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SELF-SIMILAR SOLUTIONS TO THE MEAN CURVATURE FLOWS ON RIEMANNIAN CONE MANIFOLDS AND SPECIAL LAGRANGIANS ON TORIC CALABI–YAU CONES

AKITO FUTAKI, KOTA HATTORI and HIKARU YAMAMOTO

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Abstract

The self-similar solutions to the mean curvature flow have been defined and studied on the Euclidean space. In this paper we propose a general treatment of the self-similar solutions to the mean curvature flow on Riemannian cone manifolds. As a typical result we extend the well-known result of Huisken about the asymptotic behavior for the singularities of the mean curvature flows. We also extend results on special Lagrangian submanifolds on $\mathbb{C}^n$ to the toric Calabi–Yau cones over Sasaki–Einstein manifolds.

1. Introduction

Let $F\colon M \times [0, T) \to V$ be a smooth family of immersions of an m-dimensional manifold M into a Riemannian manifold $(V, \bar{g})$ of dimension $m + k$. F is called a mean curvature flow if it satisfies

$$\frac{\partial F}{\partial t}(p, t) = H_t(p) \quad \text{for all} \quad (p, t) \in M \times [0, T) \tag{1}$$

where H_t is the mean curvature of the immersion $F_t := F(\cdot\,, t)\colon M \to V$.

When V is the Euclidean space $\mathbb{R}^{m+k}$ there is a well-studied important class of solutions of (1), that is, self-similar solutions. They are immersions $F\colon M \to \mathbb{R}^{m+k}$ satisfying

$$H = \lambda F^{\perp} \tag{2}$$

where λ is a constant and $F^{\perp}$ denotes the normal part of the position vector F. The solution of (2) is called shrinking, stationary (or minimal) or expanding depending on whether $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$.

The purpose of this paper is to extend the definition of the self-similar solutions from the case when V is the Euclidean spaces to the case when V is a Riemannian cone manifold. Let (N, g) be an n-dimensional Riemannian manifold. We define the Riemannian cone manifold $(C(N),\bar{g})$ over (N,g) by $C(N) = N \times \mathbb{R}^+$ and $\bar{g} = dr^2 + r^2 g$

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where r is the standard coordinate of $\mathbb{R}^+$. If $F\colon M \to C(N)$ is an immersion we define the position vector $\overrightarrow{F}$ of F at $p \in M$ by

$$\overrightarrow{F}(p) = r(F(p))\frac{\partial}{\partial r} \in T_{F(p)}C(N). \tag{3}$$

Then the self-similar solution is defined as

$$H = \lambda \overrightarrow{F}^{\perp} \tag{4}$$

where λ is a constant and $\overrightarrow{F}^{\perp}$ denotes the normal part of the position vector $\overrightarrow{F}$. In this paper we propose a general treatment of the self-similar solutions to the mean curvature flows on Riemannian cone manifolds. As a typical result we extend the well-known result of Huisken about the asymptotic behavior for the singularities of the mean curvature flows. In [9] Huisken introduced the rescaling technique and the monotonicity formula for the mean curvature flow of hypersurfaces in Euclidean space. Also in [9], using the monotonicity formula, Huisken proved that if the mean curvature flow has the type I singularity then there exists a smoothly convergent subsequence of the rescaling such that its limit satisfies the self-similar solution equation. In this paper we extend those techniques and consequences to Riemannian cone manifolds and an initial date manifold. We also give a construction of self-similar solutions on Riemannian cone manifolds.

Let us recall the definition of type I singularity and its parabolic rescaling. Let M be a manifold and $(V, \bar{g})$ a Riemannian manifold. Suppose $F\colon M \times [0, T) \to V$ is a mean curvature flow with maximal time $T < \infty$ of existence of the solution. One says that F develops a singularity of Type I as $t \to T$ if there exists a constant $C > 0$ such that

$$\sup_M |\mathrm{II}_t|^2 \leq \frac{C}{T-t} \quad \text{for all} \quad t \in [0, T),$$

where II_t is the second fundamental form with respect to the immersion $F_t\colon M \to V$. Otherwise one says that F develops a singularity of Type II.

Let M be a manifold and $(C(N), \bar{g})$ the Riemannian cone manifold over a Riemannian manifold (N, g). Take a constant $\lambda > 0$. For a map $F\colon M \times [0, T) \to C(N)$, we define the parabolic rescaling of F of scale λ as follows;

$$F^{\lambda}\colon M \times [-\lambda^2 T, 0) \to C(N);$$
$$F^{\lambda}(p, s) = \left(\pi_N\left(F\left(p, T + \frac{s}{\lambda^2}\right)\right), \lambda r\left(F\left(p, T + \frac{s}{\lambda^2}\right)\right)\right)$$

where $\pi_N\colon C(N) = N \times \mathbb{R}^+ \to N$ is the standard projection.

When the singularity does not occur at the apex of the cone one can show that the parabolic rescaling of type I singularity gives rise to a self-similar solution as shown

by Huisken. However when the singularity occurs at the apex we need some more conditions. Thus we are lead to the following definition of type I_c singularity.

DEFINITION 1.1. Let M be a manifold and (N, g) a Riemannian manifold. Suppose $F\colon M \times [0, T) \to C(N)$ is a mean curvature flow with $T < \infty$. We say that F develops a singularity of type I_c if the following three conditions are satisfied:
(a) F develops a singularity of type I as $t \to T$,
(b) $r(F_t(p)) \to 0$ for some $p \in M$ as $t \to T$ and
(c) $K_1(T - t) \leq \min_M r^2(F_t) \leq K_2(T - t)$ for all $t \in [0, T)$ where K_1 and K_2 are positive constants.

Examples of type I_c singularities are given in Example 6.1.

Theorem 1.2. *Let M be an m-dimensional compact manifold and $C(N)$ the Riemannian cone manifold over an n-dimensional Riemannian manifold (N, g). Let $F\colon M \times [0, T) \to C(N)$ be a mean curvature flow, and assume that F develops a type I_c singularity at T. Then, for any increasing sequence $\{\lambda_i\}_{i=1}^{\infty}$ of the scales of parabolic rescaling such that $\lambda_i \to \infty$ as $i \to \infty$, there exist a subsequence $\{\lambda_{i_k}\}_{k=1}^{\infty}$ and a sequence $t_{i_k} \to T$ such that the sequence of rescaled mean curvature flow $\{F^{\lambda_{i_k}}_{s_{i_k}}\}_{k=1}^{\infty}$ with $s_{i_k} = \lambda_{i_k}^2(t_{i_k} - T)$ converges to a self-similar solution $F^{\infty}\colon M_{\infty} \to C(N)$ to the mean curvature flow.*

The proof of this theorem is not substantially different from Huisken's original proof. But the merit of the idea to study on cones will be that we obtain examples of more non-trivial topology. In fact $N \cong \{r = 1\}$ in $C(N)$ is already a self-shrinker. Thus, any compact manifold can be a self-shrinker in some Riemannian cone manifold. It is also possible to study special Lagrangian submanifolds and Lagrangian self-similar solutions in Calabi–Yau cones over Sasaki–Einstein manifolds. A Sasaki manifold N is by definition an odd dimensional Riemannian manifold whose cone $C(N)$ is a Kähler manifold. If the Kähler cone manifold is toric then the Sasaki manifold is said to be toric. It is proven in [4] and [2] that a Sasaki–Einstein metric exists on a toric Sasaki manifold obtained from a toric diagram. A typical example is when N is the standard sphere of real dimension $2m + 1$. Then its cone is $\mathbb{C}^{m+1} - \{\mathbf{o}\}$. It is natural to expect that we can extend results on special Lagrangian submanifolds or self-similar solutions on $\mathbb{C}^{m+1}$ to these toric Calabi–Yau cones of height 1. In Theorem 7.5 we construct examples of complete special Lagrangian manifolds on toric Calabi–Yau cones using the ideas of [6] and [11]. This construction includes the examples given in Theorem 3.1 in III.3 of Harvey–Lawson [7]. Further construction of examples of special Lagrangians and Lagrangian self-similar solutions are given in the third author's subsequent paper [15], in which it is shown that, for any positive integer g, there are toric Calabi–Yau 3-dimensional cones including Lagrangian self-shrinkers diffeomorphic to $\Sigma_g \times S^1$ where Σ_g is a compact orientable surface of genus g.

In Section 8 we also study the infinitesimal deformations of special Lagrangian cone $C(\Sigma) \subset C(N)$ over a Legendrian submanifold Σ in a Sasaki–Einstein manifold N. We show that the parameter space $\mathcal{H}_{C(\Sigma)}$ of those infinitesimal deformations is isomorphic to

$$\mathrm{Ker}(\Delta_\Sigma - 2n) = \{\varphi \in C^\infty(\Sigma);\ \Delta_\Sigma \varphi = 2n\varphi\},$$

see Theorem 8.6. This is also proved by Lemma 3.1 of [14], although the proof in this paper is different from [14].

This paper is organized as follows. In Section 2 we show fundamental formulas on mean curvature flows in Riemannian cone manifolds. In Section 3 we show the finite time blowup of the mean curvature from a compact manifold (Theorem 3.1). Section 4 is devoted to the proof of the monotonicity formula (Theorem 4.1). In Section 5 we see that the type I singularity is preserved under parabolic rescaling. In Section 6 we see that we obtain a self-similar solution by parabolic rescaling at a type I_c singularity. In Section 7 we construct special Lagrangians in toric Calabi–Yau cones. In Section 8 we study the infinitesimal deformations of special Lagrangian cones in Calabi–Yau cones.

