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ON THE ISOMORPHISM CLASSES OF IWASAWA MODULES WITH $\lambda = 3$ AND $\mu = 0$

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Abstract

For an odd prime number p, we classify the isomorphism classes of finitely generated torsion $\Lambda = \mathbb{Z}_p[[T]]$-modules with $\lambda = 3$ and $\mu = 0$, which are free over $\mathbb{Z}_p$. We apply this classification to the Iwasawa module associated to the cyclotomic $\mathbb{Z}_p$-extension of an imaginary quadratic field.

1. Introduction

Let p be a fixed odd prime number and $\Lambda = \mathbb{Z}_p[[T]]$ the ring of power series in one variable over $\mathbb{Z}_p$. In the classical Iwasawa theory, one studies Iwasawa modules up to pseudo-isomorphism. In this paper, we study Iwasawa modules up to Λ-isomorphism. Especially, our aim is to generalize Sumida's results (cf. [11], [12]).

For a distinguished polynomial $f(T) \in \mathbb{Z}_p[T]$, Sumida introduced the set

$$\mathcal{M}_{f(T)} = \left\{ [M]_{\mathbb{Q}_p} \;\middle|\; \begin{array}{l} M \text{ is a finitely generated torsion } \Lambda\text{-module,} \\ \operatorname{char}(M) = (f(T)) \text{ and } M \text{ is free over } \mathbb{Z}_p \end{array} \right\},$$

where $[M]_{\mathbb{Q}_p}$ is the Λ-isomorphism class of M and $\operatorname{char}(M)$ is the characteristic ideal of M. Sumida showed that $\mathcal{M}_{f(T)}$ is a finite set if and only if $f(T)$ is a separable polynomial ([11], Theorem 2). Sumida and Koike determined $\mathcal{M}_{f(T)}$ in the case $\deg(f(T)) \leq 2$ ([7], Theorem 2.1 and [11], Proposition 10). In this paper, we determine the set $\mathcal{M}_{f(T)}$ for

$$f(T) = (T - \alpha)(T - \beta)(T - \gamma),$$

where α, β, γ are distinct elements of $p\mathbb{Z}_p$ (Theorem 3.5). This is a generalization of Sumida's result [12]. (Precisely speaking, we work over $\mathcal{O}[[T]]$ below where $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbb{Q}_p$.)

The motivation of this work lies in Iwasawa theory. We apply our theorem to the Iwasawa module associated to the cyclotomic $\mathbb{Z}_p$-extension of an imaginary quadratic field. Let k be an imaginary quadratic number field and k_∞/k the cyclotomic $\mathbb{Z}_p$-extension of k. For each $n \geq 0$, we denote by k_n the unique intermediate field of

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k_∞/k with $[k_n : k] = p^n$. Let A_n be the p-Sylow subgroup of the ideal class group of k_n. We put $X = \varprojlim A_n$, where the inverse limit is taken with respect to the relative norms. It is known that X is a finitely generated torsion Λ-module (cf. [5]). Moreover, it is known that X is a free $\mathbb{Z}_p$-module.

Therefore, we can apply our theorem to the Iwasawa module X. We apply our theorem in the case that $p = 3$ and $k = \mathbb{Q}(\sqrt{-d})$. In this setting, $f(T)$ can be approximately calculated by the p-adic L-functions (see Section 6).

The outline of this paper is as follows. Let E be a finite extension of $\mathbb{Q}_p$ and Λ_E the ring of power series in one variable over the ring of integers of E. In Section 2, we introduce the set $\mathcal{M}^E_{f(T)}$ which is the set of isomorphism classes of Λ_E-module satisfying some properties. In Section 3, we state our main theorem (Theorem 3.5). We define a certain equivalence relation $\sim'$ on $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathcal{O}_E$ and define $Z' = (\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathcal{O}_E)/\sim'$. We define Z to be a subset of Z' satisfying certain conditions. An element of Z' is written as $\overline{(m, n, x)}$. We also define an equivalence relation $\sim$ on Z and consider $Z/\sim$. An element of $Z/\sim$ is written as $[\overline{(m, n, x)}]$. Roughly speaking, Theorem 3.5 states that there is one to one correspondence between $\mathcal{M}^E_{f(T)}$ and the equivalence classes of $Z/\sim$. Moreover, we prove Sumida's result ([12], Theorem 1) in Corollary 3.8, using our Theorem 3.5. In Section 4, we give a proof of Theorem 3.5. Section 5 is a preparation for Section 6. In this section, we study the structure of Λ-modules. In Section 6, we apply Theorem 3.5 to the Iwasawa module associated to the cyclotomic $\mathbb{Z}_p$-extension of an imaginary quadratic number field. We apply our theorem in the case that $p = 3$ and $k = \mathbb{Q}(\sqrt{-d})$ for all d such that $1 < d < 10^5$ and $d \not\equiv 2 \bmod 3$, that is to say p does not split in k. We determine the Λ-isomorphism class of the Iwasawa module associated to k in the case $\lambda_p(k) = 3$, where $\lambda_p(k)$ is the Iwasawa λ-invariant. There are 1109 imaginary quadratic fields satisfying these properties. Among them, there are 1015 fields whose A_0 are cyclic groups. We can determine $[X]_{\mathbb{Q}_p}$ for these 1015 fields by Proposition 5.2 immediately. For remaining 94 fields whose A_0 are not cyclic groups, there are 66 fields whose $f(T)$ is reducible. We determine $[X]_{\mathbb{Q}_p}$ for these 66 fields.

After I submitted this paper, I was informed from Sumida (Takahashi) of the thesis by C. Franks where he independently obtained the classification of Λ-modules. In Remark 3.6, I will explain the difference between our method and that in Franks.

2. Preliminaries

Let p be an odd prime number. Let E be a finite extension over the field $\mathbb{Q}_p$ of p-adic numbers. Let $\mathcal{O}_E$, π, ord_E be the ring of integers in E, a prime element and the normalized additive valuation of E such that $\mathrm{ord}_E(\pi) = 1$, respectively. We put $\Lambda_E := \mathcal{O}_E[[T]]$ the ring of power series over $\mathcal{O}_E$.

Let M be a finitely generated torsion Λ_E-module. By the structure theorem of Λ_E-modules (cf. [13], Chapter 13), there is a Λ_E-homomorphism

$$\varphi\colon M \to \left(\bigoplus_i \Lambda_E/(\pi^{m_i})\right) \oplus \left(\bigoplus_j \Lambda_E/(f_j(T)^{n_j})\right)$$

with finite kernel and finite cokernel, where m_i, n_j are non-negative integers and $f_j(T) \in \mathcal{O}_E[T]$ is a distinguished irreducible polynomial. We put

$$\mathrm{char}(M) = \left(\prod_i \pi^{m_i} \prod_j f_j(T)^{n_j}\right)$$

which is an ideal in Λ_E. We define $[M]_E$ to be the Λ_E-isomorphism class of M.

As in the introduction, for a distinguished polynomial $f(T) \in \mathcal{O}_E[T]$, we consider finitely generated torsion Λ_E-modules whose characteristic ideals are $(f(T))$, and define the set $\mathcal{M}^E_{f(T)}$ by

$$\mathcal{M}^E_{f(T)} = \left\{[M]_E \;\middle|\; \begin{array}{l} M \text{ is a finitely generated torsion } \Lambda_E\text{-module,} \\ \mathrm{char}(M) = (f(T)) \text{ and } M \text{ is free over } \mathcal{O}_E \end{array}\right\}. \tag{1}$$

Sumida showed that $\mathcal{M}^E_{f(T)}$ is a finite set if and only if $f(T)$ is separable [11]. Here, we say $f(T)$ is separable when $f(T)$ has no multiple roots in an algebraic closure of E. Sumida also determined $\mathcal{M}_{f(T)}$ in the case $f(T) = (T-\alpha)(T-\beta)(T-\gamma)$, where $\alpha, \beta, \gamma \in p\mathbb{Z}_p$ satisfy $\alpha \not\equiv \beta$, $\beta \not\equiv \gamma$, $\gamma \not\equiv \alpha \bmod p^2$ (see [12], Theorem 1). We generalize this result to a general separable polynomial $f(T)$ with degree 3.

Now we put

$$f(T) = (T-\alpha)(T-\beta)(T-\gamma), \tag{2}$$

where α, β and γ are distinct elements of $\pi\mathcal{O}_E$. We determine all the elements of $\mathcal{M}^E_{f(T)}$ in the next section.

Let $[M]_E \in \mathcal{M}^E_{f(T)}$. Since M has no non-trivial finite Λ_E-submodule, there exists an injective Λ_E-homomorphism

$$\varphi\colon M \hookrightarrow \Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\beta) \oplus \Lambda_E/(T-\gamma) =: \mathcal{E}$$

with finite cokernel. We write $\mathcal{E}$ for the right hand side. The above fact implies that every class of $\mathcal{M}^E_{f(T)}$ can be represented by a Λ_E-submodule of $\mathcal{E}$.

Now we fix a notation to express such submodules in $\mathcal{E}$. First, by using the canonical isomorphism $\Lambda_E/(T-\alpha) \cong \mathcal{O}_E$ $(f(T) \mapsto f(\alpha))$, we define an isomorphism

$$\iota\colon \mathcal{E} = \Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\beta) \oplus \Lambda_E/(T-\gamma) \to \mathcal{O}_E^{\oplus 3}$$

by $(f_1(T), f_2(T), f_3(T)) \mapsto (f_1(\alpha), f_2(\beta), f_3(\gamma))$. We identify $\mathcal{E}$ with $\mathcal{O}_E^{\oplus 3}$ via ι. Thus an element in $\mathcal{E}$ is expressed as $(a_1, a_2, a_3) \in \mathcal{O}_E^{\oplus 3}$. Since the rank of M is equal to 3, we can write M in the form

$$M = \langle (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3) \rangle_{\mathcal{O}_E} \subset \mathcal{E},$$

where $\langle * \rangle_{\mathcal{O}_E}$ is the $\mathcal{O}_E$-submodule generated by $*$. Further, we can express the action of T by

$$T(a_1, a_2, a_3) = (\alpha a_1, \beta a_2, \gamma a_3),$$

using this notation.

3. Main result

Let M be an $\mathcal{O}_E$-submodule of $\mathcal{E}$ with $\mathrm{rank}(M) = 3$ of the form

$$M = \langle (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3) \rangle_{\mathcal{O}_E} \subset \mathcal{E}.$$

Let

$$\begin{aligned} s &= \min\{i \in \mathbb{Z}_{\geq 0} \mid \exists a, b \in \mathcal{O}_E \text{ s.t. } (\pi^i, a, b) \in M\}, \\ t &= \min\{i \in \mathbb{Z}_{\geq 0} \mid \exists c \in \mathcal{O}_E \text{ s.t. } (0, \pi^i, c) \in M\}, \\ u &= \min\{i \in \mathbb{Z}_{\geq 0} \mid (0, 0, \pi^i) \in M\}. \end{aligned}$$

Then we have

$$M = \langle (\pi^s, a, b), (0, \pi^t, c), (0, 0, \pi^u) \rangle_{\mathcal{O}_E}.$$

Suppose $(a_1, a_2, a_3) \in M$. Since $\mathrm{ord}_E(a_1) \geq s$, there exists $x \in \mathcal{O}_E$ such that $a_1 = x\pi^s$. So $(a_1, a_2, a_3) - x(\pi^s, a, b) = (0, a_2 - xa, a_3 - xb) \in M$. Since $\mathrm{ord}_E(a_2 - xa) \geq t$, there exists $y \in \mathcal{O}_E$ such that $a_2 - xa = y\pi^t$. Similarly by the same method as above, we get $(0, 0, a_3 - xb - yc) \in M$. Finally, there exists $z \in \mathcal{O}_E$ such that $a_3 - xb - yc = z\pi^u$. Then we have $(a_1, a_2, a_3) = x(\pi^s, a, b) + y(0, \pi^t, c) + z(0, 0, \pi^u)$.

The following lemma is a necessary and sufficient condition for an $\mathcal{O}_E$-module M to be a Λ_E-submodule.

Lemma 3.1. *Let $M = \langle (\pi^s, a, b), (0, \pi^t, c), (0, 0, \pi^u) \rangle_{\mathcal{O}_E}$. Then the following two statements are equivalent*:

(i) *The $\mathcal{O}_E$-module M is a Λ_E-submodule,*

(ii) *Integers a, b, c, s, t and u satisfy*

$$\begin{cases} t \leq \mathrm{ord}_E(\beta - \alpha) + \mathrm{ord}_E(a), \\ u \leq \mathrm{ord}_E\{(\gamma - \alpha)b - (\beta - \alpha)\pi^{-t}ac\}, \\ u \leq \mathrm{ord}_E(\gamma - \beta) + \mathrm{ord}_E(c). \end{cases}$$

Proof. We first suppose that M is a Λ_E-submodule. So M satisfies $TM \subset M$ and we have

$$\begin{aligned} T(\pi^s, a, b) &= (\alpha\pi^s, \beta a, \gamma b) \\ &= \alpha(\pi^s, a, b) + (\beta - \alpha)\pi^{-t}a(0, \pi^t, c) \\ &\quad + \{(\gamma - \alpha)b - (\beta - \alpha)\pi^{-t}ac\}\pi^{-u}(0, 0, \pi^u), \\ T(0, \pi^t, c) &= (0, \beta\pi^t, \gamma c) \\ &= \beta(0, \pi^t, c) + (\gamma - \beta)c\pi^{-u}(0, 0, \pi^u). \end{aligned}$$

Since these coefficients belong to $\mathcal{O}_E$, we get (ii). Conversely, if an $\mathcal{O}_E$-module M satisfies (ii), M is naturally an $\mathcal{O}_E[T]$-module by the action as above. We show that M becomes a Λ_E-module. For a positive integer n, we put $v_n = \sum_{k=0}^{n} d_k T^k \in \mathcal{O}_E[T]$ and $v = \sum_{n=0}^{\infty} d_n T^n \in \mathcal{O}_E[[T]]$. Then we have

$$\begin{aligned} & v_n(\pi^s, a, b) \\ &= \left(\pi^s \sum_{k=0}^{n} d_k\alpha^k, a \sum_{k=0}^{n} d_k\beta^k, b \sum_{k=0}^{n} d_k\gamma^k\right) \\ &= \sum_{k=0}^{n} d_k\alpha^k(\pi^s, a, b) + a\left(\sum_{k=0}^{n} d_k\beta^k - \sum_{k=0}^{n} d_k\alpha^k\right)\pi^{-t}(0, \pi^t, c) \\ &\quad + \left\{b\left(\sum_{k=0}^{n} d_k\gamma^k - \sum_{k=0}^{n} d_k\alpha^k\right) - \left(\sum_{k=0}^{n} d_k\beta^k - \sum_{k=0}^{n} d_k\alpha^k\right)\pi^{-t}ac\right\}\pi^{-u}(0, 0, \pi^u). \end{aligned}$$

Because M is an $\mathcal{O}_E[T]$-module, we have $v_n(\pi^s, a, b) \in M$. Thus we obtain

$$a\left(\sum_{k=0}^{n} d_k\beta^k - \sum_{k=0}^{n} d_k\alpha^k\right)\pi^{-t} \in \mathcal{O}_E$$

and

$$\left\{b\left(\sum_{k=0}^{n} d_k\gamma^k - \sum_{k=0}^{n} d_k\alpha^k\right) - \left(\sum_{k=0}^{n} d_k\beta^k - \sum_{k=0}^{n} d_k\alpha^k\right)\pi^{-t}ac\right\}\pi^{-u} \in \mathcal{O}_E.$$

Since $d_k\alpha^k, d_k\beta^k, d_k\gamma^k \to 0$ $(k \to \infty)$, $\sum_{k=0}^{\infty} d_k\alpha^k$, $\sum_{k=0}^{\infty} d_k\beta^k$ and $\sum_{k=0}^{\infty} d_k\gamma^k$ converge in $\mathcal{O}_E$. Thus we have $v(\pi^s, a, b) \in M$. For $(0, \pi^t, c)$ and $(0, 0, \pi^u)$, we can define the action of the elements of Λ_E by the same method as above. □

We use the following lemma to fix a representative of the Λ_E-isomorphism class of M.

