

Title	On free abelian extensions
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Citation	Osaka Journal of Mathematics. 1974, 11(1), p. 15-21
Version Type	VoR
URL	<a href="https://doi.org/10.18910/5104">https://doi.org/10.18910/5104</a>
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## ON FREE ABELIAN EXTENSIONS

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(Received August 9, 1973)

**Introduction.** Let  $R$  be a commutative ring, and  $G$  a finite abelian group. In [2] (see also [5]) the set of isomorphism classes of Galois extensions of  $R$  with group  $G$  that have normal bases is described cohomologically by means of Harrison's complex of  $RG$ ; to this end, Galois algebras are first classified and then Galois extensions with normal basis as a particular case. In this paper we use a different approach to classify Galois extensions which are free as  $R$ -modules; the restriction of this classification to extensions with normal basis yields the cohomological description of [2].

### Free Abelian Extensions

Let  $R$  be a commutative ring, and  $G$  a finite abelian group. Recall that a faithful  $R$ -algebra  $A$  is said to be a Galois extension of  $R$  with respect to a representation of  $G$  by  $R$ -algebra automorphisms of  $A$  if the following equivalent conditions are satisfied:

- 1)  $A^G = R$  and the map  $M_A$  from  $A \otimes_R A$  to the ring of functions from  $G$  to  $A$  defined by  $M_A(x \otimes y)(\sigma) = x\sigma(y)$  is an  $R$ -module isomorphism.
- 2)  $A^G = R$ ,  $A$  is a finitely generated projective  $R$ -module and  $L: AG \rightarrow \text{End}_R(A)$  is an  $R$ -algebra isomorphism, where  $AG$  is the twisted group ring of  $G$  over  $A$  and  $L$  is defined by  $L(a\sigma)(x) = a\sigma(x)$ .

Let  $E$  denote the ring of functions from  $G$  to  $R$ ; if we let  $G$  act on  $E$  by means of  $(\sigma f)(\eta) = f(\sigma^{-1}\eta)$  then  $E$  is Galois over  $R$  with group  $G$ ; we have  $E = \bigoplus R e_\sigma$  with  $\sum e_\sigma = 1$ ,  $e_\sigma e_\eta = \delta_{\sigma,\eta} e_\sigma$  and  $\sigma(e_\eta) = e_{\sigma\eta}$ . Clearly the condition 1) can be reformulated as follows:

- 3)  $A^G = R$  and  $M_A: A \otimes A \rightarrow E \otimes A$  defined by  $M_A(x \otimes y) = \sum e_\sigma \otimes x\sigma(y)$  is an  $R$ -module isomorphism.

Note that for  $M = M_E: E \otimes E \rightarrow E \otimes E$  we have  $M(e_\alpha \otimes e_\beta) = e_{\alpha\beta^{-1}} \otimes e_\alpha$ . Since  $EG \cong \text{End}_R(E)$  we have  $EG \otimes EG \cong \text{End}_R(E \otimes E)$ ; thus considering  $E \otimes E$  as a left module over  $EG \otimes EG$ , the  $R$ -module automorphisms of  $E \otimes E$  are produced by left multiplications by units of  $EG \otimes EG$ .

Suppose the Galois extension  $A$  is free as an  $R$ -module. Then there exists an  $R$ -module isomorphism  $j: A \rightarrow E$  and  $M^{-1} \cdot 1 \otimes j \cdot M_A \cdot j^{-1} \otimes j^{-1}: E \otimes E \rightarrow E \otimes E$  is an isomorphism of  $R$ -modules. Therefore there exists a unique  $u \in U(EG \otimes EG)$

such that  $M^{-1} \cdot 1 \otimes j \cdot M_A \cdot j^{-1} \otimes j^{-1} = L(u)$ , that is, there is a unique  $u$  such that the diagram

$$\begin{array}{ccccc} & A \otimes A & \xrightarrow{M_A} & E \otimes A & \xrightarrow{1 \otimes j} \\ j \otimes j & \downarrow & & & \searrow \\ & E \otimes E & \xrightarrow{L(u)} & E \otimes E & \xrightarrow{M} \\ & & & & \nearrow \\ & & & & E \otimes E \end{array}$$

is commutative. In particular if  $A=E$  and  $j$  is the identity then  $u=1$ .