2. Self-similar solutions to the mean curvature flows on Riemannian cone manifolds

Let $F\colon M \to V$ be an immersion of an m-dimensional manifold M into an $m+k$-dimensional Riemannian manifold $(V, \bar{g})$. Thus the differential $F_{*x}\colon T_xM \to T_{F(x)}V$ is injective for every $x \in M$, and we have a natural orthogonal decomposition of the vector bundle

$$F^*(TV) \cong TM \oplus T^\perp M$$

where $T^\perp M \to M$ is the normal bundle. Denote by $\perp$ (resp. $\top$) the projection $\perp\colon F^*(TV) \to T^\perp M$ (resp. $\top\colon F^*(TV) \to TM$). The second fundamental form II of the immersion $F\colon M \to V$ is a section of the vector bundle $T^\perp M \otimes (\otimes^2 T^*M)$ defined by $\mathrm{II}(X, Y) = (\bar{\nabla}_{F_*(X)} F_*(Y))^\perp$ for $X, Y \in \Gamma(TM)$. Here $\bar{\nabla}$ is the Levi-Civita connection of $(V, \bar{g})$. The mean curvature vector field H of $F\colon M \to V$ is a section of $T^\perp M$ defined by $H = \mathrm{tr}\,\mathrm{II}$, where the trace is taken with respect to the Riemannian metric $g := F^*(\bar{g})$ on M.

For the actual computations one often needs local expressions of the mean curvature vector. Let $x^1, \ldots, x^m$ and $y^1, \ldots, y^n$ be local coordinate charts around $p \in U \subset M$ and $F(p) \in U' \subset V$ such that $F|_U\colon U \to U'$ is an embedding. Write $F^\alpha(x^1, \ldots, x^m) = y^\alpha(F(x^1, \ldots, x^m))$. Then we have the induced metric

$$g_{ij} = \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j} \bar{g}_{\alpha\beta},$$

where $\bar{g} = \bar{g}_{\alpha\beta}\, dy^\alpha \otimes dy^\beta$ is the Riemannian metric on $U' \subset V$. Here we use the indices $i, j, k, \ldots$ to denote the coordinates on M and $\alpha, \beta, \gamma, \ldots$ to denote the coordinates

on V. The coefficients H^α of the mean curvature vector field

$$H = H^\alpha \frac{\partial}{\partial y^\alpha}$$

are given by the Gauß' formula

$$H^\alpha = g^{ij}\left(\frac{\partial^2 F^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial F^\alpha}{\partial x^k} + \bar{\Gamma}^\alpha_{\beta\gamma}\frac{\partial F^\beta}{\partial x^i}\frac{\partial F^\gamma}{\partial x^j}\right). \tag{5}$$

Next we consider a smooth family of immersions $F\colon M\times(a,b)\to V$. Namely, for every time t in $(a,b)\subset\mathbb{R}$, $F_t\colon M\to V$ given by $p\mapsto F(p,t)$ is an immersion. We denote by g_t the Riemannian metric $F_t^*(\bar{g})$ over M. For a fixed time t_0 in (a,b), the variation vector field $(\partial F/\partial t)(\cdot,t_0)$, considered as a section of $F_{t_0}^*TV$, is decomposed as

$$\frac{\partial F}{\partial t}(\,\cdot\,,t_0) = v_{t_0}^\perp + v_{t_0}^\top$$

where $v_{t_0}^\perp(p)$ and $v_{t_0}^\top$ are respectively the sections of $T^\perp M$ and TM.

We denote by ∇^t, div_t, II_t and H_t respectively the Levi-Civita connection on (M,g_t), the divergence with respect to g_t, the second fundamental form and the mean curvature vector field of the immersion $F_t\colon M\to V$.

Then following proposition is well-known as the "first variation formula".

Proposition 2.1. *For every p in M, two tangent vectors X, Y at p and a compactly supported integrable function f on M, we have*

$$\left.\frac{d}{dt}\right|_{t=t_0} g_t(X,Y) = g_{t_0}(\nabla^{t_0}_X v^\top_{t_0}, Y) + g_{t_0}(X, \nabla^{t_0}_Y v^\top_{t_0}) - 2\bar{g}(\mathrm{II}_{t_0}(X,Y), v^\perp_{t_0}(p)),$$

$$\left.\frac{d}{dt}\right|_{t=t_0}\int_M f\, dv_{g_t} = \int_M f(\mathrm{div}_{t_0}(v^\top_{t_0}) - \bar{g}(H_{t_0}, v^\perp_{t_0}))\, dv_{g_{t_0}}.$$

Let $F\colon M\times[0,T)\to V$ be evolving by mean curvature flow with initial condition $F_0\colon M\to V$:

$$\begin{aligned}&\frac{\partial F}{\partial t}(p,t) = H_t(p) \quad \text{for all} \quad (p,t)\in M\times[0,T),\\ &F(p,0) = F_0(p) \quad \text{for all} \quad p\in M.\end{aligned} \tag{6}$$

Applying the first variation formula in Proposition 2.1 to the mean curvature flows, we obtain following well-known properties for mean curvature flows.

Proposition 2.2. *If $F\colon M\times[0,T)\to V$ is a mean curvature flow then the following equation holds.*

$$\left.\frac{d}{dt}\right|_{t=t_0}\sqrt{\det((g_t)_{ij})} = -|H_{t_0}|_g^2\sqrt{\det((g_{t_0})_{ij})}. \tag{7}$$

If M is compact we also have

$$\left.\frac{d}{dt}\right|_{t=t_0}\mathrm{Vol}_{g_t}(M) = -\int_M |H_{t_0}|_{\bar g}^2\, dv_{g_{t_0}}.$$

Proof. Because we consider the mean curvature flow, $v_{t_0} = H_{t_0}$ and therefore

$$v_{t_0}^{\top} = 0$$

and

$$v_{t_0}^{\perp}(p) = H_{t_0}(p).$$

It then follows from Proposition 2.1 that

$$\left.\frac{d}{dt}\right|_{t=t_0}(g_t)_{ij} = -2\bar g((\mathrm{II}_{t_0})_{ij}, H_{t_0}).$$

Then the first formula (7) follows from the well-known formula for the derivative of the determinant. To prove second formula, simply let $f\equiv 1$ on M in the first variation formula. □

Recall that, for an n-dimensional Riemannian manifold (N, g), we define the Riemannian cone manifold $(C(N),\bar g)$ over (N,g) by $C(N) = N\times\mathbb{R}^+$ and $\bar g = dr^2 + r^2 g$ where r is the standard coordinate of $\mathbb{R}^+$. Note that $C(N)$ does not contain the apex.

The most typical example of a cone is the case when N is the standard sphere S^n in $\mathbb{R}^{n+1}$. In this case the cone is $\mathbb{R}^{n+1} - \{\mathbf{o}\}$. For a map $F\colon M\to\mathbb{R}^{n+1}$, one can consider the position vector of $F(p)$ for $p\in M$, and using it, one can define self-similar solutions

$$H = \lambda F^{\perp}$$

where λ is a constant.

We can extend this idea to maps into Riemannian cone manifolds. Namely, for a smooth map $F\colon M\to C(N)$ and p in M, we define the position vector $\overrightarrow{F}$ of F at $p\in M$ by

$$\overrightarrow{F}(p) = r(F(p))\frac{\partial}{\partial r}\in T_{F(p)}C(N).$$

With respect to the bundle decomposition of

$$F^*T_{F(p)}C(N) \cong T_pM \oplus T_p^{\perp}M,$$

we decompose $\overrightarrow{F}(p)$ as

$$\overrightarrow{F}(p) = \overrightarrow{F}^{\top}(p) + \overrightarrow{F}^{\perp}(p).$$

Then we can define self-similar solutions by

$$H = \lambda \overrightarrow{F}^{\perp}.$$

For a Riemannian cone manifold $(C(N),\bar{g})$ over an n-dimensional Riemannian manifold (N, g) and a point q in $C(N)$, local coordinates $(y^{\alpha})_{\alpha=1}^{n+1}$ around q are said to be *associated with normal local coordinates* of N when the part of coordinate $(y^{\alpha})_{\alpha=1}^{n}$ becomes normal local coordinates of (N, g) around $\pi_N(q)$ and y^{n+1} is the standard coordinate of $\mathbb{R}^+$, that is, $y^{n+1} = r$. Here, π_N is the projection of the cone manifold $C(N) \cong N \times \mathbb{R}^+$ onto the first factor N.

Note that under local coordinates associated with normal local coordinates of N, we have $r \circ F = r(F) = F^{n+1}$ for a given map $F\colon M \to C(N)$.

Let $(x^i)_{i=1}^m$ be normal local coordinates centered at p of the Riemannian manifold $(M, F^*(\bar{g}))$, and $(y^{\alpha})_{\alpha=1}^{n+1}$ local coordinates of $(C(N), \bar{g})$ associated with normal local coordinates centered at $\pi_N(F(p))$ of (N,g). Then calculating only $(n+1)$-th coefficient $H^{n+1}(p)$ of mean curvature vector at p, namely, the coefficient of $\partial/\partial y^{n+1}(= \partial/\partial r)$, for the local expression of the mean curvature vector (5), we obtain the following local expression for $H^{n+1}(p)$;

$$H^{n+1}(p) = \sum_{i=1}^{m} \frac{\partial^2 r(F)}{\partial {x^i}^2}(p) - r(F(p)) \sum_{i=1}^{m} \sum_{\alpha=1}^{n} \left(\frac{\partial F^{\alpha}}{\partial x^i}(p) \right)^2. \tag{8}$$

This easily follows from

$$\bar{\Gamma}_{\alpha\beta}^{n+1} = -r\bar{g}_{\alpha\beta}$$

for $1 \leq \alpha, \beta \leq n$.

3. Finite time singularity for mean curvature flows

If the ambient space is the Euclidean space $\mathbb{R}^{m+k}$ and an initial date manifold M is compact, then the mean curvature flow does not have a long time solution. It is a well-known result of Huisken:

Theorem 3.1 (Huisken [9]). *Let $F_0\colon M \to \mathbb{R}^{m+k}$ be an immersion of a compact m-dimensional manifold M. Then the maximal time T of existence of a solution $F\colon M \times [0, T) \to \mathbb{R}^{m+k}$ of the mean curvature flow with initial immersion F_0 is finite.*

The proof follows by applying the parabolic maximum principle to the function $f = |F|^2 + 2mt$ which satisfies the evolution equation $(d/dt)f = \Delta f$. One can show $T \leq (1/(2m)) \max|F_0|^2$, from which Theorem 3.1 follows. Using the position vector in a cone as defined in (3), we can extend this result when the ambient space is a Riemannian cone manifold as follows.

