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Q-TRIVIAL GENERALIZED BOTT MANIFOLDS

SEONJEEONG PARK and DONG YOUP SUH

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Abstract

When the cohomology ring of a generalized Bott manifold with $\mathbb{Q}$-coefficient is isomorphic to that of a product of complex projective spaces $\mathbb{C}P^{n_i}$, the generalized Bott manifold is said to be $\mathbb{Q}$-trivial. We find a necessary and sufficient condition for a generalized Bott manifold to be $\mathbb{Q}$-trivial. In particular, every $\mathbb{Q}$-trivial generalized Bott manifold is diffeomorphic to a $\prod_{i=1}^{h} \mathbb{C}P^{n_i}$-bundle over a $\mathbb{Q}$-trivial Bott manifold.

1. Introduction

A generalized Bott tower of height $h$ is a sequence of complex projective space bundles

$$B_h \xrightarrow{\pi_h} B_{h-1} \xrightarrow{\pi_{h-1}} \cdots \xrightarrow{\pi_1} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\},$$

where $B_i = P(\mathbb{C} \oplus \xi_i)$, $\mathbb{C}$ is a trivial complex line bundle, $\xi_i$ is a Whitney sum of $n_i$ complex line bundles over $B_{i-1}$, and $P(\cdot)$ stands a projectivization. Each $B_i$ is called an $i$-stage generalized Bott manifold. When all $n_i$’s are 1 for $i = 1, \ldots, h$, the sequence (1.1) is called a Bott tower of height $h$ and $B_i$ is called an $i$-stage Bott manifold.

A (h-stage) generalized Bott manifold is said to be $\mathbb{Q}$-trivial (respectively, $\mathbb{Z}$-trivial) if $H^*(B_h; \mathbb{Q}) \cong H^*((\prod_{i=1}^{h} \mathbb{C}P^{n_i}); \mathbb{Q})$ (respectively, $H^*(B_h; \mathbb{Z}) \cong H^*((\prod_{i=1}^{h} \mathbb{C}P^{n_i}); \mathbb{Z})$). It is shown in [4] that if $B_h$ is $\mathbb{Z}$-trivial, then every fiber bundle in the tower (1.1) is trivial so that $B_h$ is diffeomorphic to $\prod_{i=1}^{h} \mathbb{C}P^{n_i}$. Furthermore, Choi and Masuda show that every ring isomorphism between $\mathbb{Z}$-cohomology rings of two $\mathbb{Q}$-trivial Bott manifolds is induced by some diffeomorphism between them (see Theorem 3.1 and [2]).

We find a necessary and sufficient condition for a generalized Bott manifold to be $\mathbb{Q}$-trivial. Namely, we have the following proposition.
Proposition 1.1. An \( h \)-stage generalized Bott manifold \( B_h \) is \( \mathbb{Q} \)-trivial if and only if each vector bundle \( \xi_i \), \( i = 1, \ldots, h \), satisfies

\[
(n_i + 1)^k c_k(\xi_i) = \binom{n_i + 1}{k} c_1(\xi_i)^k
\]

for \( k = 1, \ldots, n_i + 1 \), where \( B_i = P(\mathbb{C} \oplus \xi_i) \).

Moreover, the following theorem says that a \( \mathbb{Q} \)-trivial generalized Bott manifold without \( \mathbb{C} P^1 \)-fibration is weakly equivariantly diffeomorphic to a trivial generalized Bott manifold.

Theorem 1.2. Let \( B_h \) be a generalized Bott manifold such that all \( n_i \)'s are greater than \( 1 \). Then the following are equivalent

1. \( B_h \) is \( \mathbb{Q} \)-trivial,
2. total Chern class \( c(\xi_i) \) is trivial for each \( i = 1, \ldots, h \),
3. \( B_h \) is \( \mathbb{Z} \)-trivial, and
4. \( B_h \) is diffeomorphic to the product of projective spaces \( \prod_{i=1}^{h} \mathbb{C} P^{n_i} \).

In the light of Theorem 1.2, we have a natural question.

Question 1.1. Let \( B_h \) and \( B'_h \) be generalized Bott manifolds with \( n_i > 1 \), \( i = 1, \ldots, h \). Is \( H^*(B_h; \mathbb{Z}) \) isomorphic to \( H^*(B'_h; \mathbb{Z}) \) if \( H^*(B_h; \mathbb{Q}) \cong H^*(B'_h; \mathbb{Q}) \)?

