Kasue, A.
Osaka J. Math.
18 (1981), 577-582

ON RIEMANNIAN MANIFOLDS ADMITTING CERTAIN STRICTLY CONVEX FUNCTIONS

ATSUSHI KASUE

(Received March 18, 1980)

1. Introduction. Let M be an m-dimensional connected complete Riemannian manifold with metric g. For a smooth function f on M, the *Hessian* D^2f of f is defined by $D^2f(X,Y)=X(Yf)-D_XY\cdot f(X,Y\in TM)$. By a theorem of H.W. Wissner ([5; Satz. II. 1.3]), if there is a smooth function f on M such that $D^2f=g$ on M, then M is isometric to Euclidean space. In this note, we shall prove that if the Hessian of a smooth function f on M is close enough to g, then M is quasi-isometric to Euclidean space in the following sense: There exists a diffeomorphism $F: M\rightarrow R^m$ and some positive constant μ such that for each tangent vector X on M, $\mu^{-1}||X||_M\leqq||F_*X||_{R^m}\leqq||X||_M$. Our result contains the above theorem by Wissner as a special case ($\mu=1$), and generalises Yagi's theorem ([7]). Our theorem is stated as follows.

Theorem. *Let M be an m-dimensional connected complete Riemannian manifold with metric g. Suppose there exists a smooth function f on M which satisfies the following conditions*:

$$\text{(i)}\quad (1-H_1(f(x)))g(X,X)\leqq\frac{1}{2}D^2f(X,X)\leqq(1+H_2(f(x)))g(X,X),$$

where $X\in T_xM(x\in M)$ and each H_i $(i=1,2)$ is a nonnegative continuous function on R,

$$\text{(ii)}\quad 1-H_1(t)>0 \textit{ for } t\in R \textit{ and } \lim_{t\to\infty}H_i(t)=0\ (i=1,2),$$

$$\text{(iii)}\quad \begin{cases}\int^\infty H_i(s)/s\,ds<+\infty\,,\\ \int^\infty\left(\int_0^s H_i(u)du/s^2\right)ds<+\infty\ (i=1,2)\,.\end{cases}$$

Then M is quasi-isometric *to Euclidean space.*

2. Proof of theorem and corollaries. Let M be an m-dimensional connected complete Riemannian manifold with metric g.

Lemma 1. *Let M and g be as above. Let f be a smooth function on M such* *that the eigenvalues of $D^2 f$ are bounded from below by some positive constant 2ν outside a compact subset C. Then f is an exhaustion function, that is, $\{x\in M: f(x) \leqq t\}$ is compact for each $t\in R$. In particular, f takes the minimum on M.*

Proof. Suppose $f \gneqq \lambda = \inf\{f(x): x\in M\}$ $(-\infty \leqq \lambda < \infty)$. Then there is a divergent sequence $\{p_n\}_{n\in N}$ in M with $\lim_{n\to\infty} f(p_n)=\lambda$. Fix any point $o\in M$. Let γ_n: $[0, 1_n]\to M$ be a minimizing geodesic joining o to p_n for each $n\in N$, where $1_n=\mathrm{dis}(o, p_n)$. Then by the assumption of Lemma 1, we can choose sufficiently large N and T so that $\gamma_n(t)\in M-C$ and $f(\gamma_n)''(t)=D^2f(\dot\gamma_n, \dot\gamma_n)(t)\geqq \nu$ for any $n\geqq N$ and $t\in[T, 1_n]$. This implies $f(\gamma_n)(t)\geqq f(\gamma_n)(T)+f(\gamma_n)'(T)(t-T)+\frac{\nu}{2}(t-T)^2$ for $t\in[T,1_n]$. Taking $t=1_n$, we have $f(p_n)\geqq f(\gamma_n)(T)+f(\gamma_n)'(T)(1_n-T)+\frac{\nu}{2}(1_n-T)^2$. Since the distance between o and $\gamma_n(T)$ equals T for each $n\geqq N$, $\{f(\gamma_n)'(T)\}_{n\in N}$ is bounded. In the preceding inequality, the left side tends to λ and the right side goes to infinity as $n\to\infty$. This is a contradiction. Therefore f takes the minimum at some points. By the same way, we see that f is an exhaustion function on M. This completes the proof of Lemma 1.

