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MEHLER FORMULA AND CAPACITIES FOR INIFNITE DIMENSIONAL ORNSTEIN-UHLENBECK PROCESSES WITH GENERAL LINEAR DRIFT

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0. Introduction

Let E be a locally convex topological vector space and μ a mean zero Gaussian Radon measure on its Borel σ-algebra $\mathscr{B}(E)$. Define the corresponding *Mehler semigroup* by

$$p_t f(z) := \int f(e^{-t}z + \sqrt{1-e^{-2t}}z')\mu(dz'), \quad z \in E, \quad t > 0, \tag{0.1}$$

where $f\colon E \to [0,\infty[$ is $\mathscr{B}(E)$-measurable. Let $C_{r,p}, r>0, p>1$ denote the corresponding (r,p)-capacities defined via the Gamma transform of $(p_t)_{t>0}$ (cf. the beginning of Section 1 below). It is well-known that each $C_{r,p}$ is *tight*, i.e., there exist compact sets $K_n \subset E$, $n \in N$, such that $C_{r,p}(E\backslash K) \to 0$ as $n \to \infty$. This was first proved by H. Sugita [50] if E is a separable Banach space, and subsequently extended by *D.* Feyel and A. de La Pradelle [19] to more general cases. The first main result of this paper (Theorem 2.2) is that each $C_{r,p}$ is in fact *strongly tight*, i.e., the compact sets K_n, $n \in N$, can be chosen metrizable for every E as above. This result was first announced in [13]. In this paper we give a detailed proof (which is also different from and shorter than the one indicated in [13]). Our proof depends on the well-known result by H.Sato [45] and B.S.Tsirelson [51] that there exist metrizable compact sets $K_n \subset E$, $n \in N$, such that $\mu(E\backslash K_n) \to 0$ as $n \to \infty$.

Before we describe our second main result we note that if H denotes the reproducing kernel Hilbert space of μ then H is separable and $H \subset E$ continuously (and also densely if μ is non-degenerate). Reversing view-points we now start with a separable Hilbert space H and assume that we are given a nonnegative definite self-adjoint linear operator A with domain $D(A)$ on H. We construct another separable real Hilbert space E carrying a mean zero Gaussian probability measure μ whose covariance is specified by a given bounded linear operator B on H which commutes with $e^{-tA}, t>0$, such that the following holds: $(e^{-tA})_{t>0}$ extends to a strongly continuous contraction semigroup on E (cf. Theorem 3.1; in fact the embedding $H \subset E$ is Hilbert-Schmidt). H will be dense in E so that this extension will be unique. In particular, for any such E we can define the corresponding Mehler semigroup by

$$(0.2) \qquad p_t f(z) := \int f(e^{-tA}z + \sqrt{1-e^{-2tA}}z')\mu(dz'), \quad z \in E, \quad t > 0,$$

where $f: E \to [0, \infty[$ is $\mathscr{B}(E)$-measurable. Note that this is a "true" semigroup of probability kernels satisfying the semigroup property for every rather than μ-a.e. point z in E (cf. Corollary 3.6). The question, whether such a space E exists for any given H, A, μ, was open. In [47] (see also [48, Example 5.1]) it was shown, that if such E exists and if $B = \mathrm{id}_H$, then the famous *Meyer equivalence* (cf. [33] and also [49,34,35]) originaly only proved for $A := \mathrm{id}_H$ has an analogue for general A as above. (The Meyer equivalence is the equivalence of the graph norms of powers of the square root of the $L^2(E;\mu)$-generator of $(p_t)_{t>0}$ and the corresponding Sobolev norms; cf. [32]). We would like to emphasize that the existence of such E is essentialy trivial if A has discrete spectrum and is much easier to show, if one does not insist on E being a Banach space, but provided (0.2) still makes sense. We use the term *state space* for E below.

Our third main result treats the question whether for E as above and $(p_t)_{t>0}$ as in (0.2), the corresponding (r,p)-capacities $C_{r,p}$ are tight. The answer to this question is negative even if $r=1$, $p=2$. We prove that $C_{1,2}$ is tight (on E) if and only if $(p_t)_{t>0}$ is the transition semigroup of a diffusion on E which in turn (surprisingly) is the case if and only if $(p_t)_{t>0}$ is the transition semigroup of a right continuous normal Markov process on E (cf. Theorem 6.3). The proof is based on a thorough study of the Dirichlet form correspondind to $(p_t)_{t>0}$ in (0.2) (which exists since $(p_t)_{t>0}$ is μ-symmetric) and results in [31]. We give a counterexample (even with $B = \mathrm{id}_H$ and A having discrete spectrum) where $C_{1,2}$ is not tight (Example 6.6(ii)), but we also prove that one can always obtain tightness of all $C_{r,p}$ replacing E by a properly chosen larger space E_1 (Theorem 6.7) at least if E is constructed as in the proof of Theorem 3.1 and $B = \mathrm{id}_H$. The question, whether such a space E_1 exists, was left open in [3,4].

Let us now briefly describe how this paper is organized and summarize other results contained in it.

In Section 1 we recall the notion of general (r,p)-capacties coming from an L^p-contraction semigroup which was first introduced by M. Fukushima and H. Kaneko [24] (see also [28] and [25]). However, we have to be a little careful, since we do not assume the state space to be metrizable. This is necessary for Section 2. We explain the respective modifications in detail. In addition, we prove some results needed in Section 2 which may be of their own interest. Section

2 contains the proof of our first main result, i.e., the strong tightness and some discussions of its consequences. Section 3 essentially consists of the proof of our second main result, including a study of the corresponding Mehler semigroup. In Section 4 we consider the associated Markov processes on E. In particular, we prove a formula by which it is possible to recalculate the starting measures P_z, $z \in E$, from $P_\mu := \int_E P_z \mu(dz)$ (Theorem 4.2). In Section 5 we study the associated Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$. First we calculate the L^2-generator L of $(p_t)_{t>0}$ given by (0.2) (cf. Theorem 5.3) on nice functions and show that it is essentially self-adjoint there (cf. Proposition 5.4). We emphasize that the calculation of L is quite trivial on an *informal* level, but is technically difficult to carry out rigorously, because of domain problems related to the unboundedness of A. Therefore, we assume the state space E to also satisfy property (iii) of Theorem 3.1 (which was verified for the special state space constructed in Section 3). Using the special form of L it is possible to derive a representation for $(\mathscr{E}, D(\mathscr{E}))$; more precisely, we show that $(\mathscr{E}, D(\mathscr{E}))$ is of "gradient type" (Theorem 5.5), which is crucial for the proof of our third main result. In addition, we discuss finite dimensional examples as well as the intensively studied *free field Dirichlet form* (cf. Examples 5.6). Finally, in Section 6 we prove our third main result and the already mentioned related results. We also give an analytic sufficient condition for $C_{1,2}$-tightness which is easy to check in applications (Proposition 6.5).

In case $C_{1,2}$ is tight, the diffusion process mentioned above with transition semigroup (0.2) weakly solves a stochastic differential equation on E of type

$$dX_t = dW_t + A(X_t)dt, \tag{0.3}$$

where $(W_t)_{t\geq 0}$ is an E-valued Brownian motion with covariance determined by $<,>_H$, A, B. For particular A, B this result has been derived in [9, Subsection 7.I.] from a general theorem about equations of type (0.3) with not-necessarily linear drifts. The linear equation (0.3) has been extensively studied. In a forthcoming paper we shall investigate (0.3) by our approach and its connections to already existing results in the literature in more detail.

In [20,21] questions related to our second and third main results have been addressed. But there, E is in general not a Banach space and the emphasis is put on measurable rather than continuous extensions. The resulting Mehler formula, therefore, does not give a "true" semigroup of probability kernels (cf. above). Nevertheless, a sufficient condition for tightness is derived in these two papers. For the connection of our results with the so-called *second quantization* we refer to [9, Subsection 7.I.].

Finally, we note that this paper was strongly motivated by I.Shigekawa's beautiful work [47]. We can now assert that the assumptions on the state space made there can be always fulfilled. Furthermore, the results and techniques in Sections 3–6 of this article were strongly influenced by earlier work of one of the authors (cf. [39,40]). Since the present paper addresses both probabilists and analysts, we have tried to choose a style in writing which should be understandable to both groups. This is also the reason why we have included a quite extensive list of references.

1. General capacities

The set-up and the terminology in this section is adopted from [24]. Let X be a topological space and m a positive measure on its Borel σ-algebra $\mathscr{B}(X)$. Let $p \in [1, \infty[$ and let $(T_t)_{t>0}$ be a strongly continuous contraction semigroup of linear operators on (real) $L^p := L^p(X; m)$ such that each T_t is *sub-Markovian*, i.e., $0 \leq T_t u \leq 1$ m-a.e. if $0 \leq u \leq 1$ m-a.e.. Consider its *Gamma transform*

$$V_r := \frac{1}{\Gamma(r/2)} \int_0^\infty t^{r/2-1} e^{-t} T_t dt, \quad r > 0.$$

$(V_r)_{r>0}$ is also a sub-Markovian contraction semigroup of linear operators on L^p. Consequently, each V_r is injective, and hence the space $F_{r,p} := V_r(L^p)$ with norm

$$\|u\|_{r,p} := \| V_r^{-1} u \|_{L^p}, \quad u \in F_{r,p},$$

is a Banach space. Obviously,

$$F_{r',p} \subset F_{r,p} \quad \text{continuously,} \tag{1.1}$$

if $r' > r$.

REMARK 1.1. Suppose $m(X) = 1$ and that $(T_t)_{t>0}$ is also a strongly continuous contraction semigroup on $L^{p'}$ for $p' > p$. Then we also have that

$$F_{r',p'} \subset F_{r,p} \quad \text{continuously,} \tag{1.2}$$

if $r' > r$.

The *(r,p)-capacity* for $(T_t)_{t>0}$ is defined as follows. For $U \subset X$, U open, let

$$C_{r,p}(U) := \inf\{\|u\|_{r,p}^p | u \in F_{r,p},\ u \geq 1 \ m\text{-a.e. on } U\}$$

and for arbitrary $A \subset X$

$$C_{r,p}(A) := \inf\{C_{r,p}(U) | A \subset U \subset X,\ U \text{ open}\}.$$

Proposition 1.2.

1. *If $A \subset X$, then $m(A) \leq C_{r,p}(A)$ and $C_{r,p}(A) \leq C_{r',p}(A)$ if $r \leq r'$.*
2. *If $A \subset B \subset X$, then $C_{r,p}(A) \leq C_{r,p}(B)$.*
3. *If $A_n \subset X$, $n \in N$, then $C_{r,p}(\cup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty C_{r,p}(A_n)$.*
4. *For every $U \subset X$, U open, with $C_{r,p}(U) < \infty$ there exists a unique element*

$e_U \geq 1$ *m-a.e. on* U and $\|e_U\|_{r,p}^p = C_{r,p}(U)$. *Furthermore, there exists* $f \in L^p$, $f \geq 0$, *such that* $e_U = V_r f$.

Proof. (i) is obvious from the definition and (1.1). (ii) is trivial. For (iii), (iv) see [24, Lemma 1 and Theorem 1] and note that the assumption made there that X is a metric space is not used in the proof. □

Let $\mathscr{B}_b(X)$, $C_{(b)}(X)$ denote the set of all bounded $\mathscr{B}(X)$-measurable resp. (bounded) continuous functions on X. As usual we shall not distinguish below between classes of functions in L^p and their various m-versions provided no confusion is possible. Fix $r > 0$. An increasing sequence $(F_n)_{n \in N}$ of closed subsets of X is called a $C_{r,p}$*-nest* if $\lim_{n\to\infty} C_{r,p}(F_n^c) = 0$ where for $A \subset X$, $A^c := X \backslash A$. $C_{r,p}$ is called *tight* resp. *strongly tight* if there exists a $C_{r,p}$-nest consisting of compact resp. metrizable compact sets. Given a $C_{r,p}$-nest $(F_n)_{n \in N}$ we define

$$(1.3) \qquad C(\{F_n\}) := \{f: A \to \boldsymbol{R} \mid \bigcup_{n \in N} F_n \subset A \subset X, f|_{F_n}| \text{ is continuous for all } n \in N\}$$

where $f_{|F_n}$ denotes the restriction of f to F_n. A property of points in X is said to hold $C_{r,p}$*-quasi-everywhere* (abbreviated $C_{r,p}$*-q.e.*) if it holds outside a set N of $C_{r,p}$-capacity zero (i.e., $C_{r,p}(N) = 0$). A $C_{r,p}$-q.e. defined function f on X is called $C_{r,p}$*-quasi-continuous* if $f \in C(\{F_n\})$ for some $C_{r,p}$-nest $(F_n)_{n \in N}$.

Proposition 1.3. *Suppose that* $C_{r,p}$ *is strongly tight and that the following condition holds.*

$$(1.4) \qquad \begin{array}{c}\textit{There exists a dense subset } D \textit{ of } (F_{r,p}, \| \ \|_{r,p}) \textit{ such} \\ \textit{that each } u \in D \textit{ has a } C_{r,p}\textit{-quasi-continuous } m\textit{-version } \tilde{u}.\end{array}$$

Then for A, $A_n \subset X$, $n \in N$, *with* $A_n \uparrow A$

$$\sup_{n \in N} C_{r,p}(A_n) = C_{r,p}(A).$$

REMARK 1.4. (1.4) takes the role of the following stronger condition in [24]

$$(1.5) \qquad F_{r,p} \cap C(X) \text{ is dense in } (F_{r,p}, \| \ \|_{r,p}).$$

For the proof of 1.3 we need the following lemma whose analogue is also contained in [24]. Since X is not assumed to be a separable metric space and m not to have full support as in [24], we have to take a little care here, so we include the proof.

Lemma 1.5.

