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ON RIEMANNIAN MANIFOLDS ADMITTING CERTAIN STRICTLY CONVEX FUNCTIONS

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1. Introduction. Let M be an m-dimensional connected complete Riemannian manifold with metric g. For a smooth function f on M, the *Hessian* D^2f of f is defined by $D^2f(X,Y)=X(Yf)-D_XY\cdot f(X,Y\in TM)$. By a theorem of H.W. Wissner ([5; Satz. II. 1.3]), if there is a smooth function f on M such that $D^2f=g$ on M, then M is isometric to Euclidean space. In this note, we shall prove that if the Hessian of a smooth function f on M is close enough to g, then M is quasi-isometric to Euclidean space in the following sense: There exists a diffeomorphism $F: M\to R^m$ and some positive constant μ such that for each tangent vector X on M, $\mu^{-1}||X||_M\leqq||F_*X||_{R^m}\leqq||X||_M$. Our result contains the above theorem by Wissner as a special case ($\mu=1$), and generalises Yagi's theorem ([7]). Our theorem is stated as follows.

Theorem. *Let M be an m-dimensional connected complete Riemannian manifold with metric g. Suppose there exists a smooth function f on M which satisfies the following conditions*:

$$\text{(i)}\quad (1-H_1(f(x)))g(X,X)\leqq\frac{1}{2}D^2f(X,X)\leqq(1+H_2(f(x)))g(X,X)\,,$$

where $X\in T_xM(x\in M)$ and each H_i $(i=1,2)$ is a nonnegative continuous function on R,

$$\text{(ii)}\quad 1-H_1(t)>0 \text{ for } t\in R \text{ and } \lim_{t\to\infty}H_i(t)=0\ (i=1,2),$$

$$\text{(iii)}\quad \begin{cases} \displaystyle\int^{\infty}H_i(s)/s\,ds<+\infty\,, \\ \displaystyle\int^{\infty}\Big(\int_0^s H_i(u)du/s^2\Big)ds<+\infty\ (i=1,2)\,. \end{cases}$$

Then M is quasi-isometric to Euclidean space.

2. Proof of theorem and corollaries. Let M be an m-dimensional connected complete Riemannian manifold with metric g.

Lemma 1. *Let M and g be as above. Let f be a smooth function on M such* *that the eigenvalues of D^2f are bounded from below by some positive constant 2ν outside a compact subset C. Then f is an exhaustion function, that is, $\{x\in M: f(x) \leqq t\}$ is compact for each $t\in R$. In particular, f takes the minimum on M.*

Proof. Suppose $f \gneqq \lambda=\inf\{f(x): x\in M\}$ $(-\infty\leqq\lambda<\infty)$. Then there is a divergent sequence $\{p_n\}_{n\in N}$ in M with $\lim_{n\to\infty} f(p_n)=\lambda$. Fix any point $o\in M$. Let $\gamma_n: [0, 1_n]\to M$ be a minimizing geodesic joining o to p_n for each $n\in N$, where $1_n=\mathrm{dis}(o, p_n)$. Then by the assumption of Lemma 1, we can choose sufficiently large N and T so that $\gamma_n(t)\in M-C$ and $f(\gamma_n)''(t)=D^2f(\dot\gamma_n, \dot\gamma_n)(t)\geqq\nu$ for any $n\geqq N$ and $t\in[T, 1_n]$. This implies $f(\gamma_n)(t)\geqq f(\gamma_n)(T)+f(\gamma_n)'(T)(t-T)+\frac{\nu}{2}(t-T)^2$ for $t\in[T, 1_n]$. Taking $t=1_n$, we have $f(p_n)\geqq f(\gamma_n)(T)+f(\gamma_n)'(T)(1_n-T)+\frac{\nu}{2}(1_n-T)^2$. Since the distance between o and $\gamma_n(T)$ equals T for each $n\geqq N$, $\{f(\gamma_n)'(T)\}_{n\in N}$ is bounded. In the preceding inequality, the left side tends to λ and the right side goes to infinity as $n\to\infty$. This is a contradiction. Therefore f takes the minimum at some points. By the same way, we see that f is an exhaustion function on M. This completes the proof of Lemma 1.

