \theta_{2k-1} - 2\pi$. Then it holds that

$$\bigcup_{j=0}^{2n-1} \Omega_j = \mathbf{R}^2 \setminus \{(0, 0)\}$$

and that

$$\begin{aligned}c &= \iint \frac{\exp(\alpha m + \beta M)}{\prod_{i=1}^n (1 + \exp(\alpha + \beta x_i))} d\alpha d\beta \\ &= \iint \frac{\exp(\alpha m + \beta M)}{\sum_S \exp(\alpha \sharp S + \beta \sum_{i \in S} x_i)} d\alpha d\beta \\ &= \sum_{j=0}^{2n-1} \iint_{\Omega_j} \frac{1}{\sum_S \exp((\sharp S - m)\alpha + (\sum_{i \in S} x_i - M)\beta)} d\alpha d\beta \\ &\leq \sum_{j=0}^{2n-1} \iint_{\Omega_j} \exp((m - \alpha_j)\alpha + (M - \beta_j)\beta) d\alpha d\beta.\end{aligned}$$

Since $|\theta_j - \phi| \leq \tau < \pi/2$ for any $(r \cos \phi, r \sin \phi) \in \Omega_j$, we have

$$(\alpha_j - m)\alpha + (\beta_j - M)\beta \geq c_0 r \cos \tau$$

for any $(\alpha, \beta) = (r \cos \phi, r \sin \phi) \in \Omega_j$. Thus,

$$c \leq \sum_{j=0}^{2k-1} \iint_{\Omega_j} \exp(-c_0 r \cos \tau) r dr d\phi < \infty.$$

Now we prove the “only if” part. Assume that $P \in \partial\Omega$. That is, P is one of the vertices of the polygon $Q_0 Q_1 \cdots Q_{2n-1} Q_0$. Let $P = Q_j$ and γ be the angle $Q_{j-1} P Q_{j+1}$ in the region of Ω . Then $\gamma \leq \pi$. Therefore, it is possible to take a half line $l = \{(m + r \cos \theta, M + r \sin \theta); r \geq 0\}$ satisfying that $\angle Q_{j-1} P l \leq \pi/2$ and

$\angle Q_{j+1}Pl \leq \pi/2$. This implies that $\angle Q(S)Pl \leq \pi/2$ for any $S \subset \{1, \dots, n\}$ with $Q(S) := (\#S, \sum_{i \in S} x_i) \neq P$.

Let

$$\Gamma := \{(\alpha, \beta) \in \mathbf{R}^2; |\alpha \sin \theta - \beta \cos \theta| \leq 1, \alpha \cos \theta + \beta \sin \theta < 0\}.$$

Then, for any $(\alpha, \beta) \in \Gamma$ and $S \subset \{1, \dots, n\}$, it holds that

$$\alpha(u - m) + \beta(v - M) \leq \rho,$$

where $(u, v) := Q(S)$ and ρ is the diameter of Ω . Thus, we have

$$\begin{aligned} c &= \iint \frac{\exp(\alpha m + \beta M)}{\prod_{i=1}^n (1 + \exp(\alpha + \beta x_i))} d\alpha d\beta \\ &= \iint \frac{\exp(\alpha m + \beta M)}{\sum_S \exp(\alpha \#S + \beta \sum_{i \in S} x_i)} d\alpha d\beta \\ &= \iint \frac{1}{\sum_S \exp(\alpha(u - m) - \beta(v - M))} d\alpha d\beta \\ &\geq \iint_{\Gamma} \frac{1}{\sum_S \exp(\alpha(u - m) - \beta(v - M))} d\alpha d\beta \\ &\geq \iint_{\Gamma} \frac{1}{2^n e^\rho} d\alpha d\beta \\ &= 2^{-n} e^{-\rho} \iint_{\Gamma} d\alpha d\beta = \infty. \end{aligned}$$