1. *Let $u \in F_{r,p}$ such that u has a $C_{r,p}$-quasi-continuous m-version $\tilde{u}$. Then for all $R \in]0,\infty[$*

$$C_{r,p}(\{|\tilde{u}|>R\}) \leq R^{-p}\|u\|_{r,p}^p. \tag{1.6}$$

2. *Let $u_n \in F_{r,p}$ which have $C_{r,p}$-quasi-continuous m-versions $\tilde{u}_n$, $n \in N$, such that $u_n \underset{n\to\infty}{\to} u \in F_{r,p}$ w.r.t. $\| \ \|_{r,p}$. Then there exists a subsequence $(u_{n_k})_{k\in N}$ and a $C_{r,p}$-quasi-continuous m-version $\tilde{u}$ of u such that $(\tilde{u}_{n_k})_{k\in N}$ converges $C_{r,p}$-quasi-uniformly to $\tilde{u}$ (i.e., there exists a $C_{r,p}$-nest $(F_n)_{n\in N}$* such that *$\tilde{u}_{n_k} \underset{k\to\infty}{\to} \tilde{u}$ uniformly on each F_n). In particular,* (1.4) *implies that every $u \in F_{r,p}$ has a $C_{r,p}$-quasi-continuous m-version,*
3. *Let u be a $C_{r,p}$-quasi-continuous function on X. If $u \geq 0$ m-a.e. on an open set $U \subset X$, then $u \geq 0$ $C_{r,p}$-q.e. on U.*

Proof. (i): Let $\delta > 0$ and $F \subset X$, F closed, such that $C_{r,p}(F^c) < \delta^{1/p}$ and $\tilde{u}|_F$ is continuous. Let $R \in]0,\infty[$. Then $\{|\tilde{u}|>R\} \cup F^c$ is open. It follows by Proposition 1.2 that

$$\begin{aligned} C_{r,p}(\{|\tilde{u}|>R\}) &\leq C_{r,p}(\{|\tilde{u}|>R\} \cup F^c) \\ &\leq (R^{-1}\|\tilde{u}\|_{r,p} + \|e_{F_c}\|_{r,p})^p \\ &= (R^{-1}\|\tilde{u}\|_{r,p} + C_{r,p}(F^c)^{1/p})^p \\ &\leq (R^{-1}\|\tilde{u}\|_{r,p} + \delta)^p. \end{aligned}$$

Since δ was arbitrary, the assertion follows.
The first part of (ii) can be proved entirely analogously as Proposition 3.5 in [31, Chap.III]. The second part is then trivial.
(iii): See [15, Proposition 8.1.6]. □

Proof of 1.3. By Lemma 1.5 the proof of Theorem 2 in [24] carries over to our more general case and proves the assertion. □

Recall that a Hausdorff topological space Y is called *Souslin* if it is the continuous image of a Polish space (cf. [46, Chap.II, Section 1]). A subset $Y \subset X$ is called a *Souslin support* of $C_{r,p}$ if Y is Souslin and $C_{r,p}(X \setminus Y) = 0$. Recall also that a kernel $\pi: X \times \mathscr{B}(X) \to [0,\infty[$ is called *sub-Markovian* if $\pi(x,X) \leq 1$ for all $x \in X$ and *strongly Feller* if πf $(:= \int f(y)\pi(\cdot,dy))$ is continuous for all $f \in \mathscr{B}_b(X)$. We call π *$C_{r,p}$-quasi-strongly* Feller if πf is $C_{r,p}$-quasi-continuous for all $f \in \mathscr{B}_b(X)$. We shall make use of the following result.

Proposition 1.6. *Assume that $m(X)<\infty$ and that there exists a $C_{r,p}$-nest $(F_n)_{n\in N}$ consisting of metrizable sets. Suppose that there exists a semigroup $(p_t)_{t>0}$ of sub-Markovian kernels on $(X,\mathscr{B}(X))$ having the following properties:*

(1.7) $\quad p_t f$ *is an m-version of* $T_t f$ *for all* $t>0$, $f\in\mathscr{B}_b(X)$.

(1.8) *There exists an algebra* $C\subset C_b(X)$ *containing* 1, *dense in* L^p, *such that* $p_t C\subset C_b$ *for all* $t>0$ *and such that* C *separates the points of a Souslin support* Y *of* $C_{r,p}$ *with the property that* $C_{r,p}$*-q.e.* $p_t 1_{Y^c}=0$ $\forall t>0$.

Set

$$v_r := \int_0^\infty t^{r/2-1}e^{-t}p_t dt.$$

Then (1.4) *holds,* $v_r f$ *is a* $C_{r,p}$*-quasi-continuous m-version of* $V_r f$ *for all* $f\in L^p$ *and thus* v^r *is* $C_{r,p}$*-quasi-strongly Feller. In particular,*

(1.9) $$C_{r,p}(\{v_r f>R\})\leq R^{-p}\|f\|_{L^p}^p \quad \textit{for all } f\in L^p,\ R\in]0,\infty[.$$

Proof. Clearly, $v_r f$ is an m-version of $V_r f$ for all $f\in L^p$. Let $R\in]0,\infty[$ and $f\in C$. By Lebesgue's dominated convergence theorem $v_r f$ is sequentially continuous, hence continuous on each F_n, i.e., $v_r f$ is $C_{r,p}$-quasi-continuous. This implies that (1.4) holds. By Lemma 1.5 (ii) and a monotone class argument it follows that $v_r f$ is $C_{r,p}$-quasi-continuous for all bounded $\sigma(C)$-measurable functions f on X and hence for all $f\in\mathscr{B}_b(X)$ by (1.8). Indeed, since $\sigma(C)\cap Y=\mathscr{B}(Y)$ (e.g. by [46, Proposition 4, p.105 and Lemma 18, p.108]), there exists a bounded $\sigma(C)$-measurable function g on X such that $f=g$ on Y. By (1.8) it follows that $v_r 1_{Y_c}=0$ $C_{r,p}$-q.e., hence $v_r(f\text{-}g)=0$ $C_{r,p}$-q.e..
Another monotone class argument implies the assertion for $f\in L^p$. □

REMARK 1.7.

1. Condition (1.8) (with $Y=X$) is, of course, fulfilled if X is polish and $(p_t)_{t>0}$ is a *Feller semigroup*, i.e., $p_t C_b(X)\subset C_b(X)$ for all $t>0$. This case is well-known. We have proved Proposition 1.6 in this generality since we want to apply it in the next section in the case where m is a Gaussian measure on an arbitrary locally convex space X and $(p_t)_{t>0}$ is the corresponding Mehler semigroup.

2. The assumption $m(X)<\infty$ in Proposition 1.6 was only made for simplicity. There is similar result without this assumption. Furthermore, condition (1.8) is not optimal. There is a modification using only $C_{r,p}$-quasi-continuous rather than continuous function.

We shall now use the following condition on $(T_t)_{t>0}$

(1.10) $t \mapsto T_t$, $t>0$ is analytic as a map taking values in the Banach space of all bounded operators on L^p.

By a result of E.M. Stein (1.16) is e.g. fulfilled if $(T_t)_{t>0}$ "comes from" a symmetric strongly continuous contraction semigroup on L^2 (see e.g. [37, Theorem X.55]). The following is contained in [28] (see the proof of Lemma 1). We include the proof for completeness.

Lemma 1.8. *Assume that* (1.10) *holds. Then T_t is a continuous map from L^p to $F_{r,p}$ for all $t>0$.*

Proof. Let $t>0$, $n\in N$ with $n>r/2$ and $f\in L^p$. Let L be the generator of $(T_t)_{t>0}$ on L^p. Then $V_2(I\text{-}L)=I$ on $D(L)$. Since by (1.10) $T_tf\in\bigcap_{m\in N}D(L^m)$ it follows that

$$T_tf=(V_2)^n(1-L)^nT_tf=V_r(V_{2n-r}(1-L)^nT_tf),$$

and hence $T_tf\in F_{r,p}$. Since by (1.10) $\|L^nT_tf\|_{L^p}\le\frac{k(n,p)}{t^n}\|f\|_{L^p}$ for some $k(n,p)\in]0,\infty[$ independent of f and t (cf. [37, Corollary 2 of Theorem X.52], it follows that T_t is continuous from L^p to $F_{r,p}$. □

By Lemma 1.8 the proof of the following result is the same as that of Proposition 1.6 with $(T_t)_{t>0}$ replacing $(V_r)_{r>0}$

Proposition 1.9. *Assume that $m(X)<\infty$, that there is a $C_{r,p}$-nest consisting of metrizable sets and that* (1.10) *holds. Suppose there exists $(p_t)_{t>0}$ as in Theorem 1.6 satisfying* (1.7), (1.8). *Then* (1.4) *holds and for all $t>0$, p_tf is a $C_{r,p}$-quasi-continuous version of $(T_t)_{t>0}$ for all $f\in L^p$ and thus p_t is $C_{r,p}$-quasi-strongly Feller. In particular, there exists $k(r,p,t)\in]0,\infty[$ such that*

(1.11) $$C_{r,p}(\{p_tf>R\})\le k(r,p,t)\in R^{-p}\|f\|^p_{L^p}\quad \text{for all } f\in L^p,\ R\in]0,\infty[.$$

REMARK 1.10. (i) By the proof of Lemma 1.8 we can choose $k(r,p,t)=k(n,p)e^tt^{-n}$ for $n\in N$ with $n>r/2$.

(ii) For conditions for the existence of $(p_t)_{t>0}$ as in Propositions 1.6, 1.9 we refer to [28]. If $(p_t)_{t>0}$ is as constructed in [28], then a stronger estimate than (1.11) holds (see [28, Lemma 1(iii)]).

(iii) Let $(p_t)_{t>0}$ be an m-symmetric semigroup satisfying

(1.12) $$p_tf\underset{t\downarrow 0}{\to}f \text{ in } m\text{-measure for } f \text{ in a dense subset } C \text{ of } L^2.$$

Then $(p_t)_{t>0}$ gives rise to an analytic strongly continuous contraction semigroup $(T_t)_{t>0}$ on L^2 (cf. e.g. [31, Chap.II, Subsection 4a)]), hence on every L^p, $p>1$, and the above applies. The m-symmetry can as usual be replaced by a sector condition (cf. [31, Chap.I, Section 2, and Chap.II, Subsection 4a)].

2. The Gaussian case

In this section we consider the case where X is a locally convex topological vector space over $\boldsymbol{R}$ (abbreviated LCS) and m is a Gaussian measure, denoted by γ. More precisely, if X^* denotes the (topological) dual of X then γ is a Radon (cf. [46]) probability measure on $\mathscr{B}(X)$ such that $\gamma\circ l^{-1}$ is a mean zero Gaussian measure on $\boldsymbol{R}$ (or a Dirac measure at zero) for all $l\in X^*$. Here $\gamma\circ l^{-1}$ denotes the image measure of γ under l. The *reproducing kernel Hilbert space* $H=H(\gamma)$ of γ consists of all vectors $h\in X$ such that the measures γ and γ_h, where $\gamma_h(B)=\gamma(B+h)$, $B\in\mathscr{B}(X)$, are equivalent. H has a natural Hilbert norm $\|\ \|_H$ w.r.t. which it is isomorphic to the dual of the completion X^*_γ of X^* in (real) $L^2:=L^2(X;\gamma)$ in such a way that for each $h\in H$ there exists a unique $l\in X^*_\gamma$ such that

$$l'(h)=\int l'(x)l(x)\gamma(dx) \quad \text{for all } l'\in X^*,\ \|h\|_H=\|l\|_{L^2}.$$

Consequently, since L^2 is separable by [45] or [51], H is separable. We now define for $f\in\mathscr{B}_b(X)$

$$p_tf(x)=\int f(e^{-t}x+\sqrt{1-e^{-2t}}x')\gamma(dx'),\ t>0,\ x\in X. \tag{2.1}$$

Then $(p_t)_{t>0}$ is a semigroup of Markovian kernels on $\mathscr{B}(X)$. $(p_t)_{t>0}$ is the well-known *Mehler semigroup* associated with on X. Define

$$\mathscr{F}C_b^\infty:=\{f(l_1,\cdots,l_m)|m\in\boldsymbol{N},\ f\in C_b^\infty(\boldsymbol{R}^m),\ l_1,\cdots,l_m\in X^*\}.$$

We need the following

Lemma 2.1. *Let $C:=$ linear span of $\{\sin l|l\in X^*\}$. Then C and hence $\mathscr{F}C_b^\infty$ is dense in L^2.*

Proof. By [45] or [51] there exists a Souslin subset E of X such that $\gamma(E)=1$. Hence it suffices to prove that $C|_E$ is dense in $L^2(E;\gamma|_E)$. But by the Hahn-Banach theorem C separates the points of E. Consequently, by [46, Proposition 4, p.105 and lemma 18, p.108] $\sigma(C)|_E=\mathscr{B}(E)$. Now a standard monotone class arguments implies that $C|_E$ is dense in $L^2(E;\gamma)$. $\square$

REMARK. In fact Lemma 2.1 is known to hold for all (not necessarily Gaussian) Radon measures on X (cf. [46]).

Clearly, $(p_t)_{t>0}$ is a Feller semigroup and well-known to be γ-symmetric (which is in fact easy to see using Fourier transforms). Since, $(p_t)_{t>0}$ obviously satisfies (1.12) with C as in Lemma 2.1, the first part of Corollary 1.11 implies that $(p_t)_{t>0}$ gives rise to an analytic strongly continuous contraction semigroup on every L^p, $p>1$. Let V_r resp. $v_r, r>0$, be the Gamma transforms of $(T_t)_{t>0}$ resp. $(p_t)_{t>0}$ and $C_{r,p}$, $r>0$, $p>1$, the corresponding (r,p)-capacities (cf. Section 1). The following is the main result of this section.

Theorem 2.2. *$C_{r,p}$ is strongly tight.*

The proof will be given in two steps. First we need a lemma which is proved in [38, Corollary 5.3]. We give a direct and shorter proof here.

Lemma 2.3. *Let K_n be a metrizable compact set in X. Then its absolutely convex closed hull S is metrizable. If X is sequentially complete, S is compact.*

Proof. First assume that X is complete. According to classical results (see [18, Corollary 8.13.3]) S is compact and, moreover, $S=\{\int_K xm(dx),\ m\in M_1(K)\}$, where $M_1(K)$ denotes the space of Radon measures on K with total variation less or equal to 1 and integrals are taken in Pettis's (weak) sense. It follows by assumption that $M_1(K)$ with the topology $\sigma(M_1(K), C(K))$ is compact and metrizable. Hence the same is true for S since the map $m\mapsto\int_K xm(dx)$ from $M_1(K)$ to X with the weak topology is continuous and the weak topology coincides on S with the initial one. Now to get the first assertion it suffices to use the existence of completions for locally convex spaces. The second follows from the fact that in a sequentially complete space closed metrizable subsets are complete. □

Step 1 of the proof of Theorem 2.2

Assume first that X is sequentially complete.