Unfortunately, Example 3.1 shows that the answer to the question is negative. From the proposition, we can deduce the following theorem.

Theorem 1.3. Every \( \mathbb{Q} \)-trivial generalized Bott manifold is diffeomorphic to a \( \prod_{n_i > 1} \mathbb{C} P^{n_i} \)-bundle over a \( \mathbb{Q} \)-trivial Bott manifold.

The remainder of this paper is organized as follows. In Section 2, we recall general facts on a generalized Bott manifold and deal with its cohomology ring. In Section 3, we prove Proposition 1.1, Theorems 1.2 and 1.3.

2. Cohomology ring of a generalized Bott manifold

Let \( B \) be a smooth manifold and let \( E \) be a complex vector bundle over \( B \). Let \( P(E) \) denote the projectivization of \( E \). Let \( y \in H^2(P(E)) \) be the negative of the first Chern class of the tautological line bundle over \( P(E) \). Then \( H^*(P(E)) \) can be viewed as an algebra over \( H^*(B) \) via \( \pi^* \colon H^*(B) \to H^*(P(E)) \), where \( \pi \colon P(E) \to B \) denotes the projection. When \( H^*(B) \) is finitely generated and torsion free (this is the case when
\( B \) is a toric manifold, \( \pi^* \) is injective and \( H^*(P(E)) \) as an algebra over \( H^*(B) \) is known to be described as

\[
H^*(P(E)) = H^*(B)[y] \left/ \left( \sum_{k=0}^{n} c_k(E)y^{n-k} \right) \right.
\]

where \( n \) denotes the complex dimension of the fiber of \( E \) (see [1]).

For a generalized Bott manifold \( B_h \) in (1.1), since \( \pi_j^*: H^*(B_{j-1}) \rightarrow H^*(B_j) \) is injective, we regard \( H^*(B_{j-1}) \) as an element of \( H^*(B_j) \) for each \( j \) so that we have a filtration

\[
H^*(B_h) \supset H^*(B_{h-1}) \supset \cdots \supset H^*(B_1).
\]

Let \( x_j \in H^2(B_j) \) denote minus the first Chern class of the tautological line bundle over \( B_j = P(C \oplus \xi_j) \). We may think of \( x_j \) as an element of \( H^2(B_i) \) for \( i \geq j \). Then the repeated use of (2.1) shows that the ring structure of \( H^*(B_h) \) can be described as

\[
H^*(B_h) = \mathbb{Z}[x_1, \ldots, x_h]/\langle x_i^{n_i+1} + c_1(\xi_i)x_i^{n_i} + \cdots + c_{n_i}(\xi_i)x_i \mid i = 1, \ldots, h \rangle.
\]

Let \( \xi_{2,1} \) be the tautological line bundle over \( B_1 = \mathbb{C}P^{n_1} \) and let \( \xi_{3,1} = \pi_2^*(\xi_{2,1}) \) the pull-back bundle of the tautological line bundle over \( B_1 \) to \( B_2 \) via the projection \( \pi_2: B_2 \rightarrow B_1 \). In general, let \( \xi_{j,j-1} \) be the tautological line bundle over \( B_{j-1} \) and we define inductively

\[
\xi_{j,j-k} = \pi_{j-1}^* \circ \cdots \circ \pi_{j-k+1}^*(\xi_{j-k+1,j-k})
\]

for \( k = 2, \ldots, j-1 \). Then one can see that the Whitney sum of complex line bundles \( \xi_i \) over \( B_{i-1} \) in the sequence (1.1) can be written as

\[
\xi_i := \left( \xi_{i,1}^{a_{i,1}} \otimes \cdots \otimes \xi_{i,i-1}^{a_{i,i-1}} \right) \oplus \cdots \oplus \left( \xi_{1,1}^{a_{1,1}} \otimes \cdots \otimes \xi_{i,i-1}^{a_{i,i-1}} \right)
\]

for some integers \( a_{1,1}, \ldots, a_{n_i,i-1} \). Note that \( \xi_1 = (\mathbb{C})^{y_1} \). Hence, the total Chern class of \( \xi_i \) is

\[
c(\xi_i) = \prod_{j=1}^{n_i} \left( 1 + \sum_{k=1}^{i-1} a_{j,k}^{j,k}x_k \right).
\]