Theorem 3.2. *Let $(C(N),\bar{g})$ be the Riemannian cone manifold over a Riemannian manifold (N, g) of dimension n, M a compact manifold of dimension m and $F\colon M \times [0, T) \to C(N)$ a mean curvature flow with initial condition $F_0\colon M \to C(N)$. Then the maximal time T of existence of the mean curvature flow is finite.*

Before the proof of this theorem, we want to prepare some lemmas.

Lemma 3.3. *Let $(C(N), \bar{g})$ be a Riemannian cone manifold over a Riemannian manifold (N, g) of dimension n and $F\colon M \to C(N)$ an immersion of a manifold M of dimension m. Then the following equation holds.*

$$\Delta(r^2(F)) = 2(\bar{g}(H, \overrightarrow{F}) + m),$$

where Δ is the Laplacian on $(M, F^*(\bar{g}))$.

Proof. Fix a point p in M. We take normal local coordinates $(x^i)_{i=1}^m$ of $(M, F^*(\bar{g}))$ centered at p and local coordinates $(y^\alpha)_{\alpha=1}^{n+1}$ of $(C(N), \bar{g})$ associated with normal local coordinates of (N, g) centered at $\pi_N(F(p))$. Note that under these coordinates, $y^{n+1} = r$ and $F^{n+1} = r \circ F = r(F)$. First of all, by the local expression of $H^{n+1}(p)$ in (8), we have the following equalities;

$$\begin{aligned} \bar{g}(H(p), \overrightarrow{F}(p)) &= H^{n+1}(p) r(F(p)) \\ &= r(F(p)) \sum_{i=1}^{m} \frac{\partial^2 r(F)}{\partial {x^i}^2}(p) - r(F(p))^2 \sum_{i=1}^{m} \sum_{\alpha=1}^{n} \left(\frac{\partial F^\alpha}{\partial x^i}(p) \right)^2. \end{aligned} \tag{9}$$

Since $(F^*\bar{g})(\partial/\partial x^i, \partial/\partial x^i) = 1$ at p, we have

$$\begin{aligned} m &= \sum_{i=1}^{m} (F^*\bar{g}) \left(\frac{\partial}{\partial x^i}(p), \frac{\partial}{\partial x^i}(p) \right) \\ &= r(F(p))^2 \sum_{i=1}^{m} \sum_{\alpha=1}^{n} \left(\frac{\partial F^\alpha}{\partial x^i}(p) \right)^2 + \sum_{i=1}^{m} \left(\frac{\partial r(F)}{\partial x^i}(p) \right)^2. \end{aligned} \tag{10}$$

Adding above two equations (9) and (10), we have

$$\bar{g}(H(p), \overrightarrow{F}(p)) + m = r(F(p)) \sum_{i=1}^{m} \frac{\partial^2 r(F)}{\partial {x^i}^2}(p) + \sum_{i=1}^{m} \left(\frac{\partial r(F)}{\partial x^i}(p) \right)^2. \tag{11}$$

Since we took $(x^i)_{i=1}^m$ as normal local coordinates of $(M, F^*(\bar{g}))$ centered at p, the Laplacian Δ is $\sum_{i=1}^m (\partial/\partial x^i)^2$, and thus we have at p

$$\begin{aligned}\Delta r^2(F) &= \sum_{i=1}^m \frac{\partial^2 r^2(F)}{\partial {x^i}^2} \\ &= 2\left(r(F) \sum_{i=1}^m \frac{\partial^2 r(F)}{\partial {x^i}^2} + \sum_{i=1}^m \left(\frac{\partial r(F)}{\partial x^i} \right)^2 \right).\end{aligned} \tag{12}$$

Thus from (11) and (12) we have shown that $\Delta r^2(F) = 2(\bar{g}(H, \overrightarrow{F}) + m)$. □

Lemma 3.4. *Let $(C(N), \bar{g})$ be a Riemannian cone manifold over an n-dimensional Riemannian manifold (N, g), M an m-dimensional manifold and $F\colon M \times [0, T) \to C(N)$ be a mean curvature flow with initial condition $F_0\colon M \to C(N)$. Then for any fixed time t in $[0, T)$ the following equality holds*;

$$2\bar{g}(H_t, \overrightarrow{F}_t) = \frac{\partial}{\partial t} r^2(F_t). \tag{13}$$

Proof. Fix a point p in M. Take local coordinates $(y^\alpha)_{\alpha=1}^{n+1}$ of $C(N)$ associated with normal local coordinates of N. Note that under these coordinates, $y^{n+1} = r$ and $F_t^{n+1} = r(F_t)$. Since F satisfies the mean curvature flow condition (6), the following equalities hold;

$$\begin{aligned}\bar{g}(H_t(p), \overrightarrow{F}_t(p)) &= \bar{g}\left(\frac{\partial F}{\partial t}(p, t), \overrightarrow{F}_t(p) \right) \\ &= r(F_t(p)) \frac{\partial}{\partial t} r(F_t(p)) = \frac{1}{2} \frac{\partial}{\partial t} r^2(F_t(p)),\end{aligned}$$

from which (13) follows. □

Now we are in a position to prove Theorem 3.2.

Proof of Theorem 3.2. Let $f\colon M \times [0, T) \to \mathbb{R}$ be a function defined by

$$f(p, t) = r^2(F_t(p)) + 2mt.$$

For a fixed time t in $[0, T)$, by Lemma 3.3 and Lemma 3.4,

$$\begin{aligned}\frac{\partial f}{\partial t} &= 2\bar{g}(H_t, \overrightarrow{F}_t) + 2m \\ &= \Delta_t r^2(F_t) = \Delta_t f(\cdot, t)\end{aligned}$$

where Δ_t is the Laplacian with respect to the metric ${F_t}^*(\bar{g})$ on M. Since M is compact, there is a maximum of $f(\cdot, 0)$ $(= r^2(F_0))$ on M, which we denote by C_0. By

applying the maximum principle to the function f, it follows that $f(p,t) = r^2(F_t(p)) + 2mt \leq C_0$ on $M \times [0, T)$. Therefore we obtain the following inequalities;

$$t \leq \frac{C_0 - r^2(F_t(p))}{2m} \leq \frac{C_0}{2m}$$

for all t in $[0, T)$. This means that the maximal time T is finite. □

4. Monotonicity formula

Next we turn to the monotonicity formula. For a fixed time T in $\mathbb{R}$, we define the backward heat kernel $\rho_T : \mathbb{R} \times (-\infty, T) \to \mathbb{R}$ as follows;

$$\rho_T(y, t) = \frac{1}{(4\pi(T-t))^{m/2}} \exp\left(-\frac{y^2}{4(T-t)}\right).$$

To simplify the notations, we use following abbreviation;

$$\int_{M_t} \rho_T := \int_M \rho_T(r(F_t(p)), t)\, dv_{g_t},$$
$$\int_{M_t} \rho_T \left| \frac{\overrightarrow{F}^{\perp}}{2(T-t)} + H \right|_{\bar{g}}^2 := \int_M \rho_T(r(F_t(p)), t) \left| \frac{\overrightarrow{F}_t^{\perp}(p)}{2(T-t)} + H_t(p) \right|_{\bar{g}}^2 dv_{g_t}.$$

Then Huisken's monotonicity formula for a cone is the following.

Theorem 4.1 (Monotonicity formula)**.** *Let M be a compact m-dimensional manifold without boundary, $(C(N), \bar{g})$ the Riemannian cone manifold over an n-dimensional Riemannian manifold (N, g) and $F : M \times [0, T) \to C(N)$ the mean curvature flow with initial condition $F_0 : M \to C(N)$. Then the following equation holds*;

$$\frac{d}{dt} \int_{M_t} \rho_T = -\int_{M_t} \rho_T \left| \frac{\overrightarrow{F}^{\perp}}{2(T-t)} + H \right|_{\bar{g}}^2. \tag{14}$$

Proof. First we calculate the left term of (14) using (7).

$$\begin{aligned} &\frac{d}{dt} \int_M \rho_T(r(F_t(p)), t)\, dv_{g_t} \\ &= \frac{d}{dt} \int_M \frac{1}{(4\pi(T-t))^{m/2}} \exp\left(-\frac{r^2(F_t(p))}{4(T-t)}\right) \sqrt{\det(g_{t,ij})}\, dx^1 \wedge \cdots \wedge dx^m \\ &= \int_M \rho_T(r(F_t(p)), t) \left(\frac{m}{2(T-t)} - \frac{r^2(F_t(p))}{4(T-t)^2} \right. \\ &\qquad \left. - \frac{r(F_t(p))((\partial/\partial t) r(F_t(p)))}{2(T-t_0)} - |H_t(p)|_{\bar{g}}^2 \right) dv_{g_t}. \end{aligned} \tag{15}$$

It is clear that

$$
\begin{aligned}
|\overrightarrow{F}_t(p)|^2_{\bar{g}} &= \bar{g}\left(r(F_t(p))\frac{\partial}{\partial r}, r(F_t(p))\frac{\partial}{\partial r}\right) \\
&= r^2(F_t(p)).
\end{aligned} \tag{16}
$$

Substituting (13) and (16) in (15), we have following formula;

$$
\begin{aligned}
&\frac{d}{dt}\int_M \rho_T(r(F_t(p)), t)\, dv_{g_t} \\
&= \int_M \rho_T(r(F_t(p)), t)\left(\frac{m}{2(T-t)} - \frac{|\overrightarrow{F}_t(p)|^2_{\bar{g}}}{4(T-t)^2}\right. \\
&\qquad\qquad \left. - \frac{\bar{g}(H_t(p), \overrightarrow{F}_t(p))}{2(T-t)} - |H_t(p)|^2_{\bar{g}}\right) dv_{g_t}.
\end{aligned} \tag{17}
$$