By [45,51] there exists a metrizable compact set K with $\gamma(K)>0$. By Rieffel's lemma above the absolutely closed convex hull of K, denoted by S, is metrizable and compact. Let E be the linear span of S. Since $E=\bigcup_{n\in N} nS$, we have that $E\in\mathscr{B}(X)$. Denote by g the Minkowski functional of S (we set $g=\infty$ on $X\backslash E$). By the 0–1 law for Gaussian measures on locally convex spaces (cf. e.g. [27]) $\gamma(E)=1$. Thus, g is a measurable seminorm and according to Fernique's theorem belongs to all L^p, $p>1$. For all $x,y\in E$ and $t>0$ we have:

$$2g(e^{-t}x)\leq g(e^{-t}x+\sqrt{1-e^{-2t}}y)+g(e^{-t}x-\sqrt{1-e^{-2t}}y).$$

Integrating this estimate in y and using the symmetry of γ we get:

$$2g(e^{-t}x) \leq 2T_t g(x).$$

Integration in t with the weight $\Gamma(r/2)^{-1} t^{-1+r/2} e^{-t}$ gives the estimate

$$g(x) \leq 2^{-r/2} V_r g(x). \tag{2.2}$$

Since $g/n \geq 1$ on $E \backslash nS$, that is, γ-a.e. on the open set $X \backslash nS$, (2.2) implies that

$$C_{r,p}(X \backslash nS) \leq \|2^{-r/2} n^{-1} V_r g\|_{r,p}^p = 2^{-rp/2} n^{-p} \|g\|_{L^p}^p,$$

which implies that $C_{r,p}(X \backslash nS) \underset{n \to \infty}{\to} 0$.

Now the results of Section 1 apply in this particular case, more precisely we have

Corollary 2.4. *Assume that X is sequentially complete. Then both Propositon 1.6 and 1.9 apply. In particular, both p_t, $t>0$, and v_r are $C_{r,p}$-quasi-storongly Feller for all $r>0$, $p>1$ resp. all $p>1$.*

Proof. We only have to show that (1.8) holds. Let C be as in Lemma 2.1. It is easy to check that (1.8) holds (with C) except for its last part. But by the preceding proof there exists a $C_{r,p}$-nest $(K_n)_{n \in N} = (nS)_{n \in N}$ with S an absolutely convex metrizable compact subset of X. Since $Y := \bigcup_{n \in N} K_n$ is Souslin, it suffices to prove that for $t>0$

$$p_t 1_{K_n^c}(x) \downarrow 0 \text{ as } n \to \infty \text{ for all } x \in Y. \tag{2.3}$$

Let $t>0$ and $n \in N$. For $k \in N$ choose $n_k \in N$ such that $n_k \geq \max[(1+e^{-t})^{1/2}(1-e^{-t})^{-1/2}k, n_{k-1}]$. Then $y \in K_k$ implies that both $y_0 := (1+e^{-t})^{1/2}(1-e^{-t})^{-1/2} y \in K_{n_k}$ and $e^{-t}x + \sqrt{1-e^{-2t}}\,y = e^{-t}x + (1-e^{-t}) y_0 \in K_{n_k}$ provided $x \in K_{n_k}$ (since $K_m = mS$, $m \in N$, and S is absolutely convex). Hence we have for $x \in K_{n_k}$ that

$$\begin{aligned} p_t 1_{K_{n_k}^c}(x) &= \int_{K_k^c} 1_{K_{n_k}^c}(e^{-t}x + \sqrt{1-e^{-2t}}\,y)\gamma(dy) \\ &\leq \gamma(K_k^c). \end{aligned}$$

Since $(p_t 1_{K_n^c})_{n \in N}$ is decreasing, (2.3) follows and the proof is complete. □

Step 2 of the proof of Theorem 2.2.
Let now X be arbitrary.
Let Z denote the completion (see [18]) of X. Consider γ, $(p_t)_{t>0}$ $C_{r,p}$, $r>0$, $p>1$, on Z. Let D be a compact subset of X of strictly positive γ-measure and E the linear span of D. Then E is σ-compact, since it is the union of the compact sets

$$D_{k,n}:=\{t_1d_1+\cdots+t_nd_n|d_1,\cdots,d_n\in D;|t_1|,\cdots,|t_n|\le k\};\ k,n\in \boldsymbol{N}.$$

Clearly, $M:=Z\backslash E\in\mathscr{B}(Z)$, $\gamma(E)=1$ (by the 0–1 law) and so $v_r1_M=1_M$. Corollary 2.4 implies that

$$C_{r,p}(M)=C_{r,p}(\{1_M=1\})=C_{r,p}(\{v_r1_M=1\})\le \|1_M\|_{L^p}^p=0.$$

Consequently, for each $n\in \boldsymbol{N}$ there exists a metrizable compact Q_n and an open set $U_n\subset Z$ such that

$$M\subset U,\ C_{r,p}(U_n)<\frac{1}{n},\ C_{r,p}(Z\backslash Q_n)<\frac{1}{n}.$$

Set $K_n:=Q_n\cap(Z\backslash U_n)$, $n\in\boldsymbol{N}$. Then $K_n\subset E\subset X$, K_n is compact in X and $C_{r,p}(X\backslash K_n)\le C_{r,p}(Z\backslash K_n)<\frac{2}{n}$, $n\in\boldsymbol{N}$, and the proof is complete.

REMARK 2.5. It was possible in Step 1 to consider only the case $X=\boldsymbol{R}$, where $\mathscr{B}(\boldsymbol{R}^\infty)=\sigma(\boldsymbol{R}^\infty)$ and then in Step 2 to reduce the general case to $\boldsymbol{R}^\infty$. This reduction is especially simple if there exist $l_n\in Z^*$, $n\in\boldsymbol{N}$, separating the points of Z, because we can take a continuous injection of Z in $\boldsymbol{R}^\infty$ and apply the invariance theorem (cf. [4, Theorem 1.1]). If there are no separating sequences in Z^* we can use these arguments for the space $Z_1=X\cap(\text{span } Q)$, where $Q\subset Z$ is the absolutely convex metrizable compact set, constructed above. It is clear that for Z_1 one can always find a separating sequence from X^*. This way was pointed out in [13], but mistakenly it was asserted there that a separating sequence can be chosen for the whole of Z. Certainly, this is not true in general as can be seen from the trivial example $Z=\{f:[0,1]\to\boldsymbol{R}^1|f(0)=0\}$ with the pointwise topology and γ the Wiener measure.

Now we can show that the assumption in Corollary 2.4 is superfluous.

Corollary 2.6. *Both Propositions* 1.6 *and* 1.9 *apply to any* X and γ *as above. In particular, both* $p_t, t>0$, *and* v_r *are* $C_{r,p}$*-quasi-storongly Feller for all* $r>0$, $p>1$, *resp.* $p>1$.

Proof. Let Z be the completion of X. Let $\gamma^\#$ be the extension of γ to Z and let $(p_t^\#)_{t>0}$, $(v_r^\#)_{r>0}$, $C_{r,p}^\#$, $r>0$, $p>1$, be defined corresponding to $\gamma^\#$ on $(Z,\mathscr{B}(Z))$. Fix $r>0, p>0$. Since the topology on Z restricted to X coincides with the initial one, we have that

$$C_{r,p}^\#(\{A)=C_{r,p}(A\cap X)\quad\text{for all } A\subset X.$$

Hence for $R>0$ by Corollary 2.4

$$C_{r,p}(\{v_rf>R\})=C_{r,p}^\#(\{v_r^\#f^\#>R\}\cap X)$$

$$\leq \frac{1}{R}\|f^{\#}\|^p_{L^p(Z;\gamma^{\#})} = \frac{1}{R}\|f\|^p_{L^p}$$

for every $f \in L^p$ and its trivial extension $f^{\#}$ to Z, i.e., (1.9) holds. Correspondingly, one shows that (1.10) holds and the assertion follows. □

REMARK 2.7. (i) The tightness of $C_{r,p}$ was studied previously by *D.* Feyel and A. de La Pradelle (cf. [19, Theoreme 9]) by a different method (extending the well-known result of H.Sugita [50] for the case where X is a separable Banach space). So, the essential new fact in Theorem 2.2 is that the compact sets can be taken metrizable in arbitrary locally convex spaces. We can also prove Theorem 2.2 based on similar arguments as in [31, Chap. , Remark 3.2(iii)].

(ii) The statement that $p_t f, t>0$, are $C_{r,p}$-quasi-continuous for all $f \in L^p$, $r>0$, $p>1$, is contained in [19, Theoreme 5].

We conclude this section by further consequences of Theorem 2.2.

Corollary 2.8.

1. *Let $B \in \mathscr{B}(X)$ (or even only a Souslin set in X). Then* $C_{r,p}(B) = \sup\{C_{r,p}(K) | K \subset B,\ K \text{ compact and metrizable}\}$
2. *Let L be a γ-measurable linear subspace of X with $\gamma(L)=1$. Then*
$$C_{r,p}(X \backslash L) = 0.$$
3. *$C_{r,p}$ does not change if the topology on X is replaced by a weaker locally convex topology.*

Proof. (i): Let $\varepsilon > 0$. Then there exists a metrizable compact set $Q \subset X$ with $C_{r,p}(X \backslash Q) < \varepsilon$. Hence

$$C_{r,p}(B) \leq C_{r,p}(B \cap Q) + \varepsilon.$$

But since $B \cap K$ is Souslin, by [16, Chap.IX, Section 6, Def.9, Théorème 6 and Proposition 10] there exists a compact set $K \subset B \cap Q$ such that

$$C_{r,p}(B \cap Q) \leq C_{r,p}(K) + \varepsilon.$$

Since K is, of course, metrizable, (i) follows.

(ii): As we see from the proof of Theorem 2.1, the compact metrizable set S in Step 1 and K_n, $n \in N$, in Step 2 can be constructed to be subsets of L.

(iii): is an immediate consequence of Theorem 2.2 and [4, Theorem 1.1.]. □

3. Mehler formula for Ornstein- Uhlenbeck processes with general linear drift

Let $\mathscr{L}(E)$ denote the space of continuous linear operators on a Banach space

E. The main result of this section is the following.

Theorem 3.1. *Let H be a separable real Hilbert space and A a nonnegative definite self-adjoint operator on H with domain $D(A)$. Then there exists a continuous norm q on H coming from an inner product and possessing the following properties:*

(i) The embedding of H into the completion E of H with respect to q is Hilbert-Schmidt.

(ii)$q(\exp(\text{-}tA)x) \leq q(x)$ for all $x \in H$ and all $t \geq 0$, and thus the operators $\exp(\text{-}tA)$ on H admit continuous extensions to operators from E to E with norms less or equal to 1 and these form a strongly continuous contraction semigroup on E.

(iii) If E' denotes the dual space of E then $E' \subset H' \equiv H \subset E$ continuously and densely. Furthermore, there exists a linear subspace $K \subset E'$, dense w.r.t. to the natural (operator) norm $\| \ \|_{E'}$, such that $K \subset D(A)$, $A(K) \subset K$, $e^{-tA}(K) \subset K$, $t > 0$, and $Al(z) = \lim_{t \downarrow 0} \frac{1}{t}(l - e^{-tA} l)(z)$ for all $z \in E$, $l \in K$. If, in addition $\|\exp(-A)\|_{\mathscr{L}(H)} < 1$, then the extensions can be taken with $\|\exp(-tA)\|_{\mathscr{L}(E)} < 1$ for all $t > 0$.

Before we prove this theorem, we discuss some related work.

REMARK 3.2. (i) Recall that in the situation of Theorem 3.1 the standard Gaussian cylinder measure γ^* on H lifts to a Gaussian (Radon) measure γ on E (cf. [26,29]).

(ii) If in the situation of 3.1, A is strictly positive, hence $\|\exp(-A)\|_{\mathscr{L}(H)} < 1$, Theorem 3.1 implies that all conditions on (E,H,γ) and A imposed in [47] are fulfilled. Therefore, the analogue of the well-known Meyer equivalence (cf. [34,35] and the references therein), which was proved in [47], holds for any given separable real Hilbert space H and operator A as above, if E is as in Theorem 3.1.

(iii) The case with E a properly constructed dual of a nuclear space was treated in [12].

(iv) In [21] *measurable extensions* have been discussed. But it follows from their method that these extensions are typically not continuous.

(v) Since $e^{-t(A+1)}K \subset K$ and K is dense in H, it follows by [37, Theorem X.49] that K in 3.1 (iii) is dense in $D(A)$ w.r.t. $\|(A+1)\cdot\|_H$, hence by the nonnegativity of A w.r.t. $\|\sqrt{A}\cdot\|_H$.

Now we give some examples which among other things show that in many cases the problem of extending $\exp(-tA), t \geq 0$, to a space E supporting γ becomes trivial if we do not insist on E being a Banach space.

EXAMPLES 3.3. (i) Assume that A has an orthonormal basis of eigenvectors e_n with corresponding eigenvalues a_n. Then the operators $\exp(\text{-}tA)$ are diagonal with eigenvalues e^{-ta_n} and trivially each weighted Hilbert space $X = \{(x_n) \in \boldsymbol{R}^\infty | \sum \sigma_n^2 x_n^2 < \infty\}$ with $\sum \sigma_n^2 < \infty$ is convenient (see also Example 6.5 (ii) below). If we take $\boldsymbol{R}^\infty$ for

X then A itself becomes continuous. This case was discussed in detail in [10] with indications to possible generalizations.

(ii) Let $H=L^2(\boldsymbol{R}^d;dx)$, $dx=$Ledesgue measure, and $A=\Delta$ with its natural domain. Then choosing $S'(\boldsymbol{R}^d)$ for X we obtain that all operators under question are continuous. The same holds for more general nonnegative elliptic operators A provided the coefficients of A are smooth. It is difficult to construct Hilbert or Banach supports E for γ in these cases since the spectra are continuous. This has, however, been done in [39,40] for the operator $A=-\Delta+1$ on $H=L^2(\boldsymbol{R}^d;dx)$. For the construction of E special subspaces of scaled Sobolev spaces have been introduced there. The proof of Theorem 3.1 given below was strongly influenced by the construction of E in [39,40].