Therefore, the cohomology ring of \( B_h \) is

\[
H^*(B_h; \mathbb{Z}) = \mathbb{Z}[x_1, \ldots, x_h]/\langle x_i^{n_i+1} + c_1(\xi_i)x_i^{n_i} + \cdots + c_{n_i}(\xi_i)x_i \mid i = 1, \ldots, h \rangle
\]

\[
= \mathbb{Z}[x_1, \ldots, x_h] \left/ \left\langle x_i \prod_{j=1}^{n_i} \left( \sum_{k=1}^{i-1} a_{j,k}x_k + x_i \right) \mid i = 1, \ldots, h \right\rangle \right.
\]
Remark 1. We can associate a generalized Bott manifold $B_h$ with an $h \times h$ vector matrix $A$ as follows:

\begin{equation}
A^T = \begin{pmatrix}
1 & a_1 & 1 \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
n & a_h & a_2 & \cdots & 1
\end{pmatrix},
\end{equation}

where

\[
a_k = \begin{pmatrix}
1 \\
\vdots \\
1 \\
1 \\
a_{1k} & a_{2k} & \cdots & a_{nk}
\end{pmatrix}
\]

and

\[
1 = \begin{pmatrix}
1 \\
\vdots \\
1 \\
1
\end{pmatrix}.
\]

Moreover we can consider $B_h$ as a quasitoric manifold over the product of simplices $\prod_{i=1}^h \Delta^{n_i}$ with the reduced characteristic matrix $\Lambda_\ast = -A^T$.

3. Q-trivial generalized Bott manifolds

As we mentioned in the introduction, Choi and Masuda classify Q-trivial Bott manifolds as follows.

**Theorem 3.1** ([2]). (1) A Bott manifold $B_h$ is Q-trivial if and only if for each $i = 1, \ldots, h$, each line bundle $\xi_i$ satisfies $c_1(\xi_i)^2 = 0$ in $H^*(B_h; \mathbb{Z})$.

(2) Every ring isomorphism $\varphi$ between two Q-trivial Bott manifolds $B_h$ and $B_h'$ is induced by some diffeomorphism $B_h \to B_h'$.

In this section we shall prove Proposition 1.1 and Theorem 1.2. To prove them, we need the following lemmas.

**Lemma 3.2.** If a generalized Bott manifold $B_h$ is Q-trivial, then there exist linearly independent primitive elements $z_1, \ldots, z_h$ in $H^2(B_h; \mathbb{Z})$ such that $z_i^{n_i}$ is not zero but $z_i^{n_i+1}$ is zero in $H^*(B_h; \mathbb{Z})$ for $i = 1, \ldots, h$.

Proof. Let $H^*(B_h; \mathbb{Z})$ be generated by $x_1, \ldots, x_h$ as in (2.3) and let

\[
H^\ast \left( \prod_{i=1}^h B_h; \mathbb{Q} \right) = \mathbb{Q}[y_1, \ldots, y_h]/(y_i^{n_i+1} | i = 1, \ldots, h).
\]
Since both \( \{x_1, \ldots, x_h\} \) and \( \{y_1, \ldots, y_h\} \) are sets of generators of \( H^2(B_h; \mathbb{Q}) \), we can write

\[
y_i = \sum_{j=1}^{h} c_{ij} x_j \quad \text{for} \quad i = 1, \ldots, h \quad \text{and} \quad c_{ij} \in \mathbb{Q},
\]

where the determinant of the matrix \( C = (c_{ij})_{h \times h} \) is non-zero. We may assume that \( c_{ij} \)'s are irreducible fractions. Multiplying \( (c_{i,1}, \ldots, c_{i,h}) \) by the least common denominator \( r_i \) of a set \( \{c_{i,1}, \ldots, c_{i,h}\} \), we can get a primitive element \( z_i = r_i y_i = r_i \sum_{j=1}^{h} c_{ij} x_j \) in \( H^2(B_h; \mathbb{Z}) \) such that \( z_i^{n_i+1} \) is zero in \( H^*(B_h; \mathbb{Z}) \) for each \( i = 1, \ldots, h \).

Since the elements \( y_1, \ldots, y_h \) are linearly independent, the elements \( z_1, \ldots, z_h \) are also linearly independent. Since \( y_i^{n_i} \) is not zero in \( H^*(B_h; \mathbb{Q}) \), \( z_i^{n_i} \) cannot be zero in \( H^*(B_h; \mathbb{Z}) \). This proves the lemma.