EXAMPLE 1. We consider the case where $x_1 = x_2 = \dots = x_{n_1} = u \neq v = x_{n_1+1} = x_{n_1+2} = \dots = x_{n_1+n_2}$ and

$$\sum_{i=1}^{n_1} y_i = m_1, \quad \sum_{i=n_1+1}^{n_1+n_2} y_i = m_2$$

with $0 < m_1 < n_1$ and $0 < m_2 < n_2$. Then we have

$$\begin{aligned} c &= \iint \frac{\exp(m_1(\alpha + u\beta))}{(1 + \exp(\alpha + u\beta))^{n_1}} \frac{\exp(m_2(\alpha + v\beta))}{(1 + \exp(\alpha + v\beta))^{n_2}} d\alpha d\beta \\ &= \frac{1}{|u - v|} B(n_1 - m_1, m_1) B(n_2 - m_2, m_2). \end{aligned}$$

3. Proof of Theorem 2

Note that

$$\begin{aligned}\frac{\partial G_n}{\partial \alpha} &= \sum_{i=1}^k \left(\frac{t_i}{n} - 1 + \frac{1}{1 + \exp(\alpha + \beta x_i)} \right) =: g_1(\alpha, \beta) \\ \frac{\partial G_n}{\partial \beta} &= \sum_{i=1}^k \left(x_i \left(\frac{t_i}{n} - 1 \right) + \frac{x_i}{1 + \exp(\alpha + \beta x_i)} \right) =: g_2(\alpha, \beta).\end{aligned}$$

Since

$$\begin{aligned}\frac{\partial g_1(\alpha, \beta)}{\partial \alpha} &= - \sum_{i=1}^k \frac{\exp(\alpha + \beta x_i)}{(1 + \exp(\alpha + \beta x_i))^2} < 0 \\ g_1(-\infty, \beta) &= \sum_{i=1}^k \frac{t_i}{n_i} > 0 \\ g_1(\infty, \beta) &= \sum_{i=1}^k \left(\frac{t_i}{n_i} - 1 \right) < 0\end{aligned}$$

for any α, β , there exists a unique $\bar{\alpha} = \bar{\alpha}(\beta)$ for any β such that $g_1(\bar{\alpha}, \beta) \equiv 0$.

Then since

$$\begin{aligned}\frac{d\bar{\alpha}}{d\beta} &= - \frac{\partial g_1 / \partial \beta}{\partial g_1 / \partial \alpha} \\ &= - \frac{\sum_{i=1}^k \{x_i \exp(\bar{\alpha} + \beta x_i) / [1 + \exp(\bar{\alpha} + \beta x_i)]^2\}}{\sum_{i=1}^k \{\exp(\bar{\alpha} + \beta x_i) / [1 + \exp(\bar{\alpha} + \beta x_i)]^2\}},\end{aligned}$$

we have

$$\begin{aligned}\frac{dg_2(\bar{\alpha}, \beta)}{d\beta} &= \frac{\partial g_2}{\partial \alpha} \frac{d\bar{\alpha}}{d\beta} + \frac{\partial g_2}{\partial \beta} \\ &= \left(\sum_{i=1}^k \frac{\exp(\bar{\alpha} + \beta x_i)}{[1 + \exp(\bar{\alpha} + \beta x_i)]^2} \right)^{-2} \\ &\quad \times \left\{ \left(\sum_{i=1}^k \frac{\exp(\bar{\alpha} + \beta x_i)}{[1 + \exp(\bar{\alpha} + \beta x_i)]^2} \right) \right. \\ &\quad \left. \left(\sum_{i=1}^k \frac{x_i^2 \exp(\bar{\alpha} + \beta x_i)}{[1 + \exp(\bar{\alpha} + \beta x_i)]^2} \right) \right. \\ &\quad \left. - \left(\sum_{i=1}^k \frac{x_i \exp(\bar{\alpha} + \beta x_i)}{[1 + \exp(\bar{\alpha} + \beta x_i)]^2} \right)^2 \right\} \\ &< 0\end{aligned}$$

by the Cauchy-Schwarz inequality.

We consider $\bar{\alpha}/\beta$ as $\beta \rightarrow \infty$. Let $p \in [-\infty, +\infty]$ be any one of limit points of $\bar{\alpha}/\beta$ as $\beta \rightarrow \infty$. We denote by $\lim_{\beta* \rightarrow \infty}$ the limit as $\beta \rightarrow \infty$ along a subset such that $\bar{\alpha}/\beta \rightarrow p$.