Proof of Theorem 3.1. Let $T:=\exp(-A)$. T is self-adjoint and bounded. Hence since H is separable, we may assume that T has a cyclic vector. The general case can be trivially reduced to this case, since by [36, Lemma 2, p. 226] T is a direct sum of operators $T_n: H_n \to H_n$, $H=\oplus_{n=1}^N H_n$, $N\in \boldsymbol{N}\cup\{+\infty\}$, each having a cyclic vector. For E one can then take the weighted Hilbert sum of the corresponding E_n with weights 2^{-n}, i.e., $(h,h)_E:=\sum_{n=1}^\infty 2^{-n}(h_n,h_n)_{E_n}$ with $h=(h_n)_{n\in\boldsymbol{N}}$. But if T has a cyclic vector, by the spectral theorem T is unitary equivalent to an operator of multiplication by $g(s):=s$ on $L^2([0,1];m)$ for some finite positive measure m on $\mathscr{B}([0,1])$. The support of this measure is equal to the spectrum $\sigma(T)$ of T. Note that proving our assertion for this particular "model" of H is enough since E will be the completion of H w.r.t. some norm. In fact E with norm q will be a (closed) subspace of the dual of a real Hilbert space W that one obtains by completing $C_0^\infty(]0,1[)$ (i.e, the set of all infinitely differentiable real functions on $]0,1[$with compact support) w.r.t. to the norm

$$\|f\|_W^2 := \int_0^\infty \|s^\alpha f\|_{W_0^1,2}^2 d\alpha + \int_0^1 f^2(4s^2|\ln s|^3)^{-1}ds, \quad f\in C_0^\infty(]0,1[). \tag{3.1}$$

Here $d\alpha$, ds denote Lebesgue measure on $\boldsymbol{R}^1$ and for $f\in C_0^\infty(]0,1[)$

$$\|s^\alpha f\|_{W_0^1,2}^2 := \int_0^1 [(s^\alpha f(s))']^2 ds \tag{3.2}$$

$$= \int_0^1 [\alpha^2 s^{2\alpha-2} f^2(s) + \alpha s^{2\alpha-1}(f^2(s))' + s^{2\alpha}(f'(s))^2] ds$$

$$= \int_0^1 [\alpha(1-\alpha)s^{2\alpha-2}f^2(s) + s^{2\alpha}(f'(s))^2]ds,$$

where we integrated by parts in the last step. Therefore, integrating w.r.t. we obtain that

$$\|f\|_W^2 = \int_0^1 f^2\rho_1 ds + \int_0^1 (f')^2\rho_2 ds \tag{3.3}$$

where for $s \in]0,1[$

$$\rho_1(s) := \frac{1}{4(s\ln s)^2}$$

and

$$\rho_2(s) := -\frac{1}{2\ln s^2}.$$

In particular, we see from (3.3) that $\|f\|_W^2 < \infty$ and that

$$\|f\|_\infty \leq \int_0^1 |f'|ds \leq \left(\int_0^1 |f'|^2\rho_2 ds\right)^{1/2}\left(\int_0^1 \rho_2^{-1}ds\right)^{1/2} \tag{3.4}$$

for all $f \in C_0^\infty(]0,1[)$. Since $\int_0^1 \rho_2^{-1}ds < \infty$, it follows that if $C[0,1]$ denotes the set of all continuous functions on $[0,1]$, then

$$W \subset C[0,1] \text{ continuously and densely.} \tag{3.5}$$

If $j: C[0,1] \to L^2([0,1];m)$ is the linear map associating each $f \in C[0,1]$ with its corresponding m-class, then $j: W \to L^2([0,1];m)$ is Hilbert-Schmidt.
This follows by a well-known result from (3.5). For the convenience of the reader we include the (in our case trivial) proof: let $W_{\rho_2}^{1,2}$ be the completion of $C_0^\infty(]0,1[)$ w.r.t. the norm

$$f \mapsto \left(\int_0^1 (f')^2\rho_2 ds\right)^{1/2}.$$

Let $(e_n)_{n\in N}$ be an orthonormal basis in $W_{\rho_2}^{1,2}$. Then $(e_n')_{n\in N}$ is an orthonormal system in $L^2([0,1];\rho_2 dt)$. Hence

$$\sum_{n\geq 1}\int e_n^2 dm = \int \sum_{n\geq 1}\left(\int_0^1 1_{[0,x[}(s)\rho_2^{-1}(s)e_n'(s)\rho_2(s)ds\right)^2 m(dx)$$

$$\leq \int \| 1_{[0,x[}\rho_2^{-1} \|^2_{L^2(]0,1[;\rho_2 ds)} m(dx)$$

which is finite since $\int_0^1 \rho_2^{-1} ds < \infty$. Now it follows that $j: W \to L^2([0,1];m)$ is Hilbert-Schmidt. Let S denote the operator on $C[0,1]$ given by multiplication with $g(s) := s$. Clearly, by the definition of $\| \ \|_W$ we have that $S(W) \subset W$ and that $\|S\|_{\mathscr{L}(W)} \leq 1$. Furthermore, an elementary calculation shows that for all $f \in C_0^\infty(]0,1[)$

$$\|(1-S)f\|_W^2 - \|f\|_W^2 = \int_0^1 (2-s)s(\ln s)^{-1}(f'(s))^2 ds$$

$$+\frac{1}{4}\int_0^1 (4s^{-1} - 2 - \ln s)(\ln s)^{-3} f^2(s) ds.$$

Since both summands on the right hand side are negative we obtain that

$$\|1-S\|_{\mathscr{L}(W)} \leq 1. \tag{3.6}$$

Hence the following series converges for all $t \in [0,\infty[$ in operator norm

$$S^t (=(1+S-1)^t) := \sum_{n=0}^{\infty} \binom{t}{n} (S-1)^n \tag{3.7}$$

where $\binom{t}{0} := \prod_{k=1}^{n} \frac{t-k+1}{k}$, $n \in N$. We obviously have that

$$S^t f(s) = s^t f(s), \quad s \in [0,1], \ f \in W. \tag{3.8}$$

Furthermore, by the definition of $\| \ \|_W$

$$\|S^t\|_{\mathscr{L}(W)} \leq 1 \quad \text{for all } t \in [0,\infty[\tag{3.9}$$

and we also have that

$$\lim_{t \downarrow 0} \|S^t f - f\|_W = 0 \quad \text{for all } f \in W. \tag{3.10}$$

To show this we first observe that because of (3.9) it sufficies to show (3.10) for all $f \in C_0^\infty(]0,1[)$. So, let $f \in C_0^\infty(]0,1[)C(]0,1[)$. Then by (3.8)

$$\|(1-S^t)f\|_W^2 = \int_0^\infty \|S^\alpha f\|^2_{W_0^1,2} d\alpha + \int_t^\infty \|S^\alpha f\|^2_{W_0^1,2} d\alpha$$

$$-2\int_0^\infty \int_0^1 s^t((s^\alpha f(s))')^2 ds\, d\alpha$$

$$-t\int_0^\infty\int_0^1 s^{t-1}((s^\alpha f(s))^2)' ds\,d\alpha.$$

But since supp $f \subset]0,1[$, the right hand side converges to zero as $t \downarrow 0$.

Now assume that $\|T\|<1$ and fix $t\in]0,\infty[$. Then $\sigma(T)\subset[0,a]$ for some $a\in[0,1[$. Replacing $[0,1]$ above by $[0,a]$ and keeping the above notation, by an elementary calculation we obtain that

$$\|S^t f\|_W^2=\int_0^a s^{2t}(f'(s))^2\rho_2(s)ds+\int_0^a F(s)f^2(s)(4s^2|\ln s|^3)^{-1}ds \tag{3.11}$$

$$\text{for all } f\in C_0^\infty(]0,a[).$$

Here

$$F(s):=s^{2t}(1+2t\ln s+2t(1-t)(\ln s)^2),\quad s\in[0,a].$$

which is dominated by $s\mapsto s^{2t}$ if $t\geq 1$. If $t<1$, it is straightforward to check that F attains its maximum either at zero or a. In any case then $F(s)\leq a^{2t}$ for all $s\in[0,a]$. Hence

$$\|S^t\|_{\mathscr{L}(W)}<1. \tag{3.12}$$

Since $j\colon W\to L^2([0,1];m)$ is continuous with a dense image, identifying $L^2([0,1];m)$ with its dual we obtain that the adjoint map

$$j'\colon L^2([0,1];m)\to W' \tag{3.13}$$

is continuous and one-to-one. Here the dual W' of W is equipped with its natural norm $\|\ \|_{W'}$. Set

$$E:=\text{closure of } j(L^2([0,1];m)\to \text{w.r.t. } \|\ \|_{W'} \tag{3.14}$$

and denote the restriction of $\|\ \|_{W'}$ to E by q. It is then straightforward to check that the adjoint S' of S satisfies

$$S'(E)\subset E \quad\text{and}\quad S'\circ j'=T. \tag{3.15}$$

Since $(S^t)'=(S')^t$, it now follows easily by (3.7) that all properties proved for S^t carry over to $(S')^t$, $t\in]0,\infty[$. Thus $((S')^t)_{t\geq 0}$ is the desired extension, and the proof of (i), (ii) and the last part of the assertion is complete. To prove (iii) we first note that identifying H with its dual H' we trivially have that $E'\subset H\subset E$ continuously and densely and that

$$E' = \left\{ h \in H \,\middle|\, \sup_{\tilde{h} \in E} \frac{\langle h, \tilde{h} \rangle}{\|\tilde{h}\|_E} < \infty \right\}. \tag{3.16}$$

(3.16) immediately implies, that $K := j(C_0^\infty(]0,1[))$ is a dense linear subspace of E'. Since $e^{-tA}f(s) = s^t f(s)$ for all $f \in L^2(]0,1[;m)$, it follows that $e^{-tA}(K) \subset K, t>0$, and by Lebesgue's dominated convergence theorem that $K \subset D(A)$ and $Al = (\text{-}\ln s)l \in K$ for all $l \in K$. Since each $l \in K$ has its support in $]0,1[$, it is easily seen from (3.3) that in fact $\frac{1}{t}(1-s^t)l \underset{t\downarrow 0}{\rightarrow} (-\ln s)l$ w.r.t. $\| \; \|_W$. Hence for all $z \in E$, $Al(z) = {}_W\langle Al, z\rangle_{W'}$ $= \lim_{t\downarrow 0} {}_W\langle \frac{1}{t}(1-e^{-tA})l, z\rangle_{W'} = \lim_{t\downarrow 0} \frac{1}{t}(1-e^{-tA})l(z)$ and also (iii) is proved. □

REMARK 3.4. (i) We note that we chose the construction of W presented in the proof of Theorem 3.1, because then the fact that $\|S^t\|_{\mathscr{L}(W)} \leq 1$ for all $t>0$ is obvious. The price is that we had to deal with weighted Sobolev spaces. It is possible to relpace W by the classical Sobolev space $W_0^{1,2}$ on $]0,1[$ which makes some of the elementary calculations above easier to some extent. However, it is a little harder to show that $\|S^t\|_{\mathscr{L}(W_0^{1;2})} \leq 1$ for all $t>0$: Let $f \in C_0^\infty(]0,1[)$, $t>0$. Then by (3.1)

$$\begin{aligned} \|S^t f\|^2_{W_0^{1,2}} &= \int_0^1 [(s^t f(s))']^2 ds \\ &= \int_0^1 [t(1-t)s^{2t-2}f^2(s) + s^{2t}(f'(s))^2] ds. \end{aligned}$$

But by the Cauchy-Schwarz inequality

$$\begin{aligned} \int_0^1 s^{2t-2} f(s)^2 ds &\leq \int_0^1 s^{2t-1}\left(\int_0^s (f'(y))^2 dy\right) ds \\ &= \int_0^1\int_y^1 s^{2t-1} ds (f'(y))^2 dy \\ &= \frac{1}{2t}\int_0^1 (1-y^{2t})(f'(y))^2 dy. \end{aligned}$$

Consequently,

$$\|S^t f\|^2_{W_0^{1,2}} \leq \frac{1-t}{2}\int_0^1 (1-s^{2t})(f'(s))^2 ds + \int_0^1 s^{2t}(f'(s))^2 ds$$

$$\leq \|f\|^2_{W_0^{1,2}},$$

since $\frac{1-t}{2}<1$. We would like to thank G. Metafune for pointing out these estimates to us.

(ii) If one merely wants E in Theorem 3.1 to be a Banach space and only to satisfy 3.1 (i),(ii), it is possible to take E as the dual of the space W consisting of all functions on $[0,1]$ admitting holomorphic extensions F to the domain $U:=\{|z|<1\}\cap\{|z-1|<1\}$, continuous on its closure. W is equipped with the norm $\|f\|_W:=\|F\|_\infty$.

(iii) Clearly, the generator of the strongly continuous contraction semigroup constructed on E above extends A. Therefore, from now on we shall also denote the extended semigroup by e^{-tA}, $t\geq 0$. Note, however, that e^{-tA}, $t\geq 0$, in our proof of Theorem 3.1 are not symmetric operators.

From now on we fix H, A as in Theorem 3.1 and consider any Hilbert space (E,q) satisfying 3.1 (i) and (ii).

Lemma 3.5. *The operators* $\sqrt{1-e^{-2tA}}$, $t\geq 0$, *are well-defined as continuous maps on* E *such that for all* $z\in E$

$$q(\sqrt{1-e^{-2tA}}(z))\underset{t\downarrow 0}{\rightarrow}0.$$

Proof. First of all we note that $\sqrt{1-e^{-2tA}}$ is well-defined by a series because $\|e^{-tA}\|_{\mathscr{L}(E)}\leq 1$ for all $t\in[0,\infty[$. So, fix $z\in E$ and let $N\in\mathbf{N}$. Then

$$q((1-e^{-2tA})^{1/2}z)\leq q\left(\sum_{n=0}^{N}\binom{\frac{1}{2}}{n}(-e^{-2tA})^n z\right)+\sum_{n=N+1}^{\infty}\left|\binom{\frac{1}{2}}{n}\right|q(z). \tag{3.17}$$

The first summand converges as $t\downarrow 0$ to

$$q\left(\sum_{n=0}^{N}\binom{\frac{1}{2}}{n}(-1)^n z\right)=\left|\sum_{n=N+1}^{\infty}\binom{\frac{1}{2}}{n}(-1)^n\right|q(z)\quad(\text{since } \textstyle\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(-1)^n=0)$$

Hence by (3.17)

$$\lim_{t\downarrow 0} q((1-e^{-2tA})^{1/2}z)\leq 2\sum_{n=N+1}^{\infty}\left|\binom{\frac{1}{2}}{n}\right|q(z)$$

which can be made arbitrarily small for large N (since $|\binom{\frac{1}{2}}{n}|\sim\frac{1}{n^{1+1/2}}$), and the assertion follows. □

Corollary 3.6. **(Mehler formula)** *Let μ be a (Gaussian) probability measure on $(E,\mathscr{B}(E))$ such that*

$$\int_E \exp[il(z)]\mu(dz)=\exp\left[-\frac{1}{2}\|Bl\|_H^2\right],\quad l\in E'(\subset H),$$

for some bounded linear operator B on H which commutes with all $\exp(-tA)$, *$t\in[0,\infty[$. Let*

$$p_t f(z):=\int f(e^{-tA}z+\sqrt{1-e^{-2tA}}z')\mu(dz'),\quad z\in E,\ t>0$$

where $f:E\to[0,\infty[$ is $\mathscr{B}(E)$-measurable. Then $(p_t)_{t>0}$ is a semigroup of probability kernels on $(E,\mathscr{B}(E))$ which is Feller (i.e., $p_t f\in C_b(E)$, if $f\in C_b(E)$, $t>0$), is μ-symmetric, and has the property that

$$\lim_{t\downarrow 0} p_t f(z)=f(z)\quad \textit{for all } f\in C_b(E),\ z\in E. \tag{3.18}$$

Proof. Below we shall use the following

Claim: Let $T\in\mathscr{L}(H)$, T self-adjoint, and $S\in\mathscr{L}(E)$ such that $S|_H=T$. Let $S':E'\to E'$ be the adjoint of S. Then $S'l=Tl$ for all $l\in E'(\subset H)$.