Proof of Theorem. By (ii) in Theorem and Lemma 1, we can see that f is a strictly convex exhaustion function on M. Let λ be the minimum of f on M and $o\in M$ be the only one point such that $f(o)=\lambda$. Set $\tilde f(x)=f(x)-\lambda$, $k(x)=\tilde f(x)^{1/2}$, and $h_i(t)=H_i(t^2+\lambda)$ $(i=1,2)$. Then the conditions (i)~(iii) in Theorem can be rewritten as follows:

$$\text{(i)}'\quad (1-h_1(k(x)))g(X,X)\leqq \frac{1}{2}D^2\tilde f(X,X)\leqq(1+h_2(k(x)))g(X,X)\,,$$

where $X\in T_xM$ $(x\in M)$ and each $h_i(t)$ $(i=1,2)$ is a nonnegative continuous function on $[0,\infty)$,

$$\text{(ii)}'\quad 1-h_1(t)>0 \text{ for } t\in 0 \text{ and } \lim_{t\to\infty} h_i(t)=0\ (i=1,2)$$

$$\text{(iii)}'\quad \begin{cases} \displaystyle\int^\infty h_i(s)/s\, ds<+\infty\,, \\ \displaystyle\int^\infty \Big(\int_0^s u\cdot h_i(u)\,du/s^3\Big)\, ds<+\infty\ (i=1,2)\,. \end{cases}$$

Since $o\in M$ is a nondegenerate critical point of $\tilde f$, there exists a coordinate system x: $U\to R^m$, where U is a neighborhood of o, with $x(o)=(0,\cdots,0)$ and $\tilde f(p)=\sum_{i=1}^m x_i(p)^2$ for all $p\in U$ where $x(p)=(x_1(p),\cdots,x_m(p))$ (cf. [3; p. 6]). Let δ be a positive number such that $\{(x_1,\cdots,x_m)\in R^m: \sum_{i=1}^m x_i^2<\delta\}\subset x(U)$. We construct a metric $\tilde g$ on M with $\tilde g\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\Big)=\delta_{ij}$ on $U_{\delta/2}=\Big\{p\in U: \sum_{i=1}^m x_i(p)^2<\frac{\delta}{2}\Big\}$ and

$\tilde{g}=g$ on $M-U$; such a metric can be constructed by the standard partition-of-unity extention process. By the construction of $\tilde{g}$, it suffices to prove that M with the metric $\tilde{g}$ is quasi-isometric to Euclidean space. Let $a>0$ be such that $k^{-1}(a)=\{x\in M: k(x)=a\}\subset U_\delta$. For each $p\in k^{-1}(a)$, let $\lambda_p(t)$ be the maximal integral curve of grad $k/\|\text{grad } k\|^2$ with $\lambda_p(a)=p$. Then we have $\frac{d}{dt}k(\lambda_p(t))=1$ and hence $k(\lambda_p(t))=t\ (t>0)$. Define $F_1: k^{-1}(a)\times(0,\infty)\to M-\{o\}$ by $F_1(p,t)=\lambda_p(t)$, and $F_2: k^{-1}(a)\times(0,\infty)\to R^m-\{(0,\cdots,0)\}$ by $F_2(p,t)=\frac{t}{a}(x_1(p),\cdots,x_m(p))$. It follows that F_1 and F_2 are diffeomorphisms and $F_2\circ F_1^{-1}$ can be extended to the diffeomorphism $F: M\to R^m$ (cf. [3; p. 221]). We shall now show that F is a required quasi-isometry. Let $\lambda: [0,\varepsilon]\to k^{-1}(a)$ be any smooth regular curve. Define a smooth map $G: [0,\varepsilon]\times[0,\infty)\to M-\{o\}$ by $G(t,s)=F_1(\lambda(t),s)$, and vector fields X and Y along G by $X=F^*\left(\frac{\partial}{\partial s}\right)=\text{grad } k/\|\text{grad } k\|^2$ and $Y=F_*\left(\frac{\partial}{\partial t}\right)$. Fix any $b>0$ such that $k^{-1}(b)\subset M-U_\delta$. Then we have for $s\geqq b$,

$$(1)\qquad \begin{aligned}\frac{\partial}{\partial s}\|Y\|(t,s) &= (D_XY, Y)/\|Y\|(t,s)\\ &= (D_YX, Y)/\|Y\|(t,s)\\ &= (D_Y\text{grad } k, Y)/\|Y\|\ \|\text{grad } k\|^2(t,s)\\ &= D^2k(Y,Y)/\|Y\|\ \|\text{grad } k\|^2(t,s)\end{aligned}$$

By the assumption (i)′, we have on $\{x\in M: k(x)\geqq b\}$

$$\{(1-h_1(k))\cdot g-dk\cdot dk\}/k\leqq D^2k\leqq\{(1+h_2(k))\cdot g-dk\cdot dk\}/k\,.$$