Let  $A, A'$  be Galois extensions,  $j: A \rightarrow E, j': A' \rightarrow E$   $R$ -module isomorphisms, and  $f: A' \rightarrow A$  an isomorphism of Galois extensions. Since  $jj'j'^{-1}: E \rightarrow E$  is an  $R$ -isomorphism, there exists  $v \in U(EG)$  such that  $jj'j'^{-1} = L(v)$ . If  $u, u' \in U(EG \otimes EG)$  are obtained from  $j$  and  $j'$  respectively, it follows from  $1 \otimes f \cdot M_{A'} = M_A \cdot f \otimes f$  that

$$(2) \quad L(u) \cdot L(v \otimes v) = M^{-1} \cdot 1 \otimes L(v) \cdot M \cdot L(u').$$

Let us now define  $R$ -algebra homomorphisms  $\Delta_0, \Delta_1, \Delta_2: EG \rightarrow EG \otimes EG$  by means of  $\Delta_0(x) = 1 \otimes x, \Delta_2(x) = x \otimes 1$  and  $\Delta_1(\sum a_\sigma \sigma) = \sum a_\sigma \sigma \otimes \sigma$ . Then  $M^{-1} \cdot 1 \otimes L(v) \cdot M = L(\Delta_1(v))$  and (2) becomes

$$(3) \quad u \cdot \Delta_0(v) \Delta_2(v) = \Delta_1(v) u'.$$

Note that if  $u \in U(EG \otimes EG)$  is obtained from a Galois extension  $A$  by means of  $j: A \rightarrow E$  and  $v \in U(EG)$  then the element of  $U(EG \otimes EG)$  obtained from  $j' = L(v^{-1})j: A \rightarrow E$  is  $u' = \Delta_1(v^{-1}) \cdot u \cdot v \otimes v$ .

Consider the following relation in  $U(EG \otimes EG)$ : if  $u, u' \in U(EG \otimes EG)$  then  $u \sim u'$  if there exists  $v \in U(EG)$  such that  $\Delta_1(v) \cdot u' = u \cdot v \otimes v$ . It is easy to verify that this is an equivalence relation. Let  $\mathcal{C}$  be the quotient set. (The previous remark shows that if  $u \in U(EG \otimes EG)$  is obtained from a Galois extension, then every element in its equivalence class is obtained from the same extension.)

Let  $E_G^f(R)$  be the set of Galois isomorphism classes  $[A]$  of Galois extensions  $A$  of  $R$  with group  $G$  that are free as  $R$ -modules. The above construction defines a map  $\psi: E_G^f(R) \rightarrow \mathcal{C}$  and we have:

**Proposition 1.**  $\psi: E_G^f(R) \rightarrow \mathcal{C}$  is injective.

Proof. Let  $[A], [B] \in E_G^f(R), j: A \rightarrow E, j': B \rightarrow E$  and  $u, u' \in U(EG \otimes EG)$  the units associated to  $A$  and  $B$  by means of  $j$  and  $j'$  respectively. Assume  $u \sim u'$ . As remarked before, we may suppose  $u = u'$ . We then have a commutative diagram

$$\begin{array}{ccccc} & A \otimes A & \xrightarrow{M_A} & E \otimes A & \xrightarrow{1 \otimes j} \\ j \otimes j & \downarrow & & & \searrow \\ & E \otimes E & \xrightarrow{L(u)} & E \otimes E & \xrightarrow{M} \\ j' \otimes j' & \uparrow & & & \nearrow \\ & B \otimes B & \xrightarrow{M_B} & E \otimes B & \xrightarrow{1 \otimes j'} \end{array} E \otimes E$$

If  $f=j'^{-1}\cdot j$  then the commutativity of the diagram

$$f \otimes f \quad \begin{array}{ccc} A \otimes A & \xrightarrow{M_A} & E \otimes A \\ \downarrow & & \downarrow \\ B \otimes B & \xrightarrow{M_B} & E \otimes B \end{array} \quad 1 \otimes f$$

shows that  $f(x)\sigma f(y)=f(x\sigma(y)) \forall x, y \in A, \sigma \in G$ . Thus  $f$  is an  $R$ -algebra isomorphism. In particular,  $f(1)=1$  and  $f(\sigma(y))=\sigma f(y)$ ; then  $f: A \rightarrow B$  is an isomorphism of Galois extensions.

