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## THE TWISTOR SPACES OF A PARA-QUATERNIONIC KÄHLER MANIFOLD

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### Abstract

We develop the twistor theory of  $G$ -structures for which the (linear) Lie algebra of the structure group contains an involution, instead of a complex structure. The twistor space  $Z$  of such a  $G$ -structure is endowed with a field of involutions  $\mathcal{J} \in \Gamma(\text{End } TZ)$  and a  $\mathcal{J}$ -invariant distribution  $\mathcal{H}_Z$ . We study the conditions for the integrability of  $\mathcal{J}$  and for the (para-)holomorphicity of  $\mathcal{H}_Z$ . Then we apply this theory to para-quaternionic Kähler manifolds of non-zero scalar curvature, which admit two natural twistor spaces  $(Z^\epsilon, \mathcal{J}, \mathcal{H}_Z)$ ,  $\epsilon = \pm 1$ , such that  $\mathcal{J}^2 = \epsilon \text{Id}$ . We prove that in both cases  $\mathcal{J}$  is integrable (recovering results of Blair, Davidov and Muškarov) and that  $\mathcal{H}_Z$  defines a holomorphic ( $\epsilon = -1$ ) or para-holomorphic ( $\epsilon = +1$ ) contact structure. Furthermore, we determine all the solutions of the Einstein equation for the canonical one-parameter family of pseudo-Riemannian metrics on  $Z^\epsilon$ . In particular, we find that there is a unique Kähler-Einstein ( $\epsilon = -1$ ) or para-Kähler-Einstein ( $\epsilon = +1$ ) metric. Finally, we prove that any Kähler or para-Kähler submanifold of a para-quaternionic Kähler manifold is minimal and describe all such submanifolds in terms of complex ( $\epsilon = -1$ ), respectively, para-complex ( $\epsilon = +1$ ) submanifolds of  $Z^\epsilon$  tangent to the contact distribution.

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## 1. Introduction

Twistor methods were originally introduced by Penrose with the aim of providing a mathematical framework which could lead to a synthesis of quantum theory and relativity [13, 14]. They have proven very fruitful for the construction and systematic study of various geometric objects governed by non-linear partial differential equations such as Yang-Mills connections, Einstein metrics, harmonic maps and minimal submanifolds.

Given a geometric problem on a real differentiable manifold  $M$  endowed with certain geometric structure  $S$ , the twistor approach is to try to translate the given problem into a problem of complex geometry on a complex manifold  $Z$ , called the twistor space, which is the total space of a bundle over  $M$ . In most cases,  $Z$  can be defined as the bundle of all complex structures in the tangent spaces of  $M$  which are compatible with the geometric structure  $S$  and it comes with a natural almost complex structure  $\mathcal{J}$ , the integrability of which has to be derived from the properties of the structure  $S$ .

In the case of a four-dimensional oriented Riemannian manifold  $M$ , for instance, the fibre at  $p \in M$  of the twistor bundle  $Z \rightarrow M$  consists of all skew-symmetric complex structures in  $T_p M$ , which induce the given orientation [4]. It is identified with the Riemann sphere  $\mathbb{C}P^1$  and, thus, carries a natural complex structure. On the other hand, the Levi-Civita connection of  $M$  induces a horizontal (i.e. transversal to the fibers) distribution  $\mathcal{H}_Z \subset TZ$  and the horizontal spaces carry a tautological complex structure. Putting the complex structures on vertical and horizontal spaces together, one obtains a canonical almost complex structure  $\mathcal{J}$  on  $Z$ . By the results of Atiyah, Hitchin and Singer,  $\mathcal{J}$  is integrable if and only if the Weyl curvature tensor of  $M$  is self-dual and, in that case, self-dual Yang-Mills vector bundles on  $M$  correspond to certain holomorphic vector bundles on  $Z$ . Salamon et al. have extended these constructions from four to higher dimensions, with the role of the self-dual four-dimensional Riemannian manifold played by a quaternionic Kähler manifold [15, 9]. In [3] the twistor method was used to construct (minimal) Kähler submanifolds of quaternionic Kähler manifolds.

A  $G$ -structure is called of *twistor type* if the (linear) Lie algebra  $\mathfrak{g} = \text{Lie } G$  of the structure group contains a complex structure, i.e. an element  $J$  such that  $J^2 = -\text{Id}$ . The twistor theory of  $G$ -structures of twistor type is developed in [2], see also references therein. This includes the case of quaternionic Kähler manifolds, for which the

structure group is  $G = \text{Sp}(1)\text{Sp}(n)$ .

In this paper, we develop a similar theory for  $G$ -structures of *para-twistor type*, i.e. for which  $\mathfrak{g}$  contains an involution  $J$ , rather than a complex structure. Let  $P \rightarrow M$  be such a  $G$ -structure and denote by  $K = Z_G(J)$  the centralizer of the involution  $J$ . For any principal connection  $\omega$  on  $P$ , we define the twistor space of  $(P, \omega)$  as the total space of the bundle  $Z = P/K \rightarrow P/G = M$ , which we endow with a  $K$ -structure  $P \rightarrow Z$ , a field of involutions  $\mathcal{J} \in \Gamma(\text{End } TZ)$  and a  $\mathcal{J}$ -invariant horizontal distribution  $\mathcal{H}_Z$ , see Definition 11. We express the integrability of  $\mathcal{J}$  and the (para-)holomorphicity of  $\mathcal{H}_Z$  as equations for the curvature and torsion of  $\omega$ , which generalize the self-duality equation for the Weyl curvature of a pseudo-Riemannian metric of signature  $(2, 2)$ , see Theorem 1.

A *para-quaternionic structure* on a vector space  $V$  is a Lie subalgebra  $Q \subset \text{End } V$  which admits a basis  $(J_1, J_2, J_3)$  such that  $J_3 = J_1J_2$  and  $J_\alpha^2 = \epsilon_\alpha \text{Id}$ , where  $(\epsilon_1, \epsilon_2, \epsilon_3) = (-1, 1, 1)$ . A pseudo-Riemannian manifold  $(M, g)$  of dimension  $> 4$  endowed with a parallel field  $M \ni p \mapsto Q_p \subset \text{End } T_pM$  of  $g$ -skew-symmetric para-quaternionic structures is called a *para-quaternionic Kähler manifold*. The metric  $g$  has signature  $(2n, 2n)$  and is Einstein [1]. Moreover, para-quaternionic Kähler manifolds are related to certain supersymmetric field theories on space-times with a positive definite rather than a Lorentzian metric [11].

For a para-quaternionic Kähler manifold  $(M, g, Q)$ , Blair et al. [6, 7] have defined two twistor spaces  $Z^\epsilon := \{A \in Q \mid A^2 = \epsilon\}$ ,  $\epsilon = \pm 1$ , and endowed them with an integrable structure  $\mathcal{J} \subset \text{End } TZ^\epsilon$  such that  $\mathcal{J}^2 = \epsilon \text{Id}$ . We recover these results by considering the twistor space associated to the underlying  $G$ -structure, which is of twistor type, as well as of para-twistor type. More precisely, we consider

$$J \in \mathfrak{sl}(2, \mathbb{R}) \subset \mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sp}(\mathbb{R}^{2n}) \subset \mathfrak{gl}(\mathbb{R}^2 \otimes \mathbb{R}^{2n}) = \mathfrak{gl}(4n, \mathbb{R}).$$

Under the assumption that the scalar curvature of  $g$  is non-zero, we prove, in addition, that the horizontal distribution  $\mathcal{H}_Z$  defines a holomorphic (respectively, para-holomorphic) contact structure on  $Z$  and that  $(Z^\epsilon, \mathcal{J})$  admits a Kähler-Einstein (respectively, para-Kähler-Einstein) metric and determine all Einstein metrics in the canonical one-parameter family of pseudo-Riemannian metrics, see Theorem 3. It turns out that there is always a second Einstein metric.

Finally, we generalize the twistor construction of Kähler submanifolds of a quaternionic Kähler manifold (see [3]) to the case of Kähler and para-Kähler submanifolds (see Definition 14) of a para-quaternionic Kähler manifold  $(M, g, Q)$ . We prove that any Kähler or para-Kähler submanifold of a para-quaternionic Kähler manifold  $(M, g, Q)$  is minimal (Corollary 7). All such submanifolds can be obtained as projections of complex ( $\epsilon = -1$ ), respectively, para-complex ( $\epsilon = +1$ ) submanifolds of  $Z^\epsilon$  which are tangent to the contact distribution, see Theorem 4. It follows that the maximal dimension of a Kähler or para-Kähler submanifold of  $(M, g, Q)$  is  $(1/2) \dim M$  and that maximal Kähler (respectively, para-Kähler) submanifolds of  $(M, g, Q)$  correspond to Legendrian

submanifolds of the complex (respectively, para-complex) contact manifold  $(Z^\epsilon, \mathcal{H}_Z)$ .