Case 1: If $-p < x_1$, then

$$\begin{aligned} 0 &= \lim_{\beta* \rightarrow \infty} g_1(\bar{\alpha}, \beta) \\ &= \sum_{i=1}^k \left(\frac{t_i}{n} - 1 \right) < 0, \end{aligned}$$

which is absurd.

Case 2: If $-p > x_k$, then

$$\begin{aligned} 0 &= \lim_{\beta* \rightarrow \infty} g_1(\bar{\alpha}, \beta) \\ &= \sum_{i=1}^k \frac{t_i}{n} > 0, \end{aligned}$$

which is absurd.

Case 3: If there exists x_{i_0} such that $x_{i_0} < -p < x_{i_0+1}$, then we have

$$\begin{aligned} 0 &= \lim_{\beta* \rightarrow \infty} g_1(\bar{\alpha}, \beta) \\ &= \sum_{i=i_0+1}^k \left(\frac{t_i}{n} - 1 \right) + \sum_{i=1}^{i_0} \frac{t_i}{n}. \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{\beta* \rightarrow \infty} g_2(\bar{\alpha}, \beta) &= \sum_{i=i_0+1}^k x_i \left(\frac{t_i}{n} - 1 \right) + \sum_{i=1}^{i_0} x_i \frac{t_i}{n} \\ &< x_{i_0} \left[\sum_{i=i_0+1}^k \left(\frac{t_i}{n} - 1 \right) + \sum_{i=1}^{i_0} \frac{t_i}{n} \right] = 0. \end{aligned}$$

Case 4: If $p = x_{i_0}$ for some $i_0 = 1, 2, \dots, k$, then

$$\begin{aligned} 0 &= \lim_{\beta* \rightarrow \infty} g_1(\bar{\alpha}, \beta) \\ &= \sum_{i=i_0+1}^k \left(\frac{t_i}{n} - 1 \right) + \sum_{i=1}^{i_0-1} \frac{t_i}{n} + \lim_{\beta* \rightarrow \infty} \frac{1}{1 + \exp(\bar{\alpha} + p\beta)}. \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{\beta \rightarrow \infty} g_2(\bar{\alpha}, \beta) &= \\ &\sum_{i=i_0+1}^k x_i \left(\frac{t_i}{n} - 1 \right) + \sum_{i=1}^{i_0-1} x_i \frac{t_i}{n} + x_{i_0} \lim_{\beta \rightarrow \infty} \frac{1}{1 + \exp(\bar{\alpha} + p\beta)} \\ &< x_{i_0} \left[\sum_{i=i_0+1}^k \left(\frac{t_i}{n} - 1 \right) + \sum_{i=1}^{i_0-1} \frac{t_i}{n} + \lim_{\beta \rightarrow \infty} \frac{1}{1 + \exp(\bar{\alpha} + \lambda\beta)} \right] = 0. \end{aligned}$$

Thus, $\lim_{\beta \rightarrow \infty} g_2(\bar{\alpha}, \beta) < 0$.

In the same way, we can prove that $\lim_{\beta \rightarrow -\infty} g_2(\bar{\alpha}, \beta) > 0$. Therefore, there exists a unique $\hat{\beta}_n$ such that $g_2(\bar{\alpha}, \hat{\beta}_n) = 0$. Putting $\hat{\alpha}_n = \bar{\alpha}(\hat{\beta}_n)$, we have proved that $(\hat{\alpha}_n, \hat{\beta}_n)$ is the unique point which maximizes the function $G_n(\alpha, \beta)$.

4. Proof of Theorem 3

The unique existence of $(\hat{\alpha}, \hat{\beta})$ can be proved exactly in the same way as for that of $(\hat{\alpha}_n, \hat{\beta}_n)$.