Let t and p be fixed. We take normal local coordinates $(x^i)_{i=1}^m$ centered at p with respect to the Riemannian metric g_t $(= F_t^*(\bar{g}))$ and local coordinates $(y^\alpha)_{\alpha=1}^{n+1}$ around $F_t(p)$ associated with normal local coordinates of (N, g). Under these coordinates, the Laplacian Δ_t with respect to g_t is $\partial^2/\partial x^{1^2} + \cdots + \partial^2/\partial x^{m^2}$ at p. Under these coordinates we have following equations at the fixed t and p;

$$
\begin{aligned}
\Delta_t \rho_T(r(F_t), t) &= \sum_{i=1}^m \frac{\partial^2}{\partial x^{i^2}}\bigg|_{x=p} \rho_T(r(F_t), t) \\
&= \sum_{i=1}^m \frac{\partial}{\partial x^i}\bigg|_{x=p}\left(\frac{\partial}{\partial x^i}\rho_T(r(F_t), t)\right) \\
&= \sum_{i=1}^m \frac{\partial}{\partial x^i}\bigg|_{x=p}\left(\rho_T(r(F_t), t)\left(-\frac{r(F_t)((\partial/\partial x^i)r(F_t))}{2(T-t)}\right)\right) \\
&= \rho_T(r(F_t), t)\left(\frac{r^2(F_t)((\partial/\partial x^i)r(F_t))^2}{4(T-t)^2}\right. \\
&\qquad\qquad \left. - \frac{((\partial/\partial x^i)r(F_t))^2}{2(T-t)} - \frac{r(F_t)((\partial^2/\partial x^{i^2})r(F_t))}{2(T-t)}\right).
\end{aligned} \tag{18}
$$

Furthermore we want to express $\overrightarrow{F}_t^{\top}(p)$ under these coordinates. Now by our choice of the local coordinates of $(x^i)_{i=1}^m$, it is clear that

$$
\bar{g}\left(F_{t*}(p)\left(\frac{\partial}{\partial x^i}\right), F_{t*}(p)\left(\frac{\partial}{\partial x^j}\right)\right) = \delta_{ij}. \tag{19}
$$

Note that $y^{n+1} = r$ and $F_t^{n+1} = r(F_t)$. The following equalities hold;

$$\begin{aligned}
\overrightarrow{F}_t^{\top}(p) &= \sum_{i=1}^{m} \bar{g}\left(\overrightarrow{F}_t^{\top}(p),\, F_{t*}(p)\left(\frac{\partial}{\partial x^i}\right)\right) F_{t*}(p)\left(\frac{\partial}{\partial x^i}\right) \\
&= \sum_{i=1}^{m} \bar{g}\left(\overrightarrow{F}_t(p),\, F_{t*}(p)\left(\frac{\partial}{\partial x^i}\right)\right) F_{t*}(p)\left(\frac{\partial}{\partial x^i}\right) \\
&= \sum_{i=1}^{m} \bar{g}\left(r(F_t(p))\frac{\partial}{\partial r},\, \sum_{\alpha=1}^{n+1} \frac{\partial F_t^{\alpha}(p)}{\partial x^i}\frac{\partial}{\partial y^{\alpha}}\right) F_{t*}(p)\left(\frac{\partial}{\partial x^i}\right) \\
&= r(F_t(p)) \sum_{i=1}^{m} \frac{\partial r(F_t(p))}{\partial x^i} F_{t*}(p)\left(\frac{\partial}{\partial x^i}\right).
\end{aligned} \tag{20}$$

Using (19) and (20), we can express the norm of $\overrightarrow{F}_t^{\top}(p)$ as follows;

$$\begin{aligned}
|\overrightarrow{F}_t^{\top}(p)|_{\bar{g}}^2 &= \bar{g}\left(\overrightarrow{F}_t^{\top}(p),\, \overrightarrow{F}_t^{\top}(p)\right) \\
&= r^2(F_t(p)) \sum_{i=1}^{m} \left(\frac{\partial r(F_t)}{\partial x^i}(p)\right)^2.
\end{aligned} \tag{21}$$

Applying (11) for F_t and using (18) and (21), we have the following equality;

$$\begin{aligned}
&\Delta_t \rho_T(r(F_t(p)), t) \\
&= \rho_T(r(F_t(p)), t)\left(\frac{|\overrightarrow{F}_t^{\top}(p)|_{\bar{g}}^2}{4(T-t)^2} - \frac{m}{2(T-t)} - \frac{\bar{g}(H_t(p),\, \overrightarrow{F}_t(p))}{2(T-t)}\right).
\end{aligned} \tag{22}$$

In this equation (22) there are no local coordinates x^i, so we have proven this equation (22) for all p in M globally. The equation (22) is equivalent to

$$\begin{aligned}
&\rho_T(r(F_t(p)), t)\frac{m}{2(T-t)} \\
&= -\Delta_t \rho_T(r(F_t(p)), t) \\
&\quad + \rho_T(r(F_t(p)), t)\left(\frac{|\overline{F_t}^{\top}(p)|_{\bar{g}}^2}{4(T-t)^2} - \frac{\bar{g}(H_t(p),\, \overline{F_t}(p))}{2(T-t)}\right).
\end{aligned} \tag{23}$$

Substituting (23) in (17), we have following equalities;

$$
\begin{aligned}
&\frac{d}{dt}\int_M \rho_T(r(F_t(p)),t)\,dv_{g_t}\\
&=-\int_M \Delta_t\rho_T(r(F_t(p)),t)\,dv_{g_t}\\
&\quad+\int_M \rho_T(r(F_t(p),t))\Bigg(\frac{|\overrightarrow{F}_t^{\top}(p)|^2_{\bar g}}{4(T-t)^2}-\frac{|\overrightarrow{F}_t(p)|^2_{\bar g}}{4(T-t)^2}\\
&\qquad\qquad -2\times\frac{\bar g(H_t(p),\overrightarrow{F}_t(p))}{2(T-t)}-|H_{t_0}(p)|^2_{\bar g}\Bigg)\,dv_{g_t} \qquad (24)\\
&=\int_M \rho_T(r(F_t(p)),t)\Bigg(-\frac{|\overrightarrow{F}_t^{\perp}(p)|^2_{\bar g}}{4(T-t)^2}\\
&\qquad\qquad -2\times\frac{\bar g(H_t(p),\overrightarrow{F}_t^{\perp}(p))}{2(T-t)}-|H_t(p)|^2_{\bar g}\Bigg)\,dv_{g_t}\\
&=-\int_M \rho_T(r(F_t(p)),t)\left|\frac{\overrightarrow{F}_t^{\perp}(p)}{2(T-t)}+H_t(p)\right|^2_{\bar g}\,dv_{g_t}.
\end{aligned}
$$

This completes the proof of Theorem 4.1. □

5. Singularities and the parabolic rescaling

In this section we see that the property that a mean curvature flow develops type I singularities is preserved under parabolic rescaling.

Proposition 5.1. *Let M be an m-dimensional manifold and $(C(N),\bar g)$ the Riemannian cone manifold over an n-dimensional Riemannian manifold (N,g). If a map $F\colon M\times[0,T)\to C(N)$ is a mean curvature flow, then the parabolic rescaling of F of scale λ is also the mean curvature flow.*

Proof. Fix (p_0,s_0) in $M\times[-\lambda^2T,0)$. Let $t=T+s/\lambda^2$ and $t_0=T+s_0/\lambda^2$. Let $(x^i)_{i=1}^m$ be local coordinates of M around p_0. Let $(y^\alpha)_{\alpha=1}^{n+1}$ be local coordinates of $C(N)$ around $F^\lambda(p_0,s_0)$ associated with local coordinates N. Put

$$(g^\lambda_{s_0})_{ij}=(F^{\lambda *}_{s_0}\bar g)\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)$$

and

$$(g_{t_0})_{ij}=(F^{*}_{t_0}\bar g)\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right).$$

Then one can easily show that

$$(g^{\lambda}_{s_0})_{ij} = \lambda^2 (g_{t_0})_{ij}. \tag{25}$$

Thus the Christoffel symbols $(\Gamma^{\lambda}_{s_0})^i_{jk}$ with respect to $g^{\lambda}_{s_0}$ and $(\Gamma_{t_0})^i_{jk}$ with respect to g_{t_0} are related by

$$(\Gamma^{\lambda}_{s_0})^i_{jk} = (\Gamma_{t_0})^i_{jk}.$$

One can also compute the Christoffel symbols of the Riemannian cone manifold $C(N)$ as follows. If $1 \le \alpha, \beta, \gamma \le n$, then $\bar{\Gamma}^{\alpha}_{\beta\gamma}(F^{\lambda}_{s_0}(p_0)) = \bar{\Gamma}^{\alpha}_{\beta\gamma}(F_{t_0}(p_0))$. If $1 \le \beta, \gamma \le n$ and $\alpha = n+1$ then $\bar{\Gamma}^{n+1}_{\beta\gamma}(F^{\lambda}_{s_0}(p_0)) = \lambda \bar{\Gamma}^{n+1}_{\beta\gamma}(F_{t_0}(p_0))$, and if $1 \le \alpha, \gamma \le n$ and $\beta = n+1$ then $\bar{\Gamma}^{\alpha}_{n+1\,\gamma}(F^{\lambda}_{s_0}(p_0)) = (1/\lambda)\bar{\Gamma}^{\alpha}_{n+1\,\gamma}(F_{t_0}(p_0))$. By using these and the formula (5), one can show that the mean curvature vectors H_{t_0} of F_{t_0} and $H^{\lambda}_{s_0}$ of $F^{\lambda}_{s_0}$ are related by

$$(H^{\lambda}_{s_0}(p_0))^{\alpha} = \frac{1}{\lambda^2}(H_{t_0}(p_0))^{\alpha}, \tag{26}$$

for $1 \le \alpha \le n$ and

$$(H^{\lambda}_{s_0}(p_0))^{n+1} = \frac{1}{\lambda}(H_{t_0}(p_0))^{n+1}. \tag{27}$$