Let us now determine the image of  $\psi$ . Let  $A$  be a free Galois extension of  $R$  with group  $G$  and  $j: A \rightarrow E$  an  $R$ -module isomorphism. Then the diagram

$$(3) \quad \begin{array}{ccccc} A \otimes A \otimes A & \xrightarrow{1 \otimes M_A} & A \otimes E \otimes A & \xrightarrow{1 \otimes p} & A \otimes A \otimes E \\ M_A \otimes 1 \downarrow & & & & \\ E \otimes A \otimes A & \xrightarrow{1 \otimes M_A} & E \otimes E \otimes A & \xrightarrow{1 \otimes 1 \otimes j} & E \otimes E \otimes E \\ & & & & \downarrow 1 \otimes 1 \otimes j \\ & & & & E \otimes E \otimes E \\ & & & & \uparrow h \\ & & & & E \otimes A \otimes E \xrightarrow{1 \otimes p} E \otimes E \otimes A \\ & & & & \uparrow M_A \otimes 1 \\ & & & & E \otimes A \otimes E \end{array}$$

is commutative, where  $p: X \otimes Y \rightarrow Y \otimes X$  is defined by  $p(x \otimes y)=y \otimes x$  and  $h: E \otimes E \otimes E \rightarrow E \otimes E \otimes E$  is defined by  $h(e_\sigma \otimes e_\eta \otimes z)=e_\sigma \otimes e_{\sigma^{-1}\eta} \otimes z$ . Let  $u$  be an element of  $U(EG \otimes EG)$  making diagram (1) commutative. Then  $M_A=1 \otimes j^{-1} \cdot M \cdot L(u) \cdot j \otimes j$  and it follows from  $h \cdot 1 \otimes 1 \otimes j \cdot 1 \otimes M_A \cdot M_A \otimes 1=1 \otimes 1 \otimes j \cdot 1 \otimes p \cdot M_A \otimes 1 \cdot 1 \otimes p \cdot 1 \otimes M_A$  that

$$(4) \quad [M^{-1} \otimes 1 \cdot 1 \otimes L(u) \cdot M \otimes 1] \cdot L(u \otimes 1) \\ = [M^{-1} \otimes 1 \cdot 1 \otimes M^{-1} \cdot h^{-1} \cdot 1 \otimes p \cdot M \otimes 1 \cdot L(u \otimes 1) \cdot 1 \otimes p \cdot 1 \otimes M] \cdot L(1 \otimes u)$$

Consider now  $\Delta_i: EG \otimes EG \rightarrow EG \otimes EG \otimes EG$  dened by  $\Delta_0(x)=1 \otimes x$ ,  $\Delta_3(x)=x \otimes 1$ ,  $\Delta_1(x \otimes y)=\Delta_1(x) \otimes y$ ,  $\Delta_2(x \otimes y)=x \otimes \Delta_1(y)$ . Then the  $\Delta_i$ 's are  $R$ -algebra homomorphisms and we can verify that

$$L(\Delta_1(u)) = M^{-1} \otimes 1 \cdot 1 \otimes L(u) \cdot M \otimes 1, \\ L(\Delta_2(u)) = M^{-1} \otimes 1 \cdot 1 \otimes M^{-1} \cdot h^{-1} \cdot 1 \otimes p \cdot M \cdot L(u \otimes 1) \cdot 1 \otimes p \cdot 1 \otimes M.$$

Thus the relation (4) is equivalent to

$$(5) \quad \Delta_1(u) \Delta_3(u) = \Delta_2(u) \Delta_0(u).$$

On the other side, if  $1_A \in A$  is the identity element of  $A$ , we have  $M_A(x \otimes 1_A) = \sum_{\sigma} e_\sigma \otimes x = 1 \otimes x$  and therefore  $L(u)(j(x) \otimes j(1_A)) = M^{-1}(1 \otimes j(x)) = j(x) \otimes 1$  for

every  $x \in A$ . Then  $u(x \otimes j(1_A)) = x \otimes 1$  for every  $x \in E$ ; therefore  $1 \otimes j(1_A) = u^{-1}(1 \otimes 1)$  and if  $m: E \otimes E \rightarrow E$  is the product map, it follows that  $j(1_A) = m(u^{-1}(1 \otimes 1))$ . Thus

$$(6) \quad u(x \otimes m(u^{-1}(1 \otimes 1))) = x \otimes 1 \quad \text{for every } x \in E.$$