**Lemma 3.3 ([4]).** Let \( B_m \) be an \( m \)-stage generalized Bott manifold. Then the set

\[
\{ bx_m + w \in H^2(B_m) \mid 0 \neq b \in \mathbb{Z}, w \in H^2(B_{m-1}), (bx_m + w)^{n_m+1} = 0 \}
\]

lies in a one-dimensional subspace of \( H^2(B_m) \) if it is non-empty.

Proof. To satisfy \( (bx_m + w)^{n_m+1} = 0 \), we need \( bc_1(\xi_m) = (n_m + 1)w \).

**Lemma 3.4 ([4]).** For an element \( z = \sum_{i=1}^{h} b_i x_i \in H^2(B_h) \), if \( b_i \) is non-zero, then \( z^{n_i} \) cannot be zero in \( H^*(B_h) \).

Proof. If we expand \( (\sum_{i=1}^{h} b_i x_i)^{n_i} \), there appears a non-zero scalar multiple of \( x_i^{n_i} \) because \( b_i \neq 0 \). Then, \( z^{n_i} \) cannot belong to the ideal generated by the polynomials \( x_i \prod_{j=1}^{i-1} (\sum_{k=1}^{i-1} a_{jk} x_k + x_i) \), hence it is not zero in \( H^*(B_h) \).

Now we can prove Proposition 1.1.

Proof of Proposition 1.1. If each vector bundle \( \xi_i \) satisfies the conditions (1.2), then \( x_i + 1/(n_i + 1)c_1(\xi_i)^{n_i+1} \) is zero in \( H^*(B_h; \mathbb{Q}) \). Since the set

\[
\left\{ x_i + \frac{1}{n_i + 1} c_1(\xi_i) \mid i = 1, \ldots, h \right\}
\]

generates \( H^*(B_h; \mathbb{Q}) \) as a graded ring, this shows that \( B_h \) is \( \mathbb{Q} \)-trivial.

Conversely, if a generalized Bott manifold is \( \mathbb{Q} \)-trivial, then there are linearly independent and primitive elements \( z_1, \ldots, z_h \) in \( H^2(B_h; \mathbb{Z}) \) such that \( z_i^{n_i+1} \) is zero but \( z_i^{n_i} \) is not zero in \( H^*(B_h) \) by Lemma 3.2. We can put \( z_i = \sum_{j=1}^{h} b_{ij} x_j \) with \( b_{ij} \in \mathbb{Z} \) for each \( i = 1, \ldots, h \).
Now, consider a map $\mu \colon \{1, \ldots, h\} \to \mathbb{N}$ given by $j \mapsto n_j$. Further assume that the image of $\mu$ is the set $\{N_1, \ldots, N_m\}$ with $N_1 < \cdots < N_m$. We will show inductively that each $z_i$ can be written as $r_i(x_i + 1/(\mu(i) + 1)c_1(\xi))$ for some $r_i \in \mathbb{Z} \setminus \{0\}$.

**CASE 1:** Assume $i \in \mu^{-1}(N_1)$. Let $\mu^{-1}(N_1) := \{i_1, \ldots, i_n\}$ with $i_1 < \cdots < i_n$. We have $z_i^{N_1+1} = 0$. Then, by Lemma 3.4, we can see that

$$z_i = \sum_{j \in \mu^{-1}(N_1)} b_{ij} x_j,$$

that is, $b_{ij} = 0$ for $j' \notin \mu^{-1}(N_1)$. Note that for each $i \in \mu^{-1}(N_1)$, one of $b_{ij}$'s is nonzero for $j \in \mu^{-1}(N_1)$ because the set $\{z_i \mid i \in \mu^{-1}(N_1)\}$ is linearly independent. For some $i_p \in \mu^{-1}(N_1)$, if $b_{ij_i}$ is nonzero, then $z_{i_p} \in H^2(B_{i_j})$ and $b_{ij} = 0$ for all $i \in \mu^{-1}(N_1) \setminus \{i_p\}$ by Lemma 3.3. Put $w_{i_p} := z_{i_p}$. If $b_{i_qi'}$ is nonzero for some $i_q \in \mu^{-1}(N_1) \setminus \{i_p\}$, then $z_{ij} \in H^2(B_{i'j'})$ and $b_{i'j'} = 0$ for all $i \in \mu^{-1}(N_1) \setminus \{i_p, i_q\}$. Now, put $w_{i'j'} := z_{ij}$. In this way, for each $i \in \mu^{-1}(N_1)$, we can obtain $w_i \in H^2(B_i)$ such that $w_i \notin H^2(B_{i'j'})$ and $w_i^{N_1+1} = 0$ in $H^*(B_{i'}B_{j'})$. Moreover, from the proof of Lemma 3.3, we can write