Now suppose that F is a mean curvature flow, so F satisfies

$$F_*(p_0, t_0)\left(\frac{\partial}{\partial t}\right) = H_{t_0}(p_0).$$

Then

$$\begin{aligned} {F^{\lambda}}_*(p_0, s_0)\left(\frac{\partial}{\partial s}\right) &= \frac{1}{\lambda^2}\sum_{\alpha=1}^{n}(H_{t_0}(p_0))^{\alpha}\frac{\partial}{\partial y^{\alpha}}(p_0) + \frac{1}{\lambda}(H_{t_0}(p_0))^{n+1}\frac{\partial}{\partial y^{n+1}}(p_0) \\ &= \sum_{\alpha=1}^{n}(H^{\lambda}_{s_0}(p_0))^{\alpha}\frac{\partial}{\partial y^{\alpha}}(p_0) + (H^{\lambda}_{s_0}(p_0))^{n+1}\frac{\partial}{\partial y^{n+1}}(p_0) \\ &= H^{\lambda}_{s_0}(p_0). \end{aligned}$$

This means that F^{λ} is the mean curvature flow. This completes the proof of Proposition 5.1. □

Proposition 5.2. *Let M be an m-dimensional manifold and $C(N)$ the Riemannian cone over an n-dimensional Riemannian manifold (N,g). Let $F\colon M\times[0,T)\to C(N)$ be a mean curvature flow. Then parabolic rescaling preserves the value of $\int_{M_t}\rho_T$. This means that for all t in $(0,T)$ the following equation holds.*

$$\int_{M_t}\rho_T = \int_{M^{\lambda}_s}\rho_0$$

where $s = \lambda^2(t-T)$. *Here we have used abbreviation for* $\int_{M_t} \rho_T$ *and* $\int_{M_s^\lambda} \rho_0$ *by*

$$\int_{M_t} \rho_T = \int_M \rho_T(r(F_t(p)), t)\, dv_{g_t},$$
$$\int_{M_s^\lambda} \rho_0 = \int_M \rho_0(r(F_s^\lambda(p)), s)\, dv_{g_t^\lambda}.$$

Proof. From the equation (25) in the proof of the Proposition 5.1, we get

$$\sqrt{\det((g_s^\lambda)_{ij})} = \lambda^m \sqrt{\det((g_t)_{ij})}$$

and

$$dv_{g_s^\lambda} = \lambda^m\, dv_{g_t}.$$

It follows that

$$\begin{aligned}\int_{M_s^\lambda} \rho_0 &= \int_M \frac{1}{(4\pi(-s))^{m/2}} \exp\left(-\frac{r^2(F_s^\lambda(p))}{4(0-s)}\right) dv_{g_s^\lambda} \\ &= \int_M \frac{1}{(4\pi(\lambda^2(T-t)))^{m/2}} \exp\left(-\frac{\lambda^2 r^2(F_t(p))}{4\lambda^2(T-t)}\right)\lambda^m\, dv_{g_t} \\ &= \int_M \frac{1}{(4\pi(T-t))^{m/2}} \exp\left(-\frac{r^2(F_t(p))}{4(T-t)}\right) dv_{g_t} = \int_{M_t} \rho_T. \qquad \square\end{aligned}$$

Proposition 5.3. *Let* M *be an* m*-dimensional manifold and* $C(N)$ *the Riemannian cone over an* n*-dimensional Riemannian manifold* (N, g). *Let* $F \times [0, T) \to C(N)$ *be a mean curvature flow. Then the parabolic rescaling preserves the condition that the mean curvature flow develops a Type* I *singularity.*

Proof. We have only to show that following two statements are equivalent.

- There exists some $c > 0$ such that $\sup_M |\mathrm{II}_t|^2 \le c/(T-t)$ for all $t \in [0, T)$.
- There exists some $c' > 0$ such that $\sup_M |\mathrm{II}_s^\lambda|^2 \le c'/-s$ for all $s \in [-\lambda^2 T, 0)$.

Here II_t and II_s^λ are the second fundamental form with respect to the immersion $F_t\colon M \to C(N)$ and $F_s^\lambda\colon M \to C(N)$ respectively.

We can find a local expression of $(\mathrm{II}_s^\lambda)_{ij}^\alpha$ and $(\mathrm{II}_t)_{ij}^\alpha$ immediately by removing the inverse of Riemannian metric tensors $(g_s^\lambda)^{ij}$ $(= (1/\lambda^2)(g_t)^{ij})$ from equalities (26) and (27). Hence, we find that $(\mathrm{II}_s^\lambda)_{ij}^\alpha = (\mathrm{II}_t)_{ij}^\alpha$ if $1 \le \alpha \le n$, and $(\mathrm{II}_s^\lambda)_{ij}^{n+1} = \lambda(\mathrm{II}_t)_{ij}^{n+1}$ if $\alpha = n+1$, where $s = \lambda^2(t-T)$. It then follows that

$$|\mathrm{II}_s^\lambda|^2(p) = \frac{1}{\lambda^2}|\mathrm{II}_t|^2(p). \tag{28}$$

Hence we get

$$(T-t)|\mathrm{II}_t|^2 = \frac{-s}{\lambda^2} \times \lambda^2 |\mathrm{II}_s^\lambda|^2 = -s|\mathrm{II}_s^\lambda|^2. \tag{29}$$

This mean that parabolic rescaling preserves the condition developing type I singularity. This completes the proof of Proposition 5.3. □

6. Self-similar solutions

This section is devoted to the proof of Theorem 1.2.

Proof of Theorem 1.2. Take any increasing sequence $\{\lambda_i\}_{i=1}^{\infty}$ of the scales of the parabolic rescaling such that $\lambda_i \to \infty$ as $i \to \infty$. Let $F^{\lambda_i} : M \times [-\lambda_i^2 T, 0) \to C(N)$ be the parabolic rescaling of the mean curvature flow $F : M \times [0, T) \to C(N)$. By Proposition 5.1, F^{λ_i} remains to be a mean curvature flow.

Since F develops type I_c singularity and in particular type I singularity, there exists a positive real number $C > 0$ such that

$$\sup_M |\mathrm{II}_t|^2 \leq \frac{C}{T-t}$$

for all t in $[0, T)$, and by Proposition 5.3 the rescaled F^{λ_i} also develops type I singularity satisfying

$$\sup_M |\mathrm{II}_s^{\lambda_i}|^2 \leq \frac{C}{-s}$$

for all s in $[-\lambda_i^2, 0)$ with the same constant $C > 0$ by (29). When s is restricted to the interval $[a, b]$, we have the following bound

$$|\mathrm{II}_s^{\lambda_i}|^2 \leq -\frac{C}{b}. \tag{30}$$

Hence we have a uniform bound of the second fundamental form, and since F^{λ_i} satisfies the mean curvature flow, all the higher derivatives of the second fundamental form are uniformly bounded on $[a, b]$ by [8].

On the other hand, by Theorem 4.1 the following monotonicity formula for F^{λ_i} holds.

$$\frac{d}{ds}\int_{M_s^{\lambda_i}} \rho_0 = -\int_{M_s^{\lambda_i}} \rho_0 \left| \frac{\overrightarrow{F^{\lambda_i}}^{\perp}}{-2s} + H^{\lambda_i} \right|_{\tilde{g}}^2.$$

Integrating the both side of the above equation on any closed interval $[a, b] \subset (-\infty, 0)$, we have

$$\int_{M_b^{\lambda_i}} \rho_0 - \int_{M_a^{\lambda_i}} \rho_0 = -\int_a^b ds \int_{M_s^{\lambda_i}} \rho_0 \left| \frac{\overrightarrow{F^{\lambda_i}}^{\perp}}{-2s} + H^{\lambda_i} \right|_{\tilde{g}}^2 \tag{31}$$

where we take i sufficiently large so that $[a, b]$ is contained in $[-\lambda_i^2 T, 0)$. By Proposition 5.2 we have

$$\int_{M_a^{\lambda_i}} \rho_0 = \int_{M_{u_i}} \rho_T$$

where $u_i = T + a/\lambda_i^2$ and

$$\int_{M_b^{\lambda_i}} \rho_0 = \int_{M_{v_i}} \rho_T$$

where $v_i = T + b/\lambda_i^2$. By the monotonicity formula, the derivative of the function $\int_{M_t} \rho_T$ is non-positive and $\int_{M_t} \rho_T \geq 0$, so for any increasing sequence $\{t_i\}_{i=1}^{\infty}$ such that $t_i \to T$ as $i \to \infty$ the sequence $\int_{M_{t_i}} \rho_T$ converges to a unique value. Now $\{u_i\}_{i=1}^{\infty}$ and $\{v_i\}_{i=1}^{\infty}$ are increasing sequences such that $u_i, v_i \to T$ as $i \to \infty$. So $\int_{M_a^{\lambda_i}} \rho_0$ and $\int_{M_b^{\lambda_i}} \rho_0$ converge to the same value as $i \to \infty$. Therefore the left hand side of the equation (31) converges to 0 as $i \to \infty$, and thus

$$\lim_{i\to\infty} \int_a^b ds \int_{M_s^{\lambda_i}} \rho_0 \left| \frac{\overrightarrow{F^{\lambda_i}}^{\perp}}{-2s} + H^{\lambda_i} \right|_{\bar{g}}^2 = 0. \tag{32}$$

From this we can take a sequence $s_i \in [a, b]$ such that we have

$$\int_{M_{s_i}^{\lambda_i}} \rho_0 \left| \frac{\overrightarrow{F^{\lambda_i}}^{\perp}}{-2s_i} + H^{\lambda_i} \right| \to 0 \tag{33}$$

as $i \to \infty$.

Suppose that p_i attains $\min_M r(F_{s_i}^{\lambda_i})$, and put

$$\gamma_i := r^2(F^{\lambda_i}(p_i, s_i)) = \lambda_i^2 r^2(F(p_i, t_i)).$$

Then p_i also attains $\min_M r(F_{t_i})$ and

$$\gamma_i = \lambda_i^2 r^2(F(p_i, t_i)) = \frac{-s_i r^2(F(p_i, t_i))}{T - t_i}. \tag{34}$$

It then follows from the condition (c) of Definition 1.1 that

$$-bK_1 \leq \gamma_i \leq -aK_2. \tag{35}$$

Thus, the image of $F^{\lambda_i}(\cdot, s_i)$ uniformly stays away from the apex, and that $F^{\lambda_i}(p_i, s_i)$ stays in a compact region in $C(N)$ for the minimum point (p_i, s_i) for $r(F^{\lambda_i})$.