Let us take $\delta > 0$ and n_0 such that for any $n \geq n_0$,

$$\delta \leq \frac{t_i}{n} \leq 1 - \delta \quad (i = 1, \dots, k).$$

Lemma 1. *Let*

$$\varphi(x, p) := px - \log(1 + e^x)$$

be a function on $x \in \mathbf{R}$ and $p \in \mathbf{R}$ with $0 < \delta \leq p \leq 1 - \delta < 1$ for some $\delta > 0$. Then, we have

$$\begin{aligned} \max_{x \in \mathbf{R}} \varphi(x, p) &= p \log p + (1 - p) \log(1 - p) \\ (i) \quad &\leq \delta \log \delta + (1 - \delta) \log(1 - \delta) < 0, \end{aligned}$$

$$(ii) \quad \max_{\delta \leq p \leq 1 - \delta} \varphi(x, p) \leq -\delta|x|$$

and

$$(iii) \quad \left| \frac{\varphi(x, p')}{\varphi(x, p)} - 1 \right| \leq C|p' - p| \text{ for some constant } C > 0.$$

Proof. (i) Since

$$\frac{\partial \varphi}{\partial x} = p - 1 + \frac{1}{1 + e^x}$$

is a monotone decreasing function in x and takes value 0 at $x = \log p - \log(1 - p)$,

we have

$$\begin{aligned}\max_{x \in \mathbb{R}} \varphi(x, p) &= \varphi(\log p - \log(1-p), p) \\ &= p \log p + (1-p) \log(1-p) \\ &\leq \delta \log \delta + (1-\delta) \log(1-\delta) < 0.\end{aligned}$$

(ii) For any $x \geq 0$, we have

$$\varphi(x, p) \leq px - \log e^x \leq -\delta x.$$

On the other hand, for any $x < 0$, we have

$$\varphi(x, p) \leq px \leq \delta x.$$

Thus we have (ii).

(iii) Since

$$\left| \frac{\partial \log \varphi}{\partial p} \right| = \left| \frac{x}{\varphi} \right| \leq \frac{1}{\delta}$$

by (ii), we have

$$|\log \varphi(x, p') - \log \varphi(x, p)| \leq \frac{1}{\delta} |p' - p|,$$

which implies (iii). □

Lemma 2. *For any $x_i \neq x_j$, there exists a constant $C > 0$ such that*

$$(\alpha + \beta x_i)^2 + (\alpha + \beta x_j)^2 \geq C(\alpha^2 + \beta^2)$$

holds for any α and β .

Proof. We have

$$\begin{aligned}(\alpha + \beta x_i)^2 + (\alpha + \beta x_j)^2 &= 2 \left(\alpha + \beta \frac{x_i + x_j}{2} \right)^2 + 2 \left(\beta \frac{x_i - x_j}{2} \right)^2 \\ &\geq C_1 \beta^2\end{aligned}$$

and

$$(\alpha + \beta x_i)^2 + (\alpha + \beta x_j)^2$$

$$\begin{aligned}
&\geq \frac{x_j^2}{x_i^2 + x_j^2}(\alpha + \beta x_i)^2 + \frac{x_i^2}{x_i^2 + x_j^2}(\alpha + \beta x_j)^2 \\
&= \frac{(\alpha x_j + \beta x_i x_j)^2 + (\alpha x_i + \beta x_i x_j)^2}{x_i^2 + x_j^2} \\
&= \frac{2\{\alpha(x_i - x_j)/2\}^2 + 2\{\alpha(x_i + x_j)/2 + \beta x_i x_j\}^2}{x_i^2 + x_j^2} \\
&\geq C_2 \alpha^2
\end{aligned}$$

with some positive constants C_1 and C_2 . Thus we have

$$(\alpha + \beta x_i)^2 + (\alpha + \beta x_j)^2 \geq C(\alpha^2 + \beta^2)$$

with $C := (1/2) \min\{C_1, C_2\} > 0$. \square

Lemma 3. *There exists a constant $D > 0$ such that*

$$G_n(\alpha, \beta) \leq -D(\alpha^2 + \beta^2)^{1/2}$$

for any $n \geq n_0$ and $(\alpha, \beta) \in \mathbf{R}^2$.

