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ON THE ISOMORPHISM CLASSES OF
IWASAWA MODULES WITH $\lambda = 3$ AND $\mu = 0$

KAZUAKI MURAKAMI

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Abstract

For an odd prime number $p$, we classify the isomorphism classes of finitely generated torsion $\Lambda = \mathbb{Z}_p[[T]]$-modules with $\lambda = 3$ and $\mu = 0$, which are free over $\mathbb{Z}_p$. We apply this classification to the Iwasawa module associated to the cyclotomic $\mathbb{Z}_p$-extension of an imaginary quadratic field.

1. Introduction

Let $p$ be a fixed odd prime number and $\Lambda = \mathbb{Z}_p[[T]]$ the ring of power series in one variable over $\mathbb{Z}_p$. In the classical Iwasawa theory, one studies Iwasawa modules up to pseudo-isomorphism. In this paper, we study Iwasawa modules up to $\Lambda$-isomorphism. Especially, our aim is to generalize Sumida’s results (cf. [11], [12]).

For a distinguished polynomial $f(T) \in \mathbb{Z}_p[T]$, Sumida introduced the set

$$M_{f(T)} = \left\{ [M]_{\mathbb{Q}_p} \mid M \text{ is a finitely generated torsion } \Lambda\text{-module, char}(M) = (f(T)) \text{ and } M \text{ is free over } \mathbb{Z}_p \right\},$$

where $[M]_{\mathbb{Q}_p}$ is the $\Lambda$-isomorphism class of $M$ and char$(M)$ is the characteristic ideal of $M$. Sumida showed that $M_{f(T)}$ is a finite set if and only if $f(T)$ is a separable polynomial ([11], Theorem 2). Sumida and Koike determined $M_{f(T)}$ in the case $\deg(f(T)) \leq 2$ ([7], Theorem 2.1 and [11], Proposition 10). In this paper, we determine the set $M_{f(T)}$ for

$$f(T) = (T - \alpha)(T - \beta)(T - \gamma),$$

where $\alpha$, $\beta$, $\gamma$ are distinct elements of $p\mathbb{Z}_p$ (Theorem 3.5). This is a generalization of Sumida’s result [12]. (Precisely speaking, we work over $\mathcal{O}[[T]]$ below where $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbb{Q}_p$.)

The motivation of this work lies in Iwasawa theory. We apply our theorem to the Iwasawa module associated to the cyclotomic $\mathbb{Z}_p$-extension of an imaginary quadratic field. Let $k$ be an imaginary quadratic number field and $k_\infty/k$ the cyclotomic $\mathbb{Z}_p$-extension of $k$. For each $n \geq 0$, we denote by $k_n$ the unique intermediate field of

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\( k_\infty/k \) with \([k_n : k] = p^n\). Let \( A_n \) be the \( p \)-Sylow subgroup of the ideal class group of \( k_n \). We put \( X = \varprojlim A_n \), where the inverse limit is taken with respect to the relative norms. It is known that \( X \) is a finitely generated torsion \( \Lambda \)-module (cf. [5]). Moreover, it is known that \( X \) is a free \( \mathbb{Z}_p \)-module.

Therefore, we can apply our theorem to the Iwasawa module \( X \). We apply our theorem in the case that \( p = 3 \) and \( k = \mathbb{Q}(\sqrt{-d}) \). In this setting, \( f(T) \) can be approximately calculated by the \( p \)-adic \( L \)-functions (see Section 6).

The outline of this paper is as follows. Let \( E \) be a finite extension of \( \mathbb{Q}_p \) and \( \Lambda_E \) the ring of power series in one variable over the ring of integers of \( E \). In Section 2, we introduce the set \( \mathcal{M}_{E,f(T)}^E \) which is the set of isomorphism classes of \( \Lambda_E \)-module satisfying some properties. In Section 3, we state our main theorem (Theorem 3.5). We define a certain equivalence relation \( \sim \) on \( \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\leq 0} \times \mathcal{O}_E \) and define \( Z' = (\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\leq 0} \times \mathcal{O}_E)/\sim' \). We define \( Z \) to be a subset of \( Z' \) satisfying certain conditions. An element of \( Z' \) is written as \((m, n, x)\). We also define an equivalence relation \( \sim \) on \( Z \) and consider \( Z/\sim \). An element of \( Z/\sim \) is written as \([m, n, x]\). Roughly speaking, Theorem 3.5 states that there is one to one correspondence between \( \mathcal{M}_{E,f(T)}^E \) and the equivalence classes of \( Z/\sim \). Moreover, we prove Sumida’s result ([12], Theorem 1) in Corollary 3.8, using our Theorem 3.5. In Section 4, we give a proof of Theorem 3.5. Section 5 is a preparation for Section 6. In this section, we study the structure of \( \Lambda \)-modules. In Section 6, we apply Theorem 3.5 to the Iwasawa module associated to the cyclotomic \( \mathbb{Z}_p \)-extension of an imaginary quadratic number field. We apply our theorem in the case that \( p = 3 \) and \( k = \mathbb{Q}(\sqrt{-d}) \) for all \( d \) such that \( 1 < d < 10^5 \) and \( d \not\equiv 2 \mod 3 \), that is to say \( p \) does not split in \( k \). We determine the \( \Lambda \)-isomorphism class of the Iwasawa module associated to \( k \) in the case \( \lambda_p(k) = 3 \), where \( \lambda_p(k) \) is the Iwasawa \( \lambda \)-invariant. There are 1109 imaginary quadratic fields satisfying these properties. Among them, there are 1015 fields whose \( A_0 \) are cyclic groups. We can determine \([X]_{\mathbb{Q}_p}\) for these 1015 fields by Proposition 5.2 immediately. For remaining 94 fields whose \( A_0 \) are not cyclic groups, there are 66 fields whose \( f(T) \) is reducible. We determine \([X]_{\mathbb{Q}_p}\) for these 66 fields.

After I submitted this paper, I was informed from Sumida (Takahashi) of the thesis by C. Franks where he independently obtained the classification of \( \Lambda \)-modules. In Remark 3.6, I will explain the difference between our method and that in Franks.

## 2. Preliminaries

Let \( p \) be an odd prime number. Let \( E \) be a finite extension over the field \( \mathbb{Q}_p \) of \( p \)-adic numbers. Let \( \mathcal{O}_E, \pi, \text{ord}_E \) be the ring of integers in \( E \), a prime element and the normalized additive valuation of \( E \) such that \( \text{ord}_E(\pi) = 1 \), respectively. We put \( \Lambda_E := \mathcal{O}_E[[T]] \) the ring of power series over \( \mathcal{O}_E \).
Let $M$ be a finitely generated torsion $\Lambda_E$-module. By the structure theorem of $\Lambda_E$-modules (cf. [13], Chapter 13), there is a $\Lambda_E$-homomorphism

$$\varphi: M \to \left( \bigoplus_i \Lambda_E/(\pi^{m_i}) \right) \oplus \left( \bigoplus_j \Lambda_E/(f_j(T)^{n_j}) \right)$$

with finite kernel and finite cokernel, where $m_i, n_j$ are non-negative integers and $f_j(T) \in \mathcal{O}_E[T]$ is a distinguished irreducible polynomial. We put

$$\text{char}(M) = \left( \prod_i \pi^{m_i} \prod_j f_j(T)^{n_j} \right)$$

which is an ideal in $\Lambda_E$. We define $[M]_E$ to be the $\Lambda_E$-isomorphism class of $M$.

As in the introduction, for a distinguished polynomial $f(T) \in \mathcal{O}_E[T]$, we consider finitely generated torsion $\Lambda_E$-modules whose characteristic ideals are $(f(T))$, and define the set $\mathcal{M}^E_{f(T)}$ by

$$\mathcal{M}^E_{f(T)} = \left\{ [M]_E \mid M \text{ is a finitely generated torsion } \Lambda_E\text{-module, } \text{char}(M) = (f(T)) \text{ and } M \text{ is free over } \mathcal{O}_E \right\}.$$  

Sumida showed that $\mathcal{M}^E_{f(T)}$ is a finite set if and only if $f(T)$ is separable [11]. Here, we say $f(T)$ is separable when $f(T)$ has no multiple roots in an algebraic closure of $E$. Sumida also determined $\mathcal{M}_{f(T)}$ in the case $f(T) = (T - \alpha)(T - \beta)(T - \gamma)$, where $\alpha, \beta, \gamma \in p\mathbb{Z}_p$ satisfy $\alpha \neq \beta, \beta \neq \gamma, \gamma \neq \alpha \mod p^2$ (see [12], Theorem 1). We generalize this result to a general separable polynomial $f(T)$ with degree 3.

Now we put

$$f(T) = (T - \alpha)(T - \beta)(T - \gamma),$$

where $\alpha, \beta$ and $\gamma$ are distinct elements of $\pi\mathcal{O}_E$. We determine all the elements of $\mathcal{M}^E_{f(T)}$ in the next section.

Let $[M]_E \in \mathcal{M}^E_{f(T)}$. Since $M$ has no non-trivial finite $\Lambda_E$-submodule, there exists an injective $\Lambda_E$-homomorphism

$$\varphi: M \hookrightarrow \Lambda_E/(T - \alpha) \oplus \Lambda_E/(T - \beta) \oplus \Lambda_E/(T - \gamma) =: \mathcal{E}$$

with finite cokernel. We write $\mathcal{E}$ for the right hand side. The above fact implies that every class of $\mathcal{M}^E_{f(T)}$ can be represented by a $\Lambda_E$-submodule of $\mathcal{E}$.

Now we fix a notation to express such submodules in $\mathcal{E}$. First, by using the canonical isomorphism $\Lambda_E/(T - \alpha) \cong \mathcal{O}_E$ ($f(T) \mapsto f(\alpha)$), we define an isomorphism

$$i: \mathcal{E} \cong \Lambda_E/(T - \alpha) \oplus \Lambda_E/(T - \beta) \oplus \Lambda_E/(T - \gamma) \to \mathcal{O}_E^{\mathbb{Z}_3}$$
by \((f_1(T), f_2(T), f_3(T)) \mapsto (f_1(\alpha), f_2(\beta), f_3(\gamma))\). We identify \(E\) with \(O_E^{33}\) via \(t\). Thus

an element in \(E\) is expressed as \((a_1, a_2, a_3) \in O_E^{33}\). Since the rank of \(M\) is equal to 3, we can write \(M\) in the form

\[ M = \langle (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3) \rangle_{O_E} \subset E, \]

where \((*)_{O_E}\) is the \(O_E\)-submodule generated by \(*\). Further, we can express the action of \(T\) by

\[ T(a_1, a_2, a_3) = (\alpha a_1, \beta a_2, \gamma a_3), \]

using this notation.

3. Main result

Let \(M\) be an \(O_E\)-submodule of \(E\) with \(\text{rank}(M) = 3\) of the form

\[ M = \langle (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3) \rangle_{O_E} \subset E. \]

Let

\[ s = \min \{ i \in \mathbb{Z}_{\geq 0} \mid \exists a, b \in O_E \text{ s.t. } (\pi^i, a, b) \in M \}, \]

\[ t = \min \{ i \in \mathbb{Z}_{\geq 0} \mid \exists c \in O_E \text{ s.t. } (0, \pi^i, c) \in M \}, \]

\[ u = \min \{ i \in \mathbb{Z}_{\geq 0} \mid (0, 0, \pi^i) \in M \}. \]

Then we have

\[ M = \langle (\pi^s, a, b), (0, \pi^t, c), (0, 0, \pi^u) \rangle_{O_E}. \]

Suppose \((a_1, a_2, a_3) \in M\). Since \(\text{ord}_E(a_1) \geq s\), there exists \(x \in O_E\) such that \(a_1 = x\pi^s\). So \((a_1, a_2, a_3) - x(\pi^s, a, b) = (0, a_2 - xa, a_3 - xb) \in M\). Since \(\text{ord}_E(a_2 - xa) \geq t\), there exists \(y \in O_E\) such that \(a_2 - xa = y\pi^t\). Similarly by the same method as above, we get \((0, 0, a_3 - xb - yc) \in M\). Finally, there exists \(z \in O_E\) such that \(a_3 - xb - yc = z\pi^u\). Then we have \((a_1, a_2, a_3) = x(\pi^s, a, b) + y(0, \pi^t, c) + z(0, 0, \pi^u)\).

The following lemma is a necessary and sufficient condition for an \(O_E\)-module \(M\) to be a \(\Lambda_E\)-submodule.

**Lemma 3.1.** Let \(M = \langle (\pi^s, a, b), (0, \pi^t, c), (0, 0, \pi^u) \rangle_{O_E}\). Then the following two statements are equivalent:

(i) The \(O_E\)-module \(M\) is a \(\Lambda_E\)-submodule,

(ii) Integers \(a, b, c, s, t, u\) satisfy

\[
\begin{align*}
    t &\leq \text{ord}_E(\beta - \alpha) + \text{ord}_E(a), \\
u &\leq \text{ord}_E((\gamma - \alpha)b - (\beta - \alpha)\pi^sac), \\
u &\leq \text{ord}_E(\gamma - \beta) + \text{ord}_E(c).
\end{align*}
\]
Proof. We first suppose that \( M \) is a \( \Lambda_E \)-submodule. So \( M \) satisfies \( TM \subset M \) and we have

\[
T(\pi^s, a, b) = (\alpha \pi^s, \beta a, \gamma b)
= \alpha(\pi^s, a, b) + (\beta - \alpha)\pi^{-l}a(0, \pi^t, c) + (\gamma - \alpha \gamma)\pi^{-m}(0, 0, \pi^u).
\]

Thus we have

\[
T(0, \pi^t, c) = (0, \beta \pi^t, \gamma c)
= \beta(0, \pi^t, c) + (\gamma - \beta)\pi^{-m}(0, 0, \pi^u).
\]

Since these coefficients belong to \( \mathcal{O}_E \), we get (ii). Conversely, if an \( \mathcal{O}_E \)-module \( M \) satisfies (ii), \( M \) is naturally an \( \mathcal{O}_E[T] \)-module by the action as above. We show that \( M \) becomes a \( \Lambda_E \)-module. For a positive integer \( n \), we put \( v_n = \sum_{k=0}^n d_k T^k \in \mathcal{O}_E[T] \) and \( v = \sum_{n=0}^\infty d_n T^n \in \mathcal{O}_E[[T]] \). Then we have

\[
v_n(\pi^s, a, b) = \left( \pi^s \sum_{k=0}^n d_k \alpha^k, a \sum_{k=0}^n d_k \beta^k, b \sum_{k=0}^n d_k \gamma^k \right)
= \sum_{k=0}^n d_k \alpha^k(\pi^s, a, b) + a \left( \sum_{k=0}^n d_k \beta^k - \sum_{k=0}^n d_k \alpha^k \right) \pi^{-l}(0, \pi^t, c)
+ \left\{ b \left( \sum_{k=0}^n d_k \gamma^k - \sum_{k=0}^n d_k \alpha^k \right) - \left( \sum_{k=0}^n d_k \beta^k - \sum_{k=0}^n d_k \alpha^k \right) \pi^{-l}ac \right\} \pi^{-m}(0, 0, \pi^u).
\]

Because \( M \) is an \( \mathcal{O}_E[T] \)-module, we have \( v_n(\pi^s, a, b) \in M \). Thus we obtain

\[
a \left( \sum_{k=0}^n d_k \beta^k - \sum_{k=0}^n d_k \alpha^k \right) \pi^{-l} \in \mathcal{O}_E
\]

and

\[
\left\{ b \left( \sum_{k=0}^n d_k \gamma^k - \sum_{k=0}^n d_k \alpha^k \right) - \left( \sum_{k=0}^n d_k \beta^k - \sum_{k=0}^n d_k \alpha^k \right) \pi^{-l}ac \right\} \pi^{-m} \in \mathcal{O}_E.
\]

Since \( d_k \alpha^k, d_k \beta^k, d_k \gamma^k \to 0 \) (\( k \to \infty \)), \( \sum_{k=0}^\infty d_k \alpha^k, \sum_{k=0}^\infty d_k \beta^k \) and \( \sum_{k=0}^\infty d_k \gamma^k \) converge in \( \mathcal{O}_E \). Thus we have \( v(\pi^s, a, b) \in M \). For \( (0, \pi^t, c) \) and \( (0, 0, \pi^u) \), we can define the action of the elements of \( \Lambda_E \) by the same method as above.