Suppose now  $u \in U(EG \otimes EG)$  verifies relations (5) and (6). We shall construct a Galois extension  $A$  with  $[A] \in E_G^1(R)$  such that  $\psi([A])$  is the class of  $u$ . Consider  $ML(u): E \otimes E \rightarrow E \otimes E$ ; if  $t \in E \otimes E$  we have  $ML(u)(t) = \sum_{\sigma} e_{\sigma} \otimes t_{\sigma}$  with  $t_{\sigma} \in E$  uniquely determined. Given  $x, y \in E$ , we set

$$x * y = (x \otimes y)_1$$

(that is,  $x * y$  is the coefficient of  $e_1 \otimes 1$  in  $ML(u)(x \otimes y)$ ). Since  $ML(u)$  is additive, it follows that  $(x, y) \rightarrow x * y$  is a distributive product.

Let  $1_A = m(u^{-1}(1 \otimes 1)) \in E$ . Then  $ML(u)(x \otimes 1_A) = M(x \otimes 1) = 1 \otimes x = \sum_{\sigma} e_{\sigma} \otimes x$ , that is

$$(7) \quad x * 1' = x \quad \text{for every } x \in E.$$

For every  $\sigma \in G$ , let  $\bar{\sigma}: E \rightarrow E$  be defined by  $\bar{\sigma}(x) = (1' \otimes x)_{\sigma}$ . Then  $ML(u)(1' \otimes x) = \sum_{\sigma} e_{\sigma} \otimes \bar{\sigma}(x)$ . Since  $ML(u)(1' \otimes 1') = 1 \otimes 1' = \sum_{\sigma} e_{\sigma} \otimes 1'$ , we have  $\bar{\sigma}(1') = 1'$  for all  $\sigma \in G$ .

Let us now show that

$$(8) \quad ML(u)(x \otimes y) = \sum_{\sigma} e_{\sigma} \otimes (x * \bar{\sigma}(y)).$$

Since  $u$  verifies (5) and therefore (4), we have

$h \cdot 1 \otimes M \cdot 1 \otimes L(u) \cdot M \otimes 1 \cdot L(u \otimes 1) = 1 \otimes p \cdot M \otimes 1 \cdot L(u) \otimes 1 \cdot 1 \otimes p \cdot 1 \otimes M \cdot 1 \otimes L(u)$ . Applying this functions to  $x \otimes 1' \otimes y$ ,  $x, y \in E$ , we obtain

$$\sum_{\sigma, \eta} e_{\sigma} \otimes e_{\sigma^{-1}\eta} \otimes (x \otimes y)_{\eta} = \sum_{\sigma, \eta} e_{\sigma} \otimes e_{\eta} \otimes (x \otimes \bar{\eta}(y))_{\sigma}$$

and therefore  $(x \otimes y)_{\eta} = (x \otimes \bar{\eta}(y))_1 = x * \bar{\eta}(y)$ .

If we apply the same relation to  $x \otimes y \otimes z$  and use (8), then we obtain

$$\sum_{\sigma, \eta} e_{\sigma} \otimes e_{\sigma^{-1}\eta} (x * \bar{\sigma}(y)) * \bar{\eta}(z) = \sum_{\sigma, \eta} e_{\sigma} \otimes e_{\eta} \otimes x * \bar{\sigma}(y * \bar{\eta}(z))$$

and it follows that

$$(9) \quad x * \bar{\sigma}(y * \bar{\tau}(z)) = (x * \bar{\sigma}(y)) * \bar{\sigma}\bar{\tau}(z).$$

In particular,  $x * \bar{\sigma}(1' * \bar{\tau}(z)) = (x * \bar{\sigma}(1')) * \bar{\sigma}\bar{\tau}(z)$ . But (8) implies  $1' * z = 1' * \bar{\tau}(z)$  and  $x * \bar{\sigma}(1') = x * 1' = x$  by (7); then  $1' * \bar{\sigma}(1' * z) = 1' * \bar{\sigma}(z)$ . Thus  $ML(u)(1' \otimes z) =$