## 2. (Almost) para-complex manifolds

### 2.1. Integrability of an almost para-complex structure.

DEFINITION 1. An (almost) para-complex structure, in the weak sense, on a differentiable manifold  $M$  is a field of endomorphisms  $J \in \text{End } TM$  such that  $J^2 = \text{Id}$ .  $J$  is called *non-trivial* if  $J \neq \pm \text{Id}$ . We say that  $J$  is an (almost) para-complex structure, in the strong sense, if the  $\pm 1$ -eigenspace distributions  $T^\pm M$  of  $J$  have the same rank. An almost para-complex structure is called *integrable*, or *para-complex structure* if the distributions  $T^\pm M$  are integrable, or, equivalently, the Nijenhuis tensor  $N_J$ , defined by

$$(2.1) \quad N_J(X, Y) = [X, Y] + [JX, JY] - J[JX, Y] - J[X, JY], \quad X, Y \in TM,$$

vanishes. An (almost) para-complex manifold  $(M, J)$  is a manifold  $M$  endowed with an (almost) para-complex structure.

Unless otherwise stated, by an (almost) para-complex structure we shall understand here an (almost) para-complex structure in the weak sense.

REMARK. The difference between weak and strong (almost) para-complex manifolds is that  $T_p^{1,0}M = \{X + eJX \mid X \in T_pM\} \subset T_pM \otimes C$ ,  $p \in M$ , is a free module over the ring  $C := \mathbb{R}[e]$ ,  $e^2 = 1$ , of para-complex numbers only in the strong case. In particular, for weak para-complex manifolds, there is no notion of para-holomorphic local coordinates  $(z^i)$  on  $M$  such that the  $(dz^i)$  form a basis of  $T_p^{1,0}M$  over  $C$ .

Let  $(V, J)$  and  $(U, J_U)$  be vector spaces endowed with constant para-complex structures. We can decompose the vector space  $C^2(U) := U \otimes \bigwedge^2 V^*$  of  $U$ -valued two-forms on  $V$  according to type

$$(2.2) \quad C^2(U) = \sum_{p+q=2} C^{p,q}(U),$$

where

$$\alpha \in C^{1,1}(U) \quad \text{if} \quad \alpha(JX, JY) = -\alpha(X, Y) \quad \text{for all} \quad X, Y \in V,$$

$$\alpha \in C^{2,0}(U) \quad \text{if} \quad \alpha(JX, Y) = \alpha(X, JY) = J_U \alpha(X, Y) \quad \text{for all} \quad X, Y \in V$$

and

$$\alpha \in C^{0,2}(U) \quad \text{if} \quad \alpha(JX, Y) = \alpha(X, JY) = -J_U \alpha(X, Y) \quad \text{for all} \quad X, Y \in V.$$

**Lemma 1.** *The projections  $\pi^{p,q}: C^2(U) \rightarrow C^{p,q}(U)$ ,  $\alpha \rightarrow \alpha^{p,q}$ , are given by:*

$$\begin{aligned}\alpha^{1,1}(X, Y) &= \frac{1}{2}(\alpha(X, Y) - \alpha(JX, JY)), \\ \alpha^{2,0}(X, Y) &= \frac{1}{4}(\alpha(X, Y) + \alpha(JX, JY) + J_U\alpha(JX, Y) + J_U\alpha(X, JY)), \\ \alpha^{0,2}(X, Y) &= \frac{1}{4}(\alpha(X, Y) + \alpha(JX, JY) - J_U\alpha(JX, Y) - J_U\alpha(X, JY)).\end{aligned}$$

For scalar valued forms ( $U = \mathbb{R}$ ) we will always assume that  $J_U = \text{Id}$ .

Let  $J$  be an almost para-complex structure on a manifold  $M$  and  $\nabla$  a linear connection which preserves  $J$ . The following lemma shows that  $J$  is integrable if and only if the  $(0, 2)$  component  $T^{0,2} = \pi^{0,2}T$  vanishes.

**Proposition 1.** *Let  $\nabla$  be a connection which preserves an almost para-complex structure  $J$  on a manifold  $M$ . Then the Nijenhuis tensor of  $J$  is given by  $N_J = -4T^{0,2}$ . In particular,  $J$  is integrable if and only if  $T^{0,2} = 0$ .*

*Proof.* Applying Lemma 1 in the case  $U = V = T_pM$ ,  $p \in M$ , we have

$$T^{0,2}(X, Y) = \frac{1}{4}(T(X, Y) + T(JX, JY) - JT(JX, Y) - JT(X, JY)), \quad X, Y \in TM.$$

Replacing  $T(X, Y)$  by  $\nabla_X Y - \nabla_Y X - [X, Y]$  in this formula, we get

$$T^{0,2}(X, Y) = -\frac{1}{4}([X, Y] + [JX, JY] - J[JX, Y] - J[X, JY]) = -\frac{1}{4}N_J(X, Y). \quad \square$$

## 2.2. Holomorphicity of distributions in almost para-complex manifolds.

**DEFINITION 2.** Let  $(M, J)$  be an almost para-complex manifold of real dimension  $n$ . A  $J$ -invariant distribution  $\mathcal{D} = \mathcal{D}_+ \oplus \mathcal{D}_- \subset T^+M \oplus T^-M = TM$  of rank  $m$  is called *para-holomorphic* if it is locally defined by equations  $\alpha_+^1 = \dots = \alpha_+^{k_+} = \alpha_-^1 = \dots = \alpha_-^{k_-} = 0$ , such that  $k_+ + k_- = n - m$ ,

$$(2.3) \quad \alpha_{\pm}^i \circ J = \pm \alpha_{\pm}^i$$

and the  $(1, 1)$ -component

$$\pi^{1,1} d\alpha_+^i = \frac{1}{2}(d\alpha_+^i - J^* d\alpha_+^i)$$

vanishes on  $\bigwedge^2(\mathcal{D}_+ \oplus T^-M)$  and the  $(1, 1)$ -component

$$\pi^{1,1} d\alpha_-^i = \frac{1}{2}(d\alpha_-^i - J^* d\alpha_-^i)$$

vanishes on  $\bigwedge^2(T^+M \oplus \mathcal{D}_-)$ .

Let  $(M, J)$  be an almost para-complex manifold of real dimension  $n$  endowed with a  $J$ -invariant distribution  $\mathcal{D} \subset TM$  of rank  $m$  and a connection  $\nabla$  which preserves  $J$  and  $\mathcal{D}$ . Then we can define a two-form with values in  $TM/\mathcal{D}$  by

$$S(X, Y) := T(X, Y) \pmod{\mathcal{D}}.$$

Since  $J$  induces a para-complex structure on the vector bundle  $TM/\mathcal{D}$ , we can decompose

$$S = S^{2,0} + S^{1,1} + S^{0,2},$$

see Lemma 1.

**Proposition 2.** *Let  $(M, J)$  be an almost para-complex manifold. A  $J$ -invariant distribution  $\mathcal{D} = \mathcal{D}_+ \oplus \mathcal{D}_- \subset T^+M \oplus T^-M = TM$  is para-holomorphic if and only if*

$$(2.4) \quad [\Gamma(\mathcal{D}_\pm), \Gamma(T^\mp M)] \subset \Gamma(T^\mp M \oplus \mathcal{D}_\pm).$$

Moreover, if  $\nabla$  is a connection which preserves  $J$  and  $\mathcal{D}$ , then (2.4) is equivalent to

$$(2.5) \quad S^{1,1}(JX, \cdot) = -JS^{1,1}(X, \cdot),$$

for all  $X \in \mathcal{D}$ .

*Proof.* First we prove that (2.5) is equivalent to the para-holomorphicity of  $\mathcal{D}$ . Let  $\mathcal{D}$  be a para-holomorphic distribution defined by one-forms  $\alpha_\pm^i$  as in Definition 2. The condition on  $\pi^{1,1} d\alpha_\pm^i$  is equivalent to

$$\begin{aligned} d\alpha_+^i(X_+, Y_-) &= 0, & X_+ \in \mathcal{D}_+, & Y_- \in T^-M, \\ d\alpha_-^i(X_+, Y_-) &= 0, & X_+ \in T^+M, & Y_- \in \mathcal{D}_-. \end{aligned}$$

Expressing the exterior derivative in terms of the covariant derivative and torsion we get

$$0 = d\alpha_+^i(X_+, Y_-) = (\nabla_{X_+} \alpha_+^i) Y_- - (\nabla_{Y_-} \alpha_+^i) X_+ + \alpha_+^i(T(X_+, Y_-)).$$

The first two terms on the right-hand side vanish. In fact, since  $\nabla$  preserves the distribution  $\mathcal{D}$ , the covariant derivative  $\nabla_X \alpha_+^i$  vanishes on  $\mathcal{D}_+ \oplus T^-M$  for all  $X \in TM$ . The last term can be written as

$$0 = \alpha_+^i(T(X_+, Y_-)) = \alpha_+^i(T^{1,1}(X_+, Y_-)),$$

which implies that  $T^{1,1}(X_+, Y_-) \in \mathcal{D}_+ \oplus T^-M$  for all  $X_+ \in \mathcal{D}_+$  and  $Y_- \in T^-M$ . A similar calculation for  $\alpha_-^i$  shows that  $T^{1,1}(X_+, Y_-) \in T^+M \oplus \mathcal{D}_-$  for all  $X_+ \in T^+M$  and  $Y_- \in \mathcal{D}_-$ . This proves that

$$\begin{aligned} S^{1,1}(\mathcal{D}_+, T^-M) &\subset (T^-M + \mathcal{D})/\mathcal{D}, \\ S^{1,1}(\mathcal{D}_-, T^+M) &\subset (T^+M + \mathcal{D})/\mathcal{D}. \end{aligned}$$

In particular,  $S^{1,1}(\mathcal{D}, \mathcal{D}) = 0$  and  $S^{1,1}(JX, \cdot) = -JS^{1,1}(X, \cdot)$  for all  $X \in \mathcal{D}$ .