We use the following lemma to fix a representative of the \( \Lambda_E \)-isomorphism class of \( M \).
Lemma 3.2 (Lemma 1 in Sumida [12]). Let $M = \langle a_1, a_2, a_3, b_1, b_2, b_3, (c_1, c_2, c_3) \rangle_{\mathcal{O}_E}$ be a $\Lambda_E$-submodule of $\mathcal{E}$ and $u_1, u_2, u_3 \in \mathcal{O}_E \setminus \{0\}$. Then we have

$$M \cong \langle (u_1a_1, u_2a_2, u_3a_3), (u_1b_1, u_2b_2, u_3b_3), (u_1c_1, u_2c_2, u_3c_3) \rangle_{\mathcal{O}_E}$$

as $\Lambda_E$-modules.

We take $M$ to be a $\Lambda_E$-submodule of $\mathcal{E}$ with finite index. Then we can write

$$M = \langle (\pi', a, b), (0, \pi', c), (0, 0, \pi^u) \rangle_{\mathcal{O}_E}$$

as we explained in the beginning of this section. By Lemma 3.2, there exist non-negative integers $m, n$ and $x \in \mathcal{O}_E$ such that there is an isomorphism

$$M \cong \langle (1, 1, 1), (0, \pi^m, x), (0, 0, \pi^n) \rangle_{\mathcal{O}_E}$$

as $\Lambda_E$-modules. In fact, by Lemma 3.2, $M$ is isomorphic to $M' = \langle (1, a, b), (0, \pi', c), (0, 0, \pi^u) \rangle_{\mathcal{O}_E}$. In the case $\ord_{\mathcal{E}}(a) \leq t$, by Lemma 3.2, $M$ is isomorphic to $\langle (1, 1, b), (0, a^{-1}\pi', c), (0, 0, \pi^u) \rangle_{\mathcal{O}_E}$. On the other hand, in the case $\ord_{\mathcal{E}}(a) > t$, since $M' = \langle (1, a + \pi', b + \pi), (0, \pi', c), (0, 0, \pi^u) \rangle_{\mathcal{O}_E}$, we can proceed by the same method as in the case $\ord_{\mathcal{E}}(a) \leq t$. Therefore $M$ is isomorphic to $M'' = \langle (1, 1, b), (0, a'\pi', c), (0, 0, \pi^u) \rangle_{\mathcal{O}_E}$ for some $a' \in \mathcal{E}$. By applying the same method as above, $M''$ is isomorphic to $\langle (1, 1, 1), (0, \pi^m, x), (0, 0, \pi^n) \rangle_{\mathcal{O}_E}$ for some non-negative integers $m, n$ and $x \in \mathcal{O}_E$.

We define $M(m, n, x)$ by

$$M(m, n, x) := \langle (1, 1, 1), (0, \pi^m, x), (0, 0, \pi^n) \rangle_{\mathcal{O}_E} \subset \mathcal{E}.$$ 

Proposition 3.3. Let $f(T) \in \mathcal{O}_E[T]$ be a distinguished polynomial. Then we have

$$\mathcal{M}^F_{f(T)} = \{ [M(m, n, x)]_{\mathcal{E}} \mid m, n, x \text{ satisfy } (*) \},$$

where $[M(m, n, x)]_{\mathcal{E}}$ is the $\Lambda_E$-isomorphism class of $M(m, n, x)$ and $(*)$ is as follows:

$$\begin{align*}
(A) \quad & 0 \leq m \leq \ord_{\mathcal{E}}(\beta - \alpha), \\
(B) \quad & 0 \leq n \leq \ord_{\mathcal{E}}(\gamma - \beta) + \ord_{\mathcal{E}}(x), \\
(C) \quad & n \leq \ord_{\mathcal{E}}((\gamma - \alpha) - (\beta - \alpha)\pi^m x).
\end{align*}$$

Proof. Let $M$ be a $\Lambda_E$-module such that $[M]_{\mathcal{E}} \in \mathcal{M}^F_{f(T)}$. Then we saw that $[M]_{\mathcal{E}} = [M(m, n, x)]_{\mathcal{E}}$ for some $m, n, x$ satisfying $(*)$ by Lemma 3.1. We will show the converse. We suppose that $m, n$ and $x$ satisfy $(*)$. By Lemma 3.1, $M(m, n, x)$ becomes a finitely generated $\Lambda_E$-module. Since $f(T) = (T-\alpha)(T-\beta)(T-\gamma)$ annihilates $M(m, n, x)$, $M(m, n, x)$ is a torsion $\Lambda_E$-module. Moreover, by the definition of $M(m, n, x)$, $M(m, n, x)$ is a free $\mathcal{O}_E$-module. Finally we show that $\text{char}(M(m, n, x)) = (f(T))$. The $\Lambda_E$-module $M(m, n, x)$
is a submodule of $\mathcal{E}$ with finite index. In fact, since $\text{rank}_{\mathcal{O}_E}(\mathcal{E}) = \text{rank}_{\mathcal{O}_E}(M(m, n, x)) = 3$, $\mathcal{E}/M(m, n, x)$ is finite. This implies that $\text{char}(M(m, n, x)) = \text{char}(\mathcal{E})$. Thus we get $[M(m, n, x)]_E \in \mathcal{M}_E$.

**Remark 3.4.** (i) If $x \equiv x' \mod \pi^n$, we have $M(m, n, x) = M(m, n, x')$ since $(0, \pi^m, x) = (0, \pi^m, x') + a(0, 0, \pi^n)$ for some $a \in \mathcal{O}_E$. In particular, if $\text{ord}_E(x) \geq n$, we have $M(m, n, x) = M(m, n, 0)$. This means that we may assume that $x = 0$ or $\text{ord}_E(x) < n$.

(ii) We have

$$
\frac{(y - \alpha)(y - \beta)}{\pi^n} = \frac{(y - \beta)x}{\pi^n} \cdot \frac{\beta - \alpha}{\pi^m} + (y - \beta) \cdot \frac{(y - \alpha) - (\beta - \alpha)\pi^{-m}x}{\pi^n}.
$$

Therefore if $(*)$ holds, we get

$$
0 \leq n \leq \text{ord}_E(y - \alpha) + \text{ord}_E(y - \beta).
$$

Let $M(m, n, x)$ and $M(m', n', x') \in \mathcal{M}_E$. We will investigate a relation among $m, m', n, n', x$ and $x'$ when $M(m, n, x)$ is isomorphic to $M(m', n', x')$ as a $\Lambda_E$-module. We note that we may assume $x = 0$ or $\text{ord}_E(x) < n$ by Remark 3.4 (i).

First of all, we prepare some notation. For $(m, n, x)$ and $(m', n', x') \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathcal{O}_E$, we define

$$(m, n, x) \sim (m', n', x') \iff m = m', n = n', \text{ and } x \equiv x' \mod \pi^n\mathcal{O}_E.
$$

We put $Z' := (\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathcal{O}_E) / \sim'$ and introduce a set

$$(3) \quad Z := \{(m, n, x) \in Z' \mid m, n, x \text{ satisfy } (*)),
$$

where $(*)$ is the inequalities $(A)$, $(B)$ and $(C)$ in Proposition 3.3 and $(m, n, x)$ is the equivalence class of $(m, n, x)$. The class $(m, n, x)$ is determined by $m, n$ and $x \mod \pi^n\mathcal{O}_E$. We note that the condition $(*)$ does not depend on the choice of a representative of $(m, n, x)$.

For an element for $x \in \mathcal{O}_E$ and $z = \overline{x} \in \mathcal{O}_E/\pi^n\mathcal{O}_E$, we define $\text{ord}_E(z) = \text{ord}_E(x \mod \pi^n)$ as follows:

$$
\text{ord}_E(z) := \begin{cases} 
\text{ord}_E(x) & \text{if } \overline{x} \neq 0, \\
\infty & \text{if } \overline{x} = 0.
\end{cases}
$$

For $(m, n, x)$ and $(m', n', x') \in Z$, let $k = \text{ord}_E(x \mod \pi^n)$ and $l = \text{ord}_E(x' - \pi^m)$. We define $(m, n, x) \sim (m', n', x')$ as follows.

(1) Suppose $m \neq 0$.

(a) When $l + k \geq n$, we define

$$(m, n, x) \sim (m', n', x') \iff m = m', n = n', \text{ and } \overline{x} = \overline{x'} \text{ in } \mathcal{O}_E/\pi^n\mathcal{O}_E.$$
(b) When \( l + k < n \), we define
\[
(m, n, x) \sim (m', n', x') \iff m = m', n = n' \quad \text{and} \quad x = \varepsilon x' \quad \text{in} \quad \mathcal{O}_E/\pi^n\mathcal{O}_E \quad \text{for some} \quad \varepsilon \in 1 + \pi^l\mathcal{O}_E.
\]

(II) Suppose \( m = 0 \). We define
\[
(m, n, x) \sim (m', n', x') \iff m = m' = 0, n = n',
\]
\[
\text{ord}_E(x \mod \pi^n) = \text{ord}_E(x' \mod \pi^n) \quad \text{and} \quad 1 - x = \varepsilon(1 - x') \quad \text{in} \quad \mathcal{O}_E/\pi^n\mathcal{O}_E \quad \text{for some} \quad \varepsilon \in \mathcal{O}_E^\times.
\]

Here, for \( s \leq 0 \), we define \( 1 + \pi^s\mathcal{O}_E = \mathcal{O}_E^\times \). We can prove that \( \sim \) is an equivalence relation. The following is our main theorem. We will prove this theorem in the next section.

**Theorem 3.5.** There is a bijection \( \Phi \):

\[
\begin{array}{ccc}
\mathcal{M}^E_{f(T)} & \longrightarrow & \mathbb{Z}/\sim \\
\psi & & \psi \\
[M(m, n, x)]_E & \longmapsto & [(m, n, x)],
\end{array}
\]

where \( \mathcal{M}^E_{f(T)} \) is defined by (1) in Section 2, \( \mathbb{Z} \) is defined by (3) after Remark 3.4, and \( \sim \) is the equivalence relation of \( \mathbb{Z} \) defined above. \( [M(m, n, x)]_E \) is the class of \( M(m, n, x) \) defined by Proposition 3.3 and \( [(m, n, x)] \) is the class of \( (m, n, x) \).

**Remark 3.6.** After we submitted this paper, we learned from Sumida the existence of the thesis by Chase Franks where he independently classified the isomorphism classes of \( \Lambda \)-modules with \( \lambda = 3 \). He also gave an algorithm to determine the \( \Lambda \)-isomorphism classes for any separable \( f(T) \) which has arbitrary degree [2]. His method is essentially the same as our paper, but there are some differences which we will explain here.

1. We give in this paper an explicit method to compute \( m \) and \( n \) using the action of \( T - \alpha, T - \beta \) etc. (cf. Lemma 4.1).

2. Our inequalities about orders of \( p \)-adic numbers ((5), (6), (7) in Section 4) are obtained from a different point of view from Franks’. He did not solve completely his equations which are essentially equivalent to our inequalities, but we solved our inequalities completely in the case \( \lambda = 3 \).

3. We explicitly give a subgroup \( H \subset \mathbb{Z}_p^\times \) such that \( M(m, n, x) \cong M(m', n', x') \) if and only if \( m = m', n = n' \) and \( x/x' \in H \) (\( H \) depends on \( \text{ord}_p(x) \)). Also, we use the higher Fitting ideals (cf. Section 5 and 6). This is a different argument from Franks’.
4. As an application, we apply our classification to the Iwasawa module associated to the cyclotomic $\mathbb{Z}_p$-extension of an imaginary quadratic field (cf. Section 6). On the other hand, Franks determined the isomorphism class of the Pontryagin dual of the $p$-Selmer group of elliptic curves over the cyclotomic $\mathbb{Z}_p$-extension for $\lambda = 2$.

5. Franks’ method has some merits. He gave an algorithm to decide whether two $\Lambda$-modules are isomorphic or not. This algorithm is to check whether some matrices he defined belong to $GL_{\lambda}(\mathbb{Z}_p)$. This algorithm works for arbitrary $\lambda$ and separable $f(T)$.

**Remark 3.7.** When $(m, n, x) \sim (m', n', x')$ and $l + k \leq n$, we have $l = \text{ord}_p(x' - \pi^m) = \text{ord}_p(x - \pi^m)$.

Sumida determined all elements of $\mathcal{M}_{f(T)}$ for $f(T) = (T - \alpha)(T - \beta)(T - \gamma)$ and $\text{ord}_p(\alpha - \beta) = \text{ord}_p(\beta - \gamma) = \text{ord}_p(\gamma - \alpha) = 1$ ([12], Theorem 1). We can also obtain the same result from our Theorem 3.5.

**Corollary 3.8.** (Sumida) Let $f(T)$ be the same as (2) in Section 2 and $E = \mathbb{Q}_p$. We assume $\text{ord}_p(\alpha - \beta) = \text{ord}_p(\beta - \gamma) = \text{ord}_p(\gamma - \alpha) = 1$. Then we have $\#\mathcal{M}_{f(T)} = 7$ and

$$\mathcal{M}^{Q_p}_{f(T)} = \{(0, 0, 0), (0, 1, 0), (1, 0, 0), (0, 1, 1), (0, 1, 2), (1, 1, 0), (0, 1, 2)\},$$

where $u = (\gamma - \alpha)/(\beta - \alpha)$ and $(m, n, x)$ means $[M(m, n, x)]_{Q_p}$.