Lemma 3.2 (Lemma 1 in Sumida [12]). *Let* $M = \langle(a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3)\rangle_{\mathcal{O}_E}$ *be a* Λ_E*-submodule of* $\mathcal{E}$ *and* $u_1, u_2, u_3 \in \mathcal{O}_E \setminus \{0\}$*. Then we have*

$$M \cong \langle(u_1a_1, u_2a_2, u_3a_3), (u_1b_1, u_2b_2, u_3b_3), (u_1c_1, u_2c_2, u_3c_3)\rangle_{\mathcal{O}_E}$$

as Λ_E*-modules.*

We take M to be a Λ_E-submodule of $\mathcal{E}$ with finite index. Then we can write

$$M = \langle(\pi^s, a, b), (0, \pi^t, c), (0, 0, \pi^u)\rangle_{\mathcal{O}_E}$$

as we explained in the beginning of this section. By Lemma 3.2, there exist non-negative integers m, n and $x \in \mathcal{O}_E$ such that there is an isomorphism

$$M \cong \langle(1, 1, 1), (0, \pi^m, x), (0, 0, \pi^n)\rangle_{\mathcal{O}_E}$$

as Λ_E-modules. In fact, by Lemma 3.2, M is isomorphic to $M' = \langle(1, a, b), (0, \pi^t, c), (0, 0, \pi^u)\rangle_{\mathcal{O}_E}$. In the case $\mathrm{ord}_E(a) \leq t$, by Lemma 3.2, M is isomorphic to $\langle(1, 1, b), (0, a^{-1}\pi^t, c), (0, 0, \pi^u)\rangle_{\mathcal{O}_E}$. On the other hand, in the case $\mathrm{ord}_E(a) > t$, since $M' = \langle(1, a+\pi^t, b+c), (0, \pi^t, c), (0, 0, \pi^u)\rangle_{\mathcal{O}_E}$, we can proceed by the same method as in the case $\mathrm{ord}_E(a) \leq t$. Therefore M is isomorphic to $M'' = \langle(1,1,b), (0, a'\pi^t, c), (0,0,\pi^u)\rangle_{\mathcal{O}_E}$ for some $a' \in E$. By applying the same method as above, M'' is isomorphic to $\langle(1,1,1), (0, \pi^m, x), (0, 0, \pi^n)\rangle_{\mathcal{O}_E}$ for some non-negative integers m, n and $x \in \mathcal{O}_E$.

We define $M(m, n, x)$ by

$$M(m, n, x) := \langle(1, 1, 1), (0, \pi^m, x), (0, 0, \pi^n)\rangle_{\mathcal{O}_E} \subset \mathcal{E}.$$

Proposition 3.3. *Let* $f(T) \in \mathcal{O}_E[T]$ *be a distinguished polynomial. Then we have*

$$\mathcal{M}^E_{f(T)} = \{[M(m, n, x)]_E \mid m,\ n,\ x \textit{ satisfy } (*)\},$$

where $[M(m,n,x)]_E$ *is the* Λ_E*-isomorphism class of* $M(m,n,x)$ *and* $(*)$ *is as follows*:

$$(*) \qquad \begin{cases} (A) & 0 \leq m \leq \mathrm{ord}_E(\beta - \alpha), \\ (B) & 0 \leq n \leq \mathrm{ord}_E(\gamma - \beta) + \mathrm{ord}_E(x), \\ (C) & n \leq \mathrm{ord}_E\{(\gamma - \alpha) - (\beta - \alpha)\pi^{-m}x\}. \end{cases}$$

Proof. Let M be a Λ_E-module such that $[M]_E \in \mathcal{M}^E_{f(T)}$. Then we saw that $[M]_E = [M(m,n,x)]_E$ for some m, n, x satisfying $(*)$ by Lemma 3.1. We will show the converse. We suppose that m,n and x satisfy $(*)$. By Lemma 3.1, $M(m,n,x)$ becomes a finitely generated Λ_E-module. Since $f(T) = (T-\alpha)(T-\beta)(T-\gamma)$ annihilates $M(m,n,x)$, $M(m,n,x)$ is a torsion Λ_E-module. Moreover, by the definition of $M(m, n, x)$, $M(m, n, x)$ is a free $\mathcal{O}_E$-module. Finally we show that $\mathrm{char}(M(m,n,x)) = (f(T))$. The Λ_E-module $M(m,n,x)$

is a submodule of $\mathcal{E}$ with finite index. In fact, since $\mathrm{rank}_{\mathcal{O}_E}(\mathcal{E}) = \mathrm{rank}_{\mathcal{O}_E}(M(m,n,x)) = 3$, $\mathcal{E}/M(m, n, x)$ is finite. This implies that $\mathrm{char}(M(m, n, x)) = \mathrm{char}(\mathcal{E})$. Thus we get $[M(m, n, x)]_E \in \mathcal{M}^E_{f(T)}$. □

REMARK 3.4. (i) If $x \equiv x' \bmod \pi^n$, we have $M(m, n, x) = M(m, n, x')$ since $(0, \pi^m, x) = (0, \pi^m, x') + a(0, 0, \pi^n)$ for some $a \in \mathcal{O}_E$. In particular, if $\mathrm{ord}_E(x) \geq n$, we have $M(m, n, x) = M(m, n, 0)$. This means that we may assume that $x = 0$ or $\mathrm{ord}_E(x) < n$.
(ii) We have

$$\frac{(\gamma-\alpha)(\gamma-\beta)}{\pi^n} = \frac{(\gamma-\beta)x}{\pi^n} \cdot \frac{\beta-\alpha}{\pi^m} + (\gamma-\beta) \cdot \frac{(\gamma-\alpha)-(\beta-\alpha)\pi^{-m}x}{\pi^n}.$$

Therefore if $(*)$ holds, we get

$$0 \leq n \leq \mathrm{ord}_E(\gamma-\alpha) + \mathrm{ord}_E(\gamma-\beta).$$

Let $M(m, n, x)$ and $M(m', n', x') \in \mathcal{M}^E_{f(T)}$. We will investigate a relation among m, m', n, n', x and x' when $M(m,n,x)$ is isomorphic to $M(m',n',x')$ as a Λ_E-module. We note that we may assume $x = 0$ or $\mathrm{ord}_E(x) < n$ by Remark 3.4 (i).

First of all, we prepare some notation. For (m,n,x) and $(m',n',x') \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathcal{O}_E$, we define

$$(m, n, x) \sim' (m', n', x') \iff m = m',\ n = n' \quad \text{and} \quad x \equiv x' \mod \pi^n\mathcal{O}_E.$$

We put $Z' := (\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathcal{O}_E)/\sim'$ and introduce a set

$$Z := \{\overline{(m, n, x)} \in Z' \mid m,\ n,\ x \text{ satisfy } (*)\}, \tag{3}$$

where $(*)$ is the inequalities (A), (B) and (C) in Proposition 3.3 and $\overline{(m, n, x)}$ is the equivalence class of (m, n, x). The class $\overline{(m, n, x)}$ is determined by m, n and x mod $\pi^n\mathcal{O}_E$. We note that the condition $(*)$ does not depend on the choice of a representative of (m, n, x).

For an element for $x \in \mathcal{O}_E$ and $z = \overline{x} \in \mathcal{O}_E/\pi^n\mathcal{O}_E$, we define $\mathrm{ord}_E(z) = \mathrm{ord}_E(x \bmod \pi^n)$ as follows:

$$\mathrm{ord}_E(z) := \begin{cases} \mathrm{ord}_E(x) & \text{if} \quad \overline{x} \neq 0, \\ \infty & \text{if} \quad \overline{x} = 0. \end{cases}$$

For $\overline{(m, n, x)}$ and $\overline{(m', n', x')} \in Z$, let $k = \mathrm{ord}_E(x \bmod \pi^n)$ and $l = \mathrm{ord}_E(x' - \pi^m)$. We define $\overline{(m, n, x)} \sim \overline{(m', n', x')}$ as follows.

(I) Suppose $m \neq 0$.

(a) When $l + k \geq n$, we define

$$\overline{(m, n, x)} \sim \overline{(m', n', x')} \iff m = m',\ n = n' \quad \text{and} \quad \overline{x} = \overline{x'} \quad \text{in} \quad \mathcal{O}_E/\pi^n\mathcal{O}_E.$$

(b) When $l+k<n$, we define

$$\overline{(m,n,x)} \sim \overline{(m',n',x')} \Longleftrightarrow m=m',\ n=n' \quad \text{and}$$
$$\overline{x}=\varepsilon\overline{x'} \quad \text{in} \quad \mathcal{O}_E/\pi^n\mathcal{O}_E \quad \text{for some} \quad \varepsilon \in 1+\pi^l\mathcal{O}_E.$$

(II) Suppose $m=0$. We define

$$\overline{(m,n,x)} \sim \overline{(m',n',x')} \Longleftrightarrow m=m'=0, \quad n=n',$$
$$\mathrm{ord}_E(x \bmod \pi^n)=\mathrm{ord}_E(x' \bmod \pi^n) \quad \text{and}$$
$$\overline{1-x}=\varepsilon\overline{(1-x')} \quad \text{in} \quad \mathcal{O}_E/\pi^n\mathcal{O}_E \quad \text{for some} \quad \varepsilon \in \mathcal{O}_E^\times.$$

Here, for $s \le 0$, we define $1+\pi^s\mathcal{O}_E=\mathcal{O}_E^\times$. We can prove that $\sim$ is an equivalence relation. The following is our main theorem. We will prove this theorem in the next section.

Theorem 3.5. *There is a bijection* Φ:

$$\begin{array}{ccc} \mathcal{M}^E_{f(T)} & \longrightarrow & Z/\sim \\ \Psi & & \Psi \\ [M(m,n,x)]_E & \longmapsto & [\overline{(m,n,x)}], \end{array}$$

where $\mathcal{M}^E_{f(T)}$ *is defined by* (1) *in Section* 2, Z *is defined by* (3) *after* Remark 3.4, *and* $\sim$ *is the equivalence relation of* Z *defined above.* $[M(m,n,x)]_E$ *is the class of* $M(m,n,x)$ *defined by* Proposition 3.3 *and* $[\overline{(m,n,x)}]$ *is the class of* $\overline{(m,n,x)}$.

REMARK 3.6. After we submitted this paper, we learned from Sumida the existence of the thesis by Chase Franks where he independently classified the isomorphism classes of Λ-modules with $\lambda=3$. He also gave an algorithm to determine the Λ-isomorphism classes for any separable $f(T)$ which has arbitrary degree [2]. His method is essentially the same as our paper, but there are some differences which we will explain here.

1. We give in this paper an explicit method to compute m and n using the action of $T-\alpha$, $T-\beta$ etc. (cf. Lemma 4.1).

2. Our inequalities about orders of p-adic numbers ((5), (6), (7) in Section 4) are obtained from a different point of view from Franks'. He did not solve completely his equations which are essentially equivalent to our inequalities, but we solved our inequalities completely in the case $\lambda=3$.

3. We explicitly give a subgroup $H \subset \mathbb{Z}_p^\times$ such that $M(m,n,x) \cong M(m',n',x')$ if and only if $m=m'$, $n=n'$ and $x/x' \in H$ (H depends on $\mathrm{ord}_p(x)$). Also, we use the higher Fitting ideals (cf. Section 5 and 6). This is a different argument from Franks'.

4. As an application, we apply our classification to the Iwasawa module associated to the cyclotomic $\mathbb{Z}_p$-extension of an imaginary quadratic field (cf. Section 6). On the other hand, Franks determined the isomorphism class of the Pontryagin dual of the p-Selmer group of elliptic curves over the cyclotomic $\mathbb{Z}_p$-extension for $\lambda = 2$.

5. Franks' method has some merits. He gave an algorithm to decide whether two Λ-modules are isomorphic or not. This algorithm is to check whether some matrices he defined belong to $\mathrm{GL}_\lambda(\mathbb{Z}_p)$. This algorithm works for arbitrary λ and separable $f(T)$.

REMARK 3.7. When $\overline{(m, n, x)} \sim \overline{(m', n', x')}$ and $l+k \le n$, we have $l = \mathrm{ord}_E(x' - \pi^m) = \mathrm{ord}_E(x - \pi^m)$.

Sumida determined all elements of $\mathcal{M}_{f(T)}$ for $f(T) = (T-\alpha)(T-\beta)(T-\gamma)$ and $\mathrm{ord}_p(\alpha-\beta) = \mathrm{ord}_p(\beta-\gamma) = \mathrm{ord}_p(\gamma-\alpha) = 1$ ([12], Theorem 1). We can also obtain the same result from our Theorem 3.5.

Corollary 3.8. (Sumida) *Let* $f(T)$ *be the same as* (2) *in* Section 2 *and* $E = \mathbb{Q}_p$. *We assume* $\mathrm{ord}_p(\alpha-\beta) = \mathrm{ord}_p(\beta-\gamma) = \mathrm{ord}_p(\gamma-\alpha) = 1$. *Then we have* $\#\mathcal{M}_{f(T)} = 7$ *and*

$$\mathcal{M}_{f(T)}^{\mathbb{Q}_p} = \{(0,0,0), (0,1,0), (1,0,0), (0,1,1), (1,2,up), (1,1,0), (0,1,2)\},$$

where $u = (\gamma-\alpha)/(\beta-\alpha)$ *and* (m, n, x) *means* $[M(m, n, x)]_{\mathbb{Q}_p}$.

Proof. We prove this corollary using Theorem 3.5. By fixing integers m and n, we put

$$Z(m, n) = \{\text{the equivalence class of } \overline{(m, n, x)} \text{ in } Z/\sim \mid \overline{(m, n, x)} \in Z\}.$$

Then, by definition, we have

$$Z/\sim = \coprod_m \coprod_n Z(m, n).$$

We determine all the elements of $Z(m, n)$ for each m and n in order to determine all the elements of $\mathcal{M}_{f(T)}$.

We first assume $[\overline{(m, n, x)}] \in Z/\sim$, where $[\overline{(m, n, x)}]$ is the equivalence class of $\overline{(m, n, x)}$. Then by Proposition 3.3, $M(m, n, x)$ is a Λ_E-module satisfying (A), (B) and (C). By the inequality (A), we have $0 \le m \le 1$. Now we investigate $\coprod_n Z(m, n)$ for $m = 0, 1$.

(I) Suppose $m = 0$.

In this case, by the inequalities (B) and (C), we have $0 \le n \le 1$. When $n \ge 2$, we get $\mathrm{ord}_p(x) = 0$ by (C). This contradicts to (B). When $n = 0$, we have $\overline{(0, 0, x)} = \overline{(0, 0, 0)}$. Therefore we get $Z(0, 0) = \{[\overline{(0, 0, 0)}]\}$. When $n = 1$, we have $Z(0, 1) =$

$\{[\overline{(0,1,0)}], [\overline{(0,1,1)}], [\overline{(0,1,2)}]\}$. By the definition of the equivalence relation, we have $\overline{(0,1,x)} \sim \overline{(0,1,x')}$ if and only if

$$\mathrm{ord}_p(x \bmod p) = \mathrm{ord}_p(x' \bmod p) \quad \text{and} \quad \overline{1-x} = \varepsilon\overline{(1-x')}$$

for some $\varepsilon \in \mathbb{Z}_p^\times$. By the definition of $\mathrm{ord}_p(x \bmod p)$, we have

$$\mathrm{ord}_p(x \bmod p) = \begin{cases} 0 & x \notin p\mathbb{Z}_p, \\ \infty & x \in p\mathbb{Z}_p. \end{cases}$$

We investigate the case $\mathrm{ord}_p(x \bmod \pi) = 0$. Suppose $x = 1$. Then we have

$$\begin{aligned} [\overline{(0,1,1)}] &= \{\overline{(0,1,x)} \mid \overline{(0,1,1)} \sim \overline{(0,1,x)}\} \\ &= \{\overline{(0,1,x)} \mid \mathrm{ord}_p(x) = 0,\ \overline{0} = \varepsilon\overline{(1-x)} \text{ for some } \varepsilon \in \mathbb{Z}_p^\times\} \\ &= \{\overline{(0,1,x)} \mid x \equiv 1 \bmod p\} \\ &= \{\overline{(0,1,1)}\}. \end{aligned}$$

Suppose $x = 2$. Then we have

$$\begin{aligned} [\overline{(0,1,2)}] &= \{\overline{(0,1,x)} \mid \mathrm{ord}_p(x) = 0,\ \overline{-1} = \varepsilon\overline{(1-x)} \text{ for some } \varepsilon \in \mathbb{Z}_p^\times\} \\ &= \{\overline{(0,1,x)} \mid x \not\equiv 0, 1\} \\ &= \{\overline{(0,1,2)}, \dots, \overline{(0,1,p-1)}\}. \end{aligned}$$

Therefore we get $Z(0,1) = \{[\overline{(0,1,0)}], [\overline{(0,1,1)}], [\overline{(0,1,2)}]\}$.