Proof of Theorem. By (ii) in Theorem and Lemma 1, we can see that f is a strictly convex exhaustion function on M. Let λ be the minimum of f on M and $o\in M$ be the only one point such that $f(o)=\lambda$. Set $\tilde f(x)=f(x)-\lambda$, $k(x)=\tilde f(x)^{1/2}$, and $h_i(t)=H_i(t^2+\lambda)$ $(i=1,2)$. Then the conditions (i)~(iii) in Theorem can be rewritten as follows:

$$\text{(i)}' \quad (1-h_1(k(x)))g(X, X)\leqq\frac{1}{2}D^2\tilde f(X, X)\leqq(1+h_2(k(x)))g(X, X),$$

where $X\in T_xM$ $(x\in M)$ and each $h_i(t)$ $(i=1,2)$ is a nonnegative continuous function on $[0,\infty)$,

$$\text{(ii)}' \quad 1-h_1(t)>0 \text{ for } t\in 0 \text{ and } \lim_{t\to\infty} h_i(t)=0 \ (i=1,2)$$

$$\text{(iii)}' \quad \begin{cases} \int^\infty h_i(s)/s\, ds<+\infty\,, \\ \int^\infty(\int_0^s u\cdot h_i(u)\,du/s^3)\,ds<+\infty \ (i=1,2)\,. \end{cases}$$

Since $o\in M$ is a nondegenerate critical point of $\tilde f$, there exists a coordinate system $x: U\to R^m$, where U is a neighborhood of o, with $x(o)=(0,\cdots,0)$ and $\tilde f(p)=\sum_{i=1}^m x_i(p)^2$ for all $p\in U$ where $x(p)=(x_1(p),\cdots,x_m(p))$ (cf. [3; p. 6]). Let δ be a positive number such that $\{(x_1,\cdots,x_m)\in R^m: \sum_{i=1}^m x_i^2<\delta\}\subset x(U)$. We construct a metric $\tilde g$ on M with $\tilde g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)=\delta_{ij}$ on $U_{\delta/2}=\left\{p\in U: \sum_{i=1}^m x_i(p)^2<\frac{\delta}{2}\right\}$ and

$\tilde{g}=g$ on $M-U$; such a metric can be constructed by the standard partition-of-unity extention process. By the construction of $\tilde{g}$, it suffices to prove that M with the metric $\tilde{g}$ is quasi-isometric to Euclidean space. Let $a>0$ be such that $k^{-1}(a)=\{x\in M: k(x)=a\}\subset U_\delta$. For each $p\in k^{-1}(a)$, let $\lambda_p(t)$ be the maximal integral curve of grad $k/\|\text{grad } k\|^2$ with $\lambda_p(a)=p$. Then we have $\frac{d}{dt}k(\lambda_p(t))=1$ and hence $k(\lambda_p(t))=t$ $(t>0)$. Define $F_1: k^{-1}(a)\times(0,\infty)\to M-\{o\}$ by $F_1(p,t)=\lambda_p(t)$, and $F_2: k^{-1}(a)\times(0,\infty)\to R^m-\{(0,\cdots,0)\}$ by $F_2(p,t)=\frac{t}{a}(x_1(p),\cdots,x_m(p))$. It follows that F_1 and F_2 are diffeomorphisms and $F_2\circ F_1^{-1}$ can be extended to the diffeomorphism $F: M\to R^m$ (cf. [3; p. 221]). We shall now show that F is a required quasi-isometry. Let $\lambda: [0,\varepsilon]\to k^{-1}(a)$ be any smooth regular curve. Define a smooth map $G: [0,\varepsilon]\times[0,\infty)\to M-\{o\}$ by $G(t,s)=F_1(\lambda(t),s)$, and vector fields X and Y along G by $X=F^*\left(\frac{\partial}{\partial s}\right)=\text{grad } k/\|\text{grad } k\|^2$ and $Y=F_*\left(\frac{\partial}{\partial t}\right)$. Fix any $b>0$ such that $k^{-1}(b)\subset M-U_\delta$. Then we have for $s\geqq b$,

$$(1)\qquad \begin{aligned}\frac{\partial}{\partial s}\|Y\|(t,s) &= (D_X Y, Y)/\|Y\|(t,s)\\ &= (D_Y X, Y)/\|Y\|(t,s)\\ &= (D_Y \text{grad } k, Y)/\|Y\|\,\|\text{grad } k\|^2(t,s)\\ &= D^2k(Y,Y)/\|Y\|\,\|\text{grad } k\|^2(t,s)\end{aligned}$$