The proof is trivial so we omit it. Clearly, each p_t is a probability kernel on $(E,\mathscr{B}(E))$. To show the semigroup property let $t,s\in]0,\infty[$, $u:E\to\boldsymbol{R}$ a bounded $\mathscr{B}(E)$-measurable function, and $z\in E$. We have to show that

$$p_{t+s}u(z)=p_t(p_s u)(z). \tag{3.19}$$

By a monotone class argument it suffices to show (3.19) for $u=\exp[il]$ for every $l\in E'$. But by the claim and the semigroup property of $(e^{-tA})_{t>0}$

$$\begin{aligned} p_{t+s}u(z)&=\exp[i\,l(e^{-(t+s)A}z)]\int\exp\left[i\,l\left(\sqrt{1-e^{-2(t+s)A}}z'\right)\right]\mu(dz')\\ &=\exp[i\,l(e^{-(t+s)A}z)]\exp\left[-\frac{1}{2}\left\|B\sqrt{1-e^{-2(t+s)A}}l\right\|_H^2\right]\end{aligned}$$

and

$$p_t(p_s u)(z)$$

$$=\int p_s u\left(e^{-tA}z+\sqrt{1-e^{-2tA}}z'\right)\mu(dz')$$

$$=\int\int u\left(e^{-(t+s)A}z+e^{-sA}\sqrt{1-e^{-2tA}}z'+\sqrt{1-e^{-2sA}}z''\right)\mu(dz'')\mu(dz')$$

$$=\exp\left[i\,l(e^{-(t+s)A}z)\right]\exp\left[-\frac{1}{2}\left\|Be^{-sA}\sqrt{1-e^{-2tA}}l\right\|_H^2\right]\cdot$$

$$\exp\left[-\frac{1}{2}\left\|B\sqrt{1-e^{-2sA}}l\right\|_H^2\right].$$

By the assumption on B a trivial calculation now implies (3.19). Since the last part of the assertion is clear by Lemma 3.5 and Lebesgue's dominated convergence theorem, it remains to show the μ-symmetry. So, let $u:=\exp[i\,l_1]$, $v=\exp[i\,l_2]$, $l_1,l_2\in E'$, and $t\in[0,\infty[$. Then similarly as above

$$\int u\,p_t v\,d\mu$$

$$=\int\exp[i\,l_1(z)]\exp[i\,l_2(e^{-tA}z)]\int\exp\left[i\,l_2\left(\sqrt{1-e^{-2tA}}z'\right)\right]\mu(dz')\mu(dz)$$

$$=\exp\left[-\tfrac{1}{2}\|B(l_1+e^{-tA}l_2)\|_H^2\right]\exp\left[-\tfrac{1}{2}\|B\sqrt{1-e^{-2tA}}l_2\|_H^2\right]$$

$$=\exp\left[-\tfrac{1}{2}(\|Bl_1\|_H^2+\|Bl_2\|_H^2+2\langle Be^{-tA}l_1,Bl_2\rangle_H)\right]$$

which is symmetric in l_1, l_2 by the assumption on B. Consequently,

$$\int u\,p_t v\,d\mu=\int v\,p_t u\,d\mu, \tag{3.20}$$

i.e., p_t is μ-symmetric. □

REMARK 3.7. (i) We sometimes call the semigroup $(p_t)_{t>0}$ of probability kernels in Corollary 3.6 Mehler semigroup corresponding to (E,H,A,μ).

(ii) Given a bounded linear operator B on H, a Gaussian measure μ as in Corollary 3.6 always exists on $(E,\mathscr{B}(E))$ since $H\subset E$ is Hilbert-Schmidt (cf. e.g. [53, Theorem 3.1]).

4. Corresponding Markov processes

Consider the situation described after Remark 3.4 and let $(p_t)_{t>0}$, μ be as in

Corollary 3.6. By the usual Kolmogrov scheme one can construct a normal Markov process $M=(\Omega,\mathscr{F},(X_t)_{t\geq 0},(P_z)_{z\in E})$ with transition semigroup $(p_t)_{t>0}$. This process, however, is only of interest if one can prove certain regularity properties of its sample paths. Unfortunately, M is in general not a *right process* (i.e., is strong Markov and has right continuous sample paths) as will be seen in Section 6 below. We now intend to give conditions which imply that the sample paths of M are even continuous (P_z-a.s. for all $z\in E$). In particular, M is then *strong Markov* (since $(p_t)_{t>0}$ is Feller), hence a *diffusion*.

By Kolmogorov's existence theorem there exists a unique probability measure $\tilde{P}$ on $(E^{[0,\infty[},\mathscr{A})$ (where $\mathscr{A}$ is the σ-algebra generated by the canonical projections $\tilde{X}_t : E^{[0,\infty[} \to E$) such that for $0<t_1<\cdots<t_n<\infty$, $n\in N$, and $A_0,\cdots,A_n\in\mathscr{B}(E)$

$$\tilde{P}[\tilde{X}_0\in A_0,\tilde{X}_{t_1}\in A_1,\cdots,\tilde{X}_n\in A_n] \tag{4.1}$$

$$\int_{A_0}\int_{A_1}\cdots\int_{A_n} p_{t_n-t_{n-1}}(z_{n-1},dz_n)\cdots p_{t_2-t_1}(z_1,dz_2)p_{t_1}(z_0,dz_1)\mu(dz_0).$$

Let $\Omega:=C([0,\infty[,E)$, $X_t:=\tilde{X}_t$ on Ω, $t\in[0,\infty[$ and $\mathscr{F}:=\sigma(X_t|t\in[0,\infty[)$. Suppose that the following condition holds:

(4.2) There exists probability measure P on $(\Omega,\mathscr{F})$ having the same finite dimensional distributions as $\tilde{P}$, i.e., P satisfies (4.1)

REMARK 4.1. By the Kolmogorov/Prohorov continuity criterion, (4.2) is easily checked to hold if $A=\mathrm{id}_H$ (cf. [41, Proof of Proposition 2.1]). In [39, 40] (4.2) was proved in the special case where $H:=L^2(\boldsymbol{R}^d;dx)$, $A:=(-\Delta+1)^{1/2}$, $B:=(-\Delta+1)^{-1/4}$, using random field techniques. In Section 6 below using the theory of Dirichlet froms we shall prove a necessary and sufficient condition for (4.2) to hold (cf. Theorem 6.3). This condition is e.g. fulfilled if $BA^{1/2}\in\mathscr{L}(H)$ (cf. Proposition 6.5, Example 6.6 (i)), i.e, is particularly fulfilled in the case studied in [39,40]. We shall also give an example where (4.2) fails to hold (cf. Example 6.6 (ii)).

Define $Y:\Omega\to\Omega$ and $T:E\to\Omega$ (componentwise) by

$$Y_t:=X_t-e^{-tA}X_0,\quad t\geq 0,$$

and for $z\in E$

$$(T_Az)_t:=e^{-tA}z,\quad t\geq 0.$$

Clearly, Y is $\mathscr{F}/\mathscr{F}$-measurable and T is $\mathscr{B}(E)/\mathscr{F}$-measurable. Define for $z\in E$ the probability measure P_z on $(\Omega,\mathscr{F})$ by

$$P_z[B]:=(P\circ Y^{-1})[B-T_A z],\quad B\in\mathscr{F}. \tag{4.3}$$

Then we have the following

Theorem 4.2. *Assume that* (4.2) *holds and that* P_z, $z\in E$, *are as in* (4.3). *Then for all* $z\in E$, $0<t_1<\cdots<t_n<\infty$, $n\in N$, *and* $A_1,\cdots,A_n\in\mathscr{B}(E)$

$$\begin{aligned} &P_z[X_{t_1}\in A_1,\cdots,X_{t_n}\in A_n] \\ &=\int_{A_1}\cdots\int_{A_n} p_{t_n-t_{n-1}}(z_{n-1},dz_n)\cdots p_{t_2-t_1}(z_1,dz_2)p_t(z,dz_1). \end{aligned}\tag{4.4}$$

In particular, $M=(\Omega,\mathscr{F},(X_t)_{t\geq 0},(P_z)_{z\in E})$ *is a (conservative) diffusion process (i.e., a conservative, normal strong Markov process with state space E and continuous sample paths) having transition probabilities* $(p_t)_{t>0}$, *and* μ *as stationary measure.*

Proof. (4.4) is easily proved by calculating the Fourier transform of $P_z\circ(X_{t_1},\cdots,X_{t_n})^{-1}$. The rest of the proof is standard (cf. e.g.[11, §42] and [17, Satz 5.10]). □

REMARK 4.3. (i) Clearly, in Theorem 4.2

$$P=P_\mu:=\int P_z\mu(dz)$$

and $(X_t)_{t>0}$ is reversible under $P=P_\mu$.

(ii) A similar study of the corresponding Martin boundary as in [40,41] can be carried out in the more general situation of Theorem 4.2 above.

(iii) Replacing Ω resp. $X_t, t\geq 0$, in condition (4.2) by the set Ω' consisting of all right continuous sample paths resp. $X_t':=\tilde{X}_t$ on Ω', Theorem 4.2 (with the same proof) obviously remains true in the sense that we obtain a conservative normal strong Markov process M' with merely right continuous sample paths. In Section 6 below we shall see, however, that in fact also in this case the sample paths are continuous P_z-a.s. for all $z\in E$.

5. The associated Dirichlet forms

We still consider fixed H, A as in Theorem 3.1 and any Hilbert space (E,q) satisfying 3.1 (i),(ii). In this and the following section we additionally assume that (E,q) also satisfies 3.1 (iii) for some K. Let $(p_t)_{t>0}$, B, μ be as in Corollary 3.6. Since $(p_t)_{t>0}$ is μ-symmetric it has associated to it a *symmetric Dirichlet form* $(\mathscr{E},D(\mathscr{E}))$ defined by

(5.1) $$D(\mathscr{E}):=\left\{u\in L^2(E;\mu)\left|\sup_{t>0}\frac{1}{t}((1-p_t)u,u)_{L^2(\mu)}<\infty\right.\right\}$$

and

(5.2) $$\mathscr{E}(u,v):=\lim_{t\downarrow 0}\frac{1}{t}((1-p_t)u,v)_{L^2(\mu)};\ u,v\in D(\mathscr{E}),$$

(cf. [22, Chap. I],[31, Chap. I]). Note that since $(p_t)_{t>0}$ is μ-symmetric (cf. (3.20)), it respects μ-classes so that $(\mathscr{E},D(\mathscr{E}))$ above is a well-defined quadratic form on $L^2(E;\mu)$.

In this section we shall calculate $(\mathscr{E},D(\mathscr{E}))$ more explicitly. To this end we first calculate the generator of $(p_t)_{t>0}$ on $\mathscr{F}C_b^2(K)$ where for $n\in N\cup\{+\infty\}$

(5.3) $$\mathscr{F}C_b^n(K):=\{f(l_1,\cdots,l_m)\,|\,m\in N,\ f\in C_b^n(\mathbf{R}^m),\ l_i\in K,\ 1\leq i\leq m\}.$$

Here $C_b^n(\mathbf{R}^m)$ denotes the set of all n-times differentiable functions with all partial derivatives bounded. We need some preparations.
For $u\in\mathscr{F}C_b^1(K)$ and $k\in E$ we set

$$\frac{\partial u}{\partial k}(z):=\frac{d}{ds}u(z+sk)|_{s=0},\quad z\in E.$$

Let $(H_0,\langle\,,\sim\rangle_{H_0})$ denote the pre-Hilbert space $(D(\sqrt{A}),\langle B\sqrt{A}\cdot,B\sqrt{A}\cdot\rangle_H)$. Clearly, $D(A)$ is a dense linear subspace of $(H_0,\langle\,,\rangle_{H_0})$ and $(H_0,\langle\,,\rangle_{H_0})$ is complete, i.e., a Hilbert space, if B^{-1} exists as a continuous operator and $A\geq c\ \mathrm{id}_H$ for some $c\in]0,\infty[$.

REMARK 5.1. Since B commutes with e^{-tA}, $t>0$, it also commutes with the semigroup $(Q_t)_{t>0}$ defined as the Bochner integral

$$Q_th:=\int_0^\infty e^{-sA}h\,\nu_t(ds),\quad t>0,\ h\in H,$$

where $(\nu_t)_{t>0}$ is the one-sided stable semigroup of order 1 on $]0,\infty[$, i.e.,

$$\nu_t(ds)=1_{]0,\infty[}(s)\frac{1}{\sqrt{4\pi s^3}}e^{-t^2/4s}ds,\quad t>0.$$

It is well-known that the generator of $(Q_t)_{t>0}$ on H is just $\sqrt{A}$. Hence by the continuity of B we obtain that

(5.4) $\langle Bh_1,BAh_2\rangle_H=\langle h_1,h_2\rangle_{H_0}$ for all $h_1\in H_0$, $h_2\in D(A)$.