To prove the converse, we assume that the torsion of  $\nabla$  satisfies (2.5). Let  $(\alpha_+^1, \dots, \alpha_+^{k_+})$  and  $(\alpha_-^1, \dots, \alpha_-^{k_-})$  be local frames of  $(\mathcal{D}_+ \oplus T^-M)^\perp$  and  $(T^+M \oplus \mathcal{D}_-)^\perp \subset T^*M$ , respectively. This implies (2.3). Since  $\pi^{1,1}\alpha(T^\pm M, T^\pm M) = 0$  for any two-form  $\alpha$ , it is sufficient to check that  $\pi^{1,1}d\alpha_+^i(\mathcal{D}_+, T^-M) = \pi^{1,1}d\alpha_-^i(T^+M, \mathcal{D}_-) = 0$ . We calculate for  $X_+ \in \mathcal{D}_+$  and  $Y_- \in T^-M$ :

$$\begin{aligned} \pi^{1,1}d\alpha_+^i(X_+, Y_-) &= d\alpha_+^i(X_+, Y_-) = (\nabla_{X_+}\alpha_+^i)Y_- - (\nabla_{Y_-}\alpha_+^i)X_+ + \alpha_+^i(T(X_+, Y_-)) \\ &= \alpha_+^i(T(X_+, Y_-)) = \alpha_+^i(T^{1,1}(X_+, Y_-)) = \alpha_+^i(S^{1,1}(X_+, Y_-)) \\ &= \alpha_+^i(S^{1,1}(JX_+, Y_-)) \stackrel{(2.5)}{=} -\alpha_+^i(JS^{1,1}(X_+, Y_-)) = -\alpha_+^i(S^{1,1}(X_+, Y_-)). \end{aligned}$$

Therefore,  $\pi^{1,1}d\alpha_+^i(X_+, Y_-) = 0$ . A similar calculation shows that  $\pi^{1,1}d\alpha_-^i(\mathcal{D}_-, T^+M) = 0$ .

Now we prove the equivalence of (2.4) and (2.5). The condition (2.5) can be written as

$$T(\mathcal{D}_\pm, T^\mp M) \subset T^\mp M \oplus \mathcal{D}_\pm.$$

Using that  $\nabla$  preserves the distributions  $\mathcal{D}_\pm$  and  $T^\pm M$ , we calculate for  $X_\pm \in \Gamma(\mathcal{D}_\pm)$  and  $Y_\mp \in \Gamma(T^\pm M)$

$$\begin{aligned} T^\mp M \oplus \mathcal{D}_\pm \ni T(X_\pm, Y_\mp) &= \nabla_{X_\pm}Y_\mp - \nabla_{Y_\mp}X_\pm - [X_\pm, Y_\mp] \\ &\equiv -[X_\pm, Y_\mp] \pmod{T^\mp M \oplus \mathcal{D}_\pm}. \end{aligned}$$

This proves the equivalence of (2.4) and (2.5). □

Let  $(M, J)$  be a para-complex manifold in the strong sense, i.e. the integrable eigendistributions  $T^\pm M$  are of the same rank. Recall [10] that a  $C$ -valued one-form  $\gamma = \alpha + e\beta$  is of *para-complex type*  $(1, 0)$ , i.e.  $J^*\gamma = e\gamma$ , if and only if  $\beta = \alpha \circ J$ . A  $(1, 0)$ -form  $\gamma$  is *para-holomorphic* if  $\bar{\partial}\gamma := \pi^{1,1}d\gamma = 0$ , which is equivalent to the para-Cauchy-Riemann equations

$$(2.6) \quad \partial_-\alpha_+ := \pi^{1,1}d\alpha_+ = \partial_+\alpha_- := \pi^{1,1}d\alpha_- = 0,$$

where  $\alpha = \alpha_+ + \alpha_-$  is the  $J$ -eigenspace decomposition of  $\alpha$ .



**Proposition 3.** *Let  $(M, J)$  be a para-complex manifold in the strong sense with eigendistributions  $T^\pm M$  of rank  $n$  and  $\mathcal{D} = \mathcal{D}_+ \oplus \mathcal{D}_- \subset T^+ M \oplus T^- M = TM$  a  $J$ -invariant distribution such that  $\mathcal{D}_\pm$  are of the same rank  $m$ . Then  $\mathcal{D}$  is para-holomorphic if and only if it is locally defined by equations  $\gamma^i = 0$  ( $i = 1, \dots, k = n - m$ ), where the  $\gamma^i$  are para-holomorphic one-forms.*

*Proof.* Let  $\mathcal{D}$  be defined by para-holomorphic one-forms  $\gamma^i = \alpha_+^i + \alpha_-^i + e(\alpha_+^i - \alpha_-^i)$ . The  $\alpha_\pm^i$  satisfy (2.6), which imply the equations in the Definition 2.

To prove the converse, we now assume that the distribution  $\mathcal{D}$  is para-holomorphic. Thanks to Proposition 2, this means that

$$[\Gamma(\mathcal{D}_\pm), \Gamma(T^\mp M)] \subset \Gamma(T^\mp M \oplus \mathcal{D}_\pm).$$

In order to construct para-holomorphic one-forms  $\gamma^i = \alpha_+^i + \alpha_-^i + e(\alpha_+^i - \alpha_-^i)$  which define  $\mathcal{D}$ , we choose locally linearly independent commuting vector fields  $Y_j^\pm \in \Gamma(T^\pm M)$  which generate distributions  $N^\pm \subset T^\pm M$  complementary to  $\mathcal{D}_\pm$ . We define one-forms  $\alpha_\pm^i$  vanishing on  $\mathcal{D}_\pm \oplus T^\mp M$  by

$$\alpha_\pm^i(Y_j^\pm) = \delta_j^i.$$

It is clear that  $\alpha_\pm^i \circ J = \pm \alpha_\pm^i$  and that  $\gamma^i := \alpha_+^i + \alpha_-^i + e(\alpha_+^i - \alpha_-^i)$  define  $\mathcal{D}$ . Now we check that the  $\gamma^i$  are para-holomorphic, i.e.  $\partial_- \alpha_+^i = \partial_+ \alpha_-^i = 0$ . It is sufficient to evaluate this equality on  $(Z^+, Z^-)$ , where  $Z^\pm = X^\pm \in \Gamma(\mathcal{D}_\pm)$  or  $Z^\pm = Y_j^\pm$ .

$$\partial_- \alpha_+^i(X^+, X^-) = X^+ \alpha_+^i(X^-) - X^- \alpha_+^i(X^+) - \alpha_+^i([X^+, X^-]) = 0,$$

since  $\alpha_+^i$  vanishes on  $\mathcal{D}_+ \oplus T^- M$  and  $[X^+, X^-] \in T^- M \oplus \mathcal{D}_+$  by (2.4). Similarly,

$$\partial_- \alpha_+^i(X^+, Y_j^-) = X^+ \alpha_+^i(Y_j^-) - Y_j^- \alpha_+^i(X^+) - \alpha_+^i([X^+, Y_j^-]) = 0.$$

Finally,

$$\partial_- \alpha_+^i(Y_j^+, Y_k^-) = Y_j^+ \alpha_+^i(Y_k^-) - Y_k^- \alpha_+^i(Y_j^+) - \alpha_+^i([Y_j^+, Y_k^-]) = 0 - Y_k^- (\delta_j^i) - 0 = 0,$$

since, by construction,  $[Y_j^+, Y_k^-] = 0$ . Similarly, one can check that  $\partial_+ \alpha_-^i = 0$ .  $\square$

### 3. Para-quaternionic manifolds and para-quaternionic Kähler manifolds

**DEFINITION 3.** Let  $(\epsilon_1, \epsilon_2, \epsilon_3) = (-1, 1, 1)$ , or a permutation thereof. An *almost para-quaternionic structure* on a differentiable manifold  $M$  (of dimension  $4n$ ) is a rank 3 subbundle  $\mathcal{Q} \subset \text{End } TM$ , which is locally generated by three anticommuting fields of endomorphisms  $J_1, J_2, J_3 = J_1 J_2$ , such that  $J_\alpha^2 = \epsilon_\alpha \text{Id}$ . Such a triple  $(J_\alpha)$  will be called a *standard local basis* of  $\mathcal{Q}$ . A linear connection which preserves  $\mathcal{Q}$  is called an *almost para-quaternionic connection*. An almost para-quaternionic structure  $\mathcal{Q}$  is called a

*para-quaternionic structure* if  $M$  admits a *para-quaternionic connection*, i.e. a torsion-free connection which preserves  $Q$ . An *(almost) para-quaternionic manifold* is a manifold endowed with an (almost) para-quaternionic structure.

An *almost para-quaternionic Hermitian manifold*  $(M, g, Q)$  is a pseudo-Riemannian manifold  $(M, g)$  endowed with a para-quaternionic structure  $Q$  consisting of skew-symmetric endomorphisms.  $(M, g, Q)$ ,  $n > 1$ , is called a *para-quaternionic Kähler manifold* if the Levi-Civita connection preserves  $Q$ .