(II) Suppose $m = 1$.

By Remark 3.4 (ii), we have $0 \le n \le 2$. When $n = 0$, we have $Z(1,0) = \{[\overline{(1,0,0)}]\}$. When $n = 1$, we have $Z(1,1) = \{[\overline{(1,1,0)}]\}$. In fact, we suppose $[\overline{(1,1,x)}] \in Z(1,1)$. Then we have $\overline{x} = 0$ by (C). When $n = 2$, we have $Z(1,2) = \{[\overline{(1,2,up)}]\}$. Indeed, we suppose $[\overline{(1,2,x)}] \in Z(1,2)$. For some $v \in \mathbb{Z}_p^\times$, we have

$$\begin{aligned} x &= \left(1 - \frac{vp^2}{\gamma - \alpha}\right)\frac{\gamma - \alpha}{\beta - \alpha} p \\ &\equiv \frac{\gamma - \alpha}{\beta - \alpha} p \bmod p^2, \end{aligned}$$

by (C). Thus,

$$Z/\!\sim\, = \{[\overline{(0,0,0)}], [\overline{(0,1,0)}], [\overline{(1,0,0)}], [\overline{(0,1,1)}], [\overline{(1,2,up)}], [\overline{(1,1,0)}], [\overline{(0,1,2)}]\}.$$

We complete the proof by Theorem 3.5. □

Corollary 3.9. *Let $f(T)$ be the same as* (2) *in* Section 2 *and $E = \mathbb{Q}_p$. We assume* $\mathrm{ord}_p(\alpha-\beta) = \mathrm{ord}_p(\beta-\gamma) = \mathrm{ord}_p(\gamma-\alpha) = 2$. *Then we have* $\#\mathcal{M}_{f(T)} = p + 18$ *and*

$$\mathcal{M}_{f(T)}^{\mathbb{Q}_p} = \left\{\begin{matrix} (0,0,0), (0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), \\ (0,2,2), (0,2,p), (0,2,p+1), (1,0,0), (1,1,0), \\ (1,1,1), (1,2,0), (1,2,p), \dots, (1,2,(p-1)p), (1,3,up), \\ (2,0,0), (2,1,0), (2,2,0), (2,3,up^2), (2,4,up^2) \end{matrix}\right\},$$

where $u = (\gamma-\alpha)/(\beta-\alpha)$ and (m,n,x) means $[M(m,n,x)]_{\mathbb{Q}_p}$.

Proof. We use the same notation as Corollary 3.8. By definition, we have

$$Z/\!\sim\; = \coprod_m \coprod_n Z(m,n).$$

We determine all the elements of $Z(m,n)$ for each m and n in order to determine all the elements of $\mathcal{M}_{f(T)}^{\mathbb{Q}_p}$.

We first assume $[\overline{(m,n,x)}] \in Z/\!\sim$, where $[\overline{(m,n,x)}]$ is the equivalence class of $\overline{(m,n,x)}$. Then $M(m,n,x)$ becomes a Λ_E-module satisfying $(A), (B)$ and (C). By the inequality (A), we have $0 \le m \le 2$. Now we investigate $\coprod_n Z(m,n)$ for each m.

(I) Suppose $m = 0$.

In this case, by the inequalities (B) and (C), we have $0 \le n \le 2$. In fact, if $\mathrm{ord}_p(x) \ge 1$, we get $n \le 2$ by (C) and if $\mathrm{ord}_p(x) = 0$, we get $n \le 2$ by (B). When $n = 0$, we have $\overline{(0,0,x)} = \overline{(0,0,0)}$ and $Z(0,0) = \{[\overline{(0,0,0)}]\}$. When $n = 1$, we have $Z(0,1) = \{[\overline{(0,1,0)}], [\overline{(0,1,1)}], [\overline{(0,1,2)}]\}$ by the same method as Corollary 3.8. When $n = 2$, we have

(4) $$Z(0,2) = \{[\overline{(0,2,0)}], [\overline{(0,2,1)}], [\overline{(0,2,2)}], [\overline{(0,2,p)}], [\overline{(0,2,p+1)}]\}.$$

In fact, we suppose $[\overline{(0,2,x)}] \in Z(0,2)$, then we have $\overline{x} = \overline{0}$ or $\mathrm{ord}_p(\overline{x}) \le 1$. We first investigate the case $\mathrm{ord}_p(x) = 0$. Then, $\overline{(0,2,x)} \sim \overline{(0,2,x')}$ if and only if

$$0 = \mathrm{ord}_p(x) = \mathrm{ord}_p(x') \quad \text{and} \quad \overline{1-x} = \varepsilon\overline{(1-x')} \quad \text{for some} \quad \varepsilon \in \mathbb{Z}_p^\times.$$

By the same method as above, we get

$$\begin{aligned} [\overline{(0,2,1)}] &= \{\overline{(0,2,1)}\}, \\ [\overline{(0,2,2)}] &= \{\overline{(0,2,x)} \mid \overline{x} \ne \overline{0}, \overline{1}\}, \\ (0,2,p+1)t &= \{\overline{(0,2,x)} \mid \mathrm{ord}_p(x) = 0,\ \overline{-p} = \varepsilon\overline{(1-x)} \text{ for some } \varepsilon \in \mathbb{Z}_p^\times\} \\ &= \{\overline{(0,2,1+x_1p)} \mid 1 \le x_1 < p\}. \end{aligned}$$

Next, we investigate the case $\mathrm{ord}_p(x) = 1$, let $x = p$. Then we have

$$[\overline{(0, 2, p)}] = \{\overline{(0, 2, x)} \mid \mathrm{ord}_p(x) = 1,\ \overline{1 - p} = \varepsilon\overline{(1 - x)} \text{ for some } \varepsilon \in \mathbb{Z}_p^\times\}$$
$$= \{\overline{(0, 2, x_1 p)} \mid 1 \le x_1 < p\}.$$

Thus we get (4).

(II) Suppose $m = 1$.

By the inequalities (B) and (C), we have $0 \le n \le 3$. If $\mathrm{ord}_p(x) \le 1$, we have $n \le 3$ by (B). If $\mathrm{ord}_p(x) > 1$, we have $n \le 2$ by (C). When $n = 0$, we have $Z(1, 0) = \{[\overline{(1, 0, 0)}]\}$. When $n = 1$, we have $Z(1, 1) = \{[\overline{(1, 1, 0)}], [\overline{(1, 1, 1)}]\}$. We suppose $[\overline{(1, 1, x)}] \in Z(1, 1)$. Then we have $\overline{x} = 0$ or $\mathrm{ord}_p(\overline{x}) = 0$. We suppose $\mathrm{ord}_p(\overline{x}) = 0$. We have $l = \mathrm{ord}_p(x - p) = 0$. This is the case (b). By the definition of the equivalence relation, $\overline{(1, 1, x)} \sim \overline{(1, 1, x')}$ if and only if

$$\overline{x} = \varepsilon\overline{x'} \quad \text{for some} \quad \varepsilon \in \mathbb{Z}_p^\times.$$

Here we note that $l = \mathrm{ord}_E(x' - p) = 0$. Then we have

$$[\overline{(1, 1, x)}] = \{\overline{(1, 1, x')} \mid \overline{x} = \varepsilon\overline{x'} \text{ for some } \varepsilon \in \mathbb{Z}_p^\times\}$$
$$= \{\overline{(1, 1, x')} \mid \overline{x'} \ne \overline{0}\}.$$

Therefore we get $Z(1, 1) = \{[\overline{(1, 1, 0)}], [\overline{(1, 1, 1)}]\}$. When $n = 2$, we have $Z(1, 2) = \{[\overline{(1, 2, x)}] \mid x = 0, p, 2p, \dots, (p - 1)p\}$. In fact, we suppose $[\overline{(1, 2, x)}] \in Z(1, 2)$. By the inequality (C), we have

$$2 \le \mathrm{ord}_p\{(\gamma - \alpha) - (\beta - \alpha)p^{-1}x\}.$$

If $\mathrm{ord}_p(x) = 0$, the order of the right hand side is 1. This is contradiction. Thus we may assume $1 \le \mathrm{ord}_p(x)$. If $\mathrm{ord}_p(x) \ge 2$, we get $[\overline{(1, 2, x)}] = \{\overline{(1, 2, 0)}\}$. We suppose $\mathrm{ord}_p(x) = 1$. Then $\overline{(1, 2, x)} \sim \overline{(1, 2, x')}$ if and only if

$$\overline{x} = \overline{x'}.$$

Here we note that this is the case (a) since $l = \mathrm{ord}_p(x' - p) \ge 1$. For each $x = \varepsilon p$, where $1 \le \varepsilon < p$, we have

$$[\overline{(1, 2, x)}] = \{\overline{(1, 2, x)}\}.$$

Thus we get $Z(1, 2) = \{[\overline{(1, 2, x)}] \mid x = 0, p, 2p, \dots, (p - 1)p\}$. When $n = 3$, we have $Z(1, 3) = \{[\overline{(1, 3, up)}]\}$. In fact, we suppose $[\overline{(1, 3, x)}] \in Z(1, 3)$. By the same method as in the case $n = 2$, we get $\mathrm{ord}_p(x) = 1$ and $\overline{(1, 3, x)} \sim \overline{(1, 3, up)}$ if and only if

$$\overline{x} = \varepsilon\overline{up} \quad \text{for some} \quad \varepsilon \in 1 + p\mathbb{Z}_p.$$

Here we note that this is the case (b) since $l = \mathrm{ord}_E(up - p) = 1$. Moreover, by (C), we have

$$x = \left(1 - \frac{vp^3}{\gamma - \alpha}\right)\frac{\gamma - \alpha}{\beta - \alpha} p \quad \text{for some} \quad v \in \mathbb{Z}_p^\times.$$

Since $1 - vp^3/(\gamma - \alpha) \in 1 + p\mathbb{Z}_p$, we have

$$[\overline{(1, 3, up)}] = \{\overline{(1, 3, x)} \mid \overline{x} = \varepsilon\overline{up} \text{ for some } \varepsilon \in 1 + p\mathbb{Z}_p\},$$

where $u = (\gamma - \alpha)/(\beta - \alpha)$. Thus we get $Z(1, 3) = \{[\overline{(1, 3, up)}]\}$.

(III) Suppose $m = 2$.

By the same method as (I) and (II), we get $Z(2, 0) = \{[\overline{(2, 0, 0)}]\}$, $Z(2, 1) = \{[\overline{(2, 1, 0)}]\}$, $Z(2,2) = \{[\overline{(2, 2, 0)}]\}$, $Z(2,3) = \{[\overline{(2, 3, up^2)}]\}$ and $Z(2,4) = \{[\overline{(2, 4, up^2)}]\}$. Thus we complete the proof. □

4. Proof of Theorem 3.5

For any $\xi \in \Lambda_E$, we define a map $\Pi_\xi = \Pi_\xi^M\colon M \to M$ by $\Pi_\xi(y) = \xi y$.

Lemma 4.1. *Let $q = \#(\mathcal{O}_E/(\pi))$ and $M = M(m, n, x)$. Then we have*

$$\#(\mathrm{Ker}(\Pi^N_{(T-\alpha)})/\mathrm{Im}(\Pi^N_{(T-\beta)})) = q^{\{\mathrm{ord}_E(\alpha-\beta)-m\}},$$
$$\#(\mathrm{Ker}(\Pi^M_{(T-\gamma)})/\mathrm{Im}(\Pi^M_{(T-\alpha)(T-\beta)})) = q^{\{\mathrm{ord}_E(\gamma-\alpha)+\mathrm{ord}_E(\gamma-\beta)-n\}},$$

where $N = \mathrm{Im}(\Pi_{(T-\gamma)})$.

Proof. We first compute $\mathrm{Ker}(\Pi_{(T-\gamma)})$. For $y \in M = M(m, n, x)$, there exist $\lambda_1, \lambda_2, \lambda_3 \in \mathcal{O}_E$ such that

$$\begin{aligned} y &= \lambda_1(1, 1, 1) + \lambda_2(0, \pi^m, x) + \lambda_3(0, 0, \pi^n) \\ &= (\lambda_1, \lambda_1 + \lambda_2\pi^m, \lambda_1 + \lambda_2 x + \lambda_3\pi^n). \end{aligned}$$

Thus we have $\Pi_{(T-\gamma)}(y) = ((\alpha - \gamma)\lambda_1, (\beta - \gamma)(\lambda_1 + \lambda_2\pi^m), 0)$. If $y \in \mathrm{Ker}(\Pi_{(T-\gamma)})$, we get $\lambda_1 = 0$ and $\lambda_1 + \lambda_2\pi^m = 0$, since α, β and γ are distinct elements of $\mathcal{O}_E$. Therefore $y = (0, 0, \lambda_3\pi^n)$ and $\mathrm{Ker}(\Pi_{(T-\gamma)}) = (0, 0, \pi^n\mathcal{O}_E)$. On the other hand, by $y = (\lambda_1, \lambda_1 + \lambda_2\pi^m, \lambda_1 + \lambda_2 x + \lambda_3\pi^n)$, we have

$$\begin{aligned} \Pi_{(T-\alpha)(T-\beta)}(y) &= \Pi_{(T-\alpha)}((\alpha - \beta)\lambda_1, 0, (\gamma - \beta)(\lambda_1 + \lambda_2 x + \lambda_3\pi^n)) \\ &= (0, 0, (\gamma - \alpha)(\gamma - \beta)(\lambda_1 + \lambda_2 x + \lambda_3\pi^n)). \end{aligned}$$

Thus we have $\mathrm{Im}(\Pi_{(T-\alpha)(T-\beta)}) = (0, 0, \pi^{\mathrm{ord}_E(\gamma-\alpha)+\mathrm{ord}_E(\gamma-\beta)}\mathcal{O}_E)$ and

$$\begin{aligned} \#(\mathrm{Ker}(\Pi_{(T-\gamma)})/\mathrm{Im}(\Pi_{(T-\alpha)(T-\beta)})) &= \#(\pi^n\mathcal{O}_E/\pi^{\mathrm{ord}_E(\gamma-\alpha)+\mathrm{ord}_E(\gamma-\beta)}\mathcal{O}_E) \\ &= q^{\{\mathrm{ord}_E(\gamma-\alpha)+\mathrm{ord}_E(\gamma-\beta)-n\}}. \end{aligned}$$

Next we put $N = \mathrm{Im}(\Pi_{(T-\gamma)})$. We have

$$\mathrm{Ker}(\Pi^{N}_{(T-\alpha)}) = (\pi^{\mathrm{ord}_E(\alpha-\gamma)+m}\mathcal{O}_E,\ 0,\ 0),$$
$$\mathrm{Im}(\Pi^{N}_{(T-\beta)}) = (\pi^{\mathrm{ord}_E(\alpha-\gamma)+\mathrm{ord}_E(\alpha-\beta)}\mathcal{O}_E,\ 0,\ 0).$$

Therefore we get

$$\#(\mathrm{Ker}(\Pi^{N}_{(T-\alpha)})/\mathrm{Im}(\Pi^{N}_{(T-\beta)})) = q^{\{\mathrm{ord}_E(\alpha-\beta)-m\}}. \qquad \square$$

Corollary 4.2. *Let* $[M]_E, [M']_E \in \mathcal{M}^{E}_{f(T)}$ *and* $M = M(m,n,x)$, $M' = M(m',n',x')$. *If* $[M]_E = [M']_E$, *then we have* $m = m'$ *and* $n = n'$.