If $(H_0, \langle\, , \rangle_{H_0})$ is complete and if $V: D(A) \times D(A) \to \boldsymbol{R}$ is a positive definite bilinear form, we define

$$\mathrm{Trace}_{H_0} V := \sum_{k=1}^{\infty} V(h_k, h_k)$$

for some orthonormal basis $(h_k)_{k \in \boldsymbol{N}}$ of H_0, which belongs to $D(A)$, whenever this is independent of the special choice of $(h_k)_{k \in \boldsymbol{N}}$. In the following Du, D^2u etc. will denote the Fréchet derivatives of $u \in \mathscr{F}C_b^2(K)$. If $u = f(l_1, \cdots, l_m) \in \mathscr{F}C_b^2(K)$, by the chain rule we have for all $z, \eta, \eta' \in E$ that

$$\begin{aligned} Du(z)(\eta) &= Df(l_1(z), \cdots, l_m(z))(l_1(\eta), \cdots, l_m(\eta)) \\ &= \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(l_1(z), \cdots, l_m(z)) l_i(\eta), \end{aligned} \tag{5.5}$$

in particular, $Du(z) \in K(\subset E' \cap D(A))$, and

$$\begin{aligned} &D^2u(z)(\eta, \eta') \\ &= D^2 f(l_1(z), \cdots, l_m(z))((l_1(\eta), \cdots, l_m(\eta)), (l_1(\eta'), \cdots, l_m(\eta'))) \\ &= \sum_{i,j=1}^{m} \frac{\partial^2 f}{\partial x_i \partial x_j}(l_1(z), \cdots, l_m(z)) l_i(\eta) l_j(\eta'), \end{aligned} \tag{5.6}$$

in particular, $D^2u(\eta, \cdot) \in K(\subset E' \cap D(A))$.

REMARK 5.2. Note that since K separates the points of E and $A(K) \subset K$ we can define Az for all $z \in E$ as follows

$$l(Az) = Al(z) \quad \text{for all } l \in K.$$

Since also $e^{-tA}(K) \subset K$, $t > 0$, it follows from 3.1 (iii) that $-A$ extends the generator of the strongly continuous contraction semigroup $(e^{-tA})_{t>0}$ on E. Therefore, we use the same symbol A.

The proof of the following theorem is fairly standard (cf. e.g.[52] where, however, the much simpler case $A = \mathrm{id}_H$ was treated). Let B^* denote the adjoint of B on H.

Theorem 5.3. *Let $u = f(l_1, \cdots, l_m) \in \mathscr{F}C_b^2(K)$. Then*

$$\begin{aligned} Lu(z) &:= \lim_{t \downarrow 0} \frac{1}{t}(p_t u(z) - u(z)) \\ &= \sum_{i,j=1}^{m} \frac{\partial^2 f}{\partial x_i \partial x_j}(l_1(z), \cdots, l_m(z)) \langle l_j, l_i \rangle_{H_0} \end{aligned} \tag{5.7}$$

$$-\sum_{i=1}^{m}\frac{\partial f}{\partial x_i}(l_1(z),\cdots,l_m(z))Al_i(z)$$

$$=\mathrm{Trace}_{H_0}\, D^2u(z)(B^*BA\cdot,B^*BA\cdot)-Du(z)(Az)$$

(*cf. Remark* 5.2) *both for all* $z\in E$ and in $L^2(E;\mu)$. *Here for the third equality we assume* $(H_0,\langle\,,\rangle_{H_0})$ *to be complete. In particular, if* $l_1,\cdots,l_m\in K\subset H_0$ *form an orthonormal systme in* H_0 *then*

$$Lu(z)=\sum_{j=1}^{m}\left[\frac{\partial^2 u}{\partial(B^*BAl_j)^2}(z)-\frac{\partial u}{\partial(B^*BAl_j)}Al_j(z)\right],\quad z\in E. \tag{5.8}$$

Proof. Let $z\in E$ and $t>0$. We have by Taylor's formula that

$$p_tu(z)-u(z)=\int\left[u(z+e^{-tA}z-z+\sqrt{1-e^{-2tA}}z'))-u(z)\right]\mu(dz')$$

$$=Du(z)(e^{-tA}z-z)+\frac{1}{2}D^2u(z)(e^{-tA}z-z,e^{-tA}z-z)$$

$$+\frac{1}{2}\int D^2u(z)(\sqrt{1-e^{-2tA}}z',\sqrt{1-e^{-2tA}}z')\mu(dz') \tag{5.9}$$

$$+\int R_2(z,e^{-tA}z-z+\sqrt{1-e^{-2tA}}z')\mu(dz'),$$

where in the second step we used (5.5),(5.6) and the fact that

$$\int l(\sqrt{1-e^{-2tA}}z')\mu(dz')=0\quad\text{for all } l\in E'$$

since μ has mean zero. Since for $\eta\in E$

$$R_2(z,\eta)=\int_0^1(1-s)(D^2u(z+s\eta)-D^2u(z))(\eta,\eta)ds,$$

the modulus of the above integral of the remainder is by (5.6) and the claim in the proof of Corollary 3.6 dominated by

$$c_t(z)\sum_{i=1}^{m}\int[l_ie^{-tA}z-z+\sqrt{1-e^{-2tA}}z')]^2\mu(dz')$$

$$\leq 2c_t(z)\sum_{i=1}^{m}([l_i(e^{-tA}z-z)]^2+\|B\sqrt{1-e^{-2tA}}l_i\|_H^2)$$

where $c_t(z) \in [0, \infty[$, with $\lim_{t \downarrow 0} c_t(z) = 0$ for all $z \in E$ and $\sup_{t,z} c_t(z) < \infty$. Since $l_i \in K$ and

$$\|B\sqrt{1-e^{-2tA}}l_i\|_H^2 = \langle Bl_i, B(1-e^{-2tA})l_i \rangle_H,$$

$1 \le i \le m$, and by 3.1 (iii) we conclude that the above integral of the remainder multiplied by t^{-1} converges to zero as $t \downarrow 0$ for all $z \in E$. But this convergence then also takes place in $L^2(E;\mu)$ since for $1 \le i \le m$, $t > 0$,

$$\frac{1}{t^2}\int [l_i(e^{-tA}z - z)]^4 d\mu = \frac{3}{t^2}\|B(1-e^{-2tA})l_i\|_H^4.$$

By similar arguments we obtain that the second summand of (5.9) multiplied by t^{-1} converges to zero as $t \downarrow 0$ pointwise and in $L^2(E;\mu)$. By 3.1 (iii) resp. (5.5) and Remark 5.2 we see that

$$\lim_{t \downarrow 0} \frac{1}{t} Du(z)(e^{-tA}z - z) = -Du(z)(Az) = -\sum_{i=1}^m \frac{\partial f}{\partial x_i}(l_1(z), \cdots, l_m(z)) A l_i(z).$$

Now we consider the third summand in (5.9). Define for $1 \le i, j \le m$

$$a_{ij}(z) := \frac{\partial^2 f}{\partial x_i \partial x_j}(l_1(z), \cdots, l_m(z)), \quad z \in E.$$

Then by (5.6) and the claim in the proof of Corollary 3.6

$$\frac{1}{2t}\int D^2u(z)(\sqrt{1-e^{-2tA}}z', \sqrt{1-e^{-2tA}}z')\mu(dz')$$

$$= \frac{1}{2t}\sum_{i,j=1}^m a_{ij}(z)\int l_i(\sqrt{1-e^{-2tA}}z') l_j(\sqrt{1-e^{-2tA}}z')\mu(dz')$$

$$= \frac{1}{2t}\sum_{i,j=1}^m a_{ij}(z)\langle Bl_i, B(1-e^{-2tA})l_j \rangle_H.$$

Since $l_j \in D(A) \subset D(\sqrt{A}) = H_0$, as $t \downarrow 0$ this converges to

$$\sum_{i,j=1}^m a_{ij}(z)\langle Bl_i, BAl_j \rangle_H = \sum_{i,j=1}^m a_{ij}(z)\langle l_i, l_j \rangle_{H_0}, \tag{5.10}$$

where the second equation follows by Remark 5.1, since $l_j \in D(A)$, $1 \le j \le m$. Since the a_{ij} are bounded this convergence also takes place in $L^2(E;\mu)$. This completes the proof of the second equality in (5.7). To see the third, we may assume that

$l_1,\cdots,l_m$ are orthonormal in H_0 (otherwise we use Gram-Schmidt orthogonalization and change f accordingly). Then the quantity in (5.10) is equal to $\Sigma_{i=1}^m a_{ii}(z)$ which in turn by (5.6) and Remark 5.1 is equal to

$$\sum_{i=1}^m D^2u(B^*BAl_i,B^*BAl_i)=\mathrm{Trace}_{H_0}D^2u(B^*BA\cdot,B^*BA\cdot),$$

provided $(H_0,\langle\,,\rangle_{H_0})$ is complete. The rest of the assertion is now trivial. □

Since each p_t is μ-symmetric (cf. (3.20)), $p_t1=1$, $t>0$, and because of (3.18), $(p_t)_{t>0}$ gives rise to a strongly continuous contraction semigroup $(T_t)_{t>0}$ on $L^2(E;\mu)$ (cf. e.g.[31, Chap. II, Subsection 4.a)]). By Theorem 5.3 we know that for its generator $(L,D(L))$ we have that $\mathscr{F}C_b^\infty(K)^\sim \subset D(L)$ and L is given by (5.7) on $\mathscr{F}C_b^\infty(K)^\sim$. (Here for a set $\mathscr{G}$ of functions on E, $\mathscr{G}^\sim$ denotes the corresponding set of μ-classes.) The Dirichlet form $(\mathscr{E},D(\mathscr{E}))$ defined in (5.1), (5.2) is exactly the one which is associated with L resp. $(T)_{t>0}$ (cf e.g.[31, Diagram 3] or [22, Chap.I]).

Proposition 5.4. *L restricted to the domain $\mathscr{F}C_b^\infty(K)^\sim$ is an essentially self-adjoint operator on $L^2(E;\mu)$.*

Proof. Let $C:=$linear span of $\{\sin l, \cos l | l\in K\}$. Since K and hence C separates the points of E, by a monotone class argument $C^\sim$ is dense in $L^2(E;\mu)$. Since $C^\sim \subset D(L)$ and $p_tu\in C$ for all $u\in C$ by the claim in the proof of Corollary 3.6, it follows by [37, Theorem X.49] that L restricted to $C^\sim$ is essentially self-adjoint on $L^2(E;\mu)$. □

Now we can prove the explicit representation of $(\mathscr{E},D(\mathscr{E}))$.

Theorem 5.5. *$\mathscr{F}C_b^2(K)^\sim \subset D(\mathscr{E})$ and*

$$\mathscr{E}(u,v)=\int\langle Du(z),Dv(z)\rangle_{H_0}\mu(dz) \quad \textit{for all } u\in\mathscr{F}C_b^2(K)^\sim. \tag{5.11}$$

Moreover, $(\mathscr{E},D(\mathscr{E}))$ is the closure of $(\mathscr{E},\mathscr{F}C_b^\infty(K)^\sim)$ on $L^2(E;\mu)$ (cf. [22], [31]), hence $(\mathscr{E},D(\mathscr{E}))$ is uniquely determined by (5.11).

Proof. First note that e.g. by [8, Proposition 5.5] we have the following integration by parts formula.

$$\begin{aligned}&\int v(z)\frac{\partial u}{\partial(B^*Bl)}(z)\mu(dz)\\ &=-\int u(z)\frac{\partial v}{\partial(B^*Bl)}(z)\mu(dz)+\int u(z)v(z)l(z)\mu(dz)\end{aligned}\tag{5.12}$$

for all $l\in E'(\subset H)$ and all $u,v\in\mathscr{F}C_b^2(K)$.

Let $u=f(l_1,\cdots,l_m), v=g(l_1,\cdots,l_m)\in\mathscr{F}C_b^2(K)$. We may assume that $l_1,\cdots,l_m$ form an orthonormal system in H_0. Then by (5.5)

$$\int\langle Du(z),Dv(z)\rangle_{H_0}\mu(dz)=\sum_{i=1}^{m}\int\frac{\partial u}{\partial(B^*BAl_i)}(z)\frac{\partial v}{\partial(B^*BAl_i)}(z)\mu(dz) \tag{5.13}$$

which (since $Al_i\in E'$, $1\le i\le m$) by (5.12) and (5.8) is equal to

$$\int(-Lu)v\,d\mu.$$

This implies that the quadratic form on the right hand side of (5.11) is closable on $L^2(E;\mu)$ (cf. e.g. [31, Chap. I, Lemma 3.4]). Let $(\mathscr{E}^0,D(\mathscr{E}^0))$ denote its closure and $(L^0,D(L^0))$ its generator. Then

$$\mathscr{F}C_b^2(K)^\sim\subset D(L^0)\quad\text{and}\quad L=L^0\text{ on }\mathscr{F}C_b^2(K)^\sim.$$

Since by Proposition 5.4 $(L,\mathscr{F}C_b^\infty(K)^\sim)$ and hence $(L,\mathscr{F}C_b^2(K)^\sim)$ is essentially self-adjoint on $L^2(E;\mu)$, it follows that $D(L)\subset D(L^0)$. Consequently, $(L,D(L))=(L^0,D(L^0))$, because both are self-adjoint. This implies that $(\mathscr{E},D(\mathscr{E}))=(\mathscr{E}^0,D(\mathscr{E}^0))$ and the assertion is proved. □

EXAMPLES 5.6. (i) Let $d\in\boldsymbol{N}$ and let B be a non-degenerate real $d\times d$-matrix. Let
Let

$$\mu(dz):=(2\pi\det B^TB)^{-1/2}\exp\left[-\frac{1}{2}\|B^{-1}z\|_{\boldsymbol{R}^d}^2\right]dz.$$

Let $\Gamma:=(\gamma_{ij})_{i,j}$ be a nonnegative definite, symmetric real $d\times d$-matrix and $(\mathscr{E},D(\mathscr{E}))$ the Dirichlet form obtained by taking the closure on $L^2(\boldsymbol{R}^d;\mu)$ of

$$\mathscr{E}(u,v)=\sum_{i,j=1}^{d}\int\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\gamma_{ij}d\mu;\ u,v\in C_b^2(\boldsymbol{R}^d). \tag{5.14}$$

Let $E'=E=H=\boldsymbol{R}^d$ with the Euclidean inner product. Let $A:=(B^{-1}\Gamma^{1/2})^2)$. By Theorem 5.5 we see that the semigroup corresponding to (5.14) is given by

$$p_tf(z)=\int f(e^{-tA}z+\sqrt{1-e^{-2tA}}z')\mu(dz'),\quad t>0,\ z\in E,$$

for $\mathscr{B}(\boldsymbol{R}^d)$-measurable, bounded $f:\boldsymbol{R}^d\to\boldsymbol{R}$, provided $AB=BA$ and $A=A^T$. It is easy to see that these two equations are equivalent. They hold, of course, if B

and Γ commute.