Proof. We prove this corollary using Theorem 3.5. By fixing integers $m$ and $n$, we put

$$Z(m, n) = \{\text{the equivalence class of } (m, n, x) \text{ in } Z/\sim \mid (m, n, x) \in Z\}.$$ 

Then, by definition, we have

$$Z/\sim = \bigsqcup_m \bigsqcup_n Z(m, n).$$

We determine all the elements of $Z(m, n)$ for each $m$ and $n$ in order to determine all the elements of $\mathcal{M}_{f(T)}$.

We first assume $[(m, n, x)] \in Z/\sim$, where $[(m, n, x)]$ is the equivalence class of $(m, n, x)$. Then by Proposition 3.3, $M(m, n, x)$ is a $\Lambda_E$-module satisfying (A), (B) and (C). By the inequality (A), we have $0 \leq m \leq 1$. Now we investigate $\bigsqcup_n Z(m, n)$ for $m = 0, 1$.

1. Suppose $m = 0$.

In this case, by the inequalities (B) and (C), we have $0 \leq n \leq 1$. When $n \geq 2$, we get $\text{ord}_p(x) = 0$ by (C). This contradicts to (B). When $n = 0$, we have $(0, 0, x) = (0, 0, 0)$. Therefore we get $Z(0, 0) = \{[(0, 0, 0)]\}$. When $n = 1$, we have $Z(0, 1) = \{[(0, 1, 0)]\}$.
By the definition of the equivalence relation, we have 
\( (0, 1, x) \sim (0, 1, x') \) if and only if 
\[
\text{ord}_p(x \mod p) = \text{ord}_p(x' \mod p) \quad \text{and} \quad 1 - x = \varepsilon (1 - x')
\]
for some \( \varepsilon \in \mathbb{Z}_p^* \). By the definition of \( \text{ord}_p(x \mod p) \), we have
\[
\text{ord}_p(x \mod p) = \begin{cases} 
0 & x \notin p\mathbb{Z}_p, \\
\infty & x \in p\mathbb{Z}_p, 
\end{cases}
\]
We investigate the case \( \text{ord}_p(x \mod \pi) = 0 \). Suppose \( x = 1 \). Then we have
\[
\{(0, 1, 1)\} = \{(0, 1, x) \mid (0, 1, 1) \sim (0, 1, x)\}
= \{(0, 1, x) \mid \text{ord}_p(x) = 0, \ 0 = \varepsilon (1 - x) \text{ for some } \varepsilon \in \mathbb{Z}_p^*\}
= \{(0, 1, x) \mid x \equiv 1 \mod p\}
= \{(0, 1, 1)\}.
\]
Suppose \( x = 2 \). Then we have
\[
\{(0, 1, 2)\} = \{(0, 1, x) \mid \text{ord}_p(x) = 0, \ -1 = \varepsilon (1 - x) \text{ for some } \varepsilon \in \mathbb{Z}_p^*\}
= \{(0, 1, x) \mid x \not\equiv 0, 1\}
= \{(0, 1, 2), \ldots, (0, 1, p - 1)\}.
\]
Therefore we get \( Z(0, 1) = \{(0, 1, 0), (0, 1, 1), (0, 1, 2)\} \).

(II) Suppose \( m = 1 \).

By Remark 3.4 (ii), we have \( 0 \leq n \leq 2 \). When \( n = 0 \), we have \( Z(1, 0) = \{(1, 0, 0)\} \).
When \( n = 1 \), we have \( Z(1, 1) = \{(1, 1, 0)\} \). In fact, we suppose \( [(1, 1, x)] \in Z(1, 1) \). Then we have \( \bar{x} = 0 \) by (C). When \( n = 2 \), we have \( Z(1, 2) = \{(1, 2, up)\} \). Indeed, we suppose \( [(1, 2, x)] \in Z(1, 2) \). For some \( v \in \mathbb{Z}_p^* \), we have
\[
x = \left(1 - \frac{vp^2}{\gamma - \alpha} \right) \frac{y - \alpha}{\beta - \alpha} p
\equiv \frac{y - \alpha}{\beta - \alpha} p \mod p^2,
\]
by (C). Thus,
\[
Z/\sim = \{(0, 0, 0), (0, 1, 0), (1, 0, 0), (0, 1, 1), (1, 2, up), (1, 1, 0), (0, 1, 2)\}.
\]
We complete the proof by Theorem 3.5. \( \square \)
Corollary 3.9. Let \( f(T) \) be the same as (2) in Section 2 and \( E = \mathbb{Q}_p \). We assume \( \text{ord}_p(\alpha - \beta) = \text{ord}_p(\beta - \gamma) = \text{ord}_p(\gamma - \alpha) = 2 \). Then we have \( \# \mathcal{M}_{f(T)} = p + 18 \) and

\[
\mathcal{M}_{f(T)}^{\mathbb{Q}_p} = \left\{(0, 0, 0), (0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (0, 2, p), (0, 2, p + 1), (1, 0, 0), (1, 1, 0), (1, 1, 1), (1, 2, 0), (1, 2, p), \ldots, (1, 2, (p - 1)p), (1, 3, up^2), (2, 0, 0), (2, 1, 0), (2, 2, 0), (2, 3, up^2), (2, 4, up^2)\right\},
\]

where \( u = (\gamma - \alpha)/(\beta - \alpha) \) and \( (m, n, x) \) means \( [M(m, n, x)]_{\mathbb{Q}_p} \).

Proof. We use the same notation as Corollary 3.8. By definition, we have

\[
Z/\sim = \bigsqcup_m \bigsqcup_n Z(m, n).
\]

We determine all the elements of \( Z(m, n) \) for each \( m \) and \( n \) in order to determine all the elements of \( \mathcal{M}_{f(T)}^{\mathbb{Q}_p} \).

We first assume \([\bar{m}, n, x] \in Z/\sim\), where \([m, n, x] \in \mathbb{Z} \) is the equivalence class of \((m, n, x)\). Then \( M(m, n, x) \) becomes a \( \Lambda_E \)-module satisfying (A), (B) and (C). By the inequality (A), we have \( 0 \leq m \leq 2 \). Now we investigate \( \bigsqcup_n Z(m, n) \) for each \( m \).

(I) Suppose \( m = 0 \).

In this case, by the inequalities (B) and (C), we have \( 0 \leq n \leq 2 \). In fact, if \( \text{ord}_p(x) \geq 1 \), we get \( n \leq 2 \) by (C) and if \( \text{ord}_p(x) = 0 \), we get \( n \leq 2 \) by (B). When \( n = 0 \), we have \((0, 0, 0) = (0, 0, 0)\) and \( Z(0, 0) = \{[(0, 0, 0)]\}\). When \( n = 1 \), we have \( Z(0, 1) = \{[(0, 1, 0)], [(0, 1, 1)], [(0, 1, 2)]\}\) by the same method as Corollary 3.8. When \( n = 2 \), we have

\[
Z(0, 2) = \{[(0, 2, 0)], [(0, 2, 1)], [(0, 2, 2)], [(0, 2, p)], [(0, 2, p + 1)]\}.
\]

In fact, we suppose \([(0, 2, x)] \in Z(0, 2)\), then we have \( x = 0 \) or \( \text{ord}_p(x) \leq 1 \). We first investigate the case \( \text{ord}_p(x) = 0 \). Then, \( (0, 2, x) \sim (0, 2, x') \) if and only if

\[
0 = \text{ord}_p(x) = \text{ord}_p(x') \quad \text{and} \quad 1 - x = \varepsilon(1 - x') \quad \text{for some} \quad \varepsilon \in \mathbb{Z}_p^\times.
\]

By the same method as above, we get

\[
[(0, 2, 1)] = \{(0, 2, 1)\},
\]

\[
[(0, 2, 2)] = \{(0, 2, x) \mid x \neq 0, 1\},
\]

\[
(0, 2, p + 1)t = \{(0, 2, x) \mid \text{ord}_p(x) = 0, -p = \varepsilon(1 - x) \quad \text{for some} \quad \varepsilon \in \mathbb{Z}_p^\times\}
\]

\[
= \{(0, 2, 1 + x_1p) \mid 1 \leq x_1 < p\}.
\]
Next, we investigate the case $\text{ord}_p(x) = 1$, let $x = p$. Then we have

$$[(0, 2, p)] = \{(0, 2, x) \mid \text{ord}_p(x) = 1, 1 - p = \varepsilon(1 - x) \text{ for some } \varepsilon \in \mathbb{Z}_p^*\} = \{(0, 2, x_1p) \mid 1 \leq x_1 < p\}.$$

Thus we get (4).

(II) Suppose $m = 1$.

By the inequalities (B) and (C), we have $0 \leq n \leq 3$. If $\text{ord}_p(x) \leq 1$, we have $n \leq 3$ by (B). If $\text{ord}_p(x) > 1$, we have $n \leq 2$ by (C). When $n = 0$, we have $Z(1, 0) = \{(1, 0, 0)\}$. When $n = 1$, we have $Z(1, 1) = \{(1, 1, 0)\}, \{(1, 1, 1)\}$. We suppose $[(1, 1, x)] \in Z(1, 1)$. Then we have $\bar{x} = 0$ or $\text{ord}_p(\bar{x}) = 0$. We suppose $\text{ord}_p(\bar{x}) = 0$. We have $l = \text{ord}_p(x' - p) = 0$. This is the case (b). By the definition of the equivalence relation, $(1, 1, x) \sim (1, 1, x')$ if and only if

$$\bar{x} = \varepsilon \bar{x}' \text{ for some } \varepsilon \in \mathbb{Z}_p^*.$$

Here we note that $l = \text{ord}_E(x' - p) = 0$. Then we have

$$[(1, 1, x)] = \{(1, 1, x') \mid \bar{x} = \varepsilon \bar{x}' \text{ for some } \varepsilon \in \mathbb{Z}_p^*\} = \{(1, 1, x') \mid \bar{x}' \neq 0\}.$$

Therefore we get $Z(1, 1) = \{(1, 1, 0)\}, \{(1, 1, 1)\}$. When $n = 2$, we have $Z(1, 2) = \{(1, 2, x) \mid x = 0, p, 2p, \ldots, (p - 1)p\}$. In fact, we suppose $[(1, 2, x)] \in Z(1, 2)$. By the inequality (C), we have

$$2 \leq \text{ord}_p((\gamma - \alpha) - (\beta - \alpha)p^{-1}x).$$

If $\text{ord}_p(x) = 0$, the order of the right hand side is 1. This is contradiction. Thus we may assume $1 \leq \text{ord}_p(x)$. If $\text{ord}_p(x) \geq 2$, we get $[(1, 2, x)] = \{(1, 2, 0)\}$. We suppose $\text{ord}_p(x) = 1$. Then $(1, 2, x) \sim (1, 2, x')$ if and only if

$$\bar{x} = \bar{x'}.$$

Here we note that this is the case (a) since $l = \text{ord}_p(x' - p) \geq 1$. For each $x = \varepsilon p$, where $1 \leq \varepsilon < p$, we have

$$[(1, 2, x)] = \{(1, 2, x)\}.$$

Thus we get $Z(1, 2) = \{(1, 2, x) \mid x = 0, p, 2p, \ldots, (p - 1)p\}$. When $n = 3$, we have $Z(1, 3) = \{(1, 3, u) \mid u \in \mathbb{Z}_p\}$. In fact, we suppose $[(1, 3, x)] \in Z(1, 3)$. By the same method as in the case $n = 2$, we get $\text{ord}_p(x) = 1$ and $(1, 3, x) \sim (1, 3, up)$ if and only if

$$\bar{x} = \varepsilon up \text{ for some } \varepsilon \in 1 + p\mathbb{Z}_p.$$
Here we note that this is the case (b) since \( l = \text{ord}_E(u^p - p) = 1 \). Moreover, by (C), we have

\[
x = \left(1 - \frac{vp^3}{y - \alpha}\right)\frac{y - \alpha}{\beta - \alpha} \quad \text{for some } v \in \mathbb{Z}_p^\times.
\]

Since \( 1 - vp^3/(y - \alpha) \in 1 + p\mathbb{Z}_p \), we have

\[
[(1, 3, up)] = [(1, 3, x) \mid \bar{x} = \varepsilon up \text{ for some } \varepsilon \in 1 + p\mathbb{Z}_p],
\]

where \( u = (y - \alpha)/(\beta - \alpha) \). Thus we get \( Z(1, 3) = \{(1, 3, up)\} \).

(III) Suppose \( m = 2 \).

By the same method as (I) and (II), we get \( Z(2, 0) = \{(2, 0, 0)\} \), \( Z(2, 1) = \{(2, 2, 1)\} \), \( Z(2, 2) = \{(2, 1, 2)\} \), \( Z(2, 3) = \{(2, 3, up^2)\} \) and \( Z(2, 4) = \{(2, 4, up^2)\} \).

Thus we complete the proof.

\[ \square \]

4. **Proof of Theorem 3.5**

For any \( \xi \in \Lambda_E \), we define a map \( \Pi_\xi = \Pi_\xi^M : M \to M \) by \( \xi(y) = \xi y \).

**Lemma 4.1.** Let \( q = \#(\mathcal{O}_E/(\pi)) \) and \( M = M(m, n, x) \). Then we have

\[
\#(\text{Ker}(\Pi_{(T - \alpha)}^N)/\text{Im}(\Pi_{(T - \beta)}^N)) = q^{[\text{ord}_E(\alpha - \beta) - m]},
\]

\[
\#(\text{Ker}(\Pi_{(T - \gamma)}^M)/\text{Im}(\Pi_{(T - \alpha)(T - \beta)}^M)) = q^{[\text{ord}_E(\gamma - \alpha) + \text{ord}_E(\gamma - \beta) - n]},
\]

where \( N = \text{Im}(\Pi_{(T - \gamma)}) \).

**Proof.** We first compute \( \text{Ker}(\Pi_{(T - \gamma)}) \). For \( y \in M = M(m, n, x) \), there exist \( \lambda_1, \lambda_2, \lambda_3 \in \mathcal{O}_E \) such that

\[
y = \lambda_1(1, 1, 1) + \lambda_2(0, \pi^m, x) + \lambda_3(0, 0, \pi^n)
\]

\[
= (\lambda_1 + \lambda_2 \pi^m, \lambda_1 + \lambda_2 x + \lambda_3 \pi^n).
\]

Thus we have \( \Pi_{(T - \gamma)}(y) = ((\alpha - \gamma)\lambda_1, (\beta - \gamma)(\lambda_1 + \lambda_2 \pi^m), 0) \). If \( y \in \text{Ker}(\Pi_{(T - \gamma)}) \), we get \( \lambda_1 = 0 \) and \( \lambda_1 + \lambda_2 \pi^m = 0 \), since \( \alpha, \beta \) and \( \gamma \) are distinct elements of \( \mathcal{O}_E \).