Proof. Since $M \cong M'$, we have $N = \mathrm{Im}(\Pi^{M}_{(T-\gamma)}) \cong \mathrm{Im}(\Pi^{M'}_{(T-\gamma)}) = N'$ and therefore

$$\mathrm{Ker}(\Pi^{N}_{(T-\alpha)})/\mathrm{Im}(\Pi^{N}_{(T-\beta)}) \cong \mathrm{Ker}(\Pi^{N'}_{(T-\alpha)})/\mathrm{Im}(\Pi^{N'}_{(T-\beta)}).$$

This implies $m = m'$ by Lemma 4.1. We get $n = n'$ by the same method. $\square$

The isomorphism

$$\iota\colon \mathcal{E} = \Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\beta) \oplus \Lambda_E/(T-\gamma) \to \mathcal{O}_E^{\oplus 3}$$

defined in Section 2, induces an isomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_E} E \xrightarrow{\sim} E^{\oplus 3}$$

such that $(f_1(T),\ f_2(T),\ {}_3(T)) \otimes y \mapsto (f_1(\alpha)y,\ f_2(\beta)y,\ f_3(\gamma)y)$.

Proposition 4.3. *Let* $[M]_E, [M']_E \in \mathcal{M}^{E}_{f(T)}$, $M = M(m, n, x)$, $M' = M(m, n, x')$ *and* $g\colon M \to M'$ *be a* Λ_E*-isomorphism. We define an* E*-linear map* F_A *by the following commutative diagram*

$$\begin{array}{ccc}
M & \xrightarrow{\ g\ } & M' \\
{\scriptstyle \varphi\otimes 1}\downarrow & & \downarrow{\scriptstyle \varphi'\otimes 1} \\
\mathcal{E}\otimes_{\mathcal{O}_E} E & \longrightarrow & \mathcal{E}\otimes_{\mathcal{O}_E} E \\
{\scriptstyle \iota\otimes 1}\downarrow & & \downarrow{\scriptstyle \iota\otimes 1} \\
E^{\oplus 3} & \xrightarrow[F_A]{} & E^{\oplus 3}.
\end{array}$$

In the diagram, φ and φ' *are natural inclusions* (Section 2). *When we take the standard basis of* $E^{\oplus 3}$, F_A *corresponds to a diagonal matrix*

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \quad \textit{for some} \quad a_1, a_2, a_3 \in \mathcal{O}_E^{\times}.$$

Proof. Consider the map $\Pi_T\colon M \to M$. Then Π_T induces a map $F_B\colon E^{\oplus 3} \to E^{\oplus 3}$ and the following commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\Pi_T} & M \\ {\scriptstyle \varphi\otimes 1}\downarrow & & \downarrow{\scriptstyle \varphi\otimes 1} \\ \mathcal{E}\otimes_{\mathcal{O}_E} E & \longrightarrow & \mathcal{E}\otimes_{\mathcal{O}_E} E \\ {\scriptstyle \iota\otimes 1}\downarrow & & \downarrow{\scriptstyle \iota\otimes 1} \\ E^{\oplus 3} & \xrightarrow[F_B]{} & E^{\oplus 3}. \end{array}$$

Thus we get

$$(\natural) \qquad F_B \circ (\iota \otimes 1) \circ (\varphi \otimes 1)(x) = (\iota \otimes 1) \circ (\varphi \otimes 1)(Tx)$$

for $x \in M$. Let A be the matrix corresponding to F_A. By the diagram, we get

$$(\sharp) \qquad F_A \circ (\iota \otimes 1) \circ (\varphi \otimes 1)(Tx) = (\iota \otimes 1) \circ (\varphi' \otimes 1)(g(Tx)).$$

By ($\natural$) and the diagrams, the left hand side of ($\sharp$) is

$$F_A \circ (\iota \otimes 1) \circ (\varphi \otimes 1)(Tx) = F_A \circ F_B \circ (\iota \otimes 1) \circ (\varphi \otimes 1)(x).$$

The right hand side of ($\sharp$) is

$$\begin{aligned} (\iota \otimes 1) \circ (\varphi' \otimes 1)(Tg(x)) &= F_B \circ (\iota \otimes 1) \circ (\varphi' \otimes 1)(g(x)) \\ &= F_B \circ F_A \circ (\iota \otimes 1) \circ (\varphi \otimes 1)(x). \end{aligned}$$

Since this holds for any $x \in M$, we have $F_A \circ F_B = F_B \circ F_A$. Taking the standard basis of $E^{\oplus 3}$, F_B corresponds to the matrix

$$B = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}.$$

Therefore we have

$$A\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}A.$$

Since α, β and γ are distinct elements and we get

$$A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \quad \text{with} \quad a_1, a_2, a_3 \in E.$$

Because $g((1, 1, 1)) = (a_1, a_2, a_3) \in M'$, we get a_1, a_2 and $a_3 \in \mathcal{O}_E$. Furthermore, by the same argument for g^{-1}, we have $a_1^{-1}, a_2^{-1}, a_3^{-1} \in \mathcal{O}_E$. So we get $a_1, a_2, a_3 \in \mathcal{O}_E^\times$. □

By the commutativity of the diagram, we obtain the following corollary.

Corollary 4.4. *Suppose that M, F_A, ι, φ and φ' are the same as* Proposition 4.3. *Then we have*

$$\langle (F_A \circ (\iota \otimes 1) \circ (\varphi \otimes 1)(M) \rangle_{\mathcal{O}_E} = \langle (\iota \otimes 1) \circ (\varphi' \otimes 1) \circ g(M) \rangle_{\mathcal{O}_E}.$$

Proposition 4.5. *Let $[M]_E, [M']_E \in \mathcal{M}_{f(T)}^E$ and $M = M(m,n,x)$, $M' = M(m,n,x')$. Then the following two statements are equivalent*:

(i) *We have $M \cong M'$ as Λ_E-modules,*

(ii) *There exist $a_1, a_2, a_3 \in \mathcal{O}_E^\times$ satisfying*

$$\operatorname{ord}_E(a_2 - a_1) \geq m, \tag{5}$$

$$\operatorname{ord}_E(a_3 x - a_2 x') \geq n, \tag{6}$$

$$\operatorname{ord}_E\{a_3 - a_1 - (a_2 - a_1)\pi^{-m} x'\} \geq n. \tag{7}$$

Proof. We first prove (i) implies (ii). If M is isomorphic to M' as a Λ_E-module, there exists a Λ_E-isomorphism g: $M \xrightarrow{\sim} M'$. By Proposition 4.3, there exists a diagonal matrix A which can be written as

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \quad \text{such that} \quad a_1, a_2, a_3 \in \mathcal{O}_E^\times,$$

which corresponds to g. We have

$$\begin{aligned} F_A \circ (\iota \otimes 1) \circ (\varphi \otimes 1)(M) &= F_A(M(m, n, x)) \\ &= \langle (a_1, a_2, a_3), (0, a_2\pi^m, a_3 x), (0, 0, a_3\pi^n) \rangle_{\mathcal{O}_{\mathcal{E}}} \end{aligned}$$

and

$$\begin{aligned} (\iota \otimes 1) \circ (\varphi' \otimes 1) \circ g(M) &= (\iota \otimes 1) \circ (\varphi' \otimes 1)(M') \\ &= \langle (1, 1, 1), (0, \pi^m, x'), (0, 0, \pi^n) \rangle_{\mathcal{O}_E}. \end{aligned}$$

By Corollary 4.4, we get

$$\langle (a_1, a_2, a_3), (0, a_2\pi^m, a_3 x), (0, 0, a_3\pi^n) \rangle_{\mathcal{O}_E} = \langle (1, 1, 1), (0, \pi^m, x'), (0, 0, \pi^n) \rangle_{\mathcal{O}_E}.$$

Because the left hand side is contained in the right hand side, we have

$$
\begin{aligned}
(a_1, a_2, a_3) &= a_1(1, 1, 1) + (a_2 - a_1)\pi^{-m}(0, \pi^m, x') \\
&\qquad + \{a_3 - a_1 - (a_2 - a_1)\pi^{-m}x'\}\pi^{-n}(0, 0, \pi^n), \\
(0, a_2\pi^m, a_3x) &= a_2(0, \pi^m, x') + (a_3x - a_2x')\pi^{-n}(0, 0, \pi^n).
\end{aligned}
$$

Since these coefficients should belong to $\mathcal{O}_E$, we have (5), (6), (7). It is easy to prove that (ii) implies (i). □

We can simplify the inequalities (5), (6), (7). The following is easy to see.

Lemma 4.6. *The followings are equivalent*:
(i) *There exist* $a_1, a_2, a_3 \in \mathcal{O}_E^\times$ *satisfying* (5), (6), (7),
(ii) *There exist* $a_1, a_2 \in \mathcal{O}_E^\times$ *satisfying*

$$\operatorname{ord}_E(a_2 - a_1) \geq m, \tag{8}$$

$$\operatorname{ord}_E(x - a_2x') \geq n, \tag{9}$$

$$\operatorname{ord}_E\{1 - a_1 - (a_2 - a_1)\pi^{-m}x'\} \geq n. \tag{10}$$

Corollary 4.7. *Let* $[M]_E, [M']_E \in \mathcal{M}^E_{f(T)}$ *and* $M = M(m,n,x)$, $M' = M(m,n,x')$. *Assume* $\operatorname{ord}_E(x) < n$. *If* $[M]_E = [M']_E$, *we have* $\operatorname{ord}_E(x) = \operatorname{ord}_E(x')$.

Proof. If $\operatorname{ord}_E(x) < \operatorname{ord}_E(x')$, by the inequality (6), we have $n \leq \operatorname{ord}_E(a_3x - a_2x') = \operatorname{ord}_E(x)$. This contradicts to the assumption $\operatorname{ord}_E(x) < n$. If we assume $\operatorname{ord}_E(x) > \operatorname{ord}_E(x')$, we would get the same contradiction. Therefore we obtain $\operatorname{ord}_E(x) = \operatorname{ord}_E(x')$. □

To prove Theorem 3.5, we prepare a lemma and some propositions.

Proposition 4.8. *The following two statements are equivalent*:
(i) *We have* $M(m, n, x) \cong M(m, n, 0)$ *as* Λ_E*-modules*,
(ii) *We have* $\overline{(m, n, x)} \sim \overline{(m, n, 0)}$, *where* $\sim$ *is the equivalence relation defined in* Section 3.

Proof. We show that (i) implies (ii). If $\operatorname{ord}_E(x) < n$, we have $\operatorname{ord}_E(x) = \operatorname{ord}_E(0)$ by Corollary 4.7, which is a contradiction. So we have $\operatorname{ord}_E(x) \geq n$ and $M(m, n, x) = M(m, n, 0)$. Then $\overline{(m, n, x)} = \overline{(m, n, 0)}$ by Remark 3.4 (i). □

Let $M = M(m,n,x)$ and $M' = M(m,n,x')$. Now we suppose that $x' \neq 0$ and the existence of $a_1, a_2 \in \mathcal{O}_E^\times$ satisfying (8), (9) and (10). By Proposition 4.5 and Lemma 4.6,

M is isomorphic to M'. From the inequalities (8) and (9), there are $s, v \in \mathcal{O}_E$ such that $a_2 - a_1 = \pi^m s$ and $x - a_2 x' = \pi^n v$. Thus we have

$$a_1 = \frac{x}{x'} - \frac{\pi^n}{x'} v - \pi^m s, \tag{11}$$

$$a_2 = \pi^m s + a_1 = \frac{x}{x'} - \frac{\pi^n}{x'} v. \tag{12}$$

By the inequality (10), we get

$$x'(x' - \pi^m)s - \pi^n v + \pi^n x' w = x' - x \tag{13}$$

for some $w \in \mathcal{O}_E$.

Lemma 4.9. *Let $m, n \neq 0$ and $\mathrm{ord}_E(x) < n$. The following two statements are equivalent*:
(i) *There exist $a_1, a_2 \in \mathcal{O}_E^\times$ satisfying* (8), (9), (10),
(ii) *We have $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$ and there exist $s, v, w \in \mathcal{O}_E$ satisfying* (13).

Proof. We have already proved that (i) implies (ii). We will prove that (ii) implies (i). We put a_1 and a_2 by the equalities (11) and (12). Since $m, n \neq 0$ and $\mathrm{ord}_E(x) = \mathrm{ord}_E(x') < n$, we have $a_1, a_2 \in \mathcal{O}_E^\times$. Then we have

$$a_2 - a_1 = \pi^m s, \quad x - a_2 x' = \pi^n v$$

and

$$1 - a_1 - (a_2 - a_1)\pi^{-m} x' = \pi^n w.$$

Therefore we get (8), (9) and (10). □

Proposition 4.10. *Let $m, n \neq 0$ and $\mathrm{ord}_E(x) < n$. Then the followings are equivalent*:
(i) *We have $M(m, n, x) \cong M(m, n, x')$ as Λ_E-modules,*
(ii) *We have $\overline{(m, n, x)} \sim \overline{(m, n, x')}$.*

Proof. We first suppose that $M(m, n, x)$ is isomorphic to $M(m, n, x')$ as a Λ_E-module. Let $k = \mathrm{ord}_E(x)$ and $l = \mathrm{ord}_E(x' - \pi^m)$. By Lemma 4.9, we have $\mathrm{ord}_E(x) = \mathrm{ord}_E(x') = k$ and there exist $s, v, w \in \mathcal{O}_E$ such that

$$x'(x' - \pi^m)s - \pi^n v + \pi^n x' w = x' - x.$$

We put $\varepsilon = x x'^{-1} \in \mathcal{O}_E^\times$. Dividing the above equality by x', we have

$$(x' - \pi^m)s - \frac{\pi^n}{x'} v + \pi^n w = 1 - \varepsilon.$$

Thus we have

$$\begin{aligned}\mathrm{ord}_E(1-\varepsilon) &\geq \min\left\{\mathrm{ord}_E((x'-\pi^m)s),\ \mathrm{ord}_E\left(-\frac{\pi^n}{x'}v\right),\ \mathrm{ord}_E(\pi^n w)\right\}\\ &\geq \min\{l, n-k, n\} = \min\{l, n-k\}.\end{aligned}$$

In the case (a) $l \geq n-k$, we have $\mathrm{ord}_E(1-\varepsilon) \geq n-k$. Thus we get $\overline{x} = \overline{\varepsilon x'} = \overline{x'}$ in $\mathcal{O}_E/\pi^n\mathcal{O}_E$. Therefore we have $\overline{(m, n, x)} \sim \overline{(m, n, x')}$. In the case (b) $l < n-k$, we have $\mathrm{ord}_E(1-\varepsilon) \geq l$ and $\overline{x} = \overline{\varepsilon x'}$ in $\mathcal{O}_E/\pi^n\mathcal{O}_E$. Therefore we get $\overline{(m, n, x)} \sim \overline{(m, n, x')}$. Conversely we assume that $\overline{(m, n, x)} \sim \overline{(m, n, x')}$. In the case (a), we have $\overline{x} = \overline{x'}$ in $\mathcal{O}_E/\pi^n\mathcal{O}_E$ and $(x'-x)/\pi^n \in \mathcal{O}_E$. Put $s = w = 0$ and $v = (x-x')/\pi^n \in \mathcal{O}_E$. Then we get

$$x'(x'-\pi^m)s - \pi^n v + \pi^n x' w = x' - x.$$

By Lemma 4.9, M and M' are isomorphic as Λ_E-modules. In the case (b), We have $\overline{x} = \varepsilon\overline{x'}$ in $\mathcal{O}_E/\pi^n\mathcal{O}_E$ for some $\varepsilon \in 1 + \pi^l\mathcal{O}_E$. Since $\mathrm{ord}_E(1-\varepsilon) \geq l$, we have $(1-\varepsilon)/(x'-\pi^m) \in \mathcal{O}_E$. Put $v = w = 0$ and $s = (1-\varepsilon)/(x'-\pi^m) \in \mathcal{O}_E$. Then we get

$$x'(x'-\pi^m)s - \pi^n v + \pi^n x' w = x' - \varepsilon x'.$$

By Lemma 4.9, we get $M(m, n, x) = M(m, n, \varepsilon x') \cong M(m, n, x')$. □

The following propositions treat the case $m = 0$ and the case $n = 0$.