**Proposition 4** ([1]). *At any point, the curvature tensor  $R$  of a para-quaternionic Kähler manifold  $(M, g, Q)$  of dimension  $4n > 4$  admits a decomposition*

$$(3.1) \quad R = \nu R_0 + W,$$

where  $\nu = \text{scal}/(4n(n + 2))$  is the reduced scalar curvature,

$$R_0(X, Y) := +\frac{1}{2} \sum_{\alpha} \epsilon_{\alpha} g(J_{\alpha} X, Y) J_{\alpha} + \frac{1}{4} \left( X \wedge Y - \sum_{\alpha} \epsilon_{\alpha} J_{\alpha} X \wedge J_{\alpha} Y \right), \quad X, Y \in T_p M,$$

is the curvature tensor of the para-quaternionic projective space of the same dimension as  $M$  and  $W$  is a trace-free  $Q$ -invariant algebraic curvature tensor, where  $Q$  acts by derivations. In particular,  $R$  is  $Q$ -invariant.

We define a *para-quaternionic Kähler manifold of dimension 4* as a pseudo-Riemannian manifold endowed with a parallel skew-symmetric para-quaternionic structure whose curvature tensor admits a decomposition (3.1).

Since the Levi-Civita connection  $\nabla$  of a para-quaternionic Kähler manifold preserves the para-quaternionic structure  $Q$ , we can write

$$(3.2) \quad \nabla J_{\alpha} = -\epsilon_{\beta} \omega_{\gamma} \otimes J_{\beta} + \epsilon_{\gamma} \omega_{\beta} \otimes J_{\gamma},$$

where  $(\alpha, \beta, \gamma)$  is a cyclic permutation of  $(1, 2, 3)$ . We shall denote by  $\rho_{\alpha} := g(J_{\alpha} \cdot, \cdot)$  the *fundamental form* associated with  $J_{\alpha}$  and put  $\rho'_{\alpha} := -\epsilon_{\alpha} \rho_{\alpha}$ .

**Proposition 5.** *The locally defined fundamental forms satisfy the following structure equations*

$$(3.3) \quad \nu \rho'_{\alpha} := -\epsilon_{\alpha} \nu \rho_{\alpha} = \epsilon_3 (d\omega_{\alpha} - \epsilon_{\alpha} \omega_{\beta} \wedge \omega_{\gamma}),$$

where  $(\alpha, \beta, \gamma)$  is a cyclic permutation of  $(1, 2, 3)$ .

*Proof.* Using Proposition 4 and the fact that

$$[J_{\alpha}, J_{\beta}] = 2\epsilon_3 \epsilon_{\gamma} J_{\gamma},$$

we calculate the action of the curvature operator  $R(X, Y)$ ,  $X, Y \in TM$ , on  $J_\alpha$ :

$$\begin{aligned} [R(X, Y), J_\alpha] &= [\nu R_0(X, Y), J_\alpha] = -\frac{\nu}{2} \sum_{\delta=1}^3 \rho'_\delta(X, Y) [J_\delta, J_\alpha] \\ &= \epsilon_3 \nu (-\epsilon_\beta \rho'_\gamma(X, Y) J_\beta + \epsilon_\gamma \rho'_\beta(X, Y) J_\gamma), \end{aligned}$$

where  $(\alpha, \beta, \gamma)$  is a cyclic permutation of  $(1, 2, 3)$ . On the other hand, using the equation (3.2), we calculate

$$\begin{aligned} [R(X, Y), J_\alpha] &= [\nabla_X, \nabla_Y] J_\alpha - \nabla_{[X, Y]} J_\alpha \\ &= \nabla_X (-\epsilon_\beta \omega_\gamma(Y) J_\beta + \epsilon_\gamma \omega_\beta(Y) J_\gamma) - \nabla_Y (-\epsilon_\beta \omega_\gamma(X) J_\beta + \epsilon_\gamma \omega_\beta(X) J_\gamma) \\ &\quad - (-\epsilon_\beta \omega_\gamma([X, Y]) J_\beta + \epsilon_\gamma \omega_\beta([X, Y]) J_\gamma) \\ &= -\epsilon_\beta d\omega_\gamma(X, Y) J_\beta + \epsilon_\gamma d\omega_\beta(X, Y) J_\gamma - \epsilon_\beta \omega_\gamma(Y) \nabla_X J_\beta + \epsilon_\gamma \omega_\beta(Y) \nabla_X J_\gamma \\ &\quad + \epsilon_\beta \omega_\gamma(X) \nabla_Y J_\beta - \epsilon_\gamma \omega_\beta(X) \nabla_Y J_\gamma. \end{aligned}$$

Applying again the equation (3.2), we finally get

$$[R(X, Y), J_\alpha] = -\epsilon_\beta (d\omega_\gamma - \epsilon_\gamma \omega_\alpha \wedge \omega_\beta)(X, Y) J_\beta + \epsilon_\gamma (d\omega_\beta - \epsilon_\beta \omega_\gamma \wedge \omega_\alpha)(X, Y) J_\gamma.$$

Comparing the two formulas for  $[R(X, Y), J_\alpha]$  we obtain the structure equations.  $\square$

#### 4. The twistor spaces of a para-quaternionic or para-quaternionic Kähler manifold

**4.1. The twistor spaces of a para-quaternionic manifold.** In the following, it will be useful to unify complex and para-complex structures in the following definition.

**DEFINITION 4.** An *almost  $\epsilon$ -complex structure*,  $\epsilon \in \{-1, 0, 1\}$ , on a differentiable manifold  $M$  of dimension  $2n$  is a field of endomorphisms  $J \in \text{End } TM$  such that  $J^2 = \epsilon \text{Id}$  and, moreover, for  $\epsilon = +1$  the eigendistributions  $T^\pm M$  are of rank  $n$  and for  $\epsilon = 0$  the two distributions  $\ker J$  and  $\text{im } J$  have rank  $n$ . In other words, an almost  $-1$ -complex structure is an almost complex structure and an almost  $+1$ -complex structure is an almost para-complex structure in the strong sense.

An  *$\epsilon$ -complex manifold* is a differentiable manifold endowed with an integrable (i.e.  $N_J = 0$ )  $\epsilon$ -complex structure  $J$ .

We shall also use the unifying adjective  *$\epsilon$ -holomorphic* as a synonym of ‘holomorphic’ or ‘para-holomorphic’, depending on whether  $\epsilon = -1$  or  $\epsilon = +1$ , respectively.

Let  $(M, Q)$  be an almost para-quaternionic manifold. We associate with  $(M, Q)$  a family of bundles  $\pi: Z^s \rightarrow M$ , with two-dimensional fibres, depending on a parameter  $s \in \mathbb{R}$  as follows:

$$Z^s := \{A \in Q \mid A \neq 0, A^2 = s\}.$$

DEFINITION 5. The fibre bundle  $\pi: Z^s \rightarrow M$  is called the  $s$ -twistor space of the almost para-quaternionic manifold  $(M, Q)$ .

**Proposition 6.** Any almost para-quaternionic connection  $\nabla$  on an almost para-quaternionic manifold  $(M, Q)$  induces a canonical almost  $\epsilon$ -complex structure  $\mathcal{J}^s = \mathcal{J}_{\nabla}^s$  on the  $s$ -twistor space  $Z^s$ , where  $\epsilon = \text{sgn}(s) \in \{-1, 0, 1\}$ .

Proof. Let  $(I, J, K)$  be a standard basis of  $Q_m$ . Then any element  $A \in Q_m$  can be written as  $A = xI + yJ + zK$  and  $A \in Z^s$  if and only if  $-x^2 + y^2 + z^2 = s$ . Hence, the fibres of  $Z^s$  are two-sheeted hyperboloids for  $s < 0$ , one-sheeted hyperboloids for  $s > 0$  and light-cones without origin for  $s = 0$ . Each fibre  $Z_m^s = \pi^{-1}(m)$  is a homogeneous space of the group  $\text{SO}(1, 2)$  with one-dimensional stabilizer  $\text{SO}(1, 2)_{A_s} = \text{SO}(2)$  if  $s < 0$ ,  $\text{SO}(1, 2)_{A_s} = \text{SO}(1, 1)$  if  $s > 0$  and  $\text{SO}(1, 2)_{A_s} \cong (\mathbb{R}, +)$  if  $s = 0$ , where  $A_s \in Z^s$ . First we define the canonical  $\text{SO}(1, 2)$ -invariant  $\epsilon$ -complex structure on  $Z_m^s$ , as follows. The three-dimensional vector space  $Q_m \subset \text{End } T_m M$  is a Lie sub-algebra isomorphic to  $\mathfrak{sl}_2(\mathbb{R})$ . The adjoint action preserves the indefinite scalar product  $\langle A, B \rangle = -(1/(4n)) \text{tr}(AB)$ ,  $4n = \dim M$ , in  $Q$  and hence identifies the Lie algebra  $Q$  with  $\mathfrak{so}(Q) = \text{Lie } \text{SO}(Q) \cong \mathfrak{so}(1, 2)$ . Let  $A \in Z_m^s \subset Q_m$ . Then  $Z_m^s = \text{SO}(Q)A$  and the tangent space to  $Z_m^s$  at  $A$  is identified with  $\mathfrak{so}(Q)A \cong \mathfrak{so}(Q)/\mathfrak{so}(Q)_A = \mathfrak{so}(Q)/\mathbb{R}A$ . It is easy to check that the adjoint action of  $(1/2)A$  on  $\mathfrak{so}(Q)/\mathbb{R}A$  defines an  $\text{SO}(Q)$ -invariant  $\epsilon$ -complex structure  $J^v$  on  $Z_m^s$ . Now we define an almost  $\epsilon$ -complex structure  $\mathcal{J}^s$  on the twistor space  $Z^s$ . We have the decomposition