Proof. Since

$$G_n(\alpha, \beta) = \sum_{i=1}^k \varphi\left(\alpha + \beta x_i, \frac{t_i}{n}\right),$$

where φ is defined in Lemma 1, we have

$$\begin{aligned}
G_n(\alpha, \beta) &\leq -\delta(|\alpha + \beta x_i| + |\alpha + \beta x_j|) \\
&\leq -\delta\{(\alpha + \beta x_i)^2 + (\alpha + \beta x_j)^2\}^{1/2} \\
&\leq -\delta C(\alpha^2 + \beta^2)^{1/2} \\
&= -D(\alpha^2 + \beta^2)^{1/2}
\end{aligned}$$

with $D = \delta C$ by Lemmas 1 and 2. \square

Now we shall complete the proof of Theorem 3, since

$$G_n(0, 0) = -k \log 2$$

and by Lemma 3, for any (α, β) with $\alpha^2 + \beta^2 > (k \log 2/C)^2$

$$G_n(\alpha, \beta) < -k \log 2,$$

it holds that

$$\hat{\alpha}_n^2 + \hat{\beta}_n^2 \leq \left(\frac{k \log 2}{C} \right)^2.$$

Since G_n converges to f uniformly in any bounded region as $n \rightarrow \infty$, for any subsequence $\{n'\}$ of $\{n\}$ such that

$$\alpha^* := \lim_{n' \rightarrow \infty} \hat{\alpha}_{n'}, \quad \beta^* := \lim_{n' \rightarrow \infty} \hat{\beta}_{n'}$$

exist, it holds that

$$\begin{aligned} \lim_{n' \rightarrow \infty} G_n(\hat{\alpha}_{n'}, \hat{\beta}_{n'}) &= \lim_{n' \rightarrow \infty} f(\hat{\alpha}_{n'}, \hat{\beta}_{n'}) \\ &= f(\alpha^*, \beta^*) \leq f(\hat{\alpha}, \hat{\beta}). \end{aligned}$$

On the other hand, since

$$\begin{aligned} &|f(\hat{\alpha}, \hat{\beta}) - G_n(\hat{\alpha}_n, \hat{\beta}_n)| \\ &= \left| \max_{\alpha^2 + \beta^2 \leq (k \log 2/D)^2} f(\alpha, \beta) - \max_{\alpha^2 + \beta^2 \leq (k \log 2/D)^2} G_n(\alpha, \beta) \right| \\ &\leq \sup_{\alpha^2 + \beta^2 \leq (k \log 2/D)^2} |f(\alpha, \beta) - G_n(\alpha, \beta)| \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, $f(\alpha^*, \beta^*) = f(\hat{\alpha}, \hat{\beta})$. The uniqueness of the (α, β) which maximizes $f(\alpha, \beta)$ implies that $(\alpha^*, \beta^*) = (\hat{\alpha}, \hat{\beta})$. This also implies that $\hat{\alpha}_n \rightarrow \hat{\alpha}$ and $\hat{\beta}_n \rightarrow \hat{\beta}$ as $n \rightarrow \infty$, which completes the proof.

EXAMPLE 2. For Example 1, we have

$$\begin{aligned} \hat{\alpha}_n &= \frac{v}{v-u} \log \frac{m_1}{n_1 - m_1} + \frac{u}{u-v} \log \frac{m_2}{n_2 - m_2} \\ \hat{\beta}_n &= \frac{1}{u-v} \log \frac{m_1(n_2 - m_2)}{m_2(n_1 - m_1)}. \end{aligned}$$

5. Proof of Theorem 4

Lemma 4. *It holds that*

$$\sum_{i=1}^k \varphi \left(\alpha + \beta x_i, \frac{t_i}{n} \right) = f(\alpha, \beta)(1 + O(\delta_n))$$

where $\delta_n := \max_i |(t_i/n) - p_i|$ and $O(\delta_n)$ is uniform in α and β as $n \rightarrow \infty$.