By the assumption (i)′, we have on $\{x\in M: k(x)\geqq b\}$

$$\{(1-h_1(k))\cdot g-dk\cdot dk\}/k\leqq D^2k\leqq\{(1+h_2(k))\cdot g-dk\cdot dk\}/k\,.$$

Therefore we get

$$(2)\qquad \{(1-h_1(k))/k\cdot\|Y\|^2\}(t,s)\leqq D^2k(Y,Y)(t,s)\leqq\{(1+h_2(k))/k\cdot\|Y\|^2\}(t,s)$$

for $s\geqq b$. Now we need the following

Lemma 2. *On $\{x\in M: k(x)\geqq b\}$, we have the following estimate*:

$$(3)\qquad \begin{aligned}&1-2\int_b^k u\cdot h_1(u)\,du/k^2+(B-4b^2)/4k^2\leqq\\ &\|\text{grad } k\|^2\leqq 1+2\int_b^k u\cdot h_2(u)\,du/k^2+(C-4b^2)/4k^2\,,\end{aligned}$$

where $B=\min\{\|\text{grad } \tilde{f}\|^2(x): x\in k^{-1}(b)\}$ and $C=\max\{\|\text{grad } \tilde{f}\|^2(x): x\in k^{-1}(b)\}$.

We leave a proof of this lemma later. By (1), (2) and (3), we have

$$(4)\qquad (1-\chi_1(s))/s\cdot\|Y\|\leqq(t,s)\leqq\frac{\partial}{\partial s}\|Y\|(t,s)\leqq(1+\chi_2(s))/s\cdot\|Y\|(t,s)\,,$$

where $\quad \chi_1(s)=(8\int_0^s u\cdot h_2(u)\,du+4s^2h_1(s)+C-4b^2)/$

$$(8\int_0^s u\cdot h_2(u)\,du+4s^2+C-4b^2)$$

and $\quad \chi_2(s)=(8\int_0^s u\cdot h_1(u)\,du+4s^2h_2(s)-B+4b^2)/$

$$(-8\int_0^s u\cdot h_1(u)\,du+4s^2+B-4b^2)\,.$$

It follows that

$$\frac{s}{b}\exp\Big(\int_b^s -\chi_1(u)/u\,du\Big)\leqq \|Y\|(t,s)/\|Y\|(t,b)\leqq \frac{s}{b}\exp\Big(\int_b^s \chi_2(u)/u\,du\Big)\,. \tag{5}$$

By the assumption (iii)$'$, there exists some positive constant ξ such that

$$\exp\Big(\int_b^s \chi_2(u)/u\,du\Big)\leqq \xi \quad\text{and}\quad \xi^{-1}\leqq \exp\Big(\int_b^s -\chi_1(u)/u\,du\Big)\,. \tag{6}$$

By (5) and (6), we get

$$\xi^{-1}\|Y\|(t,0)/b\leqq \|Y\|(t,s)/s\leqq \xi\|Y\|(t,0)/b\,. \tag{7}$$

The assumption (iii)$'$ implies that for some positive constant ζ

$$\zeta^{-1}\leqq \|\mathrm{grad}\,k\|\leqq \zeta\,. \tag{8}$$

Inequalities (7) and (8) show that $F: M\to R^m$ is quasi-isometric. This completes the proof of Theorem.

Proof of Lemma 2. For each $p\in k^{-1}(b)$, let $\gamma_p(t)$ be the maximal integral curve of $\mathrm{grad}\,\tilde f/\|\mathrm{grad}\,\tilde f\|^2$ with $\gamma_p(0)=p$. Then $\frac{d}{dt}\tilde f(\gamma_p(t))=1$ and hence $\tilde f(\gamma_p(t))=t+b^2$ $(t\geqq 0)$. From the assumption (i)$'$, we obtain the inequality:

$$(1-h_1(k))\|\mathrm{grad}\,\tilde f\|^2\leqq \frac{1}{2}D^2\tilde f(\mathrm{grad}\,\tilde f,\,\mathrm{grad}\,f)\leqq (1+h_2(k))\|\mathrm{grad}\,\tilde f\|^2$$