Therefore \( y = (0, 0, \lambda_3 \pi^n) \) and \( \text{Ker}(\Pi_{(T - \gamma)}) = (0, 0, \pi^n \mathcal{O}_E) \). On the other hand, by \( y = (\lambda_1, \lambda_1 + \lambda_3 \pi^n, \lambda_1 + \lambda_2 x + \lambda_3 \pi^n) \), we have

\[
\Pi_{(T - \alpha)(T - \beta)}(y) = \Pi_{(T - \alpha)}((\alpha - \beta)\lambda_1, 0, (\gamma - \beta)(\lambda_1 + \lambda_2 x + \lambda_3 \pi^n))
\]

\[
= (0, 0, (\gamma - \alpha)(\gamma - \beta)(\lambda_1 + \lambda_2 x + \lambda_3 \pi^n)).
\]

Thus we have \( \text{Im}(\Pi_{(T - \alpha)(T - \beta)}) = (0, 0, \pi^{[\text{ord}_E(\alpha - \beta) + \text{ord}_E(\gamma - \beta)] \mathcal{O}_E}) \) and

\[
\#(\text{Ker}(\Pi_{(T - \gamma)})/\text{Im}(\Pi_{(T - \alpha)(T - \beta)})) = \#(\pi^n \mathcal{O}_E / \pi^{[\text{ord}_E(\gamma - \alpha) + \text{ord}_E(\gamma - \beta)] \mathcal{O}_E})
\]

\[
= q^{[\text{ord}_E(\gamma - \alpha) + \text{ord}_E(\gamma - \beta) - n]}.
\]
Next we put \( N = \text{Im}(\Pi_{(T - \gamma)}) \). We have

\[
\text{Ker}(\Pi^N_{(T - \alpha)}) = (\pi^{\text{ord}_E(\alpha - \gamma) + m} \mathcal{O}_E, 0, 0),
\]

\[
\text{Im}(\Pi^N_{(T - \beta)}) = (\pi^{\text{ord}_E(\alpha - \gamma) + \text{ord}_E(\alpha - \beta)} \mathcal{O}_E, 0, 0).
\]

Therefore we get

\[
\#(\text{Ker}(\Pi^N_{(T - \alpha)})/\text{Im}(\Pi^N_{(T - \beta)})) = q^{\text{ord}_E(\alpha - \beta) - m}.
\]

\[\square\]

**Corollary 4.2.** Let \( [M]_E, [M']_E \in \mathcal{M}^E_{\mathcal{I}(T)} \) and \( M = M(m, n, x), M' = M(m', n', x') \). If \([M]_E = [M']_E\), then we have \( m = m' \) and \( n = n' \).

**Proof.** Since \( M \cong M' \), we have \( N = \text{Im}(\Pi^M_{(T - \gamma)}) \cong \text{Im}(\Pi^{M'}_{(T - \gamma)}) = N' \) and therefore

\[
\text{Ker}(\Pi^N_{(T - \alpha)})/\text{Im}(\Pi^N_{(T - \beta)}) \cong \text{Ker}(\Pi^{N'}_{(T - \alpha)})/\text{Im}(\Pi^{N'}_{(T - \beta)}).
\]

This implies \( m = m' \) by Lemma 4.1. We get \( n = n' \) by the same method. \( \square \)

The isomorphism

\[
\iota: \mathcal{E} = \Lambda_E/(T - \alpha) \oplus \Lambda_E/(T - \beta) \oplus \Lambda_E/(T - \gamma) \to \mathcal{O}_E^{\otimes 3}
\]

defined in Section 2, induces an isomorphism

\[
\mathcal{E} \otimes_{\mathcal{O}_E} E \cong E^{\otimes 3}
\]

such that \((f_1(T), f_2(T), 3(T)) \otimes y \mapsto (f_1(\alpha)y, f_2(\beta)y, f_3(\gamma)y)\).

**Proposition 4.3.** Let \([M]_E, [M']_E \in \mathcal{M}^E_{\mathcal{I}(T)}\), \( M = M(m, n, x), M' = M(m, n, x') \) and \( g: M \to M' \) be a \( \Lambda_E \)-isomorphism. We define an \( \mathcal{E} \)-linear map \( F_A \) by the following commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{g} & M' \\
\varphi \otimes 1 \downarrow & & \downarrow \varphi' \otimes 1 \\
\mathcal{E} \otimes_{\mathcal{O}_E} E & \longrightarrow & \mathcal{E} \otimes_{\mathcal{O}_E} E \\
\iota \otimes 1 \downarrow \varphi \otimes 1 & & \downarrow \varphi' \otimes 1 \\
E^{\otimes 3} & \longrightarrow & E^{\otimes 3}.
\end{array}
\]

In the diagram, \( \varphi \) and \( \varphi' \) are natural inclusions (Section 2). When we take the standard basis of \( E^{\otimes 3} \), \( F_A \) corresponds to a diagonal matrix

\[
\begin{pmatrix}
a_1 & 0 & 0 \\
0 & a_2 & 0 \\
0 & 0 & a_3
\end{pmatrix}
\]

for some \( a_1, a_2, a_3 \in \mathcal{O}_E^\times \).
Proof. Consider the map $\Pi_T: M \to M$. Then $\Pi_T$ induces a map $F_B: E^B \to E^B$ and the following commutative diagram

\[
\begin{array}{c}
M \xrightarrow{\Pi_T} M \\
\downarrow{\psi \otimes 1} \quad \downarrow{\psi \otimes 1} \\
E \otimes O_x E \xrightarrow{T} E \otimes O_x E \\
\downarrow{\iota \otimes 1} \quad \downarrow{\iota \otimes 1} \\
E^B \xrightarrow{F_B} E^B.
\end{array}
\]

Thus we get

\[(\ast)\quad F_B \circ (\iota \otimes 1) \circ (\psi \otimes 1)(x) = (\iota \otimes 1) \circ (\psi \otimes 1)(T \cdot x)\]

for $x \in M$. Let $A$ be the matrix corresponding to $F_A$. By the diagram, we get

\[(\ast\ast)\quad F_A \circ (\iota \otimes 1) \circ (\psi \otimes 1)(T \cdot x) = (\iota \otimes 1) \circ (\psi' \otimes 1)(g(T \cdot x)).\]

By (\ast) and the diagrams, the left hand side of (\ast) is

\[F_A \circ (\iota \otimes 1) \circ (\psi \otimes 1)(T \cdot x) = F_A \circ F_B \circ (\iota \otimes 1) \circ (\psi \otimes 1)(x).\]

The right hand side of (\ast\ast) is

\[(\iota \otimes 1) \circ (\psi' \otimes 1)(T g(x)) = F_B \circ (\iota \otimes 1) \circ (\psi' \otimes 1)(g(x)) = F_B \circ F_A \circ (\iota \otimes 1) \circ (\psi \otimes 1)(x).\]

Since this holds for any $x \in M$, we have $F_A \circ F_B = F_B \circ F_A$. Taking the standard basis of $E^B$, $F_B$ corresponds to the matrix

\[
B = \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \beta & 0 \\
0 & 0 & \gamma
\end{pmatrix}.
\]

Therefore we have

\[
A \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \beta & 0 \\
0 & 0 & \gamma
\end{pmatrix} = \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \beta & 0 \\
0 & 0 & \gamma
\end{pmatrix} A.
\]

Since $\alpha$, $\beta$ and $\gamma$ are distinct elements and we get

\[
A = \begin{pmatrix}
a_1 & 0 & 0 \\
0 & a_2 & 0 \\
0 & 0 & a_3
\end{pmatrix}
\]

with $a_1, a_2, a_3 \in E$. 

\[\]
Because \( g((1, 1, 1)) = (a_1, a_2, a_3) \in M' \), we get \( a_1, a_2 \) and \( a_3 \in \mathcal{O}_E \). Furthermore, by the same argument for \( g^{-1} \), we have \( a_{1}^{-1}, a_{2}^{-1}, a_{3}^{-1} \in \mathcal{O}_E \). So we get \( a_1, a_2, a_3 \in \mathcal{O}_E \).

By the commutativity of the diagram, we obtain the following corollary.

**Corollary 4.4.** Suppose that \( M, F_A, \iota, \varphi \) and \( \varphi' \) are the same as Proposition 4.3. Then we have

\[
\langle (F_A \circ (\iota \otimes 1) \circ (\varphi \otimes 1))(M) \rangle_{\mathcal{O}_E} = \langle (\iota \otimes 1) \circ (\varphi' \otimes 1) \circ g(M) \rangle_{\mathcal{O}_E}.
\]

**Proposition 4.5.** Let \( [M]_E, [M']_E \in \mathcal{M}^E_{T(J)} \) and \( M = M(m,n,x), M' = M(m,n,x') \). Then the following two statements are equivalent:

(i) We have \( M \cong M' \) as \( \Lambda_E \)-modules,

(ii) There exist \( a_1, a_2, a_3 \in \mathcal{O}_E^\times \) satisfying

\[
\begin{align*}
\text{(5)} & \quad \text{ord}_E(a_2 - a_1) \geq m, \\
\text{(6)} & \quad \text{ord}_E(a_3x - a_2x') \geq n, \\
\text{(7)} & \quad \text{ord}_E\{a_3 - a_1 - (a_2 - a_1)\pi^{-m}x\} \geq n.
\end{align*}
\]

Proof. We first prove (i) implies (ii). If \( M \) is isomorphic to \( M' \) as a \( \Lambda_E \)-module, there exists a \( \Lambda_E \)-isomorphism \( g : M \rightarrow M' \). By Proposition 4.3, there exists a diagonal matrix \( A \) which can be written as

\[
\begin{pmatrix}
a_1 & 0 & 0 \\
0 & a_2 & 0 \\
0 & 0 & a_3
\end{pmatrix}
\]

such that \( a_1, a_2, a_3 \in \mathcal{O}_E^\times \),

which corresponds to \( g \). We have

\[
F_A \circ (\iota \otimes 1) \circ (\varphi \otimes 1)(M) = F_A(M(m,n,x)) = \langle (a_1, a_2, a_3), (0, a_2\pi^m, a_3x), (0, 0, a_3\pi^n) \rangle_{\mathcal{O}_E}
\]

and

\[
(\iota \otimes 1) \circ (\varphi' \otimes 1) \circ g(M) = (\iota \otimes 1) \circ (\varphi' \otimes 1)(M') = \langle (1, 1, 1), (0, \pi^m, x'), (0, 0, \pi^n) \rangle_{\mathcal{O}_E}.
\]

By Corollary 4.4, we get

\[
\langle (a_1, a_2, a_3), (0, a_2\pi^m, a_3x), (0, 0, a_3\pi^n) \rangle_{\mathcal{O}_E} = \langle (1, 1, 1), (0, \pi^m, x'), (0, 0, \pi^n) \rangle_{\mathcal{O}_E}.
\]
Because the left hand side is contained in the right hand side, we have

\[(a_1, a_2, a_3) = a_1(1, 1, 1) + (a_2 - a_1)\pi^{-m}(0, \pi^m, x') + (a_3 - a_1)\pi^{-m}x' \pi^{-n}(0, 0, \pi^n),\]

\[(0, a_2\pi^m, a_3x) = a_2(0, \pi^m, x') + (a_3x - a_2x')\pi^{-n}(0, 0, \pi^n).\]

Since these coefficients should belong to \(\mathcal{O}_E\), we have (5), (6), (7). It is easy to prove that (ii) implies (i).

We can simplify the inequalities (5), (6), (7). The following is easy to see.

**Lemma 4.6.** The followings are equivalent:

(i) There exist \(a_1, a_2, a_3 \in \mathcal{O}_E^\times\) satisfying (5), (6), (7),

(ii) There exist \(a_1, a_2 \in \mathcal{O}_E^\times\) satisfying

\[(8) \quad \text{ord}_E(a_2 - a_1) \geq m,\]
\[(9) \quad \text{ord}_E(x - a_2x') \geq n,\]
\[(10) \quad \text{ord}_E(1 - a_1 - (a_2 - a_1)\pi^{-m}x') \geq n.\]

**Corollary 4.7.** Let \([M]_E, [M']_E \in \mathcal{M}^E_{f(T)}\) and \(M = M(m, n, x), M' = M(m, n, x').\) Assume \(\text{ord}_E(x) < n\). If \([M]_E = [M']_E\), we have \(\text{ord}_E(x) = \text{ord}_E(x').\)

Proof. If \(\text{ord}_E(x) < \text{ord}_E(x')\), by the inequality (6), we have \(n \leq \text{ord}_E(a_3x - a_2x') = \text{ord}_E(x)\). This contradicts to the assumption \(\text{ord}_E(x) < n\). If we assume \(\text{ord}_E(x) > \text{ord}_E(x')\), we would get the same contradiction. Therefore we obtain \(\text{ord}_E(x) = \text{ord}_E(x')\).

To prove Theorem 3.5, we prepare a lemma and some propositions.

**Proposition 4.8.** The following two statements are equivalent:

(i) We have \(M(m, n, x) \cong M(m, n, 0)\) as \(\Lambda_E\)-modules,

(ii) We have \((m, n, x) \sim (m, n, 0)\), where \(\sim\) is the equivalence relation defined in Section 3.

Proof. We show that (i) implies (ii). If \(\text{ord}_E(x) < n\), we have \(\text{ord}_E(x) = \text{ord}_E(0)\) by Corollary 4.7, which is a contradiction. So we have \(\text{ord}_E(x) \geq n\) and \(M(m, n, x) = M(m, n, 0)\). Then \((m, n, x) = (m, n, 0)\) by Remark 3.4 (i).

Let \(M = M(m, n, x)\) and \(M' = M(m, n, x')\). Now we suppose that \(x' \neq 0\) and the existence of \(a_1, a_2 \in \mathcal{O}_E^\times\) satisfying (8), (9) and (10). By Proposition 4.5 and Lemma 4.6,
$M$ is isomorphic to $M'$. From the inequalities (8) and (9), there are $s, v \in \mathcal{O}_E$ such that $a_2 - a_1 = \pi^m s$ and $x - a_2 x' = \pi^n v$. Thus we have

$$(11) \quad a_1 = \frac{x}{x'} - \frac{\pi^n}{x'} v - \pi^m s,$$

$$(12) \quad a_2 = \pi^m s + a_1 = \frac{x}{x'} - \frac{\pi^n}{x'} v.$$

By the inequality (10), we get

$$(13) \quad x'(x' - \pi^m)s - \pi^n v + \pi^n x' w = x' - x$$

for some $w \in \mathcal{O}_E$.

**Lemma 4.9.** Let $m, n \neq 0$ and $\text{ord}_E(x) < n$. The following two statements are equivalent:

(i) There exist $a_1, a_2 \in \mathcal{O}_E^\times$ satisfying (8), (9), (10),

(ii) We have $\text{ord}_E(x) = \text{ord}_E(x')$ and there exist $s, v, w \in \mathcal{O}_E$ satisfying (13).