Proof. Take $\delta > 0$ such that $2\delta < \min_i p_i$ and $\max_i p_i + 2\delta < 1$. Then by (1), there exists n_0 such that for any $n \geq n_0$, it holds that

$$\left| \frac{t_i}{n} - p_i \right| < \delta \quad (i = 1, \dots, k).$$

Then by (iii) of Lemma 1, there exists a constant C such that

$$\varphi \left(\alpha + \beta x_i, \frac{t_i}{n} \right) = \varphi(\alpha + \beta x_i, p_i)(1 + \xi_{i,n})$$

with $|\xi_{i,n}| \leq C|t_i/n - p_i|$ for any $i = 1, \dots, k$. Therefore, we have

$$\sum_{i=1}^k \varphi \left(\alpha + \beta x_i, \frac{t_i}{n} \right) = f(\alpha, \beta)(1 + \xi_n)$$

with

$$|\xi_n| \leq C \max_i \left| \frac{t_i}{n} - p_i \right| = O(\delta_n). \quad \square$$

To prove Theorem 4, it is sufficient to prove that for any given $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \iint_{(\hat{\alpha}-\varepsilon, \hat{\alpha}+\varepsilon) \times (\hat{\beta}-\varepsilon, \hat{\beta}+\varepsilon)} p(\alpha, \beta | t_1, \dots, t_k) d\alpha d\beta = 1.$$

Note that

$$p(\alpha, \beta | t_1, \dots, t_k) = c_n^{-1} \exp \left[n \sum_{i=1}^k \varphi \left(\alpha + \beta x_i, \frac{t_i}{n} \right) \right]$$

with

$$c_n := \iint \exp \left[n \sum_{i=1}^k \varphi \left(\alpha + \beta x_i, \frac{t_i}{n} \right) \right] d\alpha d\beta.$$

By Theorem 3, Lemmas 1 and 3,

$$(1) \quad \begin{aligned} \max_{(\alpha, \beta) \in \mathbf{R}^2} f(\alpha, \beta) &= f(\hat{\alpha}, \hat{\beta}) < 0 \\ \lim_{\alpha^2 + \beta^2 \rightarrow \infty} f(\alpha, \beta) &= -\infty. \end{aligned}$$

For any $\Delta > 0$, let

$$\Omega(\Delta) := \{(\alpha, \beta) \in \mathbf{R}^2; f(\alpha, \beta) > \Lambda - \Delta\},$$

where we put $\Lambda := f(\hat{\alpha}, \hat{\beta})$. Since by Theorem 3, $(\hat{\alpha}, \hat{\beta})$ is the unique point which maximizes f together with (3) and the fact that f is continuous, we can take Δ such that

$$(2) \quad \Omega(5\Delta) \subset (\hat{\alpha} - \varepsilon, \hat{\alpha} + \varepsilon) \times (\hat{\beta} - \varepsilon, \hat{\beta} + \varepsilon).$$

Since $\Omega(\Delta)$ is a nonempty bounded open set, it has a positive area, say $S > 0$. Moreover, by (1) and Lemma 4, there exists n_1 such that for any $n \geq n_1$ and $(\alpha, \beta) \in \Omega(\Delta)$,

$$\sum_{i=1}^k \varphi \left(\alpha + \beta x_i, \frac{t_i}{n} \right) > \Lambda - 2\Delta.$$

Hence for any $n \geq n_1$, we have

$$(3) \quad \iint_{\Omega(\Delta)} \exp \left[n \sum_{i=1}^k \varphi \left(\alpha + \beta x_i, \frac{t_i}{n} \right) \right] d\alpha d\beta \geq e^{(\Lambda - 2\Delta)n} S.$$

On the other hand, by (1), (2), (3) and Lemma 1, there exists n_2 such that for any $n \geq n_2$ and $(\alpha, \beta) \notin \Omega(5\Delta)$,

$$\sum_{i=1}^k \varphi \left(\alpha + \beta x_i, \frac{t_i}{n} \right) < \Lambda - 4\Delta.$$

Also by (1), Lemmas 3 and 4, there exists n_3 such that for any $n \geq n_3$ and $(\alpha, \beta) \in \mathbf{R}^2$,

$$\sum_{i=1}^k \varphi \left(\alpha + \beta x_i, \frac{t_i}{n} \right) < \frac{1}{2} f(\alpha, \beta) \leq -\frac{1}{2} C(\alpha^2 + \beta^2)^{1/2}.$$