$$w_i := r_i \left(x_i + \frac{1}{N_1+1}c_1(\xi)\right) \in H^2(B_i)$$

for each $i \in \mu^{-1}(N_1)$. In particular, if $N_1 = 1$, then $w_i$ is of the form either $\pm x_i$ or $\pm(x_i + c_1(\xi))$ for $i \in \mu^{-1}(N_1)$. Furthermore, without loss of generality, we may assume that $z_i = w_i$ for $i \in \mu^{-1}(N_1)$.

**CASE 2:** Assume that $z_k = r_k(x_k + 1/(\mu(k) + 1)c_1(\xi_k))$ for $N_1 \leq \mu(k) \leq N_{n-1}$ and let $l \in \mu^{-1}(N_n)$. Then we have $z_l^{N_n+1} = 0$. Then by Lemma 3.4, we can easily see that

$$z_l = \sum_{k \in \mu^{-1}(N_n)} b_{lk} x_k + \sum_{j \in \mu^{-1}(N_n)} b_{lj} x_j,$$

where $N_{n-1} = \{N_1, \ldots, N_{n-1}\}$. That is, $b_{lj} = 0$ for $j' \notin \mu^{-1}(N_{n-1})$. Since $z_l^{N_n+1}$ is zero in $H^*(B_h)$, we have

$$\left(\sum_{k \in \mu^{-1}(N_n)} b_{lk} x_k + \sum_{j \in \mu^{-1}(N_n)} b_{lj} x_j\right)^{N_n+1}$$

$$= \sum_{k \in \mu^{-1}(N_n)} f_k(x_1, \ldots, x_h)(x_k^{\mu(k)} + 1 + c_1(\xi_k)x_k^{\mu(k)} + \cdots + c_{\mu(k)}(\xi_k)x_k)$$

$$+ \sum_{j \in \mu^{-1}(N_n)} b_{lj}^{N_n+1}(x_j^{N_n+1} + c_1(\xi_j)x_j^{N_n} + \cdots + c_{N_n}(\xi_j)x_j)$$

as polynomials, where $f_k(x_1, \ldots, x_h)$ is a homogeneous polynomial of degree $N_n - \mu(k)$ for each $k \in \mu^{-1}(N_{n-1})$. Note that for each $l \in \mu^{-1}(N_n)$, one of $b_{lj}$'s is non-zero.
for \( j \in \mu^{-1}(N_n) \) from the linearity independence of the set \( \{z_i \mid i \in \mu^{-1}(N_n)\} \). Let 
\( \mu^{-1}(N_n) := \{l_1, \ldots, l_p\} \) with \( l_1 < \cdots < l_p \). Assume \( b_{l_p l_p} \) is nonzero for some \( l_p \in \mu^{-1}(N_n) \). Substituting \( l = l_p \) into (3.3) and comparing the monomials containing \( x_{l_p}^{N_n} \) as a factor on both sides of (3.3), we have

\[
(N_n + 1) \left( \sum_{k \in \mu^{-1}(N_n)} b_{l_p k} x_k + \sum_{j \in \mu^{-1}(N_n)} b_{l_p j} x_j \right) = b_{l_p l_p} c_1(\xi_{l_p}).
\]

Since \( c_1(\xi_{l_p}) \) belongs to \( H^2(B_{l_p - 1}) \), we can see that \( b_{l_p k} = 0 \) for \( k > l_p \). That is,

\[
z_{l_p} = \sum_{k \in \mu^{-1}(N_n)} b_{l_p k} x_k + \sum_{j \in \mu^{-1}(N_n)} b_{l_p j} x_j.
\]