Proof. We have already proved that (i) implies (ii). We will prove that (ii) implies (i). We put $a_1$ and $a_2$ by the equalities (11) and (12). Since $m, n \neq 0$ and $\text{ord}_E(x) = \text{ord}_E(x') < n$, we have $a_1, a_2 \in \mathcal{O}_E^\times$. Then we have

$$a_2 - a_1 = \pi^m s, \quad x - a_2 x' = \pi^n v$$

and

$$1 - a_1 - (a_2 - a_1)\pi^{-m} x' = \pi^n w.$$ 

Therefore we get (8), (9) and (10).

**Proposition 4.10.** Let $m, n \neq 0$ and $\text{ord}_E(x) < n$. Then the followings are equivalent:

(i) We have $M(m, n, x) \cong M(m, n, x')$ as $\Lambda_E$-modules,

(ii) We have $(m, n, x) \sim (m, n, x')$.

Proof. We first suppose that $M(m, n, x)$ is isomorphic to $M(m, n, x')$ as a $\Lambda_E$-module. Let $k = \text{ord}_E(x)$ and $l = \text{ord}_E(x' - \pi^m)$. By Lemma 4.9, we have $\text{ord}_E(x) = \text{ord}_E(x') = k$ and there exist $s, v, w \in \mathcal{O}_E$ such that

$$x'(x' - \pi^m)s - \pi^n v + \pi^n x' w = x' - x.$$ 

We put $\varepsilon = xx'^{-1} \in \mathcal{O}_E^\times$. Dividing the above equality by $x'$, we have

$$(x' - \pi^m)s - \frac{\pi^n}{x'} v + \pi^n w = 1 - \varepsilon.$$
Thus we have
\[ \text{ord}_E(1 - \varepsilon) \geq \min \left\{ \text{ord}_E((x' - \pi^n)s), \text{ord}_E\left(\frac{-\pi^n}{x'} v\right), \text{ord}_E(\pi^n w) \right\} \]
\[ \geq \min\{l, n - k, n\} = \min\{l, n - k\}. \]

In the case (a) \( l \geq n - k \), we have \( \text{ord}_E(1 - \varepsilon) \geq n - k \). Thus we get \( \overline{x} = \overline{\varepsilon x'} = \overline{x}' \) in \( O_E/\pi^n O_E \). Therefore we have \( (m, n, x) \sim (m, n, x') \). In the case (b) \( l < n - k \), we have \( \text{ord}_E(1 - \varepsilon) \leq l \) and \( \overline{x} = \overline{\varepsilon x'} \) in \( O_E/\pi^n O_E \). Therefore we get \( (m, n, x) \sim (m, n, x') \).

Conversely we assume that \( (m, n, x) \sim (m, n, x') \). In the case (a), we have \( \overline{x} = \overline{x}' \) in \( O_E/\pi^n O_E \) and \( (x' - x)/\pi^n \in O_E \). Put \( s = w = 0 \) and \( v = (x - x')/\pi^n \in O_E \). Then we get
\[ x'(x' - \pi^n)s - \pi^n v + \pi^n x'w = x' - x. \]

By Lemma 4.9, \( M \) and \( M' \) are isomorphic as \( \Lambda_E \)-modules. In the case (b), We have \( \overline{x} = \overline{\varepsilon x'} \) in \( O_E/\pi^n O_E \) for some \( \varepsilon \in 1 + \pi^t O_E \). Since \( \text{ord}_E(1 - \varepsilon) \geq l \), we have \( (1 - \varepsilon)/(x' - \pi^n) \in O_E \). Put \( v = w = 0 \) and \( s = (1 - \varepsilon)/(x' - \pi^n) \in O_E \). Then we get
\[ x'(x' - \pi^n)s - \pi^n v + \pi^n x'w = x' - \varepsilon x'. \]

By Lemma 4.9, we get \( M(m, n, x) = M(m, n, \varepsilon x') \cong M(m, n, x') \).

The following propositions treat the case \( m = 0 \) and the case \( n = 0 \).

**Proposition 4.11.** Suppose \( m = 0, n \neq 0 \) and \( \text{ord}_E(x) < n \). Then the followings are equivalent:

(i) We have \( M(0, n, x) \cong M(0, n, x') \) as \( \Lambda_E \)-modules,

(ii) We have \( (0, n, x) \sim (0, n, x') \).

**Proof.** Suppose that \( M(0, n, x) \) is isomorphic to \( M(0, n, x') \) as a \( \Lambda_E \)-module. By Proposition 4.5 and Lemma 4.6, there exist \( a_1, a_2 \in O_E^\times \) satisfying (9) and (10). By the inequality (9), we have \( \overline{x} = a_2 \overline{x}' \). By the inequality (10), we have \( 1 - a_2 \overline{x}' = a_1(1 - \overline{x}) \). Therefore we get
\[ \text{ord}_E(x) = \text{ord}_E(x') \quad \text{and} \quad \overline{1 - x} = a_1(1 - \overline{x'}). \]

Thus we get \( (0, n, x) \sim (0, n, x') \). Conversely we suppose that \( (0, n, x) \sim (0, n, x') \). There exists \( a_1 \in O_E^\times \) such that \( 1 - x = a_1(1 - \overline{x}) \). Put \( a_2 = x/x' \). Then we have (9) and (10). Indeed, we have \( 1 - a_1 - (a_2 - a_1)\pi^-\overline{x}' = 1 - a_1 - (a_2 - a_1)\overline{x}' = 0 \).

By Proposition 4.5 and Lemma 4.6, \( M(0, n, x) \) and \( M(0, n, x') \) are isomorphic as \( \Lambda_E \)-modules.

**Proposition 4.12.** Suppose \( n = 0 \). The followings are equivalent:
(i) We have $M(m, 0, x) \cong M(m, 0, x')$ as $\Lambda_E$-modules,
(ii) We have $(m, 0, x) \sim (m', 0, x')$.

Proof. By Remark 3.4 (i), we have $M(m, 0, x) = M(m, 0, x') = M(m, 0, 0)$ and $(m, 0, x) = (m, 0, x') = (m, 0, 0)$. 

Now we can prove Theorem 3.5.

Proof of Theorem 3.5. For $M(m, n, x) \in M_{f(T)}^E$, we may assume $x = 0$ or $\ord_E(x) < n$ by Remark 3.4 (i). At first, $\Phi$ is well-defined by Corollary 4.2 and Propositions 4.8, 4.10, 4.11 and 4.12. The surjectivity follows from Proposition 3.3 and Remark 3.4. On the other hand, $\Phi$ is injective by Propositions 4.8, 4.10, 4.11 and 4.12.

5. Complementary properties

In this section, we show some propositions in order to determine the Iwasawa module associated to an imaginary quadratic field in the next section.

For a non-negative integer $n$, we put $M_{f(T)}^E \in \mathcal{M}_{f(T)}^E$. Then the natural map

$$\psi : \mathcal{M}_{f(T)}^E \to \mathcal{M}_{f(T)}^E \quad ([M] \mapsto [M \otimes \Lambda \Lambda_E]_E),$$

is injective.

Proof. We suppose that $M \not\cong M'$ as $\Lambda$-modules. Since $M$ is a finitely generated $\Lambda$-module, $M$ is a profinite module and we have $M = \lim_{\leftarrow} M/m^n M$ where $m = (\pi, T)$. Since $M \not\cong M'$, there exists a positive integer $l$ such that $M/m^l M \not\cong M'/m^l M'$ ([11], Proposition 5). Because both $M/m^l M$ and $M'/m^l M'$ are of finite length, we can decompose these modules into indecomposable modules

$$M/m^l M = \bigoplus_i N_i^{\oplus e_i}, \quad M'/m^l M' = \bigoplus_i N_i^{\oplus e'_i},$$

where $N_i$'s are indecomposable modules, $N_i \not\cong N_j$ ($i \neq j$) and $e_i, e'_i$ are non-negative integers. By Krull–Remak–Schmidt’s theorem, there exists $i$ such that $e_i \neq e'_i$. Furthermore we have

$$(M/m^l M)^n = \bigoplus_i N_i^{\oplus n e_i}, \quad (M'/m^l M')^n = \bigoplus_i N_i^{\oplus n e'_i}.$$
Thus we get $n e_i 
eq n e'_i$ for some $i$. By Krull–Remak–Schmidt’s theorem, we have $(M/m^iM)^n \neq (M'/m^iM')^n$. This implies $M^n \neq M'^n$. This contradicts to our assumption. □

Let $f(T) \in \mathbb{Z}_p[T]$ be a distinguished polynomial, $E$ the splitting field of $f(T)$ and we put

$$f(T) = (T - \alpha)(T - \beta)(T - \gamma),$$

where $\alpha$, $\beta$ and $\gamma \in \pi \mathcal{O}_E$ as in Section 2.

**Proposition 5.2.** Let $E$ and $f(T)$ be as above and $[M]_E \in \mathcal{M}_{f(T)}$. If $M$ is a cyclic $\Lambda_E$-module, then we have

$$M \cong M(\text{ord}_E(\beta - \alpha), \text{ord}_E(\gamma - \alpha) + \text{ord}_E(\gamma - \beta), u \pi^{\text{ord}_E(\beta - \alpha)})$$

as $\Lambda_E$-modules, where $u = (\gamma - \alpha)/(\beta - \alpha)$.

**Proof.** Let $M \cong M(m, n, x) \subset \mathcal{E}$. Suppose that $M$ is cyclic and put

$$M = \langle (a, b, c) \rangle_{\Lambda_E} \subset \mathcal{E}$$

for some $a, b, c \in \mathcal{O}_E$. Because $(1, 1, 1) \in \langle (a, b, c) \rangle_{\Lambda_E}$, we have $(1, 1, 1) = h(T)(a, b, c) = (h(\alpha)a, h(\beta)b, h(\gamma)c)$ for some $h(T) \in \Lambda_E$. Therefore we get $a, b, c \in \mathcal{O}_E^*$. Since $(0, \pi^m, x)$ and $(0, 0, \pi^n) \in \langle (a, b, c) \rangle_{\Lambda_E}$, we have

$$(0, \pi^m, x) = q(T)(a, b, c) = (q(\alpha)a, q(\beta)b, q(\gamma)c),$$

$$(0, 0, \pi^n) = r(T)(a, b, c) = (r(\alpha)a, r(\beta)b, r(\gamma)c)$$

for some $q(T)$ and $r(T) \in \Lambda_E$. Since $(T - \alpha) | q(T)$ and $(T - \alpha)(T - \beta) | r(T)$, we get $m = \text{ord}_E(q(\beta)) \geq \text{ord}_E(\beta - \alpha)$ and $n = \text{ord}_E(r(\gamma)) \geq \text{ord}_E(\gamma - \alpha) + \text{ord}_E(\gamma - \beta)$. On the other hand, by Proposition 3.3 and Remark 3.4, we have $m \leq \text{ord}_E(\beta - \alpha)$ and $n \leq \text{ord}_E(\gamma - \alpha) + \text{ord}_E(\gamma - \beta)$. Therefore we obtain $m = \text{ord}_E(\beta - \alpha)$ and $n = \text{ord}_E(\gamma - \alpha) + \text{ord}_E(\gamma - \beta)$. Furthermore,

$$(T - \alpha)(1, 1, 1) = (0, \beta - \alpha, \gamma - \alpha)$$

$$= (\beta - \alpha)\pi^{-m}(0, \pi^m, x) + \{\gamma - \alpha - (\beta - \alpha)\pi^{-m}x\}\pi^{-n}(0, 0, \pi^n).$$

Because $\text{ord}_E\{\gamma - \alpha - (\beta - \alpha)\pi^{-m}x\} \geq n$, we have $x = (\gamma - \alpha)/(\beta - \alpha)\pi^m(1 - \pi^n\psi/(\gamma - \alpha))$ for some $\psi \in \mathcal{O}_E$. By Remark 3.4 (i), we get

$$M(m, n, x) = M(\text{ord}_E(\beta - \alpha), \text{ord}_E(\gamma - \alpha) + \text{ord}_E(\gamma - \beta), u \pi^{\text{ord}_E(\beta - \alpha)}).$$

□
Proposition 5.3. Let \( f(T) \) be as above. Assume \( \text{ord}_E(\alpha - \beta) = \text{ord}_E(\beta - \gamma) = \text{ord}_E(\gamma - \alpha) = 1 \) and \( \text{ord}_E(\alpha) \geq \text{ord}_E(\beta) \geq \text{ord}_E(\gamma) \). Then, we have

\[
M_{f(T)}^E = \{(0, 0, 0), (0, 1, 0), (1, 0, 0), (0, 1, 1), (1, 2, u\pi), (1, 1, 0), (0, 1, 2)\},
\]

where \( u = (\gamma - \alpha)/(\beta - \alpha) \) and \((m, n, x)\) means \( [M(m, n, x)]_E \). The following is the table of the structure of \( \mathcal{O}_E\)-modules \( M/\omega_0M \) for \( \Lambda_E \)-modules \( M \):

<table>
<thead>
<tr>
<th>( M )</th>
<th>( M/\omega_0M )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M(0, 0, 0) )</td>
<td>( \mathcal{O}_E/(\alpha) \oplus \mathcal{O}_E/(\beta) \oplus \mathcal{O}_E/(\gamma) )</td>
</tr>
<tr>
<td>( M(0, 1, 0) )</td>
<td>( \mathcal{O}_E/(\beta) \oplus \mathcal{O}_E/(\alpha \gamma) )</td>
</tr>
<tr>
<td>( M(0, 1, 1) )</td>
<td>( \mathcal{O}_E/(\alpha) \oplus \mathcal{O}_E/(\beta \gamma) )</td>
</tr>
<tr>
<td>( M(0, 1, 2) )</td>
<td>( \mathcal{O}_E/(\beta) \oplus \mathcal{O}_E/(\alpha \gamma) )</td>
</tr>
<tr>
<td>( M(1, 0, 0) )</td>
<td>( \mathcal{O}_E/(\gamma) \oplus \mathcal{O}_E/(\alpha \beta) )</td>
</tr>
<tr>
<td>( M(1, 1, 0) )</td>
<td>( \mathcal{O}_E/(\gamma) \oplus \mathcal{O}_E/(\alpha \beta) )</td>
</tr>
<tr>
<td>( M(1, 2, u\pi) )</td>
<td>( \mathcal{O}_E/(\alpha \beta \gamma) )</td>
</tr>
</tbody>
</table>

Proof. The former is Corollary 3.8. We show the latter. Let \( [M]_E^E \in M_{f(T)}^E \). There exist \( m, n \) and \( x \) such that

\[
M = \langle (1, 1, 1), (0, \pi^m, x), (0, 0, \pi^n) \rangle_{\mathcal{O}_E}
\]

and we have

\[
\omega_0M = \langle (\alpha, \beta, \gamma), (0, \beta \pi^m, \gamma x), (0, 0, \gamma \pi^n) \rangle_{\mathcal{O}_E}.
\]

Since \( \mathcal{O}_E \) is a principal ideal domain, we can use the structure theorem over the principal ideal domain. We consider the map \( \Pi_{\omega_0}: M \to M \) and take \( (1, 1, 1), (0, \pi^m, x) \) and \( (0, 0, \pi^n) \) as a basis of \( M \). Then we have

\[
T(1, 1, 1) = \alpha(1, 1, 1) + (\beta - \alpha)\pi^{-m}(0, \pi^m, x)
\]

\[
+ \{\gamma - \alpha - (\beta - \alpha)\pi^{-m}x\}\pi^{-n}(0, 0, \pi^n),
\]

\[
T(0, \pi^m, x) = (0, \beta \pi^m, \gamma x)
\]

\[
= \beta(0, \pi^m, x) + (\gamma - \beta)x\pi^{-n}(0, 0, \pi^n).
\]

By the equalities (14) and (15), the matrix corresponding to \( \Pi_{\omega_0} \) is

\[
\begin{pmatrix}
\alpha & 0 & 0 \\
(\beta - \alpha)\pi^{-m} & \beta & 0 \\
{\gamma - \alpha - (\beta - \alpha)\pi^{-m}x} & (\gamma - \beta)x\pi^{-n} & \gamma
\end{pmatrix}.
\]

In order to verify the table, we have only to transform this matrix by elementary row
and column operations. For example, the case $M = M(0, 1, 0)$, we get the matrix

$$
\begin{pmatrix}
\alpha & 0 & 0 \\
\beta - \alpha & \beta & 0 \\
(\gamma - \alpha)\pi^{-1} & 0 & \gamma
\end{pmatrix}.
$$

By the elementary row and column operations, we have

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & \beta & 0 \\
0 & 0 & \alpha \gamma
\end{pmatrix}.
$$

So we get $M / \mathcal{O}_M \cong \mathcal{O}_E/(\beta) \oplus \mathcal{O}_E/(\alpha \gamma)$. The rest of the table can be checked by the same method.