Hence, for any η with $0 < \eta < 1$, $(\alpha, \beta) \notin \Omega(5\Delta)$, and $n \geq n_4 := n_2 \vee n_3$ we have

$$\sum_{i=1}^k \varphi \left(\alpha + \beta x_i, \frac{t_i}{n} \right) \leq -\frac{1}{2} C \eta (\alpha^2 + \beta^2)^{1/2} + (1 - \eta)(\Lambda - 4\Delta).$$

Therefore, taking a small $\eta > 0$ such that

$$(1 - \eta)(\Lambda - 4\Delta) < \Lambda - 3\Delta,$$

we have

$$\sum_{i=1}^k \varphi \left(\alpha + \beta x_i, \frac{t_i}{n} \right) \leq -C'(\alpha^2 + \beta^2)^{1/2} + \Lambda - 3\Delta$$

for any $(\alpha, \beta) \notin \Omega(5\Delta)$ and $n \geq n_4$ with some constant $C' > 0$. Hence, we have

$$\begin{aligned}
 & \iint_{\mathbf{R}^2 \setminus \Omega(5\Delta)} \exp \left[n \sum_{i=1}^k \varphi \left(\alpha + \beta x_i, \frac{t_i}{n} \right) \right] d\alpha d\beta \\
 & \leq \iint \exp \left[-C'n(\alpha^2 + \beta^2)^{1/2} + (\Lambda - 3\Delta)n \right] d\alpha d\beta \\
 & \leq e^{(\Lambda-3\Delta)n} \iint \exp \left[-C'(\alpha^2 + \beta^2)^{1/2} \right] d\alpha d\beta \\
 (4) \quad & \leq C'' e^{(\Lambda-3\Delta)n}
 \end{aligned}$$

for any $n \geq n_4$ with some constant $C'' > 0$.

Let

$$I_n := \iint_{(\hat{\alpha}-\varepsilon, \hat{\alpha}+\varepsilon) \times (\hat{\beta}-\varepsilon, \hat{\beta}+\varepsilon)} p(\alpha, \beta | t_1, \dots, t_k) d\alpha d\beta.$$

Then by (4), we have

$$\begin{aligned}
 I_n & \geq \iint_{\Omega(5\Delta)} p(\alpha, \beta | t_1, \dots, t_k) d\alpha d\beta \\
 & = c_n^{-1} \iint_{\Omega(5\Delta)} \exp \left[n \sum_{i=1}^k \varphi \left(\alpha + \beta x_i, \frac{t_i}{n} \right) \right] d\alpha d\beta.
 \end{aligned}$$

Putting

$$(5) \quad J(j) := \iint_{\Omega(j\Delta)} \exp \left[n \sum_{i=1}^k \varphi \left(\alpha + \beta x_i, \frac{t_i}{n} \right) \right] d\alpha d\beta$$

and

$$(6) \quad L(j) := \iint_{\mathbf{R}^2 \setminus \Omega(j\Delta)} \exp \left[n \sum_{i=1}^k \varphi \left(\alpha + \beta x_i, \frac{t_i}{n} \right) \right] d\alpha d\beta,$$

we have

$$\begin{aligned}
 I_n & \geq c_n^{-1} J(5) = \frac{J(5)}{J(5) + L(5)} \\
 & \geq \frac{J(1)}{J(1) + L(5)} = \frac{1}{1 + \{L(5)/J(1)\}}.
 \end{aligned}$$

Let $n_0 := n_1 \vee n_4$. Then, for any $n \geq n_0$, we have by (5) and (6) that

$$J(1) \geq e^{(\Lambda-2\Delta)n} S \text{ and } L(5) \leq C'' e^{(\Lambda-3\Delta)n}.$$

Thus,

$$I_n \geq \frac{1}{1 + C'' S^{-1} e^{-\Delta n}}$$

from which $\lim_{n \rightarrow \infty} I_n = 1$ follows. \square

Lehmann gave conditions B(1)–B(4) for the asymptotic normality in [5]. The condition B(1) follows Theorem 4, the other conditions B(2)–B(4) are verified easily. Thus we have Corolloary 1.

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