$\sum_{\sigma} e_{\sigma} \otimes (1' * \bar{\sigma}(z)) = \sum_{\sigma} e_{\sigma} \otimes (1' * \bar{\sigma}(1' * z)) = ML(u)(1' \otimes (1' * z))$ ; therefore  $1' \otimes z = 1' \otimes (1' * z)$ . Thus  $z \otimes 1' = (1' * z) \otimes 1'$  and from (7) we obtain  $1' * z = z$  for all  $z \in E$ ; therefore  $1'$  is the identity element for the product  $(x, y) \rightarrow x * y$ . Moreover  $z = 1' * z = \tilde{1}' * 1(z)$  and then  $\tilde{1}(z) = z$  for every  $z \in E$ . Now (9) shows the associativity of the product, and we have  $\sigma(x * y) = \bar{\sigma}(x) * \bar{\sigma}(y)$  and  $\bar{\sigma}\bar{\tau}(z) = \bar{\sigma}\tau(z)$ . If  $r, s \in R$  then  $ML(u)(r1' \otimes s1') = ML(u)(1' \otimes rs1') = \sum_{\sigma} e_{\sigma} \otimes (1' * \bar{\sigma}(rs1')) = \sum_{\sigma} e_{\sigma} \otimes \bar{\sigma}(rs1')$ , and so we have  $r1' * s1' = \tilde{1}(rs1') = rs1'$ . Since  $r1' * x = x * r1' = rx$ , the map  $R \rightarrow E$  ( $r \rightarrow r1'$ ) defines on  $E$  with product  $*$  a structure of  $R$ -algebra, which we shall denote by  $A$ . Also  $\sum_{\sigma} e_{\sigma} \otimes \bar{\sigma}(rs1') = ML(u)(r1' \otimes s1') = ML(u)(rs1' \otimes 1') = \sum_{\sigma} e_{\sigma} \otimes rs1' * \bar{\sigma}(1') = \sum_{\sigma} e_{\sigma} \otimes rs1'$  and therefore  $\bar{\sigma}(r1') = r1', \forall \sigma \in G, r \in R$ . Thus  $G$  acts on  $A$  as a group of  $R$ -algebra automorphisms. Note that  $r1' * x = x * r1' = rx$  implies that  $A$  is free over  $R1'$ . If  $x \in A^G$  we have  $ML(u)(1' \otimes x) = \sum_{\sigma} e_{\sigma} \otimes x = 1 \otimes x = ML(u)(x \otimes 1')$ .

Therefore  $1' \otimes x = x \otimes 1'$ , and we must have  $x \in R1'$ , thus  $A^G = R1'$ .

Since  $M_A: A \otimes A \rightarrow E \otimes A$  is  $ML(u)$  by (8), we conclude that  $A$  is Galois extension of  $R$  with group  $G$ . Clearly the diagram (1) with  $j = \text{identity}$  shows that  $\psi$  (class of  $A$ ) is the class of  $u$ .

We have remarked that if  $u \in U(EG \otimes EG)$  is obtained from a Galois extension by means of the diagram (1) then every  $v \in U(EG \otimes EG)$  equivalent to  $u$  is obtained in that way. It follows therefore that if  $u$  verifies (5) and (6) and  $v$  is equivalent to  $u$  then  $v$  also satisfies (5) and (6). Let  $H$  be the subset of  $C$  of classes whose elements satisfy (5) and (6). Then we have

**Theorem.**  $\psi: E_G^L(R) \rightarrow H$  is bijective.

REMARKS.