**Proposition 5.4.** Let $f(T) = (T - \alpha)g(T)$, where $\alpha \in p\mathbb{Z}_p$ and $g(T) \in \mathbb{Z}_p[T]$ is a distinguished irreducible polynomial of degree 2. Let $E$ be the splitting field of $g(T)$ over $\mathbb{Q}_p$. If $[M(m, n, x)]_E \in \text{Image}(\Psi: \mathcal{M}^{\mathbb{Q}_p}_{f(T)} \to \mathcal{M}^{\mathbb{E}}_{f(T)}([M] \mapsto [M \otimes \Lambda \Lambda_E]_E))$, we have

$$
\text{ord}_E(x) = m.
$$

Proof. Let $[M] \in \mathcal{M}_{f(T)}^{\mathbb{Q}_p}$ and $M \otimes \Lambda_E \cong M(m, n, x) \subset \mathcal{E}$. There is a natural injective map

$$
M \to \Lambda/(f(T)) \to \Lambda/(T - \alpha) \oplus \Lambda/(g(T))
$$

([13], Lemma 13.8). By this injective map, we have

$$
M = \langle (a_1, b_1 T + c_1), (a_2, b_2 T + c_2), (a_3, b_3 T + c_3) \rangle_{\mathbb{Z}_p} \subset \Lambda/(T - \alpha) \oplus \Lambda/(g(T))
$$

for some $a_i, b_i$ and $c_i \in \mathbb{Z}_p$. Because $M \otimes \Lambda \Lambda_E = \langle (a_1, b_1 T + c_1), (a_2, b_2 T + c_2), (a_3, b_3 T + c_3) \rangle_{\mathcal{O}_E}$, by the same argument before Lemma 3.1, we can write

$$
M \otimes \Lambda \Lambda_E = \langle (a'_1, b'_1 T + c'_1), (0, b'_2 T + c'_2), (0, c'_3) \rangle_{\mathcal{O}_E}
$$

for some $a'_i, b'_i$ and $c'_i \in \mathbb{Z}_p$. Furthermore there is an injective map ([13], Lemma 13.8)

$$
\Lambda_E/(T - \alpha) \oplus \Lambda_E/(g(T)) \to \mathcal{E}, \quad (s(t), u(t)) \mapsto (s(\alpha), u(\beta), u(\gamma)),
$$

where $\beta$ and $\gamma$ are the roots of $g(T)$ in $E$. By this map, $M \otimes \Lambda \Lambda_E$ is isomorphic to the module

$$
M' = \langle (a'_1, b'_1 \beta + c'_1, b'_1 \gamma + c'_1), (0, b'_2 \beta + c'_2, b'_2 \gamma + c'_2), (0, c'_3, c'_3) \rangle_{\mathcal{O}_E} \subset \mathcal{E}.
$$
Since \( \beta \) and \( \gamma \) are conjugate, we have \( \text{ord}_E(b_1' \beta + c_1') = \text{ord}_E(b_1' \gamma + c_1') \) and \( \text{ord}_E(b_2' \beta + c_2') = \text{ord}_E(b_2' \gamma + c_2') \). By the same arguments after Lemma 3.2, we get

\[
M' \cong \langle (1, 1, 1), (0, \pi^m, x), (0, 0, \pi^n) \rangle_{\mathcal{O}_E}
\]

for some \( m, n, x \) which satisfy \( m = \text{ord}_E(x) \). Indeed, we may assume \( \text{ord}_E(b_2' \beta + c_2') \leq \text{ord}_E(c_2') \). By Lemma 3.2, we have

\[
M' \cong \langle (1, b_1' + c_1', b_1' \gamma + c_1'), (0, b_2' \beta + c_2', b_2' \gamma + c_2'), (0, c_3', c_3') \rangle_{\mathcal{O}_E}.
\]

In the case \( \text{ord}_E(b_1' \beta + c_1') \leq \text{ord}_E(b_2' \beta + c_2') \), we have

\[
M' \cong \langle (1, 1, b_1' \gamma + c_1'), (0, b_2' \beta + c_2' - c_3', b_2' \gamma + c_2'), (0, 0, c_3') \rangle_{\mathcal{O}_E}.
\]

Since \( \text{ord}_E(b_1' \beta + c_1') \leq \text{ord}_E(b_2' \beta + c_2') \leq \text{ord}_E(c_3') \), we get

\[
M' \cong \langle (1, 1, 1), (0, b_2' \beta + c_2' - c_3'), b_2' \gamma + c_2'), (0, 0, c_3') \rangle_{\mathcal{O}_E}
\]

Thus we get

\[
m = \text{ord}_E \left( \frac{b_2' \beta + c_2'}{b_1' \beta + c_1'} \right), \quad n = \text{ord}_E \left( \frac{c_3'}{b_1' \beta + c_1'} - \frac{c_3'}{b_2' \beta + c_2'} \cdot \frac{b_2' \gamma + c_2'}{b_1' \gamma + c_1'} \right),
\]

\[
x = \pi^{-m} \cdot \frac{b_1' \beta + c_1'}{b_2' \beta + c_2'} \cdot \frac{b_2' \gamma + c_2'}{b_1' \gamma + c_1'}.
\]

Therefore we obtain \( m = \text{ord}_E(x) \). On the other hand, in the case \( \text{ord}_E(b_1' \beta + c_1') > \text{ord}_E(b_2' \beta + c_2') \), we have

\[
M' = \langle a_1' , (b_1' - b_2') \beta + (c_1' - c_2'), (b_1' - b_2') \gamma + (c_1' - c_2'), (0, b_2' \beta + c_2' - c_3') \rangle_{\mathcal{O}_E}.
\]

Because \( \text{ord}_E(b_1' \beta + c_1' - (b_2' \beta + c_2')) = \text{ord}_E(b_2' \beta + c_2') \), we get the same conclusion as in the case \( \text{ord}_E(b_1' \beta + c_1') \leq \text{ord}_E(b_2' \beta + c_2') \).

**Proposition 5.5.** Let \( f(T) = (T - \alpha)g(T) \), where \( \alpha \in p\mathbb{Z}_p \) and \( g(T) \in \mathbb{Z}_p[T] \) is an irreducible polynomial of degree \( 2 \). Let \( E \) be the splitting field of \( g(T) \) over \( \mathbb{Q}_p \). We assume \( \text{ord}_E(\alpha - \beta) = \text{ord}_E(\beta - \gamma) = \text{ord}_E(\gamma - \alpha) = 1 \).

\[
M/\omega_dM \cong \mathbb{Z}/p^i \mathbb{Z} \oplus \mathbb{Z}/p^j \mathbb{Z} \quad (i, j \in \mathbb{Z}_{\geq 1})
\]

and \( E/\mathbb{Q}_p \) is a totally ramified extension. Then we have

\[
\Psi(M) = M \otimes \Lambda_E \cong M(0, 1, 1) \cong \Lambda_E/(T - \alpha) \oplus \Lambda_E/(T - \beta)(T - \gamma).
\]
Suppose $p/BH$ corresponding to the table of the Proposition 5.3. The isomorphism $M(0, 1, 1) \cong \Lambda_E/(T - \alpha) \oplus \Lambda_E/(T - \beta)(T - \gamma)$ is Lemma 3 in Sumida [12].

**Corollary 5.6.** Let $f(T), g(T)$ and $E$ be as in Propositions 5.5 and $[M]_{Q_p} \in M_{f(T)}$. We assume the same conditions of Proposition 5.5 and we put $g(T) = T^2 + c_1T + c_0$. Then

(a) Suppose $p \geq 5$. For $n \geq 0$, we have

$$\#(M/\omega_nM \otimes \Lambda_E) = p^{\text{ord}_E(\omega_n(\alpha)\omega_n(\beta)\omega_n(\gamma))} = p^{6n+2+\text{ord}_E(\alpha)}.$$ 

(b) Suppose $p = 3$. For $n \geq 1$, we have

$$\#(M/\omega_nM \otimes \Lambda_E) = \begin{cases} p^{\text{ord}_E(\omega_n(\alpha)\omega_n(\beta)\omega_n(\gamma))} = p^{6n+\text{ord}_E(\alpha)+4\text{ord}_3(c_0-3)-2} & \text{if } \text{ord}_3(c_0-3) \leq \text{ord}_3(c_1-3), \\ p^{\text{ord}_E(\omega_n(\alpha)\omega_n(\beta)\omega_n(\gamma))} = p^{6n+\text{ord}_E(\alpha)+4\text{ord}_3(c_0-3)-2} & \text{if } \text{ord}_3(c_0-3) > \text{ord}_3(c_1-3). \end{cases}$$

Proof. Put $N = \langle (1, 1, 1), (0, 1, 1), (0, 0, \pi) \rangle_{\mathcal{O}_E} \subseteq \mathcal{E}$. By Proposition 5.5, we have $M \otimes_{\Lambda} \Lambda_E \cong N$ as $\Lambda_E$-modules. Thus we have

$$M/\omega_nM \otimes \Lambda_E \cong (M \otimes_{\Lambda} \Lambda_E)/\omega_n(M \otimes_{\Lambda} \Lambda_E) \cong N/\omega_n N$$

as $\Lambda_E/\omega_n \Lambda_E$-modules. By the same method as Proposition 5.3, we consider the map $\Pi_{\omega_n}: N \to N$ and take $(1, 0, 0), (0, 1, 1)$ and $(0, 0, \pi)$ as a basis of $N$. The matrix corresponding to $\Pi_{\omega_n}$ is

$$\begin{pmatrix} \omega_n(\alpha) & 0 & 0 \\ 0 & \omega_n(\beta) & 0 \\ 0 & (\omega_n(\beta) - \omega_n(\gamma))\pi^{-1} & \omega_n(\gamma) \end{pmatrix}.$$ 

We first consider the case (a). We have $\text{ord}_E(\omega_n(\beta) - \omega_n(\gamma)) = \text{ord}_E(\beta - \gamma) + n \text{ord}_E(3) = 2n + 1$ (cf. [7], Lemma 2.5). Furthermore, we have $\text{ord}_E(\omega_n(\alpha)) = 2n + \text{ord}_E(\alpha)$, and we get $\text{ord}_E((\omega_n(\beta) - \omega_n(\gamma))\pi^{-1}) = 2n < \text{ord}_E(\omega_n(\beta))$ since $\text{ord}_E(\omega_n(\beta)) = \text{ord}_E(\omega_n(\gamma)) = 2n + 1$. Thus we can transform the above matrix into

$$\begin{pmatrix} \pi^{2n+\text{ord}_E(\alpha)} & 0 & 0 \\ 0 & \pi^{2n} & 0 \\ 0 & 0 & \pi^{2n+2} \end{pmatrix}.$$
This implies $N/\omega_0 N \cong \mathcal{O}_E/(\pi^{2n+\text{ord}_E(\alpha)}) \oplus \mathcal{O}_E/(\pi^{2n}) \oplus \mathcal{O}_E/(\pi^{2n+2})$.

Next, we prove the case (b). For $n \geq 1$, we have

$$\text{ord}_E(\omega_n(\beta)) = \begin{cases} 2 \text{ord}_3(c_0 - 3) + 2n - 1 & \text{if } \text{ord}_3(c_0 - 3) \leq \text{ord}_3(c_1 - 3), \\
2 \text{ord}_3(c_1 - 3) + 2n & \text{if } \text{ord}_3(c_0 - 3) > \text{ord}_3(c_1 - 3).
\end{cases}$$

On the other hand, for $n \geq 1$, we have

$$\text{ord}_E(\omega_n(\beta) - \omega_n(\gamma)) \begin{cases} = 2 \text{ord}_3(c_0 - 3) + 2n - 1 & \text{if } \text{ord}_3(c_0 - 3) \leq \text{ord}_3(c_1 - 3), \\
> 2 \text{ord}_3(c_1 - 3) + 2n & \text{if } \text{ord}_3(c_0 - 3) > \text{ord}_3(c_1 - 3)
\end{cases}$$

(cf. [7], Lemma 2.5). The rest can be proved by the same method as the case (a). \(\square\)

In order to determine the structure of $X$, we will use the higher Fitting ideals. For a commutative ring $R$ and a finitely presented $R$-module $M$, we consider the following exact sequence

$$R^n \xrightarrow{f} R^m \to M \to 0,$$

where $m$ and $n$ are positive integers. For an integer $i \geq 0$ such that $0 \leq i < n$, the $i$-th Fitting ideal of $M$ is defined to be the ideal of $R$ generated by all $(n-i) \times (n-i)$ minors of the matrix corresponding to $f$. This definition does not depend on the choice of the above exact sequence (see [9]).

**Proposition 5.7.** Let $f(T) = (T - \alpha)g(T)$, where $\alpha \in p\mathbb{Z}_p$ and $g(T) \in \mathbb{Z}_p[T]$ is an irreducible polynomial of degree 2. Let $E$ be the splitting field of $g(T)$ over $\mathbb{Q}_p$. Let $[M]_E \in \mathcal{M}^L_{f(T)}$, and $M = M(m, n, x)$.