Put $\gamma := -bK_1$. Let W be the manifold obtained from $C(N)$ by cutting out the portion $\{r \leq \sqrt{\gamma}/2\}$, and let V be the manifold obtained by gluing W and $-W$ smoothly

along their boundaries. This V contains $C(N) - \{r \leq \sqrt{\gamma}\}$ and the image of $F^{\lambda_i}|_{(M,s_i)}$ is included in that part.

Since the higher derivatives of the second fundamental form are bounded as shown above, we can apply Theorem 1.2 in [3] (see also [1]) by taking (M_k, p_k) to be (M, p_k), (N_k, h_k, x_k) to be $(V, h, F^{\lambda_k}(p_k, s_k))$ and F_k to be F^{λ_k}, where the metric h is chosen so that h coincides with the cone metric on $C(N) - \{r \leq \sqrt{\gamma}\}$. Then we obtain a limit $F_\infty : M_\infty \to N_\infty$ which satisfies the equation of self-similar solution to the mean curvature flow by (33). But since $x_i = F^{\lambda_i}(p_i, s_i)$ stays in a compact region we have $N_\infty = V$. The limiting self similar solution then defines a flow in the cone $C(N)$ satisfying the mean curvature equation. This completes the proof of Theorem 1.2. □

EXAMPLE 6.1 (Examples of type I_c singularities). Here we show a simple example of the mean curvature flow developing the type I_c singularity. For $-\infty < a < b \leq +\infty$, assume that there exists a mean curvature flow $\Phi : M \times [a, b) \to N$ on (N, g), namely Φ satisfies $(\partial/\partial s)\Phi(\cdot, s) = H_s^N$, where H_s^N is the mean curvature vector with respect to the embedding $\Phi(\cdot, s) : M \to N$. Then $F : M \times [0, T(1 - e^{-2m(b-a)})) \to C(N)$ defined by

$$
\begin{aligned}
&F(p, t) := (\Phi(p, \alpha(t)), \beta(t)) \in N \times \mathbb{R}^+,\\
&\alpha(t) := a - \frac{1}{2m} \log\left(1 - \frac{t}{T}\right),\\
&\beta(t) := \sqrt{2m(T - t)},
\end{aligned}
$$

becomes a solution for mean curvature flow equation with initial data $F_0 = \Phi_0 : M \to N \times \{\sqrt{2mT}\} \subset C(N)$, where $m = \dim M$. The second fundamental form $\mathrm{II}_t^{C(N)}$ of the embedding $F(\cdot, t) : M \to C(N)$ is given by

$$
\mathrm{II}_t^{C(N)} = \mathrm{II}_{\alpha(t)}^N - r(F(p, t))g|_{M_t} \otimes \frac{\partial}{\partial r},
$$

where $\mathrm{II}_{\alpha(t)}^N$ is the second fundamental form of the embedding $M_t = \Phi(M, \alpha(t)) \subset N$. Then we obtain

$$
|\mathrm{II}_t^{C(N)}|_{\bar{g}}^2 \leq \frac{m}{2(T - t)}\left(1 + \frac{1}{m^2} \sup_{p \in M} |\mathrm{II}_{\alpha(t)}^N(p)|_g^2\right),
$$

since $|\mathrm{II}_{\alpha(t)}^N|_{\bar{g}} = r(F(p, t))^{-1}|\mathrm{II}_{\alpha(t)}^N|_g$. Hence F develops a type I singularity at $t = T$, if $b = +\infty$ and

$$
\sup_{p \in M, s \geq a} |\mathrm{II}_s^N(p)|_g < \infty.
$$

The condition (b) and (c) of Definition 1.1 are obviously satisfied since $r(F(p, t)) = \sqrt{2m(T - t)}$.

7. Special Lagrangian submanifolds in toric Calabi–Yau cones

In this section we construct special Lagrangian submanifolds in toric Calabi–Yau cones. Let V be a Ricci-flat Kähler manifold with a Kähler form ω and of $\dim_{\mathbb{C}} V = n$. Then the canonical line bundle K_V is flat. V is said to be a Calabi–Yau manifold if in addition K_V is trivial and V admits a parallel holomorphic n-form Ω. This implies that, with a suitable normalization of Ω, we have

$$\frac{\omega^n}{n!} = (-1)^{n(n-1)/2}\left(\frac{\sqrt{-1}}{2}\right)^n \Omega \wedge \overline{\Omega}.$$

Let L be a real oriented n-dimensional submanifold of V. Then L is called a special Lagrangian submanifold of V if $\omega|_L = 0$ and $\operatorname{Im}\Omega|_L = 0$.

Toric Calabi–Yau cones are exactly the Kähler cones over Sasaki–Einstein manifolds. They are described as toric Kähler cones obtained from toric diagram of height 1. This result was obtained in [4] and [2], which we outline below.

DEFINITION 7.1 (Good rational polyhedral cones, cf. [12]). Let $\mathfrak{g}^*$ be the dual of the Lie algebra $\mathfrak{g}$ of an n-dimensional torus G. Let $\mathbb{Z}_{\mathfrak{g}}$ be the integral lattice of $\mathfrak{g}$, that is the kernel of the exponential map $\exp\colon \mathfrak{g} \to G$. A subset $C \subset \mathfrak{g}^*$ is a rational polyhedral cone if there exists a finite set of vectors $\lambda_i \in \mathbb{Z}_{\mathfrak{g}}$, $1 \le i \le d$, such that

$$C = \{y \in \mathfrak{g}^* \mid \langle y, \lambda_i\rangle \ge 0 \text{ for } i = 1, \dots, d\}.$$

We assume that the set λ_i is minimal in that for any j

$$C \ne \{y \in \mathfrak{g}^* \mid \langle y, \lambda_i\rangle \ge 0 \text{ for all } i \ne j\}$$

and that each λ_i is primitive, i.e. λ_i is not of the form $\lambda_i = a\mu$ for an integer $a \ge 2$ and $\mu \in \mathbb{Z}_{\mathfrak{g}}$. (Thus d is the number of facets if C has non-empty interior.) Under these two assumptions a rational polyhedral cone C with nonempty interior is said to be good if the following condition holds. If

$$\{y \in C \mid \langle y, \lambda_{i_j}\rangle = 0 \text{ for all } j = 1, \dots, k\}$$

is a non-empty face of C for some $\{i_1, \dots, i_k\} \subset \{1, \dots, d\}$, then $\lambda_{i_1}, \dots, \lambda_{i_k}$ are linearly independent over $\mathbb{Z}$ and generates the subgroup $\left\{\sum_{j=1}^k a_j \lambda_{i_j} \mid a_j \in \mathbb{R}\right\} \cap \mathbb{Z}_{\mathfrak{g}}$.

DEFINITION 7.2 (Toric diagrams of height l, cf. [2]). An n-dimensional toric diagram with height l is a collection of $\lambda_i \in \mathbb{Z}^n \cong \mathbb{Z}_{\mathfrak{g}}$ which define a good rational polyhedral cone and $\gamma \in \mathbb{Q}^n \cong (\mathbb{Q}_{\mathfrak{g}})^*$ such that
(1) l is a positive integer such that $l\gamma$ is a primitive element of the integer lattice $\mathbb{Z}^n \cong \mathbb{Z}^*_{\mathfrak{g}}$.

(2) $\langle \gamma, \lambda_i \rangle = -1$.

We say that a good rational polyhedral cone C is associated with a toric diagram of height l if there exists a rational vector γ satisfying (1) and (2) above.

The reason why we use the terminology "height l" is because using a transformation by an element of $SL(n, \mathbb{Z})$ we may assume that

$$\gamma = \begin{pmatrix} -\frac{1}{l} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and the first component of λ_i is equal to l for each i.

Theorem 7.3 ([4], [2]). *Toric Sasaki–Einstein manifolds are exactly those whose Kähler cones are obtained by the Delzant construction from toric diagram of fixed height and applying the volume minimization of Martelli–Sparks–Yau* [13]. *Equivalently, Toric Ricci-flat Kähler manifolds are exactly those obtained by the Delzant construction from toric diagram of fixed height and applying the volume minimization of Martelli–Sparks–Yau* [13].

For a Ricci-flat toric Kähler cone V obtained from a toric diagram of height l, there exists a parallel holomorphic section of $K_V^{\otimes l}$. In particular if $l = 1$ the Kähler cone manifold V is a Calabi–Yau manifold. From now on we assume $l = 1$. Then it is shown in [2] that the parallel holomorphic n-form is given in the form

$$\Omega = e^{-\sum_{i=1}^n \gamma_i z^i} dz^1 \wedge \cdots \wedge dz^n$$

where $z^1, \ldots, z^n$ are holomorphic logarithmic coordinates. Since V is obtained from a toric diagram of height 1 we may assume $\gamma = {}^t(-1, 0, \ldots, 0)$. In this case we have

$$\Omega = e^{z^1}\, dz^1 \wedge \cdots \wedge dz^n.$$

We now apply a method used in [6] and [11]. Their method is summarized in [11] as follows.

Proposition 7.4 ([11]). *Let (V, J, ω, Ω) be a Calabi–Yau manifold of complex dimension n, and H be a compact connected Lie group of real dimension $n-1$ acting effectively on V preserving the Calabi–Yau structure. Suppose there exist a moment map $\mu\colon V \to \mathfrak{h}^*$ and a H-invariant $(n-1)$-form α such that for any $X_1, \ldots, X_{n-1} \in \mathfrak{h}$ we have*

$$\operatorname{Im} \Omega(\cdot\,, X_1, \ldots, X_{n-1}) = d(\alpha(X_1, \ldots, X_{n-1}))$$

where $X_i \in \mathfrak{h}$ are identified with vector fields on V. Then for any $c \in Z_{\mathfrak{h}^*}$, $c' \in \mathbb{R}$ and any basis $\{Y_1, \ldots, Y_{n-1}\} \subset \mathfrak{h}$, the set

$$L_{c,c'} = \mu^{-1}(c) \cap (\alpha(Y_1, \ldots, Y_{n-1}))^{-1}(c')$$

is a H-invariant special Lagrangian submanifold of V.