(ii) Let $H:=L^2(\boldsymbol{R}^d;dx)$ with the usual inner product, $A=(-\Delta+1)^{1/2}$, $B=(-\Delta+1)^{-1/4}$. Let (E,q) be any Hilbert space satisfying 3.1 (i)-(iii) (e.g. the one constructed in 3.1 or in [39, 40]). The measure μ in this case is just the free time zero field in quantum field theory (cf. the references in [39, 40]). We call the corresponding Dirichlet form given by (5.11) (where $H_0=H$) the *free field-Dirichlet form*. It has been studied intensively (cf. e.g.[1,2,6,8,9]). We note that the same Dirichlet form is obtained if one starts with $H:=H^{1/2}(\boldsymbol{R}^d;dx)=$ Sobolev class of order $\frac{1}{2}$ in $L^2(\boldsymbol{R}^d;dx)$, considers $A=(-\Delta+1)^{1/2}$ as a self-adjoint operator on $H^{1/2}(\boldsymbol{R}^d;dx)$, and takes $B=\mathrm{id}_H$. This approach was taken in [44, Example 3.5]). Below we shall refer to it as the "second approach to the free field Dirichlet form".

6. Tightness of the corresponding capacities

We recall the following notion from [31]:

DEFINITION 6.1. *A Dirichlet form $(\mathscr{E},D(\mathscr{E}))$ on $L^2(E;\mu)$ (cf. [22],[31]) is called quasi-regular if*

(i) $C_{1,2}$ is tight.

(ii) There exists an $\tilde{\mathscr{E}}_1^{1/2}$-dense subset of $D(\mathscr{E})$ whose elements have $C_{1,2}$-quasi-continuous μ-versions.

(iii) There exist $u_n\in D(\mathscr{E})$, $n\in N$, having $C_{1,2}$-quasi-continuous μ-versions $\tilde{u}_n, n\in N$, and a set $N\subset E$ of $C_{1,2}$-capacity zero such that $\{\tilde{u}_n|n\in N\}$ separates the points of $E\backslash N$.

We now consider the situation as described at the beginning of the preceding section, i.e., H, A are as in Theorem 3.1, (E,q) is a Hilbert space satisfying 3.1 (i)-(iii) for some K and $(p_t)_{t>0}$, B, μ are as in Corollary 3.6. Let $C_{r,p}$, $r>0$, $p>1$, be the capacities defined as in Section 1 relative to the semigroup $(p_t)_{t>0}$. By spectral theory it is clear that

$$(D(\mathscr{E}),\mathscr{E}_1^{1/2})=(\mathscr{F}_{1,2},\|\ \|_{1,2})$$

where $\mathscr{F}_{r,p}$, $r>0$, $p>1$, are as in Section 1, the Dirichlet form $(\mathscr{E},D(\mathscr{E}))$ is the one corresponding to $(p_t)_{t>0}$ (cf. Section 5), and $\mathscr{E}_1:=\mathscr{E}+(\,,\,)_{L^2(\mu)}$. Furthermore, $C_{1,2}$ is just the (1-) capacity in [22, §3.1] resp. equal to $\mathrm{Cap}_{1,1}$ in [31, Chap. III, Section 2] associated with $(\mathscr{E},D(\mathscr{E}))$.

Proposition 6.2. *(i) $(\mathscr{E},D(\mathscr{E}))$ is quasi-regular if and only if $C_{1,2}$ is tight. In this case $(\mathscr{E},D(\mathscr{E}))$ has the local property, i.e., $\mathscr{E}(u,v)=0$ for all $u,v\in D(\mathscr{E})$ with* $\mathrm{supp}\,[|u|\cdot\mu]\cap\mathrm{supp}\,[|v|\cdot\mu]=\emptyset$.

(ii) If $(H_0,\langle\,,\,\rangle_{H_0})$ is complete, then $(\mathscr{E},D(\mathscr{E}))$ also has the local property.

Proof. (i): Since $\mathscr{F}C_b^\infty(K)$ separates the points and $\mathscr{F}C_b^\infty(K)^\sim$ is dense in $D(\mathscr{E})$ w.r.t $\mathscr{E}_1^{1,2}$, the assertion follows (cf. [31, Cahp. IV, Subsection 4b)]). Now assume that $(\mathscr{E},D(\mathscr{E}))$ is quasi-regular. Define $\Gamma:\mathscr{F}C_b^\infty(K)^\sim \times \mathscr{F}C_b^\infty(K)^\sim \to L^1(E;\mu)$ by

$$\Gamma(u,v) := \langle Du, Dv\rangle_{H_0}.$$

Then by Theorem 5.5 extends uniquely to a continuous symmetric bilinear map $\Gamma := D(\mathscr{E}) \times D(\mathscr{E}) \to L^1(E;\mu)$ (where $D(\mathscr{E})$ is equipped with the norm $\mathscr{E}_1^{1/2}$) and

$$\mathscr{E}(u,v) = \int \Gamma(u,v)d\mu \quad \text{for all } u,v \in D(\mathscr{E}).$$

Clearly, by approximation and the product rule for D we have that if $u,v,w \in D(\mathscr{E})$ are bounded, then

$$\Gamma(uw,v) = u\Gamma(w,v) + w\Gamma(u,v).$$

In particular, since $(\mathscr{E},D(\mathscr{E}))$ is quasi-regular, by [31, Chap. V, Proposition 1.7] we can take w so that $0 \le w \le 1_{E\setminus \mathrm{supp}[|u|\cdot\mu]}$ and $w > 0$ μ-a.e. on $E\setminus\mathrm{supp}[|u|\cdot\mu]$. Then

$$0 = u\Gamma(w,v) + w\Gamma(u,v)$$

and hence

$$\mathrm{supp}[|\Gamma(u,v)|\cdot\mu] \subset \mathrm{supp}[|u|\cdot\mu].$$

Similarly, one obtains that

$$\mathrm{supp}[|\Gamma(u,v)|\cdot\mu] \subset \mathrm{supp}[|v|\cdot\mu].$$

Consequently, $\Gamma(u,v)=0$, hence $\mathscr{E}(u,v)=0$, if $u,v \in D(\mathscr{E})$are bounded with $\mathrm{supp}[|u|\cdot\mu] \cap \mathrm{supp}[|v|\cdot\mu] = \emptyset$. The restriction that u,v are bounded can be removed in a standard way (cf. e.g. [31, Chap. I, Proposition 4.17]). Hence $(\mathscr{E},D(\mathscr{E}))$ has the local property.

(ii): We note that by Theorem 5.5, $D:\mathscr{F}C_b^\infty(K)^\sim \to L^2(E \to H_0;\mu)$ is a closable operator on $L^2(E;\mu)$ and that the domain of its closure (also denoted by D) is just $D(\mathscr{E})$. Let $l \in K(\subset D(A) \cap E' \subset H_0 \subset H)$ and $k := B^*Bl$. By the integration by parts formula (5.12) the form

$$\mathscr{E}_k(u,v) = \int \frac{\partial u}{\partial k}\frac{\partial v}{\partial k} d\mu;\ u,v \in \mathscr{F}C_b^\infty(K)^\sim,$$

is (well-defined and) closable (cf. e.g. [31, Chap. II, Proposition 3.4]). Its closure $(\mathscr{E}_k,D(\mathscr{E}_k))$ is a symmetric Dirichlet form (cf. [31, Chap. II, Proposition 3.5] or [8, Theorem 3.10]). The operator $\frac{\partial}{\partial k}:\mathscr{F}C_b^\infty(K)^\sim \to L^2(E;\mu)$ is hence closable on $L^2(E;\mu)$ and the domain of its closure (also denoted by $\frac{\partial}{\partial k}$) is just $D(\mathscr{E}_k)$. If $k := B^*BAl$, it follows from (5.13) by approximation that

$$D(\mathscr{E}) \subset D(\mathscr{E}_k)$$

and

$$\int \langle Du,l\rangle_{H_0}\langle Dv,l\rangle_{H_0} d\mu = \int \frac{\partial u}{\partial k}\frac{\partial v}{\partial k} d\mu \quad \text{for all } u,v\in D(\mathscr{E}).$$

Hence if $\{l_i|i\in N\} \subset K$ is an orthonormal basis of H_0 (which exists by Remark 3.2 (v)) and $k_i := B^*BAl_i$, we conclude that

$$(6.1) \quad D(\mathscr{E}) \subset \bigcap_{i=1}^{\infty} D(\mathscr{E}_{k_i})$$

and

$$(6.2) \quad \int \langle Du,Dv\rangle_{H_0} d\mu = \sum_{i=1}^{\infty} \int \frac{\partial u}{\partial k_i}\frac{\partial v}{\partial k_i} d\mu \quad \text{for all } u,v\in D(\mathscr{E}).$$

But by [31, Chap. II, Subsection 4 b) and Chap. V, Example 1.12 (ii)] $(\mathscr{E}_k, D(\mathscr{E}_k))$ is a quasi-regular Dirichlet form on $L^2(E;\mu)$ having the local property for all $k := B^*Bl$, $l\in E'$. Hence (6.1),(6.2) imply that so does $(\mathscr{E}, D(\mathscr{E}))$. □

Now we are prepared to prove the following result. We adopt the notation of Section 4.

Theorem 6.3. *The following assertions are equivalent:*

(i) Condition (4.2) *holds*

(ii) $(p_t)_{t>0}$ *is the transition function of a* (*conservative*) *diffusion process*

(iii) Condition (4.2) *holds with* Ω *resp.* X_t, $t\geq 0$, *replaced by the set* Ω' *of all right continuous paths from* $[0,\infty[$ *to* E *resp.* $X'_t :=$ *evaluation at* t *on* Ω', $t\geq 0$.

(iv) $C_{1,2}$ *is tight*

(v) $(\mathscr{E}, D(\mathscr{E}))$ *is quasi-regular.*

Proof. (iv) $\Leftrightarrow$ (v): See Proposition 6.2 (i).

(i) $\Rightarrow$ (ii): This is a consequence of Theorem 4.2.

(ii) $\Rightarrow$ (iii): Trivial.

(iii) $\Rightarrow$ (v): Since $(p_t)_{t>0}$ is Feller by Corollary 3.6, it follows by [17, Satz 5.10] that the normal Markov processs M' in Remark 4.3 (iii) is strong Markov, hence a *right process.* Clearly, M' is *associated with* $(\mathscr{E}, D(\mathscr{E}))$ (cf. [31, Chap. IV, Section 2]). Therefore, [31, Chap. IV, Theorem 6.7] implies that $(\mathscr{E}, D(\mathscr{E}))$ is quasi-regular.

(v) $\Rightarrow$ (i): By Proposition 6.2 (i) and [31, Chap. V,Theorem 1.11] (see also [5]) there exists a diffusion process $M=(\Omega,\mathscr{F},(X_t)_{t\geq 0}, (P_z)_{z\in E})$ having $(p_t)_{t>0}$ as transition semigroup. (Note that the lifetime ζ is identically equal to $+\infty$, since $p_t 1=1$, $t>0$) Hence $P:=\int P_z \mu(dz)$ satisfies (4.2). □

REMARK 6.4. Note that if $(\mathscr{E},D(\mathscr{E}))$ is quasi-regular, by [31, Chap. IV, Theorem 3.5] there always exists a right process (even a special standard process) such that for its transition semigroup $(\tilde{p}_t)_{t>0}$ we have that $\tilde{p}_t$ is a $C_{1,2}$-quasi-continuous μ-version of $T_t f$ for all $f\in L^2(E;\mu)$, $t>0$. Here $(T_t)_{t>0}$ is the strongly continuous contraction semigroup associated with $(\mathscr{E},D(\mathscr{E}))$ (cf. e.g. [31, Chap. I]). Theorem 6.3, however, implies that we can even find "better versions", namely $(p_t)_{t>0}$ given by the Mehler formula and these p_t, $t>0$, are even Feller.

Proposition 6.5. *Suppose that $(H_0,\langle\,,\rangle_{H_0})$ is complete and that the following condition holds*:

(6.3) *There exists $c\in]0,\infty[$ such that*
$\|l\|_{H_0}\le c\|l\|_{E'}$ *for all* $l\in K(\subset D(A)\cap E'\subset H_0)$.

Then $C_{1,2}$ is tight.