1. Assume $m = 0$ and $(\gamma - \beta)x\pi^{-m} \in \mathcal{O}_E^\times$. Then we have

$$\text{Fitt}_1, \Lambda(M) = \begin{cases} (T - \alpha, (\alpha - \beta)(\alpha - \gamma)) & \text{if } x = 1, \\
(T - \alpha, (\alpha - \beta)(\alpha - \gamma)(1 - x)\pi^{-m}) & \text{if } x \neq 1.
\end{cases}$$

2. Assume $n = 0$ and $(\beta - \alpha)\pi^{-m} \in \mathcal{O}_E^\times$. Then we have

$$\text{Fitt}_1, \Lambda(M) = (T - \gamma, (\alpha - \gamma)(\beta - \gamma)).$$

3. \(\text{Fitt}_1, \Lambda((T - \alpha)M) = \begin{cases} (T - \beta, (\beta - \gamma)\pi^{-m}) & \text{if } n \leq \text{ord}_E(\pi^m - x), \\
(T - \beta, \frac{\gamma - \beta}{\pi^m - x}) & \text{if } n > \text{ord}_E(\pi^m - x).
\end{cases} \)
Proof. By the action of $T$, we have
\[
T(1, 1, 1) = (\alpha, \beta, \gamma) = \alpha(1, 1, 1) + (\beta - \alpha)\pi^{-m}(0, \pi^m, x) \\
+ (\gamma - \alpha - (\beta - \alpha)\pi^{-m}x)\pi^{-n}(0, 0, \pi^n),
\]

\[
T(0, \pi^m, x) = (0, \beta\pi^m, \gamma x) = \beta(0, \pi^m, x) + (\gamma - \beta)x\pi^{-n}(0, 0, \pi^n),
\]

\[
T(0, 0, \pi^n) = \gamma(0, 0, \pi^n).
\]

Then we get the following matrix
\[
\begin{pmatrix}
T - \alpha & -(\beta - \alpha)\pi^{-m} & -(\gamma - \alpha - (\beta - \alpha)\pi^{-m}x)\pi^{-n} \\
0 & T - \beta & -(\gamma - \beta)x\pi^{-n} \\
0 & 0 & T - \gamma
\end{pmatrix}.
\]

We first show (1). Under the assumption of (1), the matrix is
\[
\begin{pmatrix}
T - \alpha & -(\beta + \alpha) & -(\gamma - \alpha - (\beta - \alpha)x)\pi^{-n} \\
0 & T - \beta & -(\gamma - \beta)x\pi^{-n} \\
0 & 0 & T - \gamma
\end{pmatrix}.
\]

By elementary row and column operations, we can transform the above matrix into
\[
\begin{pmatrix}
T - \alpha & (\alpha - \gamma)(1 - x)\pi^{-n}(T - \beta) & 0 \\
0 & (T - \beta)(T - \gamma) & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Therefore we get
\[
\text{Fitt}_{1, \Lambda}(M) = (T - \alpha, (\alpha - \beta)(\alpha - \gamma), (\alpha - \beta)(\alpha - \beta)(1 - x)\pi^{-n})
\]
\[
= \begin{cases}
(T - \alpha, (\alpha - \beta)(\alpha - \gamma)) & \text{if } x = 1, \\
(T - \alpha, (\alpha - \beta)(\alpha - \gamma)(1 - x)\pi^{-n}) & \text{if } x \neq 1.
\end{cases}
\]

Next we show (2). Under the assumption of (2), the matrix is
\[
\begin{pmatrix}
T - \alpha & -(\beta - \alpha)\pi^{-m} & -(\gamma - \alpha) + (\beta - \alpha)\pi^{-m}x \\
0 & T - \beta & -(\gamma - \beta)x \\
0 & 0 & T - \gamma
\end{pmatrix}.
\]

By elementary row and column operations, we can transform the above matrix into
\[
\begin{pmatrix}
T - \alpha & 1 & 0 \\
0 & T - \beta & 0 \\
0 & 0 & T - \gamma
\end{pmatrix}.
\]
Therefore we get
\[ \text{Fitt}_{1, A}(M) = ((T - \alpha)(T - \beta), (T - \beta)(T - \gamma), (T - \alpha)(T - \gamma), (T - \gamma)) \]
\[ = (T - \gamma, (\alpha - \gamma)(\beta - \gamma)). \]

Finally we show (3). We note that
\[ (T - \alpha)M = \left\{ \begin{array}{ll}
(0, \beta - \alpha, \gamma - \alpha), & (0, (\beta - \alpha)\pi^m, (\gamma - \alpha)x), (0, 0, (\gamma - \alpha)\pi^n) \end{array} \right\}_{\mathcal{O}_E} \]
\[ = \left\{ \begin{array}{ll}
((0, \beta - \alpha, \gamma - \alpha), (0, 0, (\gamma - \alpha)\pi^n))_{\mathcal{O}_E} & \text{if } n \leq \text{ord}_E(\pi^m - x),
((0, \beta - \alpha, \gamma - \alpha), (0, 0, (\gamma - \alpha)(\pi^m - x)))_{\mathcal{O}_E} & \text{if } n > \text{ord}_E(\pi^m - x).
\end{array} \right. \]

In the case \( n \leq \text{ord}_E(\pi^m - x) \), by the action of \( T \), we have
\[ T(0, \beta - \alpha, \gamma - \alpha) = (0, \beta(\beta - \alpha), \gamma(\gamma - \alpha)) \]
\[ = \beta(0, \beta - \alpha, \gamma - \alpha) + (\gamma - \beta)\pi^{-n}(0, 0, (\gamma - \alpha)\pi^n), \]
\[ T(0, 0, (\gamma - \alpha)\pi^n) = \gamma(0, 0, (\gamma - \alpha)\pi^n). \]

Thus we get the following matrix
\[ \left( \begin{array}{cc}
T - \beta & -(\gamma - \beta)\pi^{-n} \\
0 & T - \gamma
\end{array} \right). \]

Therefore we get
\[ \text{Fitt}_{1, A}((T - \alpha)M) = (T - \beta, T - \gamma, (\gamma - \beta)\pi^{-n}) \]
\[ = (T - \beta, (\gamma - \beta)\pi^{-n}). \]

In the case \( n > \text{ord}_E(\pi^m - x) \), by the same method as above, we get the following matrix
\[ \left( \begin{array}{cc}
T - \beta & \frac{\gamma - \beta}{\pi^m - x} \\
0 & T - \gamma
\end{array} \right). \]

Therefore we get
\[ \text{Fitt}_{1, A}((T - \alpha)M) = \left( \begin{array}{cc}
T - \beta & T - \gamma, \frac{\gamma - \beta}{\pi^m - x}
\end{array} \right) \]
\[ = \left( T - \beta, \frac{\gamma - \beta}{\pi^m - x} \right). \]

6. Numerical examples

In this section, we introduce some numerical examples which were computed using Pari-Gp.
Let $p = 3$ and $k = \mathbb{Q}(\sqrt{-d})$ where $d$ is a positive square-free integer. For simplicity, let $d \not\equiv 2 \mod 3$. Our assumption $d \not\equiv 2 \mod 3$ implies that $p = 3$ is inert or ramified in $k$. This assumption is also needed to get the isomorphism (16) below. In this section, we determine the $\Lambda$-isomorphism class of the Iwasawa module associated to $k = \mathbb{Q}(\sqrt{-d})$ in the range $1 < d < 10^5$ with $\lambda_p(k) = 3$, where $\lambda_p(k)$ is the Iwasawa $\lambda$-invariant with respect to the cyclotomic $\mathbb{Z}_p$-extension. There are 1109 imaginary quadratic fields satisfying these properties.

Let $k_{\infty}/k$ be the cyclotomic $\mathbb{Z}_p$-extension of $k$. For each $n \geq 0$, we denote by $k_n$ the intermediate field of $k_{\infty}/k$ such that $k_n$ is the unique cyclic extension over $k$ of degree $p^n$. Let $A_n$ be the $p$-Sylow subgroup of the ideal class group of $k_n$. We put $X = \varprojlim A_n$, where the inverse limit is taken with respect to the relative norms. Then $X$ becomes a $\mathbb{Z}_p[[\text{Gal}(k_{\infty}/k)]]$-module. Since there is a ring isomorphism between $\Lambda = \mathbb{Z}_p[[T]]$ and $\mathbb{Z}_p[[\text{Gal}(k_{\infty}/k)]]$ which depends on the choice of a topological generator of $\text{Gal}(k_{\infty}/k)$, $X$ becomes a finitely generated torsion $\Lambda$-module. Let $f(T)$ be the distinguished polynomial which generates $\text{char}(X)$. It is known that $X$ is a free $\mathbb{Z}_p$-module so $[X]_{\mathbb{Q}_p} \in \mathcal{M}_{f(T)}$ and we can apply Theorem 3.5 to the Iwasawa module $X$.

We can calculate the polynomial $f(T) \mod p^n$ for small $n$ numerically. Let $\chi$ be the Dirichlet character associated to $k$, $\omega$ be the Teichmüller character and $f_0$ be the least common multiple of $p$ and conductor of $\chi$. By the Iwasawa main conjecture, there exists a power series $g_{\chi^{-1}\omega}(T) \in \Lambda$ such that

$$\text{char}(X) = (g_{\chi^{-1}\omega}(T)).$$

Here, $g_{\chi^{-1}\omega}(T)$ is the $p$-adic $L$-function constructed by Iwasawa. We can approximate $g_{\chi^{-1}\omega}(T)$ such as

$$g_{\chi^{-1}\omega}(T) \equiv \frac{1}{2f_0 p^n} \sum_{0 < a < f_0 p^n, (a, f_0 p^n) = 1} a\chi\omega^{-1}(a)(1 + T)^{i_a(a)} \mod \omega_n,$$

where $i_a(a)$ is the unique integer such that $a\omega^{-1}(a) \equiv (1 + p)^{i_a(a)} \mod p^{a+1}$ and $0 \leq i_a(a) < p^n$. By Weierstrass preparation theorem ([13], Theorem 7.3), there exists $u_{\chi^{-1}\omega} \in \Lambda^*$ such that $g_{\chi^{-1}\omega}(T) = f(T)u_{\chi^{-1}\omega}(T)$. Thus we can get $f(T)$ approximately ([13], Proposition 7.2). For the detail about computation of $g_{\chi^{-1}\omega}(T)$, see [1] and [4]. We computed $f(T)$ by Mizusawa’s program Iwapoly.ub ([8], Research, Programming, Approximate Computation of Iwasawa Polynomials by UBASIC), and referred Fukuda’s table for the $\lambda$-invariants of imaginary quadratic fields [3].

Now we classify the Iwasawa module $X$. There are two cases

\begin{itemize}
  \item [(I)] $A_0$ is a cyclic group.
  \item [(II)] $A_0$ is not a cyclic group.
\end{itemize}

In order to determine the structure of $X$, we use the following fact. In our case, exactly one prime is ramified in $k_{\infty}/k$ and it is totally ramified. So there are
\( \Lambda \)-isomorphisms

\[
X/\omega_n X \cong A_n
\]

for any non-negative integers ([13], Proposition 13.22).

We determine the \( \Lambda \)-isomorphism class of \( X \) by the information on the structures of \( A_n \) for some \( n \geq 0 \).

There are 1015 fields whose \( A_0 \) are cyclic groups among 1109 fields. First of all, we determine the isomorphism classes in the case (I). In this case, \( X \) becomes a \( \Lambda_E \)-cyclic module by Nakayama’s Lemma. Thus we can use Proposition 5.2 to get

\[
M \cong M(\text{ord}_E(\beta - \alpha), \text{ord}_E(\gamma - \alpha) + \text{ord}_E(\gamma - \beta), u \pi^{\text{ord}_E(\beta - \alpha)}).
\]

In the above range of \( d \), no \( f(T) \) splits completely in \( \mathbb{Q}_p[T] \), so we have to consider the minimal splitting field \( E \) of \( f(T) \), which is quadratic over \( \mathbb{Q}_p \).

**Example 6.1.** Let \( k = \mathbb{Q}(\sqrt{-886}) \). Then we have \( A_0 \cong \mathbb{Z}/9\mathbb{Z} \) (cf. [10]). By using Mizusawa’s program [8], we have

\[
f(T) \equiv (T - 195)(T^2 + 291T + 429) \mod 3^6.
\]

By Hensel’s lemma, there exist \( \alpha \in \mathbb{Z}_p \) and \( g(T) \in \mathbb{Z}_p[T] \) such that

\[
f(T) = (T - \alpha)g(T),
\]

where \( \alpha \equiv 195 \mod 3^5 \) and \( g(T) \equiv T^2 + 48T + 186 \mod 3^5 \). Since \( g(T) \) is an Eisenstein polynomial, \( E/\mathbb{Q}_p \) is a totally ramified extension. Let \( E \) be the minimal splitting field of \( g(T) \) and \( g(T) = (T - \beta)(T - \gamma) \), where \( \beta, \gamma \in E \). Because \( \beta \gamma \equiv 186 \mod 3^5 \), we get \( \text{ord}_E(\beta) = \text{ord}_E(\gamma) = 1 \) and \( \text{ord}_E(\alpha - \gamma) = \text{ord}_E(\alpha - \gamma) = 1 \). Since \( (\beta - \gamma)^2 = (\beta + \gamma)^2 - 4\beta \gamma \equiv 1560 \mod 3^5 \), we have \( \text{ord}_E(\beta - \gamma) = 1 \). By Proposition 5.1 and 5.2, we get \( X \otimes_\Lambda \Lambda_E \cong M(1, 2, u \pi) \), where \( u = (\gamma - \alpha)/(\beta - \alpha) \).

Next, we determine the isomorphism classes in the case (II). There are 94 fields whose \( A_0 \) are not cyclic groups. There are 66 fields whose \( A_0 \) are not cyclic groups and whose \( f(T) \) is reducible. We will determine \([X]_{\mathbb{Q}_p}\) for these 66 fields. We can determine the \( \Lambda \)-isomorphism class of \( X \) for 60 fields by Proposition 5.5. The following example is a case that we can determine the \( \Lambda \)-isomorphism class of \( X \) by Proposition 5.5.