We refer the reader to [11] for the proof of Proposition 7.4. We now apply Proposition 7.4 to toric Calabi–Yau manifold obtained from toric diagrams of height 1 with

$$\Omega = e^{z^1}\, dz^1 \wedge \cdots \wedge dz^n, \quad \alpha = \operatorname{Im}(e^{z^1}\, dz^2 \wedge \cdots \wedge dz^n),$$

and with $Y_j = 2\operatorname{Im}(\partial/\partial z^j)$ and H the subtorus T^{n-1} generated by $Y_1, \ldots, Y_{n-1}$. Then one easily finds that

$$\operatorname{Im}\Omega(\,\cdot\,, Y_1, \ldots, Y_{n-1}) = d(\alpha(Y_1, \ldots, Y_{n-1})),$$

and

$$\alpha(Y_1, \ldots, Y_{n-1}) = \frac{1}{i^n}(e^{z^1} + (-1)^n e^{\overline{z_1}}).$$

Thus the assumptions of Proposition 7.4 is satisfied, and we have proved the following.

Theorem 7.5. *Let V be a toric Calabi–Yau manifold obtained from a toric diagram of height* 1. *Let*

$$\Omega = e^{z^1}\, dz^1 \wedge \cdots \wedge dz^n$$

be the parallel holomorphic n-form described as above. Then there is a T^{n-1}-invariant special Lagrangian submanifold described as

$$\mu^{-1}(c) \cap \left\{ \frac{1}{i^n}(e^{z^1} + (-1)^n e^{\overline{z_1}}) = c' \right\}$$

where T^{n-1} is a subtorus generated by $\operatorname{Im}(\partial/\partial z^2), \ldots, \operatorname{Im}(\partial/\partial z^n)$ *and $\mu\colon V \to \mathfrak{h}^*$ is a moment map.*

EXAMPLE 7.6. Take V to be the flat $\mathbb{C}^n$, and let $w^1, \ldots, w^n$ be the standard holomorphic coordinates with

$$\Omega = dw^1 \wedge \cdots \wedge dw^n.$$

The logarithmic holomorphic coordinates $v^1, \ldots, v^n$ are given by $w^i = e^{v^i}$. Thus, we have

$$\Omega = e^{(v^1 + \cdots + v^n)}\, dv^1 \wedge \cdots \wedge dv^n.$$

Taking $\gamma = {}^t(-1, 0, \ldots, 0)$ amounts to changing the coordinates $z^1 = v^1 + \cdots + v^n$, $z^2 = v^2, \ldots, z^n = v^n$. Then with the new coordinates we have

$$\Omega = e^{z^1}\, dz^1 \wedge \cdots \wedge dz^n.$$

In this situation the points in $\mu^{-1}(c)$ are described as

$$|w^2|^2 - |w^1|^2 = c_2, \ldots, |w^n|^2 - |w^1|^2 = c_n.$$

If n is even then $(e^{z^1} + (-1)^n e^{\overline{z_1}})/i^n = c'$ if and only if $\Re(w^1 \cdots w^n) = c'$, and If n is odd then $(e^{z^1} + (-1)^n e^{\overline{z_1}})/i^n = c'$ if and only if $\mathrm{Im}(w^1 \cdots w^n) = c'$. This is exactly the same as Theorem 3.1 in [7].

8. Infinitesimal deformations of special Lagrangian cones

In this section we consider the infinitesimal deformations of special Lagrangian cones embedded in the cone of Sasaki–Einstein manifolds.

DEFINITION 8.1. A Riemannian manifold (N, g) is called a *Sasakian manifold* if its Riemannian cone $(C(N), \bar{g})$ is a Kähler manifold with respect to some integrable complex structure J over $C(N)$. A *Reeb vector field* ξ on the Sasakian manifold (N, g) is a Killing vector field on N given by $\xi := J(r(\partial/\partial r))$.

For a Sasakian manifold (N, g), a contact form $\eta \in \Omega^1(N)$ on N is given by $\eta := g(\xi, \cdot)$. Then the Kähler form $\omega \in \Omega^2(C(N))$ on $C(N)$ is described as $\omega = d(r^2\eta)$.

DEFINITION 8.2. For a smooth manifold N, a *cone submanifold* C *of* $C(N)$ is a submanifold of $C(N)$ which can be written as $C = C(\Sigma)$ for a submanifold $\Sigma \subset N$. For a Sasakian manifold (N, g, ξ), a cone submanifold $C \subset C(N)$ is a *Lagrangian cone* if it is a Lagrangian submanifold of $(C(N), \omega)$.

The following proposition is well-known but here we give a proof for readers' convenience.

Proposition 8.3. *A submanifold $\Sigma \subset N$ is Legendrian if and only if $C(\Sigma) = \Sigma \times \mathbb{R}^+ \subset C(N)$ is Lagrangian with respect to the Kähler form ω on $C(N)$.*

Proof. Let $\Sigma \subset N$ be a Legendrian submanifold. For any $p \in \Sigma$, open neighborhood $U \subset \Sigma$ and $u, v \in \mathcal{X}(U)$, we have

$$\omega(u, v) = d\eta(u, v) = -\eta([u, v]) = 0,$$
$$\omega\left(u, \frac{\partial}{\partial r}\right) = g(u, \xi) = \eta(u) = 0,$$

since $\eta|_\Sigma = 0$ and $[u, v] \in \mathcal{X}(U)$. Hence $C(\Sigma) \subset C(N)$ is Lagrangian. Conversely, let $C(\Sigma) \subset C(N)$ be Lagrangian and take $u \in T_p\Sigma$ arbitrarily. Then

$$\eta(u) = g(u, \xi) = \omega\left(u, \frac{\partial}{\partial r}\right) = 0,$$

which implies that $\Sigma \subset N$ is a Legendrian submanifold. □

Proposition 8.4. *Let (V, J, ω) be a Ricci-flat Kähler manifold of $\dim_{\mathbb{C}} = n$ with $H^1_{DR}(V, \mathbb{R}) = 0$, and assume that the canonical line bundle K_V is holomorphically trivial. Then there exists a holomorphic n form $\Omega \in \Omega^{(n,0)}(V)$ satisfying*

$$\frac{\omega^n}{n!} = (-1)^{n(n-1)/2}\left(\frac{\sqrt{-1}}{2}\right)^n \Omega \wedge \overline{\Omega}. \tag{36}$$

Proof. From the assumption there is a nowhere vanishing holomorphic n form $\Omega_0 \in \Omega^{(n,0)}(V)$ on V. Since Ω_0 is holomorphic, $d\Omega_0 = 0$. The Kähler form ω on V induces a hermitian metric on K_V by

$$h := h(\Omega_0, \overline{\Omega_0}) := n!(-1)^{n(n-1)/2}\left(\frac{\sqrt{-1}}{2}\right)^n \frac{\Omega_0 \wedge \overline{\Omega_0}}{\omega^n}.$$

Now we put $\Omega := h^{-\frac{1}{2}} e^{\sqrt{-1}\rho}\Omega_0$ for $\rho \in C^\infty(V, \mathbb{R})$, which satisfies the equation (36). Then it suffices to show that there exists $\rho \in C^\infty(V, \mathbb{R})$ such that $d\Omega = 0$. From $d\Omega_0 = 0$, we have

$$\begin{aligned} d\Omega &= \bar{\partial}(h^{-1/2}e^{\sqrt{-1}\rho}) \wedge \Omega_0 \\ &= h^{-1/2}e^{\sqrt{-1}\rho}\left(-\frac{1}{2}h^{-1}\bar{\partial}h + \sqrt{-1}\bar{\partial}\rho\right) \wedge \Omega_0 \\ &= h^{-1/2}e^{\sqrt{-1}\rho}\left(-\frac{1}{2}\bar{\partial}\log h + \sqrt{-1}\bar{\partial}\rho\right) \wedge \Omega_0. \end{aligned}$$

Thus the problem is reduced to show the existence of the function ρ which satisfies $\bar{\partial}(-(1/2)\log h + \sqrt{-1}\rho) = 0$.

Recall that ω is Ricci-flat Kähler form. Then the curvature form of the Hermitian connection on K_V induced from h is equal to zero, we have $dd^c \log h = 0$. Now we have assumed $H^1_{DR}(V, \mathbb{R}) = 0$, there exists $\hat{\rho} \in C^\infty(V, \mathbb{R})$ such that $d^c \log h = (\sqrt{-1}\partial - \sqrt{-1}\bar{\partial})\log h = d\hat{\rho} = (\partial + \bar{\partial})\hat{\rho}$. By comparing $(0,1)$-part, we have $\bar{\partial}(\log h - \sqrt{-1}\hat{\rho}) = 0$, consequently we obtain the assertion by putting $\hat{\rho} = 2\rho$. □

From now on suppose (N, g, ξ) is a Sasaki–Einstein manifold of dimension $2n-1$, hence the Kähler structure ω on $C(N)$ is Ricci-flat. Moreover we assume the canonical

bundle $K_{C(N)}$ is trivial. Since (N, g) is an Einstein manifold with positive Ricci curvature, then $H^1(C(N), \mathbb{R}) = H^1(N, \mathbb{R}) = 0$. Therefore we have a holomorphic n-form Ω on $C(N)$ satisfying (36).

Now we denote by $\tilde{H}$ and H the mean curvature vector of $C(\Sigma) \subset C(N)$ and $\Sigma \subset N$, respectively. Then the direct calculation gives $\tilde{H} = r^{-2}H$, therefore $C(\Sigma)$ is minimal if and only if Σ is minimal.

It is well known that the mean curvature of a Lagrangian submanifold embedded in a Calabi–Yau manifold is equal to $d\theta$ under the identification of vector fields and 1-forms by the symplectic form, where θ is the Lagrangian angle. Then the Lagrangian submanifold embedded in the Calabi–Yau manifold is minimal if and only if the Lagrangian angle is constant. In particular it is special Lagrangian if the Lagrangian angle is equal to zero. Hence $\Sigma \subset N$ is minimal Legendrian if and only if $C(\Sigma) \subset C(N)$ is Lagrangian with constant Lagrangian angle.

In [14], the infinitesimal deformation spaces of minimal Legendrian submanifolds embedded in η-Sasaki–Einstein manifolds are studied. Here we observe the infinitesimal deformation spaces of special Lagrangian cones in $C(N)$, using the results obtained in [5].