Therefore we get

$$(2)\qquad \{(1-h_1(k))/k\cdot\|Y\|^2\}(t,s)\leqq D^2k(Y,Y)(t,s)\leqq\{(1+h_2(k))/k\cdot\|Y\|^2\}(t,s)$$

for $s\geqq b$. Now we need the following

Lemma 2. *On* $\{x\in M: k(x)\geqq b\}$, *we have the following estimate*:

$$(3)\qquad \begin{aligned}&1-2\int_b^k u\cdot h_1(u)\,du/k^2+(B-4b^2)/4k^2\leqq\\ &\|\text{grad } k\|^2\leqq 1+2\int_b^k u\cdot h_2(u)\,du/k^2+(C-4b^2)/4k^2\,,\end{aligned}$$

where $B=\min\{\|\text{grad }\tilde{f}\|^2(x): x\in k^{-1}(b)\}$ and $C=\max\{\|\text{grad }\tilde{f}\|^2(x): x\in k^{-1}(b)\}$.

We leave a proof of this lemma later. By (1), (2) and (3), we have

$$(4)\qquad (1-\chi_1(s))/s\cdot\|Y\|\leqq(t,s)\leqq\frac{\partial}{\partial s}\|Y\|(t,s)\leqq(1+\chi_2(s))/s\cdot\|Y\|(t,s)\,,$$

where $\chi_1(s)=(8\int_0^s u\cdot h_2(u)\,du+4s^2h_1(s)+C-4b^2)/$
$$(8\int_0^s u\cdot h_2(u)\,du+4s^2+C-4b^2)$$
and $\chi_2(s)=(8\int_0^s u\cdot h_1(u)\,du+4s^2h_2(s)-B+4b^2)/$
$$(-8\int_0^s u\cdot h_1(u)\,du+4s^2+B-4b^2)\,.$$

It follows that

$$\frac{s}{b}\exp\left(\int_b^s -\chi_1(u)/u\,du\right)\leqq ||Y||(t,s)/||Y||(t,b)\leqq\frac{s}{b}\exp\left(\int_b^s \chi_2(u)/u\,du\right)\,. \tag{5}$$

By the assumption (iii)′, there exists some positive constant ξ such that

$$\exp\left(\int_b^s \chi_2(u)/u\,du\right)\leqq\xi \quad\text{and}\quad \xi^{-1}\leqq\exp\left(\int_b^s -\chi_1(u)/u\,du\right)\,. \tag{6}$$

By (5) and (6), we get

$$\xi^{-1}||Y||(t,0)/b\leqq||Y||(t,s)/s\leqq\xi||Y||(t,0)/b\,. \tag{7}$$

The assumption (iii)′ implies that for some positive constant ζ

$$\zeta^{-1}\leqq||\mathrm{grad}\,k||\leqq\zeta\,. \tag{8}$$

Inequalities (7) and (8) show that $F\colon M\to R^m$ is quasi-isometric. This completes the proof of Theorem.

Proof of Lemma 2. For each $p\in k^{-1}(b)$, let $\gamma_p(t)$ be the maximal integral curve of $\mathrm{grad}\,\tilde{f}/||\mathrm{grad}\,\tilde{f}||^2$ with $\gamma_p(0)=p$. Then $\frac{d}{dt}\tilde{f}(\gamma_p(t))=1$ and hence $\tilde{f}(\gamma_p(t))=t+b^2$ $(t\geqq 0)$. From the assumption (i)′, we obtain the inequality:

$$(1-h_1(k))||\mathrm{grad}\,\tilde{f}||^2\leqq\frac{1}{2}D^2\tilde{f}(\mathrm{grad}\,\tilde{f},\,\mathrm{grad}\,f)\leqq(1+h_2(k))||\mathrm{grad}\,\tilde{f}||^2$$

on $\{x\in M\colon k(x)\geqq b\}$. Noting $D^2\tilde{f}(\mathrm{grad}\,\tilde{f},\,\mathrm{grad}\,\tilde{f})=\frac{1}{2}\,\mathrm{grad}\,\tilde{f}(||\mathrm{grad}\,\tilde{f}||^2)$ we see

$$4(1-h_1(\sqrt{t+b^2}))\leqq\frac{d}{dt}||\mathrm{grad}\,\tilde{f}||^2(\gamma_p(t))\leqq 4(1+h_2(\sqrt{t+b^2}))$$

for $t\geqq 0$. Therefore we get the inequalities:

$$\begin{aligned}8\int_b^{\sqrt{t+b^2}} u(1-h_1(u))\,du+||\mathrm{grad}\,\tilde{f}||^2(p)&\leqq||\mathrm{grad}\,\tilde{f}||^2(\gamma_p(t))\\&\leqq 8\int_b^{\sqrt{t+b^2}} u(1+h_2(u))\,du+||\mathrm{grad}\,\tilde{f}||^2(p)\end{aligned}$$

for $t\geqq 0$. Since $k(\gamma_p(t))=\sqrt{t+b^2}$ and $\mathrm{grad}\,f=2\,k\cdot\mathrm{grad}\,k$ we get the required estimate. This completes the proof of Lemma 2.