Proposition 4.11. *Suppose* $m = 0, n \neq 0$ *and* $\mathrm{ord}_E(x) < n$. *Then the followings are equivalent*:
(i) *We have* $M(0, n, x) \cong M(0, n, x')$ *as* Λ_E*-modules*,
(ii) *We have* $\overline{(0, n, x)} \sim \overline{(0, n, x')}$.

Proof. Suppose that $M(0, n, x)$ is isomorphic to $M(0, n, x')$ as a Λ_E-module. By Proposition 4.5 and Lemma 4.6, there exist $a_1, a_2 \in \mathcal{O}_E^\times$ satisfying (9) and (10). By the inequality (9), we have $\overline{x} = a_2\overline{x'}$. By the inequality (10), we have $\overline{1 - a_2 x'} = \overline{a_1(1-x')}$. Therefore we get

$$\mathrm{ord}_E(x) = \mathrm{ord}_E(x') \quad \text{and} \quad \overline{1-x} = a_1\overline{(1-x')}.$$

Thus we get $\overline{(0, n, x)} \sim \overline{(0, n, x')}$. Conversely we suppose that $\overline{(0, n, x)} \sim \overline{(0, n, x')}$. There exists $a_1 \in \mathcal{O}_E^\times$ such that $\overline{1-x} = a_1\overline{(1-x')}$. Put $a_2 = x/x'$. Then we have (9) and (10). Indeed, we have $1 - a_1 - (a_2 - a_1)\pi^{-m}\overline{x'} = 1 - a_1 - (a_2 - a_1)\overline{x'} = \overline{0}$. By Proposition 4.5 and Lemma 4.6, $M(0, n, x)$ and $M(0, n, x')$ are isomorphic as Λ_E-modules. □

Proposition 4.12. *Suppose* $n = 0$. *The followings are equivalent*:

(i) *We have* $M(m, 0, x) \cong M(m, 0, x')$ *as* Λ_E*-modules,*
(ii) *We have* $\overline{(m, 0, x)} \sim \overline{(m', 0, x')}$.

Proof. By Remark 3.4 (i), we have $M(m, 0, x) = M(m, 0, x') = M(m, 0, 0)$ and $\overline{(m, 0, x)} = \overline{(m, 0, x')} = \overline{(m, 0, 0)}$. □

Now we can prove Theorem 3.5.

Proof of Theorem 3.5. For $[M(m, n, x)]_E \in \mathcal{M}^E_{f(T)}$, we may assume $x = 0$ or $\mathrm{ord}_E(x) < n$ by Remark 3.4 (i). At first, Φ is well-defined by Corollary 4.2 and Propositions 4.8, 4.10, 4.11 and 4.12. The surjectivity follows from Proposition 3.3 and Remark 3.4. On the other hand, Φ is injective by Propositions 4.8, 4.10, 4.11 and 4.12. □

5. Complementary properties

In this section, we show some propositions in order to determine the Iwasawa module associated to an imaginary quadratic field in the next section.

For a non-negative integer n, we put $\omega_n = \omega_n(T) = (1 + T)^{p^n} - 1$.

Proposition 5.1. *For a distinguished polynomial* $f(T) \in \mathbb{Z}_p[T]$*, let* E *be the splitting field of* $f(T)$ *over* $\mathbb{Q}_p$*. Then the natural map*

$$\Psi\colon \mathcal{M}^{\mathbb{Q}_p}_{f(T)} \to \mathcal{M}^E_{f(T)} \quad ([M] \mapsto [M \otimes_\Lambda \Lambda_E]_E)$$

is injective.

Proof. We suppose that $M \otimes_\Lambda \Lambda_E \cong M' \otimes_\Lambda \Lambda_E$ for $[M]$ and $[M'] \in \mathcal{M}^{\mathbb{Q}_p}_{f(T)}$. Since $M \otimes_\Lambda \Lambda_E \cong M^n$ as Λ-modules, we get $M^n \cong M'^n$ as Λ-modules, where n is the degree of the extension $E/\mathbb{Q}_p$.

We assume that $M \ncong M'$ as Λ-modules. Since M is a finitely generated Λ-module, M is a profinite module and we have $M = \varprojlim M/\mathfrak{m}^n M$ where $\mathfrak{m} = (\pi, T)$. Since $M \ncong M'$, there exists a positive integer l such that $M/\mathfrak{m}^l M \ncong M'/\mathfrak{m}^l M'$ ([11], Proposition 5). Because both $M/\mathfrak{m}^l M$ and $M'/\mathfrak{m}^l M'$ are of finite length, we can decompose these modules into indecomposable modules

$$M/\mathfrak{m}^l M = \bigoplus_i N_i^{\oplus e_i}, \quad M'/\mathfrak{m}^l M' = \bigoplus_i N_i^{\oplus e'_i},$$

where N_i's are indecomposable modules, $N_i \neq N_j$ $(i \neq j)$ and e_i, e'_i are non-negative integers. By Krull–Remak–Schmidt's theorem, there exists i such that $e_i \neq e'_i$. Furthermore we have

$$(M/\mathfrak{m}^l M)^n = \bigoplus_i N_i^{\oplus n e_i}, \quad (M'/\mathfrak{m}^l M')^n = \bigoplus_i N_i^{\oplus n e'_i}.$$

Thus we get $ne_i \neq ne'_i$ for some i. By Krull–Remak–Schmidt's theorem, we have $(M/\mathfrak{m}^l M)^n \not\cong (M'/\mathfrak{m}^l M')^n$. This implies $M^n \not\cong M'^n$. This contradicts to our assumption. □

Let $f(T) \in \mathbb{Z}_p[T]$ be a distinguished polynomial, E the splitting field of $f(T)$ and we put

$$f(T) = (T-\alpha)(T-\beta)(T-\gamma),$$

where α, β and $\gamma \in \pi\mathcal{O}_E$ as in Section 2.

Proposition 5.2. *Let E and $f(T)$ be as above and $[M]_E \in \mathcal{M}^E_{f(T)}$. If M is a cyclic Λ_E-module, then we have*

$$M \cong M(\mathrm{ord}_E(\beta-\alpha), \mathrm{ord}_E(\gamma-\alpha) + \mathrm{ord}_E(\gamma-\beta), u\pi^{\mathrm{ord}_E(\beta-\alpha)})$$

as Λ_E-modules, where $u = (\gamma-\alpha)/(\beta-\alpha)$.

Proof. Let $M \cong M(m, n, x) \subset \mathcal{E}$. Suppose that M is cyclic and put

$$M = \langle (a, b, c) \rangle_{\Lambda_E} \subset \mathcal{E}$$

for some $a, b, c \in \mathcal{O}_E$. Because $(1,1,1) \in \langle (a,b,c) \rangle_{\Lambda_E}$, we have $(1,1,1) = h(T)(a,b,c) = (h(\alpha)a, h(\beta)b, h(\gamma)c)$ for some $h(T) \in \Lambda_E$. Therefore we get $a, b, c \in \mathcal{O}_E^\times$. Since $(0, \pi^m, x)$ and $(0, 0, \pi^n) \in \langle (a, b, c) \rangle_{\Lambda_E}$, we have

$$(0, \pi^m, x) = q(T)(a, b, c) = (q(\alpha)a, q(\beta)b, q(\gamma)c),$$
$$(0, 0, \pi^n) = r(T)(a, b, c) = (r(\alpha)a, r(\beta)b, r(\gamma)c)$$

for some $q(T)$ and $r(T) \in \Lambda_E$. Since $(T-\alpha) \mid q(T)$ and $(T-\alpha)(T-\beta) \mid r(T)$, we get $m = \mathrm{ord}_E(q(\beta)) \geq \mathrm{ord}_E(\beta-\alpha)$ and $n = \mathrm{ord}_E(r(\gamma)) \geq \mathrm{ord}_E(\gamma-\alpha) + \mathrm{ord}_E(\gamma-\beta)$. On the other hand, by Proposition 3.3 and Remark 3.4, we have $m \leq \mathrm{ord}_E(\beta-\alpha)$ and $n \leq \mathrm{ord}_E(\gamma-\alpha) + \mathrm{ord}_E(\gamma-\beta)$. Therefore we obtain $m = \mathrm{ord}_E(\beta-\alpha)$ and $n = \mathrm{ord}_E(\gamma-\alpha) + \mathrm{ord}_E(\gamma-\beta)$. Furthermore,

$$\begin{aligned}(T-\alpha)(1, 1, 1) &= (0, \beta-\alpha, \gamma-\alpha) \\ &= (\beta-\alpha)\pi^{-m}(0, \pi^m, x) + \{\gamma-\alpha-(\beta-\alpha)\pi^{-m}x\}\pi^{-n}(0, 0, \pi^n).\end{aligned}$$

Because $\mathrm{ord}_E\{\gamma-\alpha-(\beta-\alpha)\pi^{-m}x\} \geq n$, we have $x = (\gamma-\alpha)/(\beta-\alpha)\pi^m(1-\pi^n v/(\gamma-\alpha))$ for some $v \in \mathcal{O}_E$. By Remark 3.4 (i), we get

$$M(m, n, x) = M(\mathrm{ord}_E(\beta-\alpha), \mathrm{ord}_E(\gamma-\alpha) + \mathrm{ord}_E(\gamma-\beta), u\pi^{\mathrm{ord}_E(\beta-\alpha)}). \qquad \square$$

Proposition 5.3. *Let* $f(T)$ *be as above. Assume* $\mathrm{ord}_E(\alpha-\beta)=\mathrm{ord}_E(\beta-\gamma)=\mathrm{ord}_E(\gamma-\alpha)=1$ *and* $\mathrm{ord}_E(\alpha)\geq\mathrm{ord}_E(\beta)\geq\mathrm{ord}_E(\gamma)$. *Then, we have*

$$\mathcal{M}^E_{f(T)}=\{(0,0,0),(0,1,0),(1,0,0),(0,1,1),(1,2,u\pi),(1,1,0),(0,1,2)\},$$

where $u=(\gamma-\alpha)/(\beta-\alpha)$ *and* (m,n,x) *means* $[M(m,n,x)]_E$. *The following is the table of the structure of* $\mathcal{O}_E$*-modules* $M/\omega_0 M$ *for* Λ_E*-modules* M.

M	$M/\omega_0 M$
$M(0,0,0)$	$\mathcal{O}_E/(\alpha)\oplus\mathcal{O}_E/(\beta)\oplus\mathcal{O}_E/(\gamma)$
$M(0,1,0)$	$\mathcal{O}_E/(\beta)\oplus\mathcal{O}_E/(\alpha\gamma)$
$M(0,1,1)$	$\mathcal{O}_E/(\alpha)\oplus\mathcal{O}_E/(\beta\gamma)$
$M(0,1,2)$	$\mathcal{O}_E/(\beta)\oplus\mathcal{O}_E/(\alpha\gamma)$
$M(1,0,0)$	$\mathcal{O}_E/(\gamma)\oplus\mathcal{O}_E/(\alpha\beta)$
$M(1,1,0)$	$\mathcal{O}_E/(\gamma)\oplus\mathcal{O}_E/(\alpha\beta)$
$M(1,2,u\pi)$	$\mathcal{O}_E/(\alpha\beta\gamma)$

Proof. The former is Corollary 3.8. We show the latter. Let $[M]_E\in\mathcal{M}^E_{f(T)}$. There exist m,n and x such that

$$M=\langle(1,1,1),(0,\pi^m,x),(0,0,\pi^n)\rangle_{\mathcal{O}_E}$$

and we have

$$\omega_0 M=\langle(\alpha,\beta,\gamma),(0,\beta\pi^m,\gamma x),(0,0,\gamma\pi^n)\rangle_{\mathcal{O}_E}.$$

Since $\mathcal{O}_E$ is a principal ideal domain, we can use the structure theorem over the principal ideal domain. We consider the map $\Pi_{\omega_0}\colon M\to M$ and take $(1,1,1)$, $(0,\pi^m,x)$ and $(0,0,\pi^n)$ as a basis of M. Then we have

$$\begin{aligned}T(1,1,1)&=\alpha(1,1,1)+(\beta-\alpha)\pi^{-m}(0,\pi^m,x)\\&\qquad+\{\gamma-\alpha-(\beta-\alpha)\pi^{-m}x\}\pi^{-n}(0,0,\pi^n),\end{aligned}\tag{14}$$

$$\begin{aligned}T(0,\pi^m,x)&=(0,\beta\pi^m,\gamma x)\\&=\beta(0,\pi^m,x)+(\gamma-\beta)x\pi^{-n}(0,0,\pi^n).\end{aligned}\tag{15}$$

By the equalities (14) and (15), the matrix corresponding to Π_{ω_0} is

$$\begin{pmatrix}\alpha&0&0\\(\beta-\alpha)\pi^{-m}&\beta&0\\\{\gamma-\alpha-(\beta-\alpha)\pi^{-m}x\}\pi^{-n}&(\gamma-\beta)x\pi^{-n}&\gamma\end{pmatrix}.$$

In order to verify the table, we have only to transform this matrix by elementary row and column operations. For example, the case $M = M(0, 1, 0)$, we get the matrix

$$\begin{pmatrix} \alpha & 0 & 0 \\ \beta - \alpha & \beta & 0 \\ (\gamma - \alpha)\pi^{-1} & 0 & \gamma \end{pmatrix}.$$

By the elementary row and column operations, we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha\gamma \end{pmatrix}.$$

So we get $M/\omega_0 M \cong \mathcal{O}_E/(\beta) \oplus \mathcal{O}_E/(\alpha\gamma)$. The rest of the table can be checked by the same method. □

Proposition 5.4. *Let $f(T) = (T - \alpha)g(T)$, where $\alpha \in p\mathbb{Z}_p$ and $g(T) \in \mathbb{Z}_p[T]$ is a distinguished irreducible polynomial of degree 2. Let E be the splitting field of $g(T)$ over $\mathbb{Q}_p$. If $[M(m, n, x)]_E \in \mathrm{Image}(\Psi\colon \mathcal{M}^{\mathbb{Q}_p}_{f(T)} \to \mathcal{M}^{E}_{f(T)}([M] \longmapsto [M \otimes_\Lambda \Lambda_E]_E))$, we have*

$$\mathrm{ord}_E(x) = m.$$

Proof. Let $[M] \in \mathcal{M}^{\mathbb{Q}_p}_{f(T)}$ and $M \otimes \Lambda_E \cong M(m, n, x) \subset \mathcal{E}$. There is a natural injective map

$$M \to \Lambda/(f(T)) \to \Lambda/(T - \alpha) \oplus \Lambda/(g(T))$$

([13], Lemma 13.8). By this injective map, we have

$$M = \langle (a_1, b_1 T + c_1), (a_2, b_2 T + c_2), (a_3, b_3 T + c_3) \rangle_{\mathbb{Z}_p} \subset \Lambda/(T - \alpha) \oplus \Lambda/(g(T))$$

for some a_i, b_i and $c_i \in \mathbb{Z}_p$. Because $M \otimes_\Lambda \Lambda_E = \langle (a_1, b_1 T + c_1), (a_2, b_2 T + c_2), (a_3, b_3 T + c_3) \rangle_{\mathcal{O}_E}$, by the same argument before Lemma 3.1, we can write

$$M \otimes_\Lambda \Lambda_E = \langle (a'_1, b'_1 T + c'_1), (0, b'_2 T + c'_2), (0, c'_3) \rangle_{\mathcal{O}_E}$$

for some a'_i, b'_i and $c'_i \in \mathbb{Z}_p$. Furthermore there is an injective map ([13], Lemma 13.8)

$$\Lambda_E/(T - \alpha) \oplus \Lambda_E/(g(T)) \to \mathcal{E}, \quad (s(t), u(t)) \mapsto (s(\alpha), u(\beta), u(\gamma)),$$

where β and γ are the roots of $g(T)$ in E. By this map, $M \otimes_\Lambda \Lambda_E$ is isomorphic to the module

$$M' = \langle (a'_1, b'_1\beta + c'_1, b'_1\gamma + c'_1), (0, b'_2\beta + c'_2, b'_2\gamma + c'_2), (0, c'_3, c'_3) \rangle_{\mathcal{O}_E} \subset \mathcal{E}.$$