$$(4.1) \quad T_z Z^s = T_z^v Z^s + H_z \cong T_z(Z_m^s) \oplus T_{\pi z} M,$$

where  $T_z^v Z^s$  is the vertical space of the bundle  $\pi: Z^s \rightarrow M$  and  $H_z$  is the horizontal space of the connection in the bundle  $\pi$  induced by the para-quaternionic connection  $\nabla$  of  $(M, Q)$ . The latter is identified with  $T_{\pi z} M$  via the projection  $Z^s \rightarrow M$ . We denote by  $J^z$  the tautological  $\epsilon$ -complex structure on  $T_{\pi z} M$  defined by  $z \in Z^s$ . With respect to the above decomposition we define

$$(4.2) \quad \mathcal{J}_z^s = J^v \oplus J^z$$

By construction,  $\mathcal{J}^s$  is an almost  $\epsilon$ -complex structure. □

**4.2. The twistor spaces of a para-quaternionic Kähler manifold.** Let  $(M, g, Q)$  be a para-quaternionic Kähler manifold with twistor spaces  $Z^s$ . The Levi-Civita connection  $\nabla = \nabla^g$  is a para-quaternionic connection and, hence, induces a canonical almost  $\epsilon$ -complex structure  $\mathcal{J}^s = \mathcal{J}_{\nabla}^s$  on  $Z^s$ .

**Proposition 7.** The twistor space  $Z^s$  of a para-quaternionic Kähler manifold  $(M, g, Q)$  admits a canonical almost  $\epsilon$ -complex structure  $\mathcal{J}^\epsilon$ , where  $\epsilon = \text{sgn}(s)$ , and a one-parameter family  $g_t^s$ ,  $t \in \mathbb{R} - \{0\}$ , of pseudo-Riemannian metrics such that the

almost  $\epsilon$ -complex structure  $\mathcal{J}^s$  is skew-symmetric, provided that  $s \neq 0$ . For  $s = 0$  there exists a canonical one-parameter family  $g_t^0$ ,  $t \in \mathbb{R} - \{0\}$ , of symmetric bilinear forms with one-dimensional (vertical) kernel such that  $\mathcal{J}^s$  is skew-symmetric. Finally, for  $s \neq 0$ , the projection  $\pi : (Z^s, g_t^s) \rightarrow (M, g)$  is a pseudo-Riemannian submersion.

Proof. We denote by  $g^v := \langle \cdot, \cdot \rangle|_{Z_m^s}$  the induced metric on the fibres  $Z_m^s \subset (Q, \langle \cdot, \cdot \rangle)$ . It is nondegenerate for  $s \neq 0$  and has one-dimensional kernel for  $s = 0$ . The  $\epsilon$ -complex structure  $J^v$  on  $Z_m^s$  is  $g^v$ -skew-symmetric. With respect to the decomposition (4.1), we define

$$(g_t^s)_z = t g^v \oplus g_{\pi z}.$$

The almost  $\epsilon$ -complex structure  $\mathcal{J}^s$  defined above is skew-symmetric with respect to the field of symmetric bilinear forms  $g_t^s$ , which is nondegenerate for  $s \neq 0$  and has one-dimensional vertical kernel for  $s = 0$ . The above formula for  $g_t^s$  shows that the decomposition of  $TZ$  into vertical and horizontal space is  $g_t^s$ -orthogonal and that the projection induces an isometry  $H_z \rightarrow T_{\pi z}M$ . This proves that  $\pi$  is a pseudo-Riemannian submersion. □

The scalar multiplication by  $|s|^{1/2} \neq 0$  in the vector bundle  $Q \rightarrow M$  induces an isometry  $(Z^\epsilon, g_t^\epsilon) \rightarrow (Z^s, g_{t/|s|}^s)$ , which preserves the almost  $\epsilon$ -complex structure, where  $\epsilon = \text{sgn}(s)$ . This shows that it is sufficient to consider only three of the above twistor spaces, namely  $Z^+ := Z^{+1}$ ,  $Z^- := Z^{-1}$  and  $Z^0$ . We will study the integrability of the almost  $\epsilon$ -complex structure  $\mathcal{J}^\epsilon$  and the holomorphicity of the horizontal distribution  $H \subset TZ^\epsilon$ , which is  $\mathcal{J}^\epsilon$ -invariant. For this we extend the  $G$ -structure approach developed in [2] to the para-case ( $\epsilon = 1$ ).

**4.3. Twistor spaces of para-quaternionic (Kähler) manifolds as bundles associated to  $G$ -structures.** In this subsection we interpret the twistor spaces  $Z^\epsilon$  ( $\epsilon = -1, 0, 1$ ) from the point of view of  $G$ -structures.

Let  $(M, Q)$  be a para-quaternionic manifold. Note that  $\tilde{Q}_m := \mathbb{R}\text{Id} + Q_m \subset \text{End } T_m M$  is an algebra isomorphic to the algebra of para-quaternions, i.e. to the matrix algebra  $\mathbb{R}(2)$ . Since any irreducible module of  $\mathbb{R}(2)$  is isomorphic to  $\mathbb{R}^2$ , the  $\tilde{Q}_m$ -module  $T_m M$  is isomorphic to the  $\mathbb{R}(2)$ -module  $\mathbb{R}^2 \otimes \mathbb{R}^n$ ,  $2n = \dim M$ , with the action on the first factor.

DEFINITION 6. Let  $(M, Q)$  be an (almost) para-quaternionic manifold. A *para-quaternionic coframe* at  $m \in M$  is an isomorphism  $\phi : T_m M \xrightarrow{\sim} \mathbb{R}^2 \otimes \mathbb{R}^n$  which maps  $\tilde{Q}_m$  into  $\mathbb{R}(2)$ , i.e.

$$\phi \circ \tilde{Q}_m \circ \phi^{-1} = \mathbb{R}(2) \otimes \text{Id}.$$

**Proposition 8.** (i) *The set  $P$  of all para-quaternionic coframes together with the natural projection  $\pi^P: P \rightarrow M$  is a  $G$ -structure, i.e. a principal subbundle of the bundle of all coframes with the structure group  $G := \mathrm{SL}_2^\pm(\mathbb{R}) \otimes \mathrm{GL}_n(\mathbb{R})$ , where*

$$\mathrm{SL}_2^\pm(\mathbb{R}) = \{A \in \mathrm{GL}_2(\mathbb{R}) \mid \det A = \pm 1\}.$$

(ii) *Let  $A \in \mathfrak{sl}_2(\mathbb{R}) \otimes \mathrm{Id} \subset \mathfrak{g} = \mathrm{Lie} G$  such that  $A^2 = \epsilon \mathrm{Id}$  and  $G_A$  the stabilizer (i.e. centralizer) of  $A$  in  $G$ . There is a canonical isomorphism of fibre bundles*

$$P/G_A \xrightarrow{\sim} Z^\epsilon.$$

Proof. (i) It is clear that any two para-quaternionic coframes are related by an element of  $\mathrm{GL}_2(\mathbb{R}) \otimes \mathrm{GL}_n(\mathbb{R}) = \mathrm{SL}_2^\pm(\mathbb{R}) \otimes \mathrm{GL}_n(\mathbb{R})$ .

(ii) Let  $\phi \in P$  be a coframe at  $m \in M$ . It induces an algebra isomorphism  $\hat{\phi}: \mathbb{R}(2) \rightarrow \hat{Q}_m$ ,  $B \mapsto \phi^{-1}B\phi$ . The image  $\hat{\phi}(A) \in \hat{Q}_m$  satisfies  $\hat{\phi}(A)^2 = \epsilon \mathrm{Id}$ , hence  $\hat{\phi}(A) \in Z_m^\epsilon$ . If  $k \in G_A$  then  $\widehat{k\phi}(A) = \phi^{-1}k^{-1}Ak\phi = \hat{\phi}(A)$ . So the map  $P \rightarrow Z^\epsilon$ ,  $\phi \mapsto \hat{\phi}(A)$ , factorizes to an isomorphism  $P/G_A \rightarrow Z^\epsilon$  of fibre bundles.  $\square$

Assume now that  $(M, g, Q)$  is a para-quaternionic Kähler manifold of dimension  $4n$ , or more generally an almost para-quaternionic Hermitian manifold. On  $\mathbb{R}^2 \otimes \mathbb{R}^{2n}$  we fix the standard scalar product  $g_{\mathrm{can}} = \omega_{\mathbb{R}^2} \otimes \omega_{\mathbb{R}^{2n}}$ , where  $\omega_{\mathbb{R}^{2n}}$  denotes the standard symplectic structure of  $\mathbb{R}^{2n}$ .