Thus, we can see that \( z_{l_p} \in H^2(B_{l_p}) \) and \( b_{l_p l_p} = 0 \) for all \( l \in \mu^{-1}(N_n) \setminus \{l_p\} \) by Lemma 3.3. Put \( w_{l_p} := z_{l_p} \). Now assume that \( b_{l_p l_p} \) is nonzero for some \( l_q \in \mu^{-1}(N_n) \setminus \{l_p\} \). Substituting \( l = l_q \) into (3.3) and comparing the monomials containing \( x_{l_q}^{N_n} \) as a factor on both sides of (3.3), we have

\[
(N_n + 1) \left( \sum_{k \in \mu^{-1}(N_n)} b_{l_p k} x_k + \sum_{j \in \mu^{-1}(N_n)} b_{l_q j} x_j \right) = b_{l_p l_q} c_1(\xi_{l_q}).
\]

Since \( c_1(\xi_{l_q}) \) belongs to \( H^2(B_{l_q - 1}) \), we can see that \( b_{l_q k} = 0 \) for \( k > l_q - 1 \), and hence,

\[
z_{l_q} = \sum_{k \in \mu^{-1}(N_n)} b_{l_q k} x_k + \sum_{j \in \mu^{-1}(N_n)} b_{l_q j} x_j.
\]

Thus, we can see that \( z_{l_q} \in H^2(B_{l_q - 1}) \) and \( b_{l_q l_q} = 0 \) for all \( l \in \mu^{-1}(N_n) \setminus \{l_p, l_q\} \) by Lemma 3.3. Now, put \( w_{l_q} := z_{l_q} \). In this way, for each \( l \in \mu^{-1}(N_n) \), we can obtain \( w_l \in H^2(B_l) \) such that \( w_l \notin H^2(B_{l - 1}) \) and \( w_l^{N_n + 1} = 0 \) in \( H^2(B_k) \). Moreover, from the proof of Lemma 3.3, \( w_l \) can be written as \( r_l (x_l + 1/(N_n + 1) c_1(\xi_l)) \). Furthermore, without loss of generality, we may assume that \( z_l = w_l \) for \( l \in \mu^{-1}(N_n) \).

By Cases 1 and 2, we can see that, for each \( i = 1, \ldots, h \), we can write

\[
z_i = r_i \left( x_i + \frac{1}{n_i + 1} c_1(\xi_i) \right)
\]
for some \( r_i \in \mathbb{Z} \setminus \{0\} \). Therefore, \((n_i + 1)x_i + c_1(\xi_i)^{n_i+1}\) is zero in \( H^*(B_h) \). From this, we can see

\[
(n_i + 1)^k c_k(\xi_i) = \binom{n_i + 1}{k} c_1(\xi_i)^k \quad \text{and} \quad c_1(\xi_i)^{n_i+1} = 0
\]

\[ k = 1, \ldots, n_i. \]

By using Proposition 1.1, we can prove Theorem 1.2.

Proof of Theorem 1.2. We first prove the implication \((1) \Rightarrow (2)\). By Proposition 1.1, we have the relation

\[
(n_i + 1)^2 c_2(\xi_i) = \frac{n_i(n_i + 1)}{2} c_1(\xi_i)^2.
\]

If \( n_i = 2 \), from (2.2) and (3.4), we have

\[
\begin{align*}
&\{ (a_{11}^i x_1 + \cdots + a_{1,i-1}^i x_{i-1}) + (a_{21}^i x_1 + \cdots + a_{2,i-1}^i x_{i-1}) \}^2 \\
= &\quad 3(a_{11}^i x_1 + \cdots + a_{1,i-1}^i x_{i-1})(a_{21}^i x_1 + \cdots + a_{2,i-1}^i x_{i-1}).
\end{align*}
\]

For \( j = 1, \ldots, i - 1 \), since \( x_j^2 \neq 0 \) in \( H^*(B_i) \), by comparing the coefficients of \( x_j^2 \) on both sides of (3.5), we have \((a_{1j}^i + a_{2j}^i)^2 = 3a_{1j}^i a_{2j}^i\) whose integer solution is only \( a_{1j}^i = a_{2j}^i = 0 \). If \( n_i = n > 2 \), then we have

\[
\begin{align*}
n \{ (a_{11}^i x_1 + \cdots + a_{1,i-1}^i x_{i-1}) + \cdots + (a_{21}^i x_1 + \cdots + a_{2,j-1}^i x_{j-1}) \}^2 \\
= &\quad 2(n + 1) \{ (a_{11}^i x_1 + \cdots + a_{1,i-1}^i x_{i-1})(a_{21}^i x_1 + \cdots + a_{2,i-1}^i x_{i-1}) + \cdots \\
+ &\quad (a_{n-1}^i x_1 + \cdots + a_{n-1,i-1}^i x_{i-1})(a_{n,1}^i x_1 + \cdots + a_{n,i-1}^i x_{i-1}) \}.
\end{align*}
\]