Since β and γ are conjugate, we have $\mathrm{ord}_E(b_1'\beta + c_1') = \mathrm{ord}_E(b_1'\gamma + c_1')$ and $\mathrm{ord}_E(b_2'\beta + c_2') = \mathrm{ord}_E(b_2'\gamma + c_2')$. By the same arguments after Lemma 3.2, we get

$$M' \cong \langle (1, 1, 1), (0, \pi^m, x), (0, 0, \pi^n) \rangle_{\mathcal{O}_E}$$

for some m, n, x which satisfy $m = \mathrm{ord}_E(x)$. Indeed, we may assume $\mathrm{ord}_E(b_2'\beta + c_2') \leq \mathrm{ord}_E(c_3')$. By Lemma 3.2, we have

$$M' \cong \langle (1, b_1'\beta + c_1', b_1'\gamma + c_1'), (0, b_2'\beta + c_2', b_2'\gamma + c_2'), (0, c_3', c_3') \rangle_{\mathcal{O}_E}.$$

In the case $\mathrm{ord}_E(b_1'\beta + c_1') \leq \mathrm{ord}_E(b_2'\beta + c_2')$, we have

$$M' \cong \left\langle (1, 1, b_1'\gamma + c_1'), \left(0, \frac{b_2'\beta + c_2'}{b_1'\beta + c_1'}, b_2'\gamma + c_2'\right), \left(0, \frac{c_3'}{b_1'\beta + c_1'}, c_3'\right) \right\rangle_{\mathcal{O}_E}.$$

Since $\mathrm{ord}_E(b_1'\gamma + c_1') \leq \mathrm{ord}_E(b_2'\gamma + c_2') \leq \mathrm{ord}_E(c_3')$, we get

$$\begin{aligned} M' &\cong \left\langle (1,1,1), \left(0, \frac{b_2'\beta + c_2'}{b_1'\beta + c_1'}, \frac{b_2'\gamma + c_2'}{b_1'\gamma + c_1'}\right), \left(0, \frac{c_3'}{b_1'\beta + c_1'}, \frac{c_3'}{b_1'\gamma + c_1'}\right) \right\rangle_{\mathcal{O}_E} \\ &= \left\langle (1,1,1), \left(0, \frac{b_2'\beta + c_2'}{b_1'\beta + c_1'}, \frac{b_2'\gamma + c_2'}{b_1'\gamma + c_1'}\right), \left(0, 0, \frac{c_3'}{b_1'\gamma + c_1'} - \frac{c_3'}{b_2'\beta + c_2'} \cdot \frac{b_2'\gamma + c_2'}{b_1'\gamma + c_1'}\right) \right\rangle_{\mathcal{O}_E}. \end{aligned}$$

Thus we get

$$\begin{aligned} &m = \mathrm{ord}_E\left(\frac{b_2'\beta + c_2'}{b_1'\beta + c_1'}\right), \quad n = \mathrm{ord}_E\left(\frac{c_3'}{b_1'\gamma + c_1'} - \frac{c_3'}{b_2'\beta + c_2'} \cdot \frac{b_2'\gamma + c_2'}{b_1'\gamma + c_1'}\right), \\ &x = \pi^{-m} \cdot \frac{b_1'\beta + c_1'}{b_2'\beta + c_2'} \cdot \frac{b_2'\gamma + c_2'}{b_1'\gamma + c_1'}. \end{aligned}$$

Therefore we obtain $m = \mathrm{ord}_E(x)$. On the other hand, in the case $\mathrm{ord}_E(b_1'\beta + c_1') > \mathrm{ord}_E(b_2'\beta + c_2')$, we have

$$M' = \langle a_1', (b_1' - b_2')\beta + (c_1' - c_2'), (b_1' - b_2')\gamma + (c_1' - c_2')), (0, b_2'\beta + c_2', b_2'\gamma + c_2'), (0, c_3', c_3') \rangle_{\mathcal{O}_E}.$$

Because $\mathrm{ord}_E(b_1'\beta + c_1' - (b_2'\beta + c_2')) = \mathrm{ord}_E(b_2'\beta + c_2')$, we get the same conclusion as in the case $\mathrm{ord}_E(b_1'\beta + c_1') \leq \mathrm{ord}_E(b_2'\beta + c_2')$. □

Proposition 5.5. *Let $f(T) = (T - \alpha)g(T)$, where $\alpha \in p\mathbb{Z}_p$ and $g(T) \in \mathbb{Z}_p[T]$ is an irreducible polynomial of degree 2. Let E be the splitting field of $g(T)$ over $\mathbb{Q}_p$. We assume* $\mathrm{ord}_E(\alpha - \beta) = \mathrm{ord}_E(\beta - \gamma) = \mathrm{ord}_E(\gamma - \alpha) = 1$,

$$M/\omega_0 M \cong \mathbb{Z}/p^i\mathbb{Z} \oplus \mathbb{Z}/p^j\mathbb{Z} \quad (i, j \in \mathbb{Z}_{\geq 1})$$

and $E/\mathbb{Q}_p$ is a totally ramified extension. Then we have

$$\Psi(M) = M \otimes_\Lambda \Lambda_E \cong M(0, 1, 1) \cong \Lambda_E/(T - \alpha) \oplus \Lambda_E/(T - \beta)(T - \gamma).$$

Proof. Since $M/\omega_0 M \cong \mathbb{Z}/p^i\mathbb{Z} \oplus \mathbb{Z}/p^j\mathbb{Z}$, we have $M/\omega_0 M \otimes_\Lambda \Lambda_E \cong \mathcal{O}_E/(\pi^{2i}) \oplus \mathcal{O}_E/(\pi^{2j})$. Since $E/\mathbb{Q}_p$ is a totally ramified extension, $\mathrm{ord}_E(\alpha) = 2\,\mathrm{ord}_p(\alpha) \geq 2$. Thus we get $\mathrm{ord}_E(\beta) = \mathrm{ord}_E(\gamma) = 1$. Because $\mathrm{ord}_E(\pi^{2i}) = 2i$ and $\mathrm{ord}_E(\pi^{2j}) = 2j$ are even, we get

$$M \otimes_\Lambda \Lambda_E \cong M(0, 1, 1)$$

by the table of the Proposition 5.3. The isomorphism $M(0, 1, 1) \cong \Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\beta)(T-\gamma)$ is Lemma 3 in Sumida [12]. □

Corollary 5.6. *Let $f(T)$, $g(T)$ and E be as in* Propositions 5.5 *and* $[M]_{\mathbb{Q}_p} \in \mathcal{M}_{f(T)}^{\mathbb{Q}_p}$. *We assume the same conditions of* Proposition 5.5 *and we put $g(T) = T^2 + c_1 T + c_0$. Then*

(a) *Suppose $p \geq 5$. For $n \geq 0$, we have*

$$\#(M/\omega_n M \otimes \Lambda_E) = p^{\mathrm{ord}_E(\omega_n(\alpha)\omega_n(\beta)\omega_n(\gamma))} = p^{6n+2+\mathrm{ord}_E(\alpha)}.$$

(b) *Suppose $p = 3$. For $n \geq 1$, we have*

$$\#(M/\omega_n M \otimes \Lambda_E) = \begin{cases} p^{\mathrm{ord}_E(\omega_n(\alpha)\omega_n(\beta)\omega_n(\gamma))} = p^{6n+\mathrm{ord}_E(\alpha)+4\,\mathrm{ord}_3(c_0-3)-2} \\ \textit{if} \quad \mathrm{ord}_3(c_0-3) \leq \mathrm{ord}_3(c_1-3), \\ p^{\mathrm{ord}_E(\omega_n(\alpha)\omega_n(\beta)\omega_n(\gamma))} = p^{6n+\mathrm{ord}_E(\alpha)+4\,\mathrm{ord}_3(c_0-3)-2} \\ \textit{if} \quad \mathrm{ord}_3(c_0-3) > \mathrm{ord}_3(c_1-3). \end{cases}$$

Proof. Put $N = \langle (1,1,1), (0,1,1), (0,0,\pi) \rangle_{\mathcal{O}_E} \subset \mathcal{E}$. By Proposition 5.5, we have $M \otimes_\Lambda \Lambda_E \cong N$ as Λ_E-modules. Thus we have

$$M/\omega_n M \otimes \Lambda_E \cong (M \otimes_\Lambda \Lambda_E)/\omega_n(M \otimes_\Lambda \Lambda_E) \cong N/\omega_n N$$

as $\Lambda_E/\omega_n\Lambda_E$-modules. By the same method as Proposition 5.3, we consider the map $\Pi_{\omega_n} \colon N \to N$ and take $(1,0,0)$, $(0,1,1)$ and $(0,0,\pi)$ as a basis of N. The matrix corresponding to Π_{ω_n} is

$$\begin{pmatrix} \omega_n(\alpha) & 0 & 0 \\ 0 & \omega_n(\beta) & 0 \\ 0 & (\omega_n(\beta) - \omega_n(\gamma))\pi^{-1} & \omega_n(\gamma) \end{pmatrix}.$$

We first consider the case (a). We have $\mathrm{ord}_E(\omega_n(\beta) - \omega_n(\gamma)) = \mathrm{ord}_E(\beta - \gamma) + n\,\mathrm{ord}_E(3) = 2n+1$ (cf. [7], Lemma 2.5). Furthermore, we have $\mathrm{ord}_E(\omega_n(\alpha)) = 2n + \mathrm{ord}_E(\alpha)$, and we get $\mathrm{ord}_E\{(\omega_n(\beta) - \omega_n(\gamma))\pi^{-1}\} = 2n < \mathrm{ord}_E(\omega_n(\beta))$ since $\mathrm{ord}_E(\omega_n(\beta)) = \mathrm{ord}_E(\omega_n(\gamma)) = 2n+1$. Thus we can transform the above matrix into

$$\begin{pmatrix} \pi^{2n+\mathrm{ord}_E(\alpha)} & 0 & 0 \\ 0 & \pi^{2n} & 0 \\ 0 & 0 & \pi^{2n+2} \end{pmatrix}.$$

This implies $N/\omega_n N \cong \mathcal{O}_E/(\pi^{2n+\mathrm{ord}_E(\alpha)}) \oplus \mathcal{O}_E/(\pi^{2n}) \oplus \mathcal{O}_E/(\pi^{2n+2})$.

Next, we prove the case (b). For $n \geq 1$, we have

$$\mathrm{ord}_E(\omega_n(\beta)) = \begin{cases} 2\,\mathrm{ord}_3(c_0-3)+2n-1 & \text{if} \quad \mathrm{ord}_3(c_0-3) \leq \mathrm{ord}_3(c_1-3), \\ 2\,\mathrm{ord}_3(c_1-3)+2n & \text{if} \quad \mathrm{ord}_3(c_0-3) > \mathrm{ord}_3(c_1-3). \end{cases}$$

On the other hand, for $n \geq 1$, we have

$$\mathrm{ord}_E(\omega_n(\beta)-\omega_n(\gamma)) \begin{cases} = 2\,\mathrm{ord}_3(c_0-3)+2n-1 & \text{if} \quad \mathrm{ord}_3(c_0-3) \leq \mathrm{ord}_3(c_1-3), \\ > 2\,\mathrm{ord}_3(c_1-3)+2n & \text{if} \quad \mathrm{ord}_3(c_0-3) > \mathrm{ord}_3(c_1-3) \end{cases}$$

(cf. [7], Lemma 2.5). The rest can be proved by the same method as the case (a). □

In order to determine the structure of X, we will use the higher Fitting ideals. For a commutative ring R and a finitely presented R-module M, we consider the following exact sequence

$$R^m \xrightarrow{f} R^n \to M \to 0,$$

where m and n are positive integers. For an integer $i \geq 0$ such that $0 \leq i < n$, the i-th Fitting ideal of M is defined to be the ideal of R generated by all $(n-i)\times(n-i)$ minors of the matrix corresponding to f. This definition does not depend on the choice of the above exact sequence (see [9]).

Proposition 5.7. *Let $f(T)=(T-\alpha)g(T)$, where $\alpha \in p\mathbb{Z}_p$ and $g(T) \in \mathbb{Z}_p[T]$ is an irreducible polynomial of degree 2. Let E be the splitting field of $g(T)$ over $\mathbb{Q}_p$. Let $[M]_E \in \mathcal{M}^E_{f(T)}$ and $M = M(m,n,x)$.*

(1) *Assume $m=0$ and $(\gamma-\beta)x\pi^{-n} \in \mathcal{O}_E^\times$. Then we have*

$$\mathrm{Fitt}_{1,\Lambda}(M) = \begin{cases} (T-\alpha,\,(\alpha-\beta)(\alpha-\gamma)) & \textit{if} \quad x=1, \\ (T-\alpha,\,(\alpha-\beta)(\alpha-\gamma)(1-x)\pi^{-n}) & \textit{if} \quad x \neq 1. \end{cases}$$

(2) *Assume $n=0$ and $(\beta-\alpha)\pi^{-m} \in \mathcal{O}_E^\times$. Then we have*

$$\mathrm{Fitt}_{1,\Lambda}(M) = (T-\gamma,\,(\alpha-\gamma)(\beta-\gamma)).$$

(3)

$$\mathrm{Fitt}_{1,\Lambda}((T-\alpha)M) = \begin{cases} (T-\beta,\,(\beta-\gamma)\pi^{-n}) & \textit{if} \quad n \leq \mathrm{ord}_E(\pi^m-x), \\ \left(T-\beta,\,\dfrac{\gamma-\beta}{\pi^m-x}\right) & \textit{if} \quad n > \mathrm{ord}_E(\pi^m-x). \end{cases}$$

Proof. By the action of T, we have

$$
\begin{aligned}
T(1,1,1) &= (\alpha,\beta,\gamma)\\
&= \alpha(1,1,1) + (\beta-\alpha)\pi^{-m}(0,\pi^m,x)\\
&\quad + \{\gamma-\alpha-(\beta-\alpha)\pi^{-m}x\}\pi^{-n}(0,0,\pi^n),\\
T(0,\pi^m,x) &= (0,\beta\pi^m,\gamma x)\\
&= \beta(0,\pi^m,x) + (\gamma-\beta)x\pi^{-n}(0,0,\pi^n),\\
T(0,0,\pi^n) &= \gamma(0,0,\pi^n).
\end{aligned}
$$

Then we get the following matrix

$$
\begin{pmatrix}
T-\alpha & -(\beta-\alpha)\pi^{-m} & -\{(\gamma-\alpha)-(\beta-\alpha)\pi^{-m}x\}\pi^{-n}\\
0 & T-\beta & -(\gamma-\beta)x\pi^{-n}\\
0 & 0 & T-\gamma
\end{pmatrix}.
$$

We first show (1). Under the assumption of (1), the matrix is

$$
\begin{pmatrix}
T-\alpha & -\beta+\alpha & -\{(\gamma-\alpha)-(\beta-\alpha)x\}\pi^{-n}\\
0 & T-\beta & -(\gamma-\beta)x\pi^{-n}\\
0 & 0 & T-\gamma
\end{pmatrix}.
$$