Proof. Let $l\in E'$ and $l_n\in K$, $n\in N$, such that $l_n\to l$ in E' as $n\to\infty$. If $l_i,k_i,i\in N$, are as in (6.1),(6–2), by (5.4),(6.3), and Fatou's Lemma we have that

$$\sum_{i=1}^{\infty} l(k_i)^2=\sum_{i=1}^{\infty}\lim_{n\to\infty} l_n(k_i)^2\le\liminf_{n\to\infty}\sum_{i=1}^{\infty}\langle l_n,l_i\rangle_{H_0}^2$$

$$=\liminf_{n\to\infty}\|l_n\|_{H_0}^2\le c^2\|l\|_{E'}^2.$$

for all $l\in E'$. Since by [7, Proposition 2.10] and Theorem 5.5, $\mathscr{F}C_b^\infty$ is $\mathscr{E}_1^{1/2}$-dense in $D(\mathscr{E})$ (cf. Section 2 above for the definition of $\mathscr{F}C_b^\infty$), the assertion now follows by the representation (6.1), (6.2) of $(\mathscr{E},D(\mathscr{E}))$ and [31, Chap. IV, Subsection 4b)]. □

EXAMPLES 6.6. (i) Clearly, (6.3) is fulfilled if $B\sqrt{A}\in\mathscr{L}^\infty(H)$. But this is not necessary. It is easily seen that in the second approach to the free field Dirichlet form where $B=\mathrm{id}_H$ and $\sqrt{A}$ is unbounded, nevertheless (6.3) holds, if e.g. E is taken as in [39,40]. In this case also $(H,\langle\,,\rangle_{H_0})$ is complete.
(ii) Let $(H,\langle\,,\rangle_H)$, $(E,\langle\,,\rangle_E)$ be separable real Hilbert spaces such that the embedding $H\subset E$ is dense and Hilbert-Schmidt. Hence we can find an orthonormal basis $\{e_n|n\in N\}$ of $(H,\langle\,,\rangle_H)$ and $\sigma_n>0$, $n\in N$, with $\Sigma_{n=1}^\infty$such that $\{e_n/\sigma_n|n\in N\}$ is an orthonormals basis of $(E,\langle\,,\rangle_E)$ and

(6.4) $$\langle e_n,h\rangle_E=\sigma_n^2\langle e_n,h\rangle_H=\sigma_n^2 e_n(h)\quad\text{for all } h\in H(\subset E),\ n\in N.$$

Clearly, $\{e_n|n\in N\}\subset E'$, if we embed E' as before (i.e., $E'\subset H'\equiv H\subset E$). Let μ

be the (standard) Gaussian measure on E with covariance given by $\langle , \rangle_H$ (i.e., $B=\mathrm{id}_H$ in our previous situation). If $z_n := \langle e_n/\sigma_n, z\rangle_E$, $z\in E$, $n\in N$, then by(6.4)

$$\int z_n^2 d\mu = \sigma_n^2 \int e_n^2 d\mu = \sigma_n^2, \quad n\in N.$$

Define a linear operator A on H as the closure of the linear operator satisfying $Ae_n := \alpha_n e_n$, $n\in N$, where α_n are fixed strictly positive numbers. Clearly, A, H, E satisfy assumptions 3.1 (i)-(iii) (with $K:=$ linear span of $\{e_n | n\in N\}$). Furthermore, if $l_n := \alpha_n^{-1/2} e_n$, $n\in N$, then obviously $\{l_n | n\in N\}$ is an orthonormal basis of $(H, \langle , \rangle_{H_0}) := (D(\sqrt{A}); \langle \sqrt{A}\cdot, \sqrt{A}\cdot\rangle_H)$. By (6.1), (6.2) we know that the corresponding Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$ is obtained as the closure of

$$\begin{aligned}\mathscr{E}(u,v) &= \sum_{n=1}^{\infty} \int \frac{\partial u}{\partial(\alpha_n^{1/2} e_n)} \frac{\partial v}{\partial(\alpha_n^{1/2} e_n)} d\mu \\ &= \sum_{n=1}^{\infty} \int (\alpha_n \sigma_n^2) \frac{\partial u}{\partial f_n} \frac{\partial v}{\partial f_n} d\mu; \ u,v \in \mathscr{F}C_b^\infty(K)^\sim\end{aligned}$$

where $f_n := e_n/\sigma_n$. By [7, Proposition 2.10] we may replace $\mathscr{F}C_b^\infty(K)$ by $\mathscr{F}C_b^\infty$ here. Consequently, we know by [43, Proposition 5.2] that the $C_{1,2}$-capacity of $(\mathscr{E}, D(\mathscr{E}))$ is *not* tight provided

$$\lim_{n\to\infty} (\alpha_n \sigma_n)^{-1} \exp[1/\sigma_n^2] = 0$$

which is e.g. the case if $\alpha_n = \sigma_n^{-2} \exp[1/\sigma_n^2]$, $n\in N$.

The following result shows that by enlarging the space E constructed in the proof of Theorem 3.1 we nevertheless can achieve tightness. More precisely, let H, A, μ be as at the beginning of this section, but with $B=\mathrm{id}_H$. (i.e., (E,H,μ) is an *abstract Wiener space*).

Theorem 6.7. *Let E be as constructed in the proof of Theorem* 3.1. *Then there exists a Banach space E_1 such that E is continuously and densely embedded into E_1, E_1 satisfies* 3.1 *(i), (ii), and the (r,p)-capacities of the Mehler semigroup corresponding to (E_1,H,A,μ) are tight for all $r>0$, $p>1$, where μ is considered as a measure on $E_1(\supset E)$.*

Proof. By [4, Corollary 1.5] (see also [3]) we only have to show that for some $s>0$,

(6.5) $e^{-sA}: E\to E$ is injective.

So, let $s>0$. If $z\in E$ with $e^{-sA}z=0$, it follows that $(e^{-sA}l)(z)=0$ for all $l\in K$. In fact, for K as in the proof of Theorem 3.1 we have that $e^{-sA}(K)=K$. Since K is dense in E' it follows that $z=0$. Hence (6.5) holds for all $s>0$. $\square$

REMARK 6.8. (i) As seen from the proof, Theorem 6.7 holds for any E satisfying 3.1 (i)-(iii) provided $e^{-sA}K=K$ for some $s>0$.
(ii) Note that the proof of [4, Corollary 1.5] also works if merely $\|e^{-tA}\|_{\mathscr{L}(E)}\leq 1$ for all $t>0$. In [4] it was assumed throughout that $\|e^{-tA}\|_{\mathscr{L}(E)}<1$ for all $t>0$.
(iii) One might ask under what conditions on A a given locally convex space E satisfies 3.1 (ii) with the continuity replaced by a measurability condition, such that the Mehler formula in Corollary 3.6 still makes sense at least μ-a.e. and the corresponding (r,p)-caparities are tight (on E). Essentially the answer was given by D.Feyel and A. de La Pradelle in [21]. But they only give sufficient conditions.

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References

[1] S. Albeverio, R. Høegh-Krohn: *Dirichlet forms and diffusion processes on rigged Hilbert spaces.* Z. Wahrsch. verw. Geb. **40**, 1–57 (1977).
[2] S. Albeverio, R. Høegh-Krohn: *Hunt processes and analytic potential theory on rigged Hilbert spaces.* Ann. Inst. Henri Poincare Sect. **B 13**, 269–291 (1977).
[3] S. Albeverio, M. Fukushima, W. Hansen, Z.M. Ma, M. Röckner: *Capacities on Wiener space: tightness and invariance.* C.R. Acad. Sci. Paris, t. **312**, Serie I, 931–935 (1991).
[4] S. Albeverio, M. Fukushima, W. Hansen, Z.M. Ma, M. Röckner: *An invariance result for capacities on Wiener space.* J. Funct. Anal. **106**, 35–49 (1992)
[5] S. Albeverio, Z.M. Ma, M. Röckner: *Local property of Dirichlet forms and diffusions on general state spaces.* Preprint (1992). Math. Ann. 296, 677–686 (1993)
[6] S. Albeverio, M. Röckner: *Classical Dirichlet forms on topological vector spaces - construction of assosiated diffusion process.* Probab. Th. Rel. Fields **83**, 405–434 (1989).
[7] S. Albeverio, M. Röckner: *On the maximality problem for Dirichlet forms on topological vector spaces.* In: Proceedings Bad Honnef Conference. June 1988, Lect. Notes Inform. Control 126, 14–31. Editors: N. Christopeit et al. Berlin: Springer 1989.
[8] S. Albeverio, M. Rökner: *Classical Dirichlet forms on topological vector spaces-chosabiliby and a Cameron-Martin formula.* J. Funct. Anal. **88**, 395–436 (1990).
[9] S. Albeverio, M. Röckner: *Stochastic differential equations in infinite dimensions: solutions via Dirichlet forms.* Probab. Th. Rel. Fields **89**, 347–886 (1991).
[10] A. Antoniadis, R. Carmona: *Eigenfunction expansions for infinite dimensional Ornstein-Uhlenbeck processes.* Probab. Th. Rel. Fields **74**, 31–54 (1987).
[11] H. Bauer: Wahrscheinlichkeitstheorie. Berlin-New York: de Gruyter 1991.
[12] Yu.M. Berezansky, Yu.G. Kondratiev: Spectral methods in infinite dimensional analysis. Kiev:

Naukova Dumka 1988. English translation: Kluwer Academic Publishers 1993.

[13] V.I. Bogachev, M. Röckner: *Capacités gaussiennes sont portées par des compacts métrisables.* C.R. Acad. Sci. Paris, t. **315**, Serie I, 197–202 (1992).

[14] C. Borell: *Gaussian Radon measures on locally convex spaces.* Math. Scand. **38**, 265–284 (1976).

[15] N. Bouleau, F. Hirsch: Dirichlet forms and analysis on Wiener space. Berlin-New York: de Gruyter 1991.

[16] N. Bourbaki: Topologie générale. Chapitres 5 à 10. Paris: Hermann 1974.

[17] E.B. Dynkin: Die Grundlagen der Theorie der Markoffschen Prozesse. Berlin-Göttingen-Heidelberg: Springer 1971.

[18] R.E. Edwards: Functional analysis. Theory and Applications, New York: Holt, Rinehart and Winston 1965.

[19] D. Feyel, A. de La Pradelle: *Capacités gaussiennes.* Ann. Inst. Fourier **41**, 49–76 (1991).

[20] D. Feyel, A. de La Pradelle: *Operateurs lineaires et espaces de Sobolev sur l'espace de Wiener.* C. R. Acad. Sci. Paris t **316**, Serie I, 227–229 (1991).

[21] D. Feyel, A. de La Pradelle: Operateurs lineaires gaussiens. Potential Analysis **3**, 89–105 (1944).

[22] M. Fukushima: Dirichlet forms and Markov processes. Amsterdam-Oxford-New York: North Holland (1980).

[23] M. Fukushima: A note on capacities in infinite dimensions. Lecture Notes Math. 1299, 80–85, Berlin: Springer 1988.

[24] M. Fukushima, H. Kaneko: *On (r,p)-capacities for general Markovian semigroups.* in: "Infinite dimensional analysis and stochastic processes". 41–47, ed. by S. Albeverio, Pitman 1985.

[25] M. Fukushima: Two topics related to Dirichlet forms: quasi everywhere convergences and additive functionals. "CIME-Lectures". Lecture Notes in Math. 1563 Berlin: Springer 1993.

[26] L. Gross: *Abstract Wiener spaces.* Proc. 5th Berkeley Symp. Math. Stat. Prob. **2**, 31–42 (1965).

[27] G. Kallianpur: *Zero-one laws for Gaussian processes.* Trans. A.M.S., **149**, 199–211 (1970).

[28] H. Kaneko: *On (r,p)-capacities for Markov processes.* Osaka J. Math. **23**, 325–336 (1986).

[29] H. Kuo: Gaussian measures in Banach spaces. Lect. Notes in Math. **463**, 1–224, Berlin-Heidelberg-New York: Springer 1975.

[30] S. Kusuoka: *Dirichlet forms and diffusion processes on Banach spaces.* J. Fac. Sci. Univ. Tokyo, sect. IA, **29**, 79–95 (1982).

[31] Z.M. Ma, M. Röckner: An introduction to the theory of (non-symmetric) Dirichlet forms. Berlin: Springer 1992.

[32] P. Malliavin: *Stochastic calculus of variation and hypoelliptic operators.* Proc. of the International Symposium on Stochastic Differential Equations Kyoto 1976, Tokyo 1978.

[33] P.A. Meyer: *Notes sur les processus d'Ornstein-Uhlenbeck.* Séminaire de Probabilités XVI. Ed.: J. Azema and M. Yor, Lect. Notes in Math. **920**, 125–133. Berlin: Springer 1982.

[34] J. Potthoff: *Littlewood-Paley theory on Gaussian spaces.* Nagoya Math. J. **109**, 47–61 (1988).

[35] J. Potthoff: *On Meyer's equivalence.* Nagoya Math. J. **111**, 99–109 (1988).

[36] M. Reed, B. Simon: Methods of modern mathematical physics I. Functional Analysis. New York-San Francisco- London: Academic Press 1972.

[37] M. Reed, B. Simon: Methods of modern mathematical physics II. Fourier Analysis, self-adjointness. New York-San Francisco-London: Academic Press 1975.

[38] M.A. Rieffel: *The Radon-Nikodym theorem for the Bochner integral.* Trans. Amer. Math. Soc. **131**, 466–487 (1968).

[39] M. Röckner: *Traces of harmonic functions and a new path space for the free quantum field.* J. Funct. Anal. **79**, 211–246 (1988).

[40] M. Röckner: *On the transition function of the infinite dimensional Ornstein-Uhlenbeck process given* by the free quantum field. In: Potential Theory. Ed. J. Kral et al., New York-London Plenum Press 1988.

[41] M. Röckner: *On the parabolic Martin boundary of the Ornstein-Uhlenbeck operator on Wiener space.* Ann. Prob. **20**, 1063–1085 (1992).

[42] M. Röckner, B. Schmuland: *Tightness of general $C_{1,p}$ capacities on Banach space.* J. Funct. Anal. **108**, 1–12 (1992).

[43] M. Röckner, B. Schmuland: *Quasi-regular Dirichlet forms: examples and counterexamples.* Canadian

J. of Math. **47**, 165–200 (1995).

[44] M. Röckner, T.S. Zhang: *Uniqueness of generalized Schrödinger operators and applications*. J. Funct. Anal. **105**, 187–231 (1992).

[45] H. Sato: *Souslin support and Fourier expansion of a gaussian Radon measure*. Lect. Notes Math. 860, 299–313. Prob. in Banach spaces Bd. III, Berlin: Springer 1981.

[46] L. Schwartz: Radon measures on arbitrary topological spaces and cylindrical measures. London: Oxford University Press 1973.

[47] I. Shigekawa: *Sobolev spaces over the Wiener space based on an Ornstein-Uhlenbeck operator*. J. Math. Kyoto Univ. **32**, 731–748 (1992).

[48] I. Shigekawa, N. Yoshida: *Littlewood-Paley-Stein inequality for a symmetric diffusion*. J. Math. Soc. Japan **44**, 251–280 (1992).

[49] H. Sugita: *Sobolev spaces of Wiener functionals and Malliavin calculus*. J. Math. Kyoto Univ. **25**, 31–48 (1985).

[50] H. Sugita: *Positive generalized Wiener functions and potential theory on abstract Wiener space*. Osaka J. Math., **25**, 665–696 (1988).

[51] B.S. Tsirelson: *Supplement to: A natural modification of a random process and its application to series of random functions and to Gaussian measures*. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), **72**, 202–211 (1977).

[52] S. Watanabe: Lectures on stochastic differential equations and Malliavin calculus. Berlin: Springer 1984.

[53] J.A. Yan: *Generalizations of Gross' and Minlos' theorems*. In: Azema, J., Meyer, P.A., Yor, M. (eds) Séminare de Probabilités. XXII(Lect. Notes Math., Vol. 1372, 395–404. Berlin-Heidelberg-New York: Springer 1989.

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