- 1) Since  $\psi([E]) = \text{class of } 1$ , we have  $\psi([E]) = \{\Delta_1(v^{-1}). v \otimes v: v \in U(EG)\}$
- 2) The group structure of  $U(EG \otimes EG)$  does not induce a group structure on  $H$ ; in fact, the inverse of an element verifying (5) and (6) does not necessarily verify (5) and (6), nor does the product of two such units. For example, if  $K$  is a field and  $G = Z_2$ , and  $x = -e_{\sigma} + \sigma, x^{-1} = e_1 + \sigma$  then  $y = \Delta_1(x) \cdot x^{-1} \otimes x^{-1}$  verifies (5) and (6) but  $y^{-1}$  does not; if  $K = Z_2$  then  $z = \Delta_1(x^{-1}) \cdot x \otimes x$  verifies (5) and (6) but  $z^2$  does not.
- 3) If  $\alpha$  is an automorphism of  $G$  and  $A$  is a Galois extension of  $R$  with group  $G$ , we can obtain a new structure of Galois extension  $A^{\alpha}$  on  $A$  by defining  $\bar{\sigma}(a) = \alpha(\sigma)(a)$ . If  $\alpha: E \rightarrow E$  is defined by  $\alpha(e_{\sigma}) = e_{\alpha(\sigma)}$ , and  $\alpha: EG \otimes EG \rightarrow EG \otimes EG$  by  $\alpha(a\sigma \otimes b\eta) = \alpha(a)\alpha(\sigma) \otimes \alpha(b)\alpha(\eta)$ , then  $\psi(A^{\alpha}) = \alpha^{-1}\psi(A)$ ; more precisely, if  $u$  is obtained from  $j: A \rightarrow E$ , then  $\alpha^{-1}(u)$  is obtained from  $\alpha^{-1}j: A \rightarrow E$ .

**Abelian Extensions with normal basis.**

If  $A$  is Galois over  $R$  with group  $G$ , we have a commutative diagram

$$\begin{array}{ccc}
A \otimes A & \xrightarrow{M_A} & E \otimes A \\
\sigma \otimes \eta \downarrow & & \downarrow \sigma \eta^{-1} \otimes \sigma \\
A \otimes A & \xrightarrow{M_A} & E \otimes A
\end{array}$$

for every  $\sigma, \eta \in G$ .

Let  $j: A \rightarrow E$  be an  $R$ -module isomorphism, and  $u \in U(EG \otimes EG)$  the corresponding element in diagram (1). Then it follows from  $M_A \cdot \sigma \otimes \eta = \sigma \eta^{-1} \otimes \sigma \cdot M_A$  (and  $M \sigma \otimes \eta = \sigma \eta^{-1} \otimes \sigma M$ ) that  $j$  is an  $RG$ -isomorphism if and only if  $L(u)$  is an  $RG \otimes RG$ -isomorphism, *i.e.* if  $u \in U(RG \otimes RG)$ . Note that there exists an  $RG$ -isomorphism  $A \rightarrow E$  if and only if  $A$  has a normal basis.

Let us recall the definition of Harrison's complex; if  $(RG)^n = RG \otimes \cdots \otimes RG$  ( $n$ -times) we define  $\Delta_i: (RG)^n \rightarrow (RG)^{n+1}$ ,  $i=0, 1, \dots, n+1$  by  $\Delta_0(x) = 1 \otimes x$ ,  $\Delta_{n+1}(x) = x \otimes 1$  and for  $i=1, \dots, n$ ,  $\Delta_i$  is the map induced by  $\Delta_i(\sigma_1 \otimes \cdots \otimes \sigma_n) = \sigma_1 \otimes \cdots \otimes \sigma_i \otimes \sigma_i \otimes \sigma_{i+1} \otimes \cdots \otimes \sigma_n$ . Then the  $\Delta_j$ 's are algebra homomorphisms and therefore  $\Delta_j: U(RG^n) \rightarrow U(RG^{n+1})$ . Since  $RG$  is commutative, setting  $\Delta(x) = \prod_{i=0}^{n+1} \Delta_i(x)^{(-1)^i}$  for  $x \in U(RG)$  we obtain group homomorphisms  $\Delta: U(RG^n) \rightarrow U(RG^{n+1})$  such that  $\Delta \Delta = 0$ ; we thus have a complex whose cohomology groups are denoted by  $H^n(R, G)$ , ([3]).