DEFINITION 7. Let  $(M, g, Q)$  be an almost para-quaternionic Hermitian manifold of dimension  $4n$ . A *para-quaternionic Hermitian coframe* at  $m \in M$  is a linear isometry  $\phi: (T_m M, g_m) \xrightarrow{\sim} (\mathbb{R}^2 \otimes \mathbb{R}^{2n}, g_{\mathrm{can}})$  which maps  $\hat{Q}_m$  into  $\mathbb{R}(2)$ .

**Proposition 9.** *The set  $P$  of all para-quaternionic Hermitian coframes together with the natural projection  $\pi^P: P \rightarrow M$  is a  $G$ -structure with  $G = G_0 \cup \xi G_0$ ,  $G_0 := \mathrm{SL}_2(\mathbb{R}) \otimes \mathrm{Sp}(\mathbb{R}^{2n})$ ,  $\xi = A \otimes B \in \mathrm{SL}_2^\pm(\mathbb{R}) \otimes \mathrm{GL}_n(\mathbb{R})$ ,  $\det A = -1$  and  $B^* \omega_{\mathbb{R}^{2n}} = -\omega_{\mathbb{R}^{2n}}$ . Moreover, the twistor space  $Z^\epsilon$  is canonically isomorphic to the bundle  $P/G_A$ , where  $0 \neq A \in \mathfrak{sl}_2(\mathbb{R})$  with  $A^2 = \epsilon \mathrm{Id}$ .*

## 5. $G$ -structures of para-twistor type and their twistor spaces: obstructions for integrability

### 5.1. Groups of para-twistor type and para-complex symmetric spaces.

DEFINITION 8. A connected linear Lie group  $G \subset \mathrm{GL}(V)$ ,  $V = \mathbb{R}^n$ , is called of *para-twistor type* if its Lie algebra contains a para-complex structure, i.e. an element  $J$  such that  $J^2 = \mathrm{Id}$ . (If  $G$  is not connected, we shall assume, in addition, that the conjugation by  $J$  preserves  $G$ .)

Since the endomorphism  $J$  is semi-simple, the adjoint operator  $\text{ad}_J$  is semi-simple and, hence, we have the direct sum  $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ , where  $\mathfrak{k} = \ker \text{ad}_J = Z_{\mathfrak{g}}(J)$  and  $\mathfrak{m} = [J, \mathfrak{g}]$ . It follows that

$$\mathfrak{m} = \{A \in \mathfrak{g} \mid \{J, A\} = AJ + JA = 0\}.$$

This implies that  $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$  and, hence, that  $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$  is a symmetric decomposition.

**Proposition 10.** *The orbit  $S := \text{Ad}_G(J) \cong G/K$ ,  $K := Z_G(J)$ , is an affine symmetric space and carries a canonical  $G$ -invariant para-complex structure  $J^S$ .*

*Proof.* The involutive automorphism  $A \mapsto JAJ^{-1} = JAJ$  of  $G$  has  $K$  as its fixed point set and defines the symmetry of  $G/K$  at the point  $eK$ .

The formula  $J_{\mathfrak{m}}A = JA = (1/2)[J, A]$ ,  $A \in \mathfrak{m}$ , defines a  $K$ -invariant para-complex structure on  $\mathfrak{m}$ , which extends to a  $G$ -invariant para-complex structure  $J^S$  on  $S$ . The structure  $J^S$  is integrable, since it is parallel under the canonical torsion-free connection of the symmetric space  $S$ .  $\square$

The projections onto  $\mathfrak{k}$  and  $\mathfrak{m}$  are given by

$$(5.1) \quad A \mapsto \frac{1}{2}J\{J, A\} = \frac{1}{2}(A + JAJ),$$

$$(5.2) \quad A \mapsto \frac{1}{2}J[J, A] = \frac{1}{2}(A - JAJ).$$

**5.2. The space of curvature tensors.** Let  $G \subset \text{GL}(V)$  be a linear Lie group of para-twistor type with Lie algebra  $\mathfrak{g}$ ,  $J \in \mathfrak{g}$  a para-complex structure and  $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$  the corresponding symmetric decomposition;  $\mathfrak{k} = Z_{\mathfrak{g}}(J)$  and  $\mathfrak{m} = [J, \mathfrak{g}]$ . Recall that  $\mathfrak{m}$  carries the para-complex structure  $J_{\mathfrak{m}}: A \mapsto JA = (1/2)[J, A]$ . For any subspace  $U \subset \text{End } V$  we denote by

$$\mathcal{R}(U) := \left\{ R \in U \otimes \bigwedge^2 V^* \mid R \text{ satisfies the first Bianchi identity} \right\}$$

the vector space of algebraic curvature tensor of type  $U$ .

The projection  $\pi_{\mathfrak{m}}: \mathfrak{g} \rightarrow \mathfrak{m}$  induces a projection

$$\pi_{\mathfrak{m}}: C^2(\mathfrak{g}) \rightarrow C^2(\mathfrak{m}).$$

According to (5.2), the projection  $\alpha^{\mathfrak{m}} := \pi_{\mathfrak{m}}\alpha \in C^2(\mathfrak{m})$  of  $\alpha \in C^2(\mathfrak{g})$  is given by

$$(5.3) \quad \alpha^{\mathfrak{m}}(X, Y) = \frac{1}{2}(\alpha(X, Y) - J\alpha(X, Y)J).$$

Recall that, since  $\mathfrak{m} \subset \text{End } V$  is endowed with the para-complex structure  $J_{\mathfrak{m}}$ , we have the decomposition (2.2)

$$C^2(\mathfrak{m}) = \sum_{p+q=2} C^{p,q}(\mathfrak{m}).$$

We put  $\pi_{\mathfrak{m}}^{p,q} := \pi^{p,q} \circ \pi_{\mathfrak{m}} : C^2(\mathfrak{g}) \rightarrow C^{p,q}(\mathfrak{m})$  and  $\mathcal{R}^{p,q}(\mathfrak{m}) := \mathcal{R}(\mathfrak{m}) \cap C^{p,q}(\mathfrak{m})$ .

The action of  $J$  as an automorphism of the tensor algebra induces involutions

$$T_J : C^2(\mathfrak{g}) \rightarrow C^2(\mathfrak{g}), \quad T_J : C^2(V) \rightarrow C^2(V).$$

We denote the  $\pm 1$ -eigenspaces of  $T_J$  on  $C^2(\mathfrak{g})$  by  $C_{\pm}^2(\mathfrak{g})$ , such that

$$C^2(\mathfrak{g}) = C_+^2(\mathfrak{g}) + C_-^2(\mathfrak{g}),$$

and put  $C_{\pm}^2(U) := C_{\pm}^2(\mathfrak{g}) \cap C^2(U)$  and  $\mathcal{R}_{\pm}(U) := C_{\pm}^2(\mathfrak{g}) \cap \mathcal{R}(U)$ , where  $U = \mathfrak{k}, \mathfrak{m}$ .

**Proposition 11.** (i) *The eigenspaces of  $T_J$  on  $C^2(\mathfrak{g})$  are given by*

$$(5.4) \quad C_+^2(\mathfrak{m}) = C^{1,1}(\mathfrak{m}),$$

$$(5.5) \quad C_-^2(\mathfrak{m}) = C^{2,0}(\mathfrak{m}) + C^{0,2}(\mathfrak{m}).$$

(ii) *The action of  $T_J$  on  $C^{p,q}(V)$  is given by*

$$T_J \alpha^{1,1} = -J \alpha^{1,1},$$

$$T_J \alpha^{2,0} = J \alpha^{2,0},$$

$$T_J \alpha^{0,2} = J \alpha^{0,2}.$$

*In particular,*

$$C^{1,1}(V) = \ker(T_J + L_J),$$

$$C^{2,0}(V) + C^{0,2}(V) = \ker(T_J - L_J),$$

where  $L_J \alpha = J \circ \alpha$ .

The action of  $J$  as a derivation on the tensor algebra induces an endomorphism

$$\alpha \mapsto J \cdot \alpha = [J, \alpha] - \alpha(J \cdot, \cdot) - \alpha(\cdot, J \cdot)$$

of  $C^2(\mathfrak{g})$ . Similarly,  $J$  acts as a derivation on  $C^2(V)$ .

**Proposition 12.** (i) *The action of  $J$  as a derivation on  $C^2(\mathfrak{g})$  is given by*

$$J \cdot \alpha^{p,q} = 2q J \alpha^{p,q} \quad \text{for all } \alpha^{p,q} \in C^{p,q}(\mathfrak{m}),$$



$$J \cdot \alpha = -2\alpha(J \cdot, \cdot) \quad \text{for all } \alpha \in C_+^2(\mathfrak{k}),$$

$$J \cdot C_-^2(\mathfrak{k}) = 0.$$

In particular, the vector space of  $J$ -invariants is given by

$$(5.6) \quad C^2(\mathfrak{g})^J = C_-^2(\mathfrak{k}) + C^{2,0}(\mathfrak{m}).$$

(ii) The action of  $J$  as a derivation on  $C^2(V)$  is given by

$$J \cdot \alpha^{2,0} = -J\alpha^{2,0},$$

$$J \cdot \alpha^{0,2} = 3J\alpha^{0,2},$$

$$J \cdot \alpha^{1,1} = J\alpha^{1,1}.$$

In particular,

$$C^{2,0}(V) = \ker(D_J + L_J),$$

$$C^{0,2}(V) = \ker(D_J - 3L_J),$$

$$C^{1,1}(V) = \ker(D_J - L_J),$$

where  $D_J\alpha = J \cdot \alpha$ .