on $\{x\in M: k(x)\geqq b\}$. Noting $D^2\tilde f(\mathrm{grad}\,\tilde f,\,\mathrm{grad}\,\tilde f)=\frac{1}{2}\,\mathrm{grad}\,\tilde f(\|\mathrm{grad}\,\tilde f\|^2)$ we see

$$4(1-h_1(\sqrt{t+b^2}))\leqq \frac{d}{dt}\|\mathrm{grad}\,\tilde f\|^2(\gamma_p(t))\leqq 4(1+h_2(\sqrt{t+b^2}))$$

for $t\geqq 0$. Therefore we get the inequalities:

$$\begin{aligned}8\int_b^{\sqrt{t+b^2}} u(1-h_1(u))\,du+\|\mathrm{grad}\,\tilde f\|^2(p)&\leqq \|\mathrm{grad}\,\tilde f\|^2(\gamma_p(t))\\ &\leqq 8\int_b^{\sqrt{t+b^2}} u(1+h_2(u))\,du+\|\mathrm{grad}\,\tilde f\|^2(p)\end{aligned}$$

for $t\geqq 0$. Since $k(\gamma_p(t))=\sqrt{t+b^2}$ and $\mathrm{grad}\,f=2\,k\cdot\mathrm{grad}\,k$ we get the required estimate. This completes the proof of Lemma 2.

By Moser's theorem ([4]), we have the following

Corollary 1. *Let M be as in Theorem. Then on M there are no positive harmonic functions other than constants. If M is in addition a Kaehler manifold, then it has no nonconstant bounded holomorphic functions.*

We shall derive the theorem of Wissner ([5; Satz. II. 1.3]) as follows.

Corollary 2. *If M is a connected complete Riemannian manifold and f is a smooth function on M whose Hessian is equal to the metric tensor on M, then M is isometric to Euclidean space.*

Proof. By Lemma 1 and the strictly convexity of f, f attains its minimum λ at the one and only one point $o\in M$. Replacing f for $\frac{1}{2}(f-\lambda)$, we may assume that f is a smooth function such that $D^2f=\frac{1}{2}g$ on M and $f(x)\gneqq f(o)=0$ for any $x\in M-\{o\}$. Let $\gamma\colon [0,\infty)\to M$ be any arc-length parametrized geodesic issuing from o. Then $D^2f(\dot{\gamma},\dot{\gamma})=f(\gamma(t))''=\frac{1}{2}$ for $t\geqq 0$ and hence $f(\gamma(t))=t^2$. That is, $f(x)$ equals $\mathrm{dis}(x,o)^2$ near $o\in M$. Therefore the same arguments as in the proof of Theorem can be applied without any change of metric and we see that the exponential mapping at $o\in M$ is an isometry. This completes the proof of Corollary 2.

EXAMPLE. Let M be C^m with the Kaehler metric g defined by $g_{i\bar{j}}=\frac{\partial^2}{\partial z_i\partial \bar{z}_j}(|z|^2+\log(1+|z|^2))$, where $(z_1,\cdots,z_m)$ is the canonical holomorphic coordinates on C^m and $|z|^2=|z_1|^2+\cdots+|z_m|^2$. Then M is a Kaehler manifold with a pole $o=(0,\cdots 0)$, that is, the exponential mapping at o induces a global diffeomorphism between T_0M and M. Let $r(x)$ be the distance between o and $x\in M$. By computing the (radial) curvatures, we can see r^2 satisfies all the conditions required in Theorem (cf. [1; Theorem C]).

Corollary 3 ([7]). *Let M be a Riemannian manifold with a pole $o\in M$ and $r(x)$ be the distance between o and $x\in M$. Suppose there exists a continuous function $h\colon [0,\infty)\to[0,\infty)$ satisfying the following conditions:*

(i) $$(1-h(r(x)))g(X,X)\leqq\frac{1}{2}D^2r^2(X,X)\leqq(1+h(r(x)))g(X,X)$$

where $X\in T_xM(x\in M)$, and

(ii) $$\int_1^\infty h(t)/t\,dt<+\infty\,.$$

Then the exponential mapping at $o \in M$ is quasi-isometric.

Proof. Noting $\|\operatorname{grad} r\| = 1$ on $M - \{o\}$, we see the result easily from the proof of Theorem.

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