Note that the  $\Delta_i$ 's defined on  $RG$  and on  $RG \otimes RG$  are the restrictions of the  $\Delta_i$ 's considered before; thus for  $u \in (RG \otimes RG)$  the condition (5) is equivalent to  $\Delta(u) = 1$ ; also if  $u, u' \in U(RG \otimes RG)$  then  $u \sim u'$  is equivalent to  $u' = u \Delta(v)$  for some  $v \in U(RG)$ . If it is known that for  $u \in U(RG \otimes RG)$  (5) implies (6), it follows that there is a one to one correspondence from the set of Galois isomorphism classes of Galois extensions of  $R$  with group  $G$  that have normal basis onto  $H^2(R, G)$ . Now, let us show that (5) implies (6) for  $u \in U(RG \otimes RG)$ : If  $\pi: RG \otimes RG \otimes RG \rightarrow RG \otimes RG$  is the map induced by  $\pi(\sigma \otimes \eta \otimes \tau) = \sigma \otimes \tau$  then  $\pi$  is an  $R$ -algebra homomorphism and from  $\Delta_2(u) \Delta_0(u) = \Delta_3(u) \Delta_1(u)$  we obtain  $u \cdot \pi \Delta_0(u) = \pi \Delta_3(u) \cdot u$ , and then  $\pi \Delta_0(u^{-1}) = \pi \Delta_3(u^{-1})$ . If  $u^{-1} = \sum_{\sigma, \eta} r_{\sigma, \eta} \sigma \otimes \eta$  we have  $\sum_{\sigma, \eta} r_{\sigma, \eta} 1 \otimes \eta = \sum_{\sigma, \eta} r_{\sigma, \eta} \sigma \otimes 1$  and therefore for  $x \in E$ ,  $L(u^{-1})(x \otimes 1) = \sum_{\sigma, \eta} r_{\sigma, \eta} \sigma(x) \otimes 1 = \sum_{\sigma, \eta} r_{\sigma, \eta} x \otimes 1 = x \otimes (\sum_{\sigma, \eta} r_{\sigma, \eta})$ . Since  $\sum_{\sigma, \eta} r_{\sigma, \eta} = m \cdot u^{-1}(1 \otimes 1)$ , we have (6).

Recall now that the set  $E_G(R)$  of Galois isomorphism classes of all Galois extensions of  $R$  with group  $G$  is a group, whose product is defined as follows: Let  $g = \{\sigma, \sigma^{-1}\} \in G \times G$ . If  $A, B \in E_G(R)$  then  $A \cdot B = (A \otimes B)^g$  with  $\sigma \in G$  acting on  $A \cdot B$  as the restriction of  $\sigma \otimes 1$ . (see [1], [4]). The subset  $E_G^c(R)$  of extensions with normal basis is a subgroup of  $E_G(R)$ ; indeed, if  $A$  and  $B$  have normal bases there exist  $RG$ -isomorphisms  $j_A: A \rightarrow E$ ,  $j_B: B \rightarrow E$ , then  $j_A \otimes j_B: (A \otimes B)^g \rightarrow (E \otimes E)^g$  and it is an  $RG$ -isomorphism. Since  $t: E \rightarrow (E \otimes E)^g$  given by  $t(e_\sigma) = \sum_{\alpha} e_{\sigma\alpha} \otimes e_{\alpha^{-1}}$  is a Galois isomorphism, we obtain that  $j_{AB} = t^{-1} \cdot j_A \otimes j_B|_{A \cdot B}$ :

$A \cdot B \rightarrow E$  is an  $RG$ -isomorphism, i.e.  $A \cdot B$  has a normal basis. On the other side, if  $u, v \in U(RG \otimes RG)$  are the cocycles associated to  $j_A$  and  $j_B$  respectively, it is easy to verify that the cocycle associated to  $j_{AB}$  is  $u \cdot v$ . Thus we have ([2], [4])

**Proposition.** *Let  $R$  be a commutative ring, and  $G$  a finite abelian group. Then there is an isomorphism  $\psi: E_G^n(R) \rightarrow H^2(R, G)$ ; if  $A$  has a normal basis and  $j: A \rightarrow E$  is an  $RG$ -isomorphism, then  $\psi([A])$  is the cohomology class of the cocycle  $u$  defined by the diagram (1).*

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### References

- [1] S. Chase, D. Harrison and A. Rosenberg: *Galois theory and cohomology of commutative rings*, Mem. Amer. Math. Soc. No. 52 (1965).
- [2] S. Chase and A. Rosenberg: *A theorem of Harrison, Kummer theory and Galois algebras*, Nagoya Math. J. **27** (1966), 663–685.
- [3] D. Harrison: *Abelian extensions of arbitrary fields*, Trans. Amer. Math. Soc. **106** (1963), 230–235.
- [4] M. Orzech: *Cohomological description of Abelian Galois extensions*, Trans. Amer. Math. Soc. **137**(1969), 481–500.
- [5] P. Wolf: *Algebraische Theorie der Galoischen Algebren*, Deutscher Verlag der Wissenschaften, Berlin, 1956.