Let $C(\Sigma)$ be a special Lagrangian submanifold in $C(N)$, and we have orthogonal decompositions $TC(N)|_{C(\Sigma)} = TC(\Sigma) \oplus NC(\Sigma)$ and $TN|_{\Sigma} = T\Sigma \oplus N\Sigma$, where $N\Sigma$, $NC(\Sigma)$ are normal bundles. Then for any $(x, r) \in C(\Sigma)$ we have the natural identification $N_{(x,r)}C(\Sigma) = N_x\Sigma$.

The infinitesimal deformations of cone submanifolds of $C(N)$ is generated by the smooth 1-parameter families of cone submanifolds $\{C(\Sigma_t) = {\pi_N}^{-1}(\Sigma_t);\ -\varepsilon < t < \varepsilon\}$, where $\{\Sigma_t;\ -\varepsilon < t < \varepsilon\}$ is the smooth families of submanifolds of N which satisfies $\Sigma_0 = \Sigma$, and $\pi_N \colon N \times \mathbb{R}^+ \to N$ is the projection onto the first component. Since the infinitesimal deformations of $\Sigma \subset N$ are parameterized by smooth sections of $N\Sigma$, the infinitesimal deformations of cone submanifolds are parameterized by

$$\mathcal{A}_{C(\Sigma)} := \{\alpha = {\pi_N}^*\alpha_0 \in \Gamma(NC(\Sigma));\ \alpha_0 \in \Gamma(N\Sigma)\}.$$

Then $\alpha_{(x,r)} \in N_{(x,r)}C(\Sigma) = N_x\Sigma$ is independent of r for each $\alpha \in \mathcal{A}_{C(\Sigma)}$.

Since $C(\Sigma)$ is Lagrangian, $NC(\Sigma)$ is identified with the cotangent bundle $T^*C(\Sigma)$ by the bundle isomorphism $\hat{\omega} \colon NC(\Sigma) \to T^*C(\Sigma)$ defined by $\hat{\omega}(v) := \iota_v\bar{\omega} = \bar{\omega}(v, \cdot)$.

By the results in [5], the infinitesimal deformations of special Lagrangian submanifolds of $C(\Sigma)$ are parameterized by harmonic 1-forms on $C(\Sigma)$. Thus the infinitesimal deformations of special Lagrangian cones of $C(\Sigma)$ are parameterized by

$$\mathcal{H}_{C(\Sigma)} := \{\hat{\omega}(\alpha) \in \Omega^1(C(\Sigma));\ \alpha \in \mathcal{A}_{C(\Sigma)},\ d\hat{\omega}(\alpha) = d * \hat{\omega}(\alpha) = 0\},$$

where $\hat{\omega}$ is the isomorphism induced by ω, and $*$ is the Hodge star with respect to the induced metric $\bar{g}|_{C(\Sigma)}$. To study the vector space $\mathcal{H}_{C(\Sigma)}$, we need the next lemma.

Lemma 8.5. *Under the natural identification* $T^*_{(x,r)}C(\Sigma) = T^*_x\Sigma \oplus T^*_r\mathbb{R}^+$, *we have*

$$\hat{\omega}(\mathcal{A}_{C(\Sigma)}) = \{\beta_{(x,r)} = r\varphi(x)\,dr + r^2\gamma_x \in \Omega^1(C(\Sigma));\ \varphi \in C^\infty(\Sigma),\ \gamma \in \Omega^1(\Sigma)\}.$$

Proof. Define a diffeomorphism $m_a = \exp(ar(\partial/\partial r))\colon C(N) \to C(N)$ by $m_a(p,r) = (p,ar)$ for $a > 0$. First of all we show that m_a is a biholomorphism. Since $(d/da)(m_a)_* J = (m_a)_* \mathcal{L}_{r(\partial/\partial r)} J$, it suffices to show $\mathcal{L}_{r(\partial/\partial r)} J = 0$. Now we may write $r(\partial/\partial r) = -J\xi$, then for any $x \in C(N)$ and open neighborhood $x \in U \subset C(N)$ and $v \in \mathcal{X}(C(N))$,

$$\begin{aligned}(\mathcal{L}_{J\xi} J)(v) &= \mathcal{L}_{J\xi}(Jv) - J(\mathcal{L}_{J\xi} v)\\ &= [J\xi,\ Jv] - J([J\xi,\ v])\\ &= -N_J(\xi,\ v) - J^2[\xi,\ v] + J[\xi,\ Jv]\\ &= -N_J(\xi,\ v) + J\{(\mathcal{L}_\xi J)(v)\},\end{aligned}$$

where N_J is the Nijenhuis tensor. Thus we have $\mathcal{L}_{J\xi} J = 0$ since J is integrable and $\mathcal{L}_\xi J = 0$, hence m_a is a biholomorphism.

Next we show that

$$\hat{\omega}(\mathcal{A}_{C(\Sigma)}) = \{\beta \in \Omega^1(C(\Sigma));\ {m_a}^*\beta = a^2\beta \text{ for all } a \in \mathbb{R}^+\}.$$

Since m_a satisfies ${m_a}^*\bar{g} = {m_a}^*(dr^2 + r^2 g) = d(ar)^2 + (ar)^2 g = a^2\bar{g}$, we obtain ${m_a}^*\omega = a^2\omega$. By the definition of $\mathcal{A}_{C(\Sigma)}$, we may write

$$\mathcal{A}_{C(\Sigma)} = \{\alpha \in \Omega^1(C(\Sigma));\ (m_a)_*\alpha = \alpha \text{ for all } a \in \mathbb{R}^+\}.$$

For any $\alpha \in \Gamma(NC(\Sigma))$, we have

$$\begin{aligned}m_a^*(\hat{\omega}(\alpha)) = m_a^*(\iota_\alpha\omega) &= \iota_{(m_a)_*^{-1}\alpha} m_a^*\omega = a^2\hat{\omega}((m_a)_*^{-1}\alpha)\\ &= a^2\hat{\omega}(\alpha) + a^2\hat{\omega}((m_a)_*^{-1}\alpha - \alpha).\end{aligned}$$

Therefore the equation $m_a^*(\hat{\omega}(\alpha)) = a^2\hat{\omega}(\alpha)$ holds for all $a \in \mathbb{R}^+$ if and only if $\alpha \in \mathcal{A}_{C(\Sigma)}$.

Now we take $\beta \in \Omega^1(C(\Sigma))$ and decompose it as $\beta_{(x,r)} = \sigma(x,r) + \tau(x,r)\,dr$ such that $\sigma(x,r) \in T^*_x\Sigma$ and $\tau \in C^\infty(C(\Sigma))$.

$$ {m_a}^*\beta = {m_a}^*\sigma + {m_a}^*\tau \cdot a\,dr,$$

then ${m_a}^*\beta = a^2\beta$ is equivalent to

$$\begin{aligned}\sigma(x,ar) &= a^2\sigma(x,r),\\ \tau(x,ar) &= a\tau(x,r).\end{aligned}$$

Thus we may put $\sigma = r^2\gamma$ and $\tau = r\varphi$ for some $\gamma \in \Omega^1(\Sigma)$ and $\varphi \in C^\infty(\Sigma)$. □

Theorem 8.6. *The vector space $\mathcal{H}_{C(\Sigma)}$ is isomorphic to*

$$\mathrm{Ker}(\Delta_\Sigma - 2n) = \{\varphi \in C^\infty(\Sigma);\ \Delta_\Sigma \varphi = 2n\varphi\},$$

where $\Delta_\Sigma = d^{*_\Sigma} d$ and d^{*_Σ} is a formal adjoint operator of d with respect to the metric $\bar{g}|_\Sigma$.

Proof. From Lemma 8.5, all $\beta \in \hat{\omega}(\mathcal{A}_{C(\Sigma)})$ can be written as $\beta = r\varphi\, dr + r^2\gamma$. Then we have

$$d\beta = r\, dr \wedge (2\gamma - d\varphi) + r^2\, d\gamma,$$

from which it follows that $d\beta = 0$ is equivalent to $2\gamma = d\varphi$.

Next we calculate $d * \beta$. Denote by vol_Σ the volume form of $g|_\Sigma$. Since the volume forms of $\bar{g}|_{C(\Sigma)}$ is given by $r^{n-1}\, dr \wedge \mathrm{vol}_\Sigma$, we can deduce

$$*\gamma = -r^{n-3}\, dr \wedge *_\Sigma \gamma,$$
$$*dr = r^{n-1}\, \mathrm{vol}_\Sigma,$$

where $*_\Sigma$ is the Hodge star operator with respect to $g|_\Sigma$. Consequently, we obtain

$$d * \beta = r^{n-1}\, dr \wedge (d *_\Sigma \gamma + n\varphi\, \mathrm{vol}_\Sigma).$$

Hence $d\beta = d * \beta = 0$ is equivalent to

$$\gamma = \frac{1}{2}\, d\varphi, \quad n\varphi\, \mathrm{vol}_\Sigma + \frac{1}{2} d *_\Sigma d\varphi = 0,$$

and the latter equation is equivalent to $d^{*_\Sigma} d\varphi = 2n\varphi$. □

In [14], the infinitesimal deformation spaces of minimal Legendrian submanifolds in Sasaki–Einstein manifolds are studied. Proposition 8.6 is also obtained from the case of η-Ricci constant A is equal to $2n - 2$ in [14]. Here we should pay attention that the dimension of infinitesimal deformation spaces obtained in [14] is equal to $1 + \dim \mathrm{Ker}(\Delta_\Sigma - 2n)$, since the deformations of $C(\Sigma)$ generated by Reeb vector field ξ is not special Lagrangian cone, but minimal Lagrangian cone whose Lagrangian angle is not equal to zero. Actually, if we put $\alpha = \xi$, then $\beta = \hat{\omega}(\alpha) = -r\, dr$ and

$$d * \beta = -d(r^n\, \mathrm{vol}_\Sigma) = -nr^{n-1}\, dr \wedge \mathrm{vol}_\Sigma \neq 0,$$

accordingly this α does not generate deformations of special Lagrangian cones.

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