The proposition shows that  $\pi_m^{1,1}C^2(\mathfrak{g})^J = \pi_m^{0,2}C^2(\mathfrak{g})^J = 0$  and  $\pi_m^{2,0}C^2(\mathfrak{g})^J = C^{2,0}(\mathfrak{m})$ .

**Proposition 13.** *The following holds*

- (i)  $\mathcal{R}(\mathfrak{g}) = \mathcal{R}_+(\mathfrak{g}) + \mathcal{R}_-(\mathfrak{g})$ ,
- (ii)  $\mathcal{R}(\mathfrak{m}) = \mathcal{R}_+(\mathfrak{m}) + \mathcal{R}_-(\mathfrak{m})$ ,
- (iii)  $\pi_m \mathcal{R}_+(\mathfrak{g}) = \pi_m^{1,1} \mathcal{R}_+(\mathfrak{g}) \supset \mathcal{R}_+(\mathfrak{m}) = \mathcal{R}^{1,1}(\mathfrak{m})$ ,
- (iv)  $\pi_m^{0,2} \mathcal{R}(\mathfrak{g}) = \mathcal{R}^{0,2}(\mathfrak{m})$ ,
- (v)  $\pi_m \mathcal{R}_-(\mathfrak{g}) = (\pi_m^{2,0} + \pi_m^{0,2}) \mathcal{R}_-(\mathfrak{g}) \supset \mathcal{R}_-(\mathfrak{m}) = \mathcal{R}^{2,0}(\mathfrak{m}) + \mathcal{R}^{0,2}(\mathfrak{m})$ .

Proof. (i) and (ii) follow from the fact that  $T_J: C^2(\mathfrak{g}) \rightarrow C^2(\mathfrak{g})$  preserves the subspaces  $\mathcal{R}(\mathfrak{m}) \subset \mathcal{R}(\mathfrak{g}) \subset C^2(\mathfrak{g})$  and (iii) follows from the equation (5.4). The equation (5.5) and (iv) imply (v). Therefore it suffices to prove (iv). For  $R \in \mathcal{R}(\mathfrak{g})$  and  $X, Y, Z \in V$  we calculate

$$\begin{aligned} (\pi_m^{0,2}R)(X, Y) &= \frac{1}{4}(R^m(X, Y) + R^m(JX, JY) - JR^m(JX, Y) - JR^m(X, JY)) \\ &= \frac{1}{8}(R(X, Y) - JR(X, Y)J + R(JX, JY) - JR(JX, JY)J \\ &\quad - JR(JX, Y) + R(JX, Y)J - JR(X, JY) + R(X, JY)J) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{8}(R(X, Y) - JR(X, Y)J - JR(JX, Y) - JR(X, JY)) \\
 &\quad + \frac{1}{8}(-JR(JX, JY)J + R(JX, JY) + R(X, JY)J + R(JX, Y)J) \\
 &= \frac{1}{8}(J(J \cdot R)(X, Y) - (J \cdot R)(JX, JY)J)
 \end{aligned}$$

and, therefore,

$$\sum_{\text{cyclic}} (\pi_m^{0,2} R)(X, Y)Z = \frac{1}{8}J \sum_{\text{cyclic}} (J \cdot R)(X, Y)Z - \frac{1}{8} \sum_{\text{cyclic}} (J \cdot R)(JX, JY)JZ = 0,$$

where the sum is over cyclic permutations of  $(X, Y, Z)$ . Here we used the fact that  $A \cdot \mathcal{R}(\mathfrak{g}) \subset \mathcal{R}(\mathfrak{g})$  for any  $A \in \mathfrak{g}$ . □

**5.3.  $G$ -structures with connection and associated  $K$ -structures.** Let  $G \subset GL(V)$ ,  $V = \mathbb{R}^n$ , be a linear Lie group.

DEFINITION 9. A  $G$ -structure on a manifold  $M$  is  $G$ -principal bundle  $\pi: P \rightarrow M$  endowed with a displacement form  $\theta$ , i.e. a  $G$ -equivariant  $V$ -valued one-form such that  $\ker \theta = T^v P := \ker d\pi$ .

We shall identify a point  $p \in P$  with the coframe

$$p: T_{\pi(p)}M \rightarrow V, \quad X \mapsto \theta_p((d\pi)_p^{-1}(X)).$$

DEFINITION 10. A principal connection in a  $G$ -principal bundle  $\pi: P \rightarrow M$  is a  $G$ -equivariant  $\mathfrak{g}$ -valued one-form  $\omega: TP \rightarrow \mathfrak{g}$  such that  $H := \ker \omega$  is a distribution transversal to the vertical distribution  $T^v P$ .

Recall that the wedge product of two one-forms  $\alpha, \beta$  with values in a Lie algebra is the Lie algebra valued two-form given by

$$[\alpha \wedge \beta](X, Y) := [\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)].$$

The curvature of a connection  $\omega$  is the  $\mathfrak{g}$ -valued  $G$ -equivariant horizontal two-form

$$\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega].$$

If  $\pi: P \rightarrow M$  is a  $G$ -structure with displacement form  $\theta$ , then the torsion of  $\omega$  is the  $V$ -valued  $G$ -equivariant horizontal two-form

$$\Theta := d\theta + [\omega \wedge \theta],$$

where the Lie bracket is taken in the affine Lie algebra  $V + \mathfrak{g}$ .

If  $\theta$  is the displacement form of a  $G$ -structure  $\pi: P \rightarrow M$  and  $\omega$  a principal connection then:

$$\kappa = \theta + \omega: TP \rightarrow V \oplus \mathfrak{g}$$

is a *Cartan connection*, i.e. a  $G$ -equivariant absolute parallelism which extends the canonical vertical parallelism  $T^v P \rightarrow \mathfrak{g}$ . The *curvature* of the Cartan connection  $\kappa$  is defined as the  $(V \oplus \mathfrak{g})$ -valued  $G$ -equivariant horizontal two-form

$$\Omega_\kappa := d\kappa + \frac{1}{2}[\kappa \wedge \kappa].$$

Notice that the  $V$  and  $\mathfrak{g}$ -components of  $\Omega_\kappa$  are exactly the torsion and curvature forms of  $\omega$ :

$$\Omega_\kappa^V = \Theta, \quad \Omega_\kappa^\mathfrak{g} = \Omega.$$

Let now  $K \subset G$  be a Lie subgroup with Lie algebra  $\mathfrak{k}$  and  $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$  a  $K$ -invariant direct decomposition of the vector space  $\mathfrak{g}$ . Accordingly, any  $\mathfrak{g}$ -valued form  $\alpha$  on  $P$  is decomposed as

$$\alpha = \alpha^\mathfrak{k} + \alpha^\mathfrak{m}.$$

**Proposition 14** ([2]). *Let  $(\pi: P \rightarrow M, \theta, \omega)$  be a  $G$ -structure with a connection and  $K \subset G$  a Lie subgroup. Then*

$$\pi': P \rightarrow Z := P/K$$

*is a  $K$ -structure with displacement form*

$$\theta' := \theta + \omega^\mathfrak{m}: TP \rightarrow V' := V \oplus \mathfrak{m}$$

*and connection*

$$\omega' := \omega^\mathfrak{k}.$$

*The curvature  $\Omega'$  and torsion  $\Theta'$  of  $\omega'$  are given by*

$$\begin{aligned} \Theta' &= (\Theta')^V + (\Theta')^\mathfrak{m} = (\Theta - [\omega^\mathfrak{m} \wedge \theta]) + \Omega^\mathfrak{m} - \frac{1}{2}[\omega^\mathfrak{m} \wedge \omega^\mathfrak{m}]^\mathfrak{m}, \\ \Omega' &= \Omega^\mathfrak{k} - \frac{1}{2}[\omega^\mathfrak{m} \wedge \omega^\mathfrak{m}]^\mathfrak{k}. \end{aligned}$$

**5.4. The twistor space of a  $G$ -structure of para-twistor type.** Let  $G \subset \text{GL}(V)$  be a linear Lie group of para-twistor type,  $J \in \mathfrak{g}$  a para-complex structure and  $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$  the corresponding symmetric decomposition;  $\mathfrak{k} = Z_{\mathfrak{g}}(J)$  and  $\mathfrak{m} = [J, \mathfrak{g}]$ . Let  $\pi: P \rightarrow M$  be a  $G$ -structure endowed with a principal connection  $\omega: TP \rightarrow \mathfrak{g}$ .  $(P, \omega)$  will be called a  $G$ -structure of para-twistor type. The vector space  $V' := V \oplus \mathfrak{m}$  has the para-complex structure  $J' = J \oplus J_{\mathfrak{m}}$ . The natural action of  $K = Z_G(J)$  on  $V'$  preserves this structure and is identified with a subgroup  $K \subset \text{GL}(V', C) := \text{Aut}(V', J')$ . This implies that the  $K$ -structure

$$\pi': P \rightarrow Z := P/K$$

is subordinated to a  $\text{GL}(V', C)$ -structure, i.e. to an almost para-complex structure  $\mathcal{J}$  on  $Z$ . At the point  $z = \pi' p \in Z$ ,  $p \in P$ , the almost para-complex structure  $\mathcal{J}$  is defined by:

$$\mathcal{J}_z = \hat{p}^{-1} \circ J' \circ \hat{p},$$

where  $\hat{p}: T_z Z \rightarrow V'$  is the coframe associated with  $p \in P$ . It is easily checked that this definition does not depend on  $p \in (\pi')^{-1}(z)$ .