By Moser's theorem ([4]), we have the following

Corollary 1. *Let M be as in Theorem. Then on M there are no positive harmonic functions other than constants. If M is in addition a Kaehler manifold, then it has no nonconstant bounded holomorphic functions.*

We shall derive the theorem of Wissner ([5; Satz. II. 1.3]) as follows.

Corollary 2. *If M is a connected complete Riemannian manifold and f is a smooth function on M whose Hessian is equal to the metric tensor on M, then M is isometric to Euclidean space.*

Proof. By Lemma 1 and the strictly convexity of f, f attains its minimum λ at the one and only one point $o \in M$. Replacing f for $\frac{1}{2}(f-\lambda)$, we may assume that f is a smooth function such that $D^2 f = \frac{1}{2} g$ on M and $f(x) \gneqq f(o) = 0$ for any $x \in M - \{o\}$. Let $\gamma\colon [0, \infty) \to M$ be any arc-length parametrized geodesic issuing from o. Then $D^2 f(\dot{\gamma}, \dot{\gamma}) = f(\gamma(t))'' = \frac{1}{2}$ for $t \geqq 0$ and hence $f(\gamma(t)) = t^2$. That is, $f(x)$ equals $\mathrm{dis}(x, o)^2$ near $o \in M$. Therefore the same arguments as in the proof of Theorem can be applied without any change of metric and we see that the exponential mapping at $o \in M$ is an isometry. This completes the proof of Corollary 2.

EXAMPLE. Let M be C^m with the Kaehler metric g defined by $g_{i\bar{j}} = \frac{\partial^2}{\partial z_i \partial \bar{z}_j}(|z|^2 + \log(1+|z|^2))$, where $(z_1, \cdots, z_m)$ is the canonical holomorphic coordinates on C^m and $|z|^2 = |z_1|^2 + \cdots + |z_m|^2$. Then M is a Kaehler manifold with a pole $o = (0, \cdots 0)$, that is, the exponential mapping at o induces a global diffeomorphism between $T_0 M$ and M. Let $r(x)$ be the distance between o and $x \in M$. By computing the (radial) curvatures, we can see r^2 satisfies all the conditions required in Theorem (cf. [1; Theorem C]).

Corollary 3 ([7]). *Let M be a Riemannian manifold with a pole $o \in M$ and $r(x)$ be the distance between o and $x \in M$. Suppose there exists a continuous function $h\colon [0, \infty) \to [0, \infty)$ satisfying the following conditions:*

$$\text{(i)} \qquad (1-h(r(x)))g(X, X) \leqq \frac{1}{2} D^2 r^2(X, X) \leqq (1+h(r(x)))g(X, X)$$

where $X \in T_x M (x \in M)$, and

$$\text{(ii)} \qquad \int_1^\infty h(t)/t \, dt < +\infty .$$

Then the exponential mapping at $o \in M$ is quasi-isometric.

Proof. Noting $\|\operatorname{grad} r\|=1$ on $M-\{o\}$, we see the result easily from the proof of Theorem.

References

[1] R.E. Greene and H. Wu: Function theory on manifolds which posses a pole, Mathematics Lecture Notes No. 699, Springer-Verlag, 1979.

[2] R.E. Greene and H. Wu: *C^∞ convex function and manifolds of positive curvature,* Acta Math. **137** (1976), 209–245.

[3] J. Milnor: Morse theory, Ann. of Math. Studies, No. 51, Princeton University Press, Princeton, 1963.

[4] J. Moser: *On Harnack's theorem for elliptic differential equations,* Comm. Pure Appl. Math. **14** (1961), 577–591.

[5] H.W. Wissner: *Geodätische Konvexität in Riemannsche Mannigfältigkeiten,* Doctoral Disseration, Berlin, 1979.

[6] H. Wu: *On a problem concerning the intrinsic characterization of C^n*, (to appear).

[7] K. Yagi: *On the Hessian of the square of the distance on a manifold with a pole,* Proc. Japan Acad. Ser. A **56** (1980), 332–337.

Department of Mathematics
University of Tokyo
Hongo, Tokyo 113
Japan