**Example 6.2.** Let \( k = \mathbb{Q}(\sqrt{-6583}) \). In this case, we have \( A_0 \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \) (cf. [10]). We have

\[
f(T) \equiv (T - 96)(T^2 + 96T + 696) \mod 3^6.
\]
By Hensel’s lemma, there exist $\alpha \in \mathbb{Z}_p$ and $g(T) \in \mathbb{Z}_p[T]$ such that

$$f(T) = (T - \alpha)g(T),$$

where $\alpha \equiv 96 \mod 3^5$ and $g(T) \equiv T^2 + 210 \mod 3^5$. Let $E$ be the minimal splitting field of $g(T)$ and $g(T) = (T - \beta)(T - \gamma)$, where $\beta, \gamma \in E$. Then, $E/\mathbb{Q}_p$ is a totally ramified extension and we get $\text{ord}_E(\alpha - \beta) = \text{ord}_E(\beta - \gamma) = \text{ord}_E(\gamma - \alpha) = 1$, $\text{ord}_E(\alpha) = 2$, and $\text{ord}_E(\beta) = \text{ord}_E(\gamma) = 1$. Therefore we get $X \otimes_A \Lambda_E \cong M(0, 1, 1)$ by Proposition 5.5.

There are remaining 6 fields which we cannot determine the structure of $X$ by Proposition 5.5. For these fields, we have to investigate the action of the group $\Gamma_1 = \text{Gal}(k_1/k)$. Explicitly, the remaining 6 fields are $\mathbb{Q}(\sqrt{-9574})$, $\mathbb{Q}(\sqrt{-30994})$, $\mathbb{Q}(\sqrt{-41631})$, $\mathbb{Q}(\sqrt{-64671})$, $\mathbb{Q}(\sqrt{-82774})$, $\mathbb{Q}(\sqrt{-92515})$.

**Example 6.3.** Let $k = \mathbb{Q}(\sqrt{-9574})$. In this case, we have $A_0 \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$ (cf. [10]) and $A_1 \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/27\mathbb{Z}$. We have

$$f(T) \equiv (T - 192)(T^2 + 1173T + 1422) \mod 3^7.$$

By Hensel’s Lemma, there exist $\alpha \in \mathbb{Z}_p$ and $g(T) \in \mathbb{Z}_p[T]$ such that

$$f(T) = (T - \alpha)g(T),$$

where $\alpha \equiv 192 \mod 3^5$ and $g(T) \equiv T^2 + 201T + 207 \mod 3^5$. Let $E$ be the splitting field of $g(T)$ and $g(T) = (T - \beta)(T - \gamma)$, where $\beta, \gamma \in E$. Because the discriminant of $g(T)$ is $3^2 \cdot 4397 \mod 3^7$ and $4397$ is a quadratic nonresidue, $E/\mathbb{Q}_p$ is an unramified extension. Since the discriminant of $f(T)$ is $2^8 \cdot 3^6 \cdot 43 \cdot 89 \cdot 1039 \mod 3^7$, we get $\text{ord}_E(\alpha - \beta) = \text{ord}_E(\beta - \gamma) = \text{ord}_E(\gamma - \alpha) = 1$ and $\text{ord}_E(\alpha) = \text{ord}_E(\beta) = \text{ord}_E(\gamma) = 1$. By checking the structures of $A_0$ and $A_1$ as $\mathcal{O}_E$-modules, we get

$$X \otimes_A \Lambda_E \cong M(0, 1, 1), M(0, 1, 2), M(1, 0, 0) \text{ or } M(1, 1, 0).$$

Now we investigate the structure of $A_1$ as a $\Gamma_1$-module. We have an isomorphism $A_1 \cong \mathbb{Z}/27\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. Furthermore, Pari-Gp gives explicit generators which give this isomorphism. Let $a_1$, $a_2$ and $a_3$ be the generators which was computed. (We do not write down $a_1$, $a_2$ and $a_3$ because they are complicated.) Let $\sigma$ be the generator of $\Gamma_1$, which was computed by Pari-Gp. We compute,

$$(\sigma - 1)a_1 = 3a_2 - a_3,$$

$$(\sigma - 1)a_2 = 6a_2,$$

$$(\sigma - 1)a_3 = 18a_1 + 6a_2.$$
There is a topological generator $\tilde{\sigma} \in \text{Gal}(k_{\infty}/k)$ such that $\tilde{\sigma}$ is an extension of $\sigma$. By this topological generator, we have an isomorphism

$$Z_p[[\text{Gal}(k_{\infty}/k)]] \cong \Lambda = Z_p[[T]]$$

such that $\tilde{\sigma} \leftrightarrow 1 + T$.

We regard $X$ as a $\Lambda$-module by this isomorphism. We note that $f(T)$ depends on the choice of $\tilde{\sigma}$, but we can easily check that $M_{f(T)}$ does not depend on the choice of $\tilde{\sigma}$. Because $Z_p[[\Gamma_1]] \cong \Lambda/\omega_1 \Lambda$, we get

$$\overline{T}a_1 = 3a_2 - a_3,$$

$$\overline{T}a_2 = 6a_2,$$

$$\overline{T}a_3 = 18a_1 + 6a_2,$$

where $\overline{T} = T \mod \omega_1$. Now we have

$$\overline{(T^2 + 18)a_1 + 6a_2} = 0,$$

$$\overline{(T - 6)a_2} = 0,$$

$$\overline{3T}a_1 = 0,$$

$$\overline{27a_1} = 0,$$

$$\overline{9a_2} = 0.$$

Therefore we can calculate the 1-st Fitting ideal of $A_1 \otimes \mathcal{O}_E$;

$$\text{Fitt}_{1, \Lambda_E/\omega_1 \Lambda_E}(A_1 \otimes \mathcal{O}_E) = (T, 3) \mod \omega_1,$$

where $\text{Fitt}_{1, \Lambda_E/\omega_1 \Lambda_E}(A_1 \otimes \mathcal{O}_E)$ is the 1-st Fitting ideal of $A_1 \otimes \mathcal{O}_E$ as a $\Lambda_E/\omega_1 \Lambda_E$-module. On the other hand, by Proposition 5.7 (1) and (2), for $M(0, 1, 2), M(1, 0, 0), M(0, 1, 1)$, we have

$$\text{Fitt}_{1, \Lambda_E/\omega_1 \Lambda_E}(M/\omega_1 M) = \begin{cases} (T, 3) \mod \omega_1 & \text{if } M = M(0, 1, 2), \\ (T - \gamma, 9) \mod \omega_1 & \text{if } M = M(1, 0, 0), \\ (T - \alpha, 9) \mod \omega_1 & \text{if } M = M(0, 1, 1). \end{cases}$$

Therefore we have

$$X \otimes_\Lambda \Lambda_E \cong M(0, 1, 2) \text{ or } M(1, 1, 0).$$

We investigate the module $(T - \alpha)(M/\omega_1 M)$. By Proposition 5.7 (3), for $M(0, 1, 2), M(1, 1, 0)$ we get

$$\text{Fitt}_{1, \Lambda_E/\omega_1 \Lambda_E}((T - \alpha)(M/\omega_1 M)) = \begin{cases} (T, 3) \mod \omega_1 & \text{if } M = M(0, 1, 2), \\ \Lambda_E/\omega_1 \Lambda_E & \text{if } M = M(1, 1, 0). \end{cases}$$

We can compute the following from the above data

$$\text{Fitt}_{1, \Lambda_E/\omega_1 \Lambda_E}((T - \alpha)A_1 \otimes \mathcal{O}_E) = (T, 3) \mod \omega_1.$$
Therefore, we get \( X \otimes_{\Lambda} \Lambda_E \cong M(0, 1, 2) \).

By the same method as above, we can determine the isomorphism classes of \( X \) of \( \mathbb{Q}(\sqrt{-30994}) \), \( \mathbb{Q}(\sqrt{-82774}) \) and \( \mathbb{Q}(\sqrt{-92515}) \). For the 3 fields, we can show that \( X \otimes_{\Lambda} \Lambda_E \cong M(0, 1, 2) \).

Finally we determine the structure of \( X \) for remaining 2 fields \( \mathbb{Q}(\sqrt{-41631}) \) and \( \mathbb{Q}(\sqrt{-64671}) \).

**Example 6.4.** Let \( k = \mathbb{Q}(\sqrt{-41631}) \). In this case, we have \( A_0 \cong \mathbb{Z}/3^3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \) (cf. [10]) and \( A_1 \cong \mathbb{Z}/3^3\mathbb{Z} \oplus \mathbb{Z}/3^2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \) computing by Pari-Gp. We have

\[
f(T) \equiv (T - 42)(T^2 - 279T + 594) \mod 3^7.
\]

By Hensel’s lemma, there exist \( \alpha \in \mathbb{Z}_p \) and \( g(T) \in \mathbb{Z}_p[T] \) such that

\[
f(T) = (T - \alpha)g(T),
\]

where \( \alpha \equiv 42 \mod 3^5 \) and \( g(T) \equiv T^2 + 36T + 108 \mod 3^5 \). Let \( E \) be the minimal splitting field of \( g(T) \) and \( g(T) = (T - \beta)(T - \gamma) \), where \( \beta, \gamma \in E \). Then \( E/\mathbb{Q}_p \) is a totally ramified extension with \( \text{ord}_E(\alpha - \beta) = \text{ord}_E(\gamma - \alpha) = 2, \text{ord}_E(\beta - \gamma) = 3, \text{ord}_E(\alpha) = 2, \text{ord}_E(\beta) = \text{ord}_E(\gamma) = 3 \). Let \( \pi \) be a prime element of \( E \). In this case, the elements \( M(m, n, x) \in \mathcal{M}^E_{f(T)} \) which satisfy the conclusion of Proposition 5.4 are

\[
\left\{
(0, 0, 0), (0, 1, 1), (0, 1, 2), (0, 2, 1), (0, 2, 2), (0, 2, 1 + \pi), (0, 3, 1),
(0, 3, 1 + \pi), (0, 3, 1 + \pi^2), (1, 0, 0), (1, 1, 0), (1, 1, 1), (1, 2, \pi),
(1, 2, 2\pi), (1, 3, \pi), (1, 3, \pi + \pi^2), (1, 3, \pi + 2\pi^2), (1, 4, u\pi),
(2, 0, 0), (2, 1, 0), (2, 2, 0), (2, 3, u\pi^2), (2, 4, u\pi^2), (2, 5, u\pi^2)
\right\},
\]

where \( u = (\gamma - \alpha)/(\beta - \alpha) \). By checking the structures of \( A_0 \) and \( A_1 \) as \( \mathcal{O}_E \)-modules, we get

\[
X \otimes_{\Lambda} \Lambda_E \cong M(0, 3, 1), M(0, 3, 1 + \pi), M(0, 3, 1 + \pi^2),
M(1, 3, \pi + \pi^2), M(1, 3, \pi + 2\pi^2) \quad \text{or} \quad M(2, 3, u\pi^2).
\]

We have an isomorphism \( A_1 \cong \mathbb{Z}/81\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \). Let \( a_1, a_2 \) and \( a_3 \) be the generators which were computed by Pari-Gp. By Pari-Gp we have:

\[
(\sigma - 1)a_1 = 54a_1 + 6a_2 + a_3,
(\sigma - 1)a_2 = 54a_1,
(\sigma - 1)a_3 = 54a_1 + 3a_2,
\]

for a certain generator \( \sigma \) of \( \Gamma_1 \). By the same method as \( k = \mathbb{Q}(\sqrt{-9574}) \), we fix a topological generator \( \tilde{\sigma} \in \text{Gal}(k_{\infty}/k) \) such that \( \tilde{\sigma} \) is an extension of \( \sigma \). Because
$\mathbb{Z}_p[[\Gamma_1]] \cong \Lambda/\omega_1 \Lambda$, we have

\[
\begin{align*}
(T^2 - 54T - 54)a_1 - 3a_2 &= 0, \\
54a_1 - Ta_2 &= 0, \\
3Ta_1 &= 0, \\
81a_1 &= 0, \\
9a_2 &= 0,
\end{align*}
\]

where $\overline{T} = T \mod \omega_1$. Therefore we get the 1-st Fitting ideal of $A_1 \otimes \mathcal{O}_E$:

\[
\text{Fitt}_{1, \Lambda_E/\omega_1 \Lambda_E}(A_1 \otimes \mathcal{O}_E) = (T, 3) \mod \omega_1.
\]

On the other hand, by Proposition 5.7 (1) and (2), we have

\[
\text{Fitt}_{1, \Lambda_E/\omega_1 \Lambda_E}(M/\omega_1 M) = \begin{cases} 
(T - \alpha, 9) \mod \omega_1 & \text{if } M = M(0, 3, 1), \\
(T, 3) \mod \omega_1 & \text{if } M = M(0, 3, 1 + \pi), \\
(T - \alpha, \pi^2) \mod \omega_1 & \text{if } M = M(0, 3, 1 + \pi^2),
\end{cases}
\]

for $M(0, 3, 1)$, $M(0, 3, 1 + \pi)$ and $M(0, 3, 1 + \pi^2)$. Therefore we have

\[
X \otimes_{\Lambda} \Lambda_E \cong M(0, 3, 1 + \pi), M(1, 3, \pi + \pi^2), M(1, 3, \pi + 2\pi^2) \text{ or } M(2, 3, u\pi^2).
\]

As in the case $k = \mathbb{Q}(\sqrt{-9574})$, we investigate the structure of $(T - \alpha)(M/\omega_1 M)$. By Proposition 5.7 (3), we get

\[
\text{Fitt}_{1, \Lambda_E/\omega_1 \Lambda_E}((T - \alpha)(M/\omega_1 M)) = \begin{cases} 
(T, 3) \mod \omega_1 & \text{if } M = M(0, 3, 1 + \pi), \\
\Lambda_E/\omega_1 \Lambda_E & \text{if } M = M(1, 3, \pi + \pi^2), \\
(T, \pi) \mod \omega_1 & \text{if } M = M(1, 3, \pi + 2\pi^2), \\
\Lambda_E/\omega_1 \Lambda_E & \text{if } M = M(2, 3, u\pi^2).
\end{cases}
\]

We can compute from the above data

\[
\text{Fitt}_{1, \Lambda_E/\omega_1 \Lambda_E}((T - \alpha)A_1 \otimes \mathcal{O}_E) = (T, 3) \mod \omega_1.
\]

Therefore we get $X \otimes_{\Lambda} \Lambda_E \cong M(0, 3, 1 + \pi)$.

We can determine the structure of $\mathbb{Q}(\sqrt{-64671})$ by the same method as above. For $\mathbb{Q}(\sqrt{-64671})$, we can show that $X \otimes_{\Lambda} \Lambda_E \cong M(0, 3, 1 + \pi)$.

The following is the table of the $X \otimes_{\Lambda} \Lambda_E$ for the fields such that $A_0$ is not cyclic and $f(T)$ is reducible. Here, $m, n, x$ represent $X \otimes_{\Lambda} \Lambda_E \cong M(m, n, x)$, and ram. /unram. means that $E/Q_{A_3}$ is ramified /unramified extension, respectively. We marked (*) on the remaining 6 fields for which we determined the structures in Example 6.3 and 6.4.
Table 1.

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<th>$d$</th>
<th>$\text{ord}_E(\alpha - \beta)$</th>
<th>$\text{ord}_E(\beta - \gamma)$</th>
<th>$\text{ord}_E(\gamma - \alpha)$</th>
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References


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