By elementary row and column operations, we can transform the above matrix into

$$
\begin{pmatrix}
T-\alpha & (\alpha-\gamma)(1-x)\pi^{-n}(T-\beta) & 0\\
0 & (T-\beta)(T-\gamma) & 0\\
0 & 0 & 1
\end{pmatrix}.
$$

Therefore we get

$$
\begin{aligned}
\mathrm{Fitt}_{1,\Lambda}(M) &= (T-\alpha,(\alpha-\beta)(\alpha-\gamma),(\alpha-\beta)(\alpha-\beta)(1-x)\pi^{-n})\\
&= \begin{cases}(T-\alpha,(\alpha-\beta)(\alpha-\gamma)) & \text{if} \quad x=1,\\ (T-\alpha,(\alpha-\beta)(\alpha-\gamma)(1-x)\pi^{-n}) & \text{if} \quad x\neq 1.\end{cases}
\end{aligned}
$$

Next we show (2). Under the assumption of (2), the matrix is

$$
\begin{pmatrix}
T-\alpha & -(\beta-\alpha)\pi^{-m} & -(\gamma-\alpha)+(\beta-\alpha)\pi^{-m}x\\
0 & T-\beta & -(\gamma-\beta)x\\
0 & 0 & T-\gamma
\end{pmatrix}.
$$

By elementary row and column operations, we can transform the above matrix into

$$
\begin{pmatrix}
T-\alpha & 1 & 0\\
0 & T-\beta & 0\\
0 & 0 & T-\gamma
\end{pmatrix}.
$$

Therefore we get

$$\begin{aligned}\mathrm{Fitt}_{1,\Lambda}(M) &= ((T-\alpha)(T-\beta), (T-\beta)(T-\gamma), (T-\alpha)(T-\gamma), (T-\gamma)) \\ &= (T-\gamma, (\alpha-\gamma)(\beta-\gamma)).\end{aligned}$$

Finally we show (3). We note that

$$\begin{aligned}(T-\alpha)M &= \langle (0, \beta-\alpha, \gamma-\alpha), (0, (\beta-\alpha)\pi^m, (\gamma-\alpha)x), (0, 0, (\gamma-\alpha)\pi^n)\rangle_{\mathcal{O}_E} \\ &= \begin{cases} \langle (0, \beta-\alpha, \gamma-\alpha), (0, 0, (\gamma-\alpha)\pi^n)\rangle_{\mathcal{O}_E} & \text{if} \quad n \le \mathrm{ord}_E(\pi^m - x), \\ \langle (0, \beta-\alpha, \gamma-\alpha), (0, 0, (\gamma-\alpha)(\pi^m - x))\rangle_{\mathcal{O}_E} & \text{if} \quad n > \mathrm{ord}_E(\pi^m - x). \end{cases}\end{aligned}$$

In the case $n \le \mathrm{ord}_E(\pi^m - x)$, by the action of T, we have

$$\begin{aligned}T(0, \beta-\alpha, \gamma-\alpha) &= (0, \beta(\beta-\alpha), \gamma(\gamma-\alpha)) \\ &= \beta(0, \beta-\alpha, \gamma-\alpha) + (\gamma-\beta)\pi^{-n}(0, 0, (\gamma-\alpha)\pi^n), \\ T(0, 0, (\gamma-\alpha)\pi^n) &= \gamma(0, 0, (\gamma-\alpha)\pi^n).\end{aligned}$$

Thus we get the following matrix

$$\begin{pmatrix} T-\beta & -(\gamma-\beta)\pi^{-n} \\ 0 & T-\gamma \end{pmatrix}.$$

Therefore we get

$$\begin{aligned}\mathrm{Fitt}_{1,\Lambda}((T-\alpha)M) &= (T-\beta, T-\gamma, (\gamma-\beta)\pi^{-n}) \\ &= (T-\beta, (\gamma-\beta)\pi^{-n}).\end{aligned}$$

In the case $n > \mathrm{ord}_E(\pi^m - x)$, by the same method as above, we get the following matrix

$$\begin{pmatrix} T-\beta & -\dfrac{\gamma-\beta}{\pi^m - x} \\ 0 & T-\gamma \end{pmatrix}.$$

Therefore we get

$$\begin{aligned}\mathrm{Fitt}_{1,\Lambda}((T-\alpha)M) &= \left(T-\beta, T-\gamma, \frac{\gamma-\beta}{\pi^m - x}\right) \\ &= \left(T-\beta, \frac{\gamma-\beta}{\pi^m - x}\right).\end{aligned}$$

□

6. Numerical examples

In this section, we introduce some numerical examples which were computed using Pari-Gp.

Let $p = 3$ and $k = \mathbb{Q}(\sqrt{-d})$ where d is a positive square-free integer. For simplicity, let $d \not\equiv 2 \bmod 3$. Our assumption $d \not\equiv 2 \bmod 3$ implies that $p = 3$ is inert or ramified in k. This assumption is also needed to get the isomorphism (16) below. In this section, we determine the Λ-isomorphism class of the Iwasawa module associated to $k = \mathbb{Q}(\sqrt{-d})$ in the range $1 < d < 10^5$ with $\lambda_p(k) = 3$, where $\lambda_p(k)$ is the Iwasawa λ-invariant with respect to the cyclotomic $\mathbb{Z}_p$-extension. There are 1109 imaginary quadratic fields satisfying these properties.

Let k_∞/k be the cyclotomic $\mathbb{Z}_p$-extension of k. For each $n \geq 0$, we denote by k_n the intermediate field of k_∞/k such that k_n is the unique cyclic extension over k of degree p^n. Let A_n be the p-Sylow subgroup of the ideal class group of k_n. We put $X = \varprojlim A_n$, where the inverse limit is taken with respect to the relative norms. Then X becomes a $\mathbb{Z}_p[[\mathrm{Gal}(k_\infty/k)]]$-module. Since there is a ring isomorphism between $\Lambda = \mathbb{Z}_p[[T]]$ and $\mathbb{Z}_p[[\mathrm{Gal}(k_\infty/k)]]$ which depends on the choice of a topological generator of $\mathrm{Gal}(k_\infty/k)$, X becomes a finitely generated torsion Λ-module. Let $f(T)$ be the distinguished polynomial which generates $\mathrm{char}(X)$. It is known that X is a free $\mathbb{Z}_p$-module so $[X]_{\mathbb{Q}_p} \in \mathcal{M}_{f(T)}^{\mathbb{Q}_p}$ and we can apply Theorem 3.5 to the Iwasawa module X.

We can calculate the polynomial $f(T) \bmod p^n$ for small n numerically. Let χ be the Dirichlet character associated to k, ω be the Teichimüler character and f_0 be the least common multiple of p and conductor of χ. By the Iwasawa main conjecture, there exists a power series $g_{\chi^{-1}\omega}(T) \in \Lambda$ such that

$$\mathrm{char}(X) = (g_{\chi^{-1}\omega}(T)).$$

Here, $g_{\chi^{-1}\omega}(T)$ is the p-adic L-function constructed by Iwasawa. We can approximate $g_{\chi^{-1}\omega}(T)$ such as

$$g_{\chi^{-1}\omega}(T) \equiv -\frac{1}{2 f_0 p^n} \sum_{0<a<f_0p^n, (a, f_0p^n)=1} a\chi\omega^{-1}(a)(1+T)^{i_n(a)} \mod \omega_n,$$

where $i_n(a)$ is the unique integer such that $a\omega^{-1}(a) \equiv (1+p)^{i_n(a)} \bmod p^{n+1}$ and $0 \leq i_n(a) < p^n$. By Weierstrass preparation theorem ([13], Theorem 7.3), there exists $u_{\chi^{-1}\omega} \in \Lambda^\times$ such that $g_{\chi^{-1}\omega}(T) = f(T)u_{\chi^{-1}\omega}(T)$. Thus we can get $f(T)$ approximately ([13], Proposition 7.2). For the detail about computation of $g_{\chi^{-1}\omega}(T)$, see [1] and [4]. We computed $f(T)$ by Mizusawa's program Iwapoly.ub ([8], Research, Programing, Approximate Computation of Iwasawa Polynomials by UBASIC), and referred Fukuda's table for the λ-invariants of imaginary quadratic fields [3].

Now we classify the Iwasawa module X. There are two cases

$$\begin{cases} \text{(I)} & A_0 \text{ is a cyclic group,} \\ \text{(II)} & A_0 \text{ is not a cyclic group.} \end{cases}$$

In order to determine the structure of X, we use the following fact. In our case, exactly one prime is ramified in k_∞/k and it is totally ramified. So there are

Λ-isomorphisms

(16) $$X/\omega_n X \cong A_n$$

for any non-negative integers ([13], Proposition 13.22).

We determine the Λ-isomorphism class of X by the information on the structures of A_n for some $n \geq 0$.

There are 1015 fields whose A_0 are cyclic groups among 1109 fields. First of all, we determine the isomorphism classes in the case (I). In this case, X becomes a Λ_E-cyclic module by Nakayama's Lemma. Thus we can use Proposition 5.2 to get

$$M \cong M(\mathrm{ord}_E(\beta - \alpha), \mathrm{ord}_E(\gamma - \alpha) + \mathrm{ord}_E(\gamma - \beta), u\pi^{\mathrm{ord}_E(\beta-\alpha)}).$$

In the above range of d, no $f(T)$ splits completely in $\mathbb{Q}_p[T]$, so we have to consider the minimal splitting field E of $f(T)$, which is quadratic over $\mathbb{Q}_p$.

EXAMPLE 6.1. Let $k = \mathbb{Q}(\sqrt{-886})$. Then we have $A_0 \cong \mathbb{Z}/9\mathbb{Z}$ (cf. [10]). By using Mizusawa's program [8], we have

$$f(T) \equiv (T - 195)(T^2 + 291T + 429) \mod 3^6.$$

By Hensel's lemma, there exist $\alpha \in \mathbb{Z}_p$ and $g(T) \in \mathbb{Z}_p[T]$ such that

$$f(T) = (T - \alpha)g(T),$$

where $\alpha \equiv 195 \mod 3^5$ and $g(T) \equiv T^2 + 48T + 186 \mod 3^5$. Since $g(T)$ is an Eisenstein polynomial, $E/\mathbb{Q}_p$ is a totally ramified extension. Let E be the minimal splitting field of $g(T)$ and $g(T) = (T - \beta)(T - \gamma)$, where $\beta, \gamma \in E$. Because $\beta\gamma \equiv 186 \mod 3^5$, we get $\mathrm{ord}_E(\beta) = \mathrm{ord}_E(\gamma) = 1$ and $\mathrm{ord}_E(\alpha - \gamma) = \mathrm{ord}_E(\alpha - \gamma) = 1$. Since $(\beta - \gamma)^2 = (\beta + \gamma)^2 - 4\beta\gamma \equiv 1560 \mod 3^5$, we have $\mathrm{ord}_E(\beta - \gamma) = 1$. By Proposition 5.1 and 5.2, we get $X \otimes_\Lambda \Lambda_E \cong M(1, 2, u\pi)$, where $u = (\gamma - \alpha)/(\beta - \alpha)$.

Next, we determine the isomorphism classes in the case (II). There are 94 fields whose A_0 are not cyclic groups. There are 66 fields whose A_0 are not cyclic groups and whose $f(T)$ is reducible. We will determine $[X]_{\mathbb{Q}_p}$ for these 66 fields. We can determine the Λ-isomorphism class of X for 60 fields by Proposition 5.5. The following example is a case that we can determine the Λ-isomorphism class of X by Proposition 5.5.

EXAMPLE 6.2. Let $k = \mathbb{Q}(\sqrt{-6583})$. In this case, we have $A_0 \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ (cf. [10]). We have

$$f(T) \equiv (T - 96)(T^2 + 96T + 696) \mod 3^6.$$

By Hensel's lemma, there exist $\alpha \in \mathbb{Z}_p$ and $g(T) \in \mathbb{Z}_p[T]$ such that

$$f(T) = (T - \alpha)g(T),$$

where $\alpha \equiv 96 \bmod 3^5$ and $g(T) \equiv T^2 + 96T + 210 \bmod 3^5$. Let E be the minimal splitting field of $g(T)$ and $g(T) = (T - \beta)(T - \gamma)$, where $\beta, \gamma \in E$. Then, $E/\mathbb{Q}_p$ is a totally ramified extension and we get $\mathrm{ord}_E(\alpha - \beta) = \mathrm{ord}_E(\beta - \gamma) = \mathrm{ord}_E(\gamma - \alpha) = 1$, $\mathrm{ord}_E(\alpha) = 2$ and $\mathrm{ord}_E(\beta) = \mathrm{ord}_E(\gamma) = 1$. Therefore we get $X \otimes_\Lambda \Lambda_E \cong M(0, 1, 1)$ by Proposition 5.5.

There are remaining 6 fields which we cannot determine the structure of X by Proposition 5.5. For these fields, we have to investigate the action of the group $\Gamma_1 = \mathrm{Gal}(k_1/k)$. Explicitly, the remaining 6 fields are $\mathbb{Q}(\sqrt{-9574})$, $\mathbb{Q}(\sqrt{-30994})$, $\mathbb{Q}(\sqrt{-41631})$, $\mathbb{Q}(\sqrt{-64671})$, $\mathbb{Q}(\sqrt{-82774})$, $\mathbb{Q}(\sqrt{-92515})$.

EXAMPLE 6.3. Let $k = \mathbb{Q}(\sqrt{-9574})$. In this case, we have $A_0 \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$ (cf. [10]) and $A_1 \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/27\mathbb{Z}$. We have

$$f(T) \equiv (T - 192)(T^2 + 1173T + 1422) \mod 3^7.$$

By Hensel's Lemma, there exist $\alpha \in \mathbb{Z}_p$ and $g(T) \in \mathbb{Z}_p[T]$ such that

$$f(T) = (T - \alpha)g(T),$$

where $\alpha \equiv 192 \bmod 3^5$ and $g(T) \equiv T^2 + 201T + 207 \bmod 3^5$. Let E be the splitting field of $g(T)$ and $g(T) = (T - \beta)(T - \gamma)$, where $\beta, \gamma \in E$. Because the discriminant of $g(T)$ is $3^2 \cdot 4397 \bmod 3^7$ and 4397 is a quadratic nonresidue, $E/\mathbb{Q}_p$ is an unramified extension. Since the discriminant of $f(T)$ is $2^8 \cdot 3^6 \cdot 43 \cdot 89 \cdot 1039 \bmod 3^7$, we get $\mathrm{ord}_E(\alpha - \beta) = \mathrm{ord}_E(\beta - \gamma) = \mathrm{ord}_E(\gamma - \alpha) = 1$ and $\mathrm{ord}_E(\alpha) = \mathrm{ord}_E(\beta) = \mathrm{ord}_E(\gamma) = 1$. By checking the structures of A_0 and A_1 as $\mathcal{O}_E$-modules, we get

$$X \otimes_\Lambda \Lambda_E \cong M(0, 1, 1),\ M(0, 1, 2),\ M(1, 0, 0) \text{ or } M(1, 1, 0).$$

Now we investigate the structure of A_1 as a Γ_1-module. We have an isomorphism $A_1 \cong \mathbb{Z}/27\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. Furthermore, Pari-Gp gives explicit generators which give this isomorphism. Let $\mathfrak{a}_1$, $\mathfrak{a}_2$ and $\mathfrak{a}_3$ be the generators which was computed. (We do not write down $\mathfrak{a}_1$, $\mathfrak{a}_2$ and $\mathfrak{a}_3$ because they are complicated.) Let σ be the generator of Γ_1, which was computed by Pari-Gp. We compute,

$$(\sigma - 1)\mathfrak{a}_1 = 3\mathfrak{a}_2 - \mathfrak{a}_3,$$

$$(\sigma - 1)\mathfrak{a}_2 = 6\mathfrak{a}_2,$$

$$(\sigma - 1)\mathfrak{a}_3 = 18\mathfrak{a}_1 + 6\mathfrak{a}_2.$$

There is a topological generator $\tilde{\sigma} \in \mathrm{Gal}(k_\infty/k)$ such that $\tilde{\sigma}$ is an extension of σ. By this topological generator, we have an isomorphism

$$\mathbb{Z}_p[[\mathrm{Gal}(k_\infty/k)]] \cong \Lambda = \mathbb{Z}_p[[T]] \quad \text{such that} \quad \tilde{\sigma} \leftrightarrow 1 + T.$$