Since \( x_j^2 \neq 0 \) in \( H^*(B_i) \) for \( j = 1, \ldots, i - 1 \), by comparing the coefficients of \( x_j^2 \) on both sides of (3.6) we have

\[
n(a_{1,j}^i + \cdots + a_{n,j}^i)^2 = 2(n + 1) \sum_{1 \leq k < l \leq n} a_{k,j}^i a_{l,j}^i.
\]

The equation (3.7) is equivalent to

\[
\sum_{m=1}^{n} (a_{m,j}^i)^2 + \sum_{1 \leq k < l \leq n} (a_{k,j}^i - a_{l,j}^i)^2 = 0.
\]

Therefore, \( a_{1j}^i = \cdots = a_{nj}^i = 0 \) for each \( j = 1, \ldots, i - 1 \), and hence, in any case, \( c(\xi_i) \) is trivial for all \( i = 1, \ldots, h \).
Let \( y(3.9) \) \( c \) we can see that \( /AC \) as polynomials. So, we can see that \( \text{line bundle over } B \text{H} \)\( /AD \)

Then the map \( \text{ Shelby } \)\( \text{fiber bundle } P \text{A} \text{B} \)\( \text{and } \)\( \text{and } \)

Therefore, all four conditions are equivalent. \( \square \)

From Theorem 1.2, we have the following corollary.

**Corollary 3.5.** Let \( M \) be a quasitoric manifold. If \( H^*(M; Q) \) is isomorphic to \( H^*(\prod_{i=1}^h C P^{n_i}; Q) \), then \( M \) is homeomorphic to \( \prod_{i=1}^h C P^{n_i} \) provided \( n_i > 1 \) for all \( i \).

**Proof.** By [3], if \( H^*(M; Q) \) is isomorphic to \( H^*(\prod_{i=1}^h C P^{n_i}; Q) \), then \( M \) is homeomorphic to a generalized Bott manifold. But a \( Q \)-trivial generalized Bott manifolds with \( n_i > 1 \) is diffeomorphic to \( \prod_{i=1}^h C P^{n_i} \). Hence, \( M \) is homeomorphic to \( \prod_{i=1}^h C P^{n_i} \). \( \square \)

The following is the counter-example of Question 1.1.

**Example 3.1.** Let \( B \) be a fiber bundle \( P(C^3 \oplus \xi) \) over \( C P^2 \) and let \( B' \) be a fiber bundle \( P(C^3 \oplus \xi^2) \) over \( C P^2 \), where \( \xi \) is the tautological line bundle over \( C P^2 \). Let \( y \) (respectively, \( Y \)) denote the negative of the first Chern class of the tautological line bundle over \( B_2 \) (respectively, \( B'_2 \)). Then their cohomology rings are

\[
H^*(B) = \mathbb{Z}[x, y]/\langle x^3, y(y^3 + xy^2) \rangle
\]

and

\[
H^*(B') = \mathbb{Z}[X, Y]/\langle X^3, Y(Y^3 + 2XY^2) \rangle.
\]

Then the map \( \phi \) defined by \( \phi(x) = 2X \) and \( \phi(y) = Y \) is an isomorphism from \( H^*(B; Q) \to H^*(B'; Q) \). But this \( \phi \) is not a \( \mathbb{Z} \)-isomorphism. Suppose that \( \psi \) is an isomorphism \( H^*(B; \mathbb{Z}) \to H^*(B'; \mathbb{Z}) \). Then there exist \( \alpha, \beta, \gamma, \delta \) in \( \mathbb{Z} \) such that

\[
\begin{pmatrix}
\psi(x) \\
\psi(y)
\end{pmatrix} = \begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} \begin{pmatrix}
X \\
Y
\end{pmatrix}
\]

and \( \alpha \delta - \beta \gamma = \pm 1 \). Since \( \psi(x) = 0 \) in \( H^*(B'; \mathbb{Z}) \), we have