We regard X as a Λ-module by this isomorphism. We note that $f(T)$ depends on the choice of $\tilde{\sigma}$, but we can easily check that $\mathcal{M}^E_{f(T)}$ does not depend on the choice of $\tilde{\sigma}$. Because $\mathbb{Z}_p[[\Gamma_1]] \cong \Lambda/\omega_1\Lambda$, we get

$$\begin{aligned}\overline{T}\mathfrak{a}_1 &= 3\mathfrak{a}_2 - \mathfrak{a}_3,\\ \overline{T}\mathfrak{a}_2 &= 6\mathfrak{a}_2,\\ \overline{T}\mathfrak{a}_3 &= 18\mathfrak{a}_1 + 6\mathfrak{a}_2,\end{aligned}$$

where $\overline{T} = T \bmod \omega_1$. Now we have

$$\begin{aligned}\overline{(T^2+18)}\mathfrak{a}_1 + \overline{6}\mathfrak{a}_2 &= 0,\\ \overline{(T-6)}\mathfrak{a}_2 &= 0,\\ \overline{3T}\mathfrak{a}_1 &= 0,\\ \overline{27}\mathfrak{a}_1 &= 0,\\ \overline{9}\mathfrak{a}_2 &= 0.\end{aligned}$$

Therefore we can calculate the 1-st Fitting ideal of $A_1 \otimes \mathcal{O}_E$;

$$\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}(A_1 \otimes \mathcal{O}_E) = (T, 3) \mod \omega_1,$$

where $\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}(A_1 \otimes \mathcal{O}_E)$ is the 1-st Fitting ideal of $A_1 \otimes \mathcal{O}_E$ as a $\Lambda_E/\omega_1\Lambda_E$-module. On the other hand, by Proposition 5.7 (1) and (2), for $M(0, 1, 2)$, $M(1, 0, 0)$, $M(0, 1, 1)$, we have

$$\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}(M/\omega_1 M) = \begin{cases}(T, 3) \bmod \omega_1 & \text{if} \quad M = M(0,1,2),\\ (T-\gamma, 9) \bmod \omega_1 & \text{if} \quad M = M(1,0,0),\\ (T-\alpha, 9) \bmod \omega_1 & \text{if} \quad M = M(0,1,1).\end{cases}$$

Therefore we have

$$X \otimes_\Lambda \Lambda_E \cong M(0, 1, 2) \quad \text{or} \quad M(1, 1, 0).$$

We investigate the module $(T-\alpha)(M/\omega_1 M)$. By Proposition 5.7 (3), for $M(0,1,2)$, $M(1, 1, 0)$ we get

$$\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}((T-\alpha)(M/\omega_1 M)) = \begin{cases}(T, 3) \bmod \omega_1 & \text{if} \quad M = M(0,1,2),\\ \Lambda_E/\omega_1\Lambda_E & \text{if} \quad M = M(1,1,0).\end{cases}$$

We can compute the following from the above data

$$\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}(\overline{(T-\alpha)}A_1 \otimes \mathcal{O}_E) = (T, 3) \mod \omega_1.$$

Therefore, we get $X \otimes_\Lambda \Lambda_E \cong M(0, 1, 2)$.

By the same method as above, we can determine the isomorphism classes of X of $\mathbb{Q}(\sqrt{-30994})$, $\mathbb{Q}(\sqrt{-82774})$ and $\mathbb{Q}(\sqrt{-92515})$. For the 3 fields, we can show that $X \otimes_\Lambda \Lambda_E \cong M(0, 1, 2)$.

Finally we determine the structure of X for remaining 2 fields $\mathbb{Q}(\sqrt{-41631})$ and $\mathbb{Q}(\sqrt{-64671})$.

EXAMPLE 6.4. Let $k = \mathbb{Q}(\sqrt{-41631})$. In this case, we have $A_0 \cong \mathbb{Z}/3^3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ (cf. [10]) and $A_1 \cong \mathbb{Z}/3^4\mathbb{Z} \oplus \mathbb{Z}/3^2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ computing by Pari-Gp. We have

$$f(T) \equiv (T - 42)(T^2 - 279T + 594) \bmod 3^7.$$

By Hensel's lemma, there exist $\alpha \in \mathbb{Z}_p$ and $g(T) \in \mathbb{Z}_p[T]$ such that

$$f(T) = (T - \alpha)g(T),$$

where $\alpha \equiv 42 \bmod 3^5$ and $g(T) \equiv T^2 + 36T + 108 \bmod 3^5$. Let E be the minimal splitting field of $g(T)$ and $g(T) = (T - \beta)(T - \gamma)$, where $\beta, \gamma \in E$. Then $E/\mathbb{Q}_p$ is a totally ramified extension with $\mathrm{ord}_E(\alpha - \beta) = \mathrm{ord}_E(\gamma - \alpha) = 2$, $\mathrm{ord}_E(\beta - \gamma) = 3$, $\mathrm{ord}_E(\alpha) = 2$, and $\mathrm{ord}_E(\beta) = \mathrm{ord}_E(\gamma) = 3$. Let π be a prime element of E. In this case, the elements $M(m, n, x) \in \mathcal{M}^E_{f(T)}$ which satisfy the conclusion of Proposition 5.4 are

$$\left\{\begin{matrix} (0, 0, 0), (0, 1, 1), (0, 1, 2), (0, 2, 1), (0, 2, 2), (0, 2, 1 + \pi), (0, 3, 1), \\ (0, 3, 1 + \pi), (0, 3, 1 + \pi^2), (1, 0, 0), (1, 1, 0), (1, 1, 1), (1, 2, \pi), \\ (1, 2, 2\pi), (1, 3, \pi), (1, 3, \pi + \pi^2), (1, 3, \pi + 2\pi^2), (1, 4, u\pi), \\ (2, 0, 0), (2, 1, 0), (2, 2, 0), (2, 3, u\pi^2), (2, 4, u\pi^2), (2, 5, u\pi^2) \end{matrix}\right\},$$

where $u = (\gamma - \alpha)/(\beta - \alpha)$. By checking the structures of A_0 and A_1 as $\mathcal{O}_E$-modules, we get

$$X \otimes_\Lambda \Lambda_E \cong M(0, 3, 1), \quad M(0, 3, 1 + \pi), \quad M(0, 3, 1 + \pi^2), \\ M(1, 3, \pi + \pi^2), M(1, 3, \pi + 2\pi^2) \quad \text{or} \quad M(2, 3, u\pi^2).$$

We have an isomorphism $A_1 \cong \mathbb{Z}/81\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. Let $\mathfrak{a}_1$, $\mathfrak{a}_2$ and $\mathfrak{a}_3$ be the generators which were computed by Pari-Gp. By Pari-Gp we have:

$$(\sigma - 1)\mathfrak{a}_1 = 54\mathfrak{a}_1 + 6\mathfrak{a}_2 + \mathfrak{a}_3,$$
$$(\sigma - 1)\mathfrak{a}_2 = 54\mathfrak{a}_1,$$
$$(\sigma - 1)\mathfrak{a}_3 = 54\mathfrak{a}_1 + 3\mathfrak{a}_2,$$

for a certain generator σ of Γ_1. By the same method as $k = \mathbb{Q}(\sqrt{-9574})$, we fix a topological generator $\tilde{\sigma} \in \mathrm{Gal}(k_\infty/k)$ such that $\tilde{\sigma}$ is an extension of σ. Because

$\mathbb{Z}_p[[\Gamma_1]] \cong \Lambda/\omega_1\Lambda$, we have

$$\begin{aligned} \overline{(T^2-54T-54)}\mathfrak{a}_1 - \overline{3}\mathfrak{a}_2 &= 0, \\ \overline{54}\ \mathfrak{a}_1 - \overline{T}\mathfrak{a}_2 &= 0, \\ \overline{3T}\mathfrak{a}_1 &= 0, \\ \overline{81}\mathfrak{a}_1 &= 0, \\ \overline{9}\mathfrak{a}_2 &= 0, \end{aligned}$$

where $\overline{T} = T \bmod \omega_1$. Therefore we get the 1-st Fitting ideal of $A_1 \otimes \mathcal{O}_E$;

$$\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}(A_1 \otimes \mathcal{O}_E) = (T, 3) \mod \omega_1.$$

On the other hand, by Proposition 5.7 (1) and (2), we have

$$\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}(M/\omega_1 M) = \begin{cases} (T-\alpha, 9) \bmod \omega_1 & \text{if} \quad M = M(0,3,1), \\ (T, 3) \bmod \omega_1 & \text{if} \quad M = M(0,3,1+\pi), \\ (T-\alpha, \pi^3) \bmod \omega_1 & \text{if} \quad M = M(0,3,1+\pi^2), \end{cases}$$

for $M(0,3,1)$, $M(0,3,1+\pi)$ and $M(0,3,1+\pi^2)$. Therefore we have

$$X \otimes_\Lambda \Lambda_E \cong M(0,3,1+\pi),\ M(1,3,\pi+\pi^2),\ M(1,3,\pi+2\pi^2) \text{ or } M(2,3,u\pi^2).$$

As in the case $k = \mathbb{Q}(\sqrt{-9574})$, we investigate the structure of $(T-\alpha)(M/\omega_1 M)$. By Proposition 5.7 (3), we get

$$\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}((T-\alpha)(M/\omega_1 M)) = \begin{cases} (T, 3) \bmod \omega_1 & \text{if} \quad M = M(0,3,1+\pi), \\ \Lambda_E/\omega_1\Lambda_E & \text{if} \quad M = M(1,3,\pi+\pi^2), \\ (T, \pi) \bmod \omega_1 & \text{if} \quad M = M(1,3,\pi+2\pi^2), \\ \Lambda_E/\omega_1\Lambda_E & \text{if} \quad M = M(2,3,u\pi^2). \end{cases}$$

We can compute from the above data

$$\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}(\overline{(T-\alpha)}A_1 \otimes \mathcal{O}_E) = (T, 3) \mod \omega_1.$$

Therefore we get $X \otimes_\Lambda \Lambda_E \cong M(0,3,1+\pi)$.

We can determine the structure of $\mathbb{Q}(\sqrt{-64671})$ by the same method as above. For $\mathbb{Q}(\sqrt{-64671})$, we can show that $X \otimes_\Lambda \Lambda_E \cong M(0,3,1+\pi)$.

The following is the table of the $X \otimes_\Lambda \Lambda_E$ for the fields such that A_0 is not cyclic and $f(T)$ is reducible. Here, m, n, x represent $X \otimes \Lambda_E \cong M(m,n,x)$, and ram. /unram. means that $E/\mathbb{Q}_3$ is ramified /unramified extension, respectively. We marked $(*)$ on the remaining 6 fields for which we determined the structures in Example 6.3 and 6.4.

Table 1.

	d	$\text{ord}_E(\alpha-\beta)$	$\text{ord}_E(\beta-\gamma)$	$\text{ord}_E(\gamma-\alpha)$	$E/\mathbb{Q}_3$	m	n	x	A_0
	6583	1	1	1	ram.	0	1	1	(3, 3)
	8751	1	1	1	ram.	0	1	1	(3, 3)
	9069	1	1	1	ram.	0	1	1	(3, 3)
(∗)	9574	1	1	1	unram.	0	1	2	$(3^2, 3)$
	12118	1	1	1	ram.	0	1	1	(3, 3)
	16627	1	1	1	ram.	0	1	1	(3, 3)
	21018	1	1	1	ram.	0	1	1	(3, 3)
	23178	1	1	1	ram.	0	1	1	(3, 3)
	24109	1	1	1	ram.	0	1	1	(3, 3)
	25122	1	1	1	ram.	0	1	1	(3, 3)
	29569	1	1	1	ram.	0	1	1	(3, 3)
	29778	1	1	1	ram.	0	1	1	(3, 3)
	29994	1	1	1	ram.	0	1	1	(3, 3)
(∗)	30994	1	1	1	unram.	0	1	2	$(3^2, 3)$
	31999	1	1	1	ram.	0	1	1	(3, 3)
	34507	1	1	1	ram.	0	1	1	(3, 3)
	34867	1	1	1	ram.	0	1	1	(3, 3)
	35539	1	1	1	ram.	0	1	1	(3, 3)
	37213	1	1	1	ram.	0	1	1	(3, 3)
	37237	1	1	1	ram.	0	1	1	(3, 3)
	38226	1	1	1	ram.	0	1	1	(3, 3)
	38553	1	1	1	ram.	0	1	1	(3, 3)
	38926	1	1	1	ram.	0	1	1	(3, 3)
	40299	1	1	1	ram.	0	1	1	(3, 3)
	41583	1	1	1	ram.	0	1	1	(3, 3)
(∗)	41631	2	3	2	ram.	0	3	$1+\pi$	$(3^3, 3)$
	41671	1	1	1	ram.	0	1	1	(3, 3)
	45210	1	1	1	ram.	0	1	1	(3, 3)
	45753	1	1	1	ram.	0	1	1	(3, 3)
	45942	1	1	1	ram.	0	1	1	(3, 3)
	46198	1	1	1	ram.	0	1	1	(3, 3)
	47199	1	1	1	ram.	0	1	1	$(3^2, 3)$
	48667	1	1	1	ram.	0	1	1	(3, 3)

Table 2.

	d	$\mathrm{ord}_E(\alpha-\beta)$	$\mathrm{ord}_E(\beta-\gamma)$	$\mathrm{ord}_E(\gamma-\alpha)$	$E/\mathbb{Q}_3$	m	n	x	A_0
	49074	1	1	1	ram.	0	1	1	(3, 3)
	51142	1	1	1	ram.	0	1	1	(3, 3)
	52858	1	1	1	ram.	0	1	1	(3, 3)
	53839	1	1	1	ram.	0	1	1	(3, 3)
	53862	1	1	1	ram.	0	1	1	(3, 3)
	54319	1	1	1	ram.	0	1	1	(3, 3)
	54853	1	1	1	ram.	0	1	1	(3, 3)
	56773	1	1	1	ram.	0	1	1	(3, 3)
	59478	1	1	1	ram.	0	1	1	(3, 3)
	59578	1	1	1	ram.	0	1	1	(3, 3)
	60099	1	1	1	ram.	0	1	1	(3, 3)
(∗)	64671	2	3	2	ram.	0	3	$1+\pi$	$(3^2, 3)$
	68314	1	1	1	ram.	0	1	1	(3, 3)
	72591	1	1	1	ram.	0	1	1	(3, 3)
	75273	1	1	1	ram.	0	1	1	(3, 3)
	75354	1	1	1	ram.	0	1	1	$(3^2, 3)$
	75790	1	1	1	ram.	0	1	1	(3, 3)
	75841	1	1	1	ram.	0	1	1	(3, 3)
	78181	1	1	1	ram.	0	1	1	$(3^2, 3)$
	80233	1	1	1	ram.	0	1	1	(3, 3)
	80242	1	1	1	ram.	0	1	1	$(3^2, 3)$
	80746	1	1	1	ram.	0	1	1	(3, 3)
(∗)	82774	1	1	1	unram.	0	1	2	$(3^2, 3)$
	87727	1	1	1	ram.	0	1	1	(3, 3)
	87979	1	1	1	ram.	0	1	1	$(3^2, 3)$
	88134	1	1	1	ram.	0	1	1	$(3^2, 3)$
	88242	1	1	1	ram.	0	1	1	(3, 3)
(∗)	92515	1	1	1	unram.	0	1	2	$(3^2, 3)$
	94998	1	1	1	ram.	0	1	1	(3, 3)
	95691	1	1	1	ram.	0	1	1	(3, 3)
	97555	1	1	1	ram.	0	1	1	(3, 3)
	98277	1	1	1	ram.	0	1	1	(3, 3)
	98929	1	1	1	ram.	0	1	1	(3, 3)

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