\[
(\alpha X + \beta Y)^3 = \alpha^3 X^3
\]
as polynomials. So, we can see that \( \beta \) is zero and \( \alpha = \pm 1 \), and hence \( \delta = \pm 1 \). Since \( \psi(y(y^3 + xy^2)) \) is zero in \( H^*(B'; \mathbb{Z}) \), we have

\[
(y X + \delta Y)^3((\alpha + \gamma)X + \delta Y) = (aX + bY)X^3 + cY(Y^3 + 2XY^2)
\]
as polynomials in \( \mathbb{Z}[X, Y] \). By comparing the coefficients of \( XY^3 \) on both sides of (3.8), we can see that

\[
2c = 3\gamma \delta^3 + (\alpha + \gamma)\delta^3 = \delta(\alpha + 4\gamma).
\]
Since the right hand side of (3.9) is odd, there is no such an integer $c$. Hence, there is no such $\mathbb{Z}$-isomorphism $\psi$.

Now consider $\mathbb{Q}$-trivial generalized Bott manifolds $B_h$ which have $\mathbb{C}P^1$-fibers, that is, $n_k = 1$ for some $k \in [h]$.

**Lemma 3.6.** Let $B_h$ and $B'_h$ be two $h$-stage generalized Bott towers. If the associated vector matrices to them are

\[
A = \begin{pmatrix}
1 \\
* & \ldots & * \\
a_1 & \ldots & a_{h-2} & 1 \\
b_1 & \ldots & b_{h-2} & 0 & 1
\end{pmatrix}
\]

and

\[
A' = \begin{pmatrix}
1 \\
* & \ldots & * \\
b_1 & \ldots & b_{h-2} & 1 \\
a_1 & \ldots & a_{h-2} & 0 & 1
\end{pmatrix}
\]

respectively, then $B_h$ and $B'_h$ are equivariantly diffeomorphic.

Proof. Note that this lemma can be seen by the fact that $B_h$ and $B'_h$ are equivariantly diffeomorphic if two associated vector matrices are conjugated by a permutation matrix, see the paper [3]. It is obvious that

\[A' = E_\sigma A E_\sigma^{-1},\]

where $\sigma := (1, \ldots, h-2, h, h-1)$ is the permutation on $[h]$ which permutes only $h-1$ and $h$.

Now, we can prove Theorem 1.3.

Proof of Theorem 1.3. Let $B_h$ be a $\mathbb{Q}$-trivial generalized Bott manifold whose associated matrix is of the form (2.4).

Consider a map $\mu: \{1, \ldots, h\} \to \mathbb{N}$ given by $j \mapsto n_j$ and assume that the image of $\mu$ is the set $\{N_1, \ldots, N_m\}$ with $1 = N_1 < N_2 < \cdots < N_m$.

For each $i \in \mu^{-1}(1)$, by Proposition 1.1, we have $c_i(\xi_1)^2 = 0$ in $H^*(B_h)$. Since $\chi_k \neq 0$ in $H^*(B_h)$ for $k \notin \mu^{-1}(1)$, we can see that $a^i_{ik} = 0$ for $k \in [i-1]$ with $n_k > 1$. 


Now suppose that $n_j > 1$. Then by Proposition 1.1, we have the relation

$$(n_j + 1)^2 c_2(\xi_j) = \frac{n_j(n_j + 1)}{2} c_1(\xi_j)^2.$$ 

Since $x_k^2 \neq 0$ in $H^*(B_h)$ for $n_k > 1$, we can show that $a_k^j = 0$ by using the same argument to the proof of Theorem 1.2.

Since $a_k^j = 0$ for all $n_k > 1$, by Lemma 3.6, $B_h$ is diffeomorphic to the Q-trivial generalized Bott manifold $B'$ whose associated matrix is of the form

$$(3.10) \quad (A')^T = \begin{pmatrix}
1 & & & \\
\vdots & \ddots & & \\
a_{11}^2 & a_{12}^2 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
a_{1r}^2 & a_{2r}^2 & \cdots & a_{r+1}^2 \\
a_{1}^r+1 & a_{2}^r+1 & \cdots & a_{r+1}^r \\
\vdots & \vdots & \ddots & \vdots \\
a_{h}^r & a_{r}^h & \cdots & a_{r}^h \\
0 & \cdots & 0 & 1
\end{pmatrix},$$

where $r$ is the cardinality of the set $\mu^{-1}(1)$, that is, $r = |\mu^{-1}(1)|$. This proves the theorem.

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\begin{center}
\textbf{References}
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