Similarly, we can associate a para-complex structure  $J_z: T_{\pi p} M \rightarrow T_{\pi p} M$  with any point  $z = Kp \in Z$  by the formula

$$J_z := p \circ J \circ p^{-1},$$

using the isomorphism  $p: T_{\pi p} M \rightarrow V$ . This allows to identify the  $G/K$ -bundle  $\pi_Z: Z = P/K \rightarrow M = P/G$  with a bundle of para-complex structures on the tangent spaces of  $M$ .

We denote by  $\mathcal{H}_Z = \pi'_* \ker \omega \subset TZ$  the projection of the horizontal distribution of  $\omega$  to  $TZ$ . We call it the *horizontal distribution* of  $Z$ .

**DEFINITION 11.** Let  $(\pi: P \rightarrow M, \omega)$  be a  $G$ -structure of para-twistor type and  $K = Z_G(J)$ . Then the induced  $K$ -structure  $\pi': P \rightarrow Z = P/K$  endowed with the induced connection  $\omega' = \pi_{\mathfrak{k}} \circ \omega$ , the horizontal distribution  $\mathcal{H}_Z$  and the almost para-complex structure  $\mathcal{J}$  is called the *twistor space* associated to the  $G$ -structure of para-twistor type  $(P, \omega)$  and to the para-complex structure  $J \in \mathfrak{g}$ .

Notice that the almost para-complex structure  $\mathcal{J}$  and the horizontal distribution  $\mathcal{H}_Z$  are invariant under the parallel transport in  $TZ$  defined by the connection  $\omega'$ . Therefore, we can apply Propositions 1 and 2.

**Theorem 1.** *Let  $(\pi: P \rightarrow M, \omega)$  be a  $G$ -structure of para-twistor type, where  $\omega$  is a principal connection with curvature form  $\Omega$  and torsion form  $\Theta$  and  $(Z, \mathcal{J}, \mathcal{H}_Z)$  the corresponding twistor space. Then*

(i) The almost para-complex structure  $\mathcal{J}$  on  $Z$  is integrable if and only if

$$(5.7) \quad \pi^{0,2} \circ \Theta = 0 \quad \text{and} \quad \pi_m^{0,2} \circ \Omega = 0,$$

(ii) The horizontal distribution  $\mathcal{H}_Z \subset TZ$  is para-holomorphic if and only if

$$\pi_m^{1,1} \circ \Omega = 0,$$

where we consider the values of the horizontal forms  $\Theta$  and  $\Omega$  at  $p \in P$  as

$$\Theta_p: \bigwedge^2 T_{\pi'_p} Z \rightarrow V \quad \text{and} \quad \Omega_p: \bigwedge^2 T_{\pi'_p} Z \rightarrow \mathfrak{g}.$$

Proof. Since  $G$  is of para-twistor type,  $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$  is a symmetric decomposition and, in particular,  $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$ . By Proposition 14, the torsion of the connection  $\omega'$  in the  $K$ -principal bundle  $\pi': P \rightarrow Z$  is given by

$$\Theta' = (\Theta')^V + (\Theta')^{\mathfrak{m}} = (\Theta - [\omega^{\mathfrak{m}} \wedge \theta]) + \Omega^{\mathfrak{m}}.$$

The second term  $[\omega^{\mathfrak{m}} \wedge \theta]_p: \bigwedge^2 T_{\pi'_p} Z \rightarrow V' = V \oplus \mathfrak{m}$ ,  $p \in P$ , on the right-hand side is of type  $(2, 0)$  since

$$\theta' = \theta + \omega^{\mathfrak{m}}: T_{\pi'_p} Z \rightarrow V'$$

is of type  $(1, 0)$ :

$$\theta' \circ \mathcal{J}_{\pi'_p} = (J \oplus J_{\mathfrak{m}}) \circ \theta'.$$

Therefore the integrability condition  $\pi^{0,2}\Theta' = 0$  of Proposition 1 reduces to (5.7).

To prove (ii), we notice that the coframe  $\hat{p}: T_{\pi'_p} Z \rightarrow V' = V \oplus \mathfrak{m}$  maps the horizontal space  $(\mathcal{H}_Z)_{\pi'_p}$  to  $V$ . Therefore the tensor

$$S = T \quad \text{mod} \quad \mathcal{H}_Z$$

corresponds to  $(\Theta')^{\mathfrak{m}} = \Omega^{\mathfrak{m}}$  and  $S^{1,1}$  corresponds to  $\pi_m^{1,1} \circ \Omega$ . The two-form  $\Omega^{\mathfrak{m}}$  on  $P$  vanishes on the vertical distribution  $T^v P = \kappa^{-1}(\mathfrak{g})$ . This implies that  $\pi_m^{1,1} \circ \Omega$  vanishes on  $\hat{p}^{-1}(\mathfrak{m})$ . Therefore the para-holomorphicity condition (2.5) of Proposition 2 reduces to  $\pi_m^{1,1} \circ \Omega|_{\mathcal{H}_Z \times \mathcal{H}_Z} = 0$ , which is equivalent to  $\pi_m^{1,1} \circ \Omega = 0$ .  $\square$

Since any  $p \in P$  is an isomorphism  $p: T_{\pi_p} M \rightarrow V$  we can identify the horizontal two-forms  $\Theta$  and  $\Omega$  with  $G$ -equivariant functions

$$T: P \rightarrow \bigwedge^2 V^* \otimes V \quad \text{and} \quad R: P \rightarrow \bigwedge^2 V^* \otimes \mathfrak{g}.$$

In particular,  $T + \pi_m \circ R: P \rightarrow \bigwedge^2 V^* \otimes V' = C^2(V') = \oplus C^{p,q}(V')$ . Now we can reformulate the theorem in terms of  $T$  and  $R_m := \pi_m \circ R$ .

**Corollary 1.** *Under the assumptions of the previous theorem, the following is true.*

- (i) *The almost para-complex structure is integrable if and only if  $T$  and  $R_m$  take values in  $C^{2,0}(V') \oplus C^{1,1}(V')$ .*
- (ii) *The horizontal distribution is para-holomorphic if and only if  $R_m$  takes values in  $C_-(\mathfrak{m}) = C^{2,0}(\mathfrak{m}) \oplus C^{0,2}(\mathfrak{m})$ .*

*Both conditions are satisfied if and only if  $R_m$  is of type  $(2, 0)$  and  $T$  is of type  $(2, 0) + (1, 1)$ .*

Now we choose a local section  $p_0: M \rightarrow P$  and identify  $P$  locally with  $M \times G$ . We denote by  $T^{(p_0)}$  and  $R^{(p_0)}$  the restrictions of  $T$  and  $R$  to  $M = M \times \{e\} \subset M \times G$ . Then

$$T_{(x,g)} = g_* T_x^{(p_0)} = g T_x^{(p_0)}(g^{-1} \cdot, g^{-1} \cdot)$$

and

$$R_{(x,g)} = g_* R_x^{(p_0)} = g R_x^{(p_0)}(g^{-1} \cdot, g^{-1} \cdot) g^{-1}.$$

This implies, for all  $u, v \in V$ ,

$$\begin{aligned} \pi_m R_{(x,g)}(u, v) &= \pi_m g R_x^{(p_0)}(g^{-1}u, g^{-1}v) g^{-1} \\ &= g \pi_{g^{-1}mg} R_x^{(p_0)}(g^{-1}u, g^{-1}v) g^{-1} = g_*(\pi_{g^{-1}mg} R_x^{(p_0)})(u, v). \end{aligned}$$

For any para-complex structure  $I = gJg^{-1} \in S = G/K$  we have the vector spaces  $\mathfrak{m}(I) = [I, \mathfrak{g}] = g\mathfrak{m}g^{-1}$  and  $V'(I) = V \oplus \mathfrak{m}(I)$  with the para-complex structures  $gJ_mg^{-1}$  and  $I' = gJ'g^{-1}$ , respectively.

The above calculation implies that the  $(p, q)$  component of  $T$  or  $R_m$ , with respect to  $(J, J')$ , vanishes if and only if the  $(p, q)$  component of  $T^{(p_0)}$  or  $\pi_{\mathfrak{m}(I)} \circ R^{(p_0)}$ , with respect to  $(I, I')$ , vanishes for all  $I \in S$ . We will use the symbol  $\pi_{\mathfrak{m}(I)}^{p,q} := \pi_I^{p,q} \circ \pi_{\mathfrak{m}(I)}$ , where  $\pi_I^{p,q}: C^2(\mathfrak{m}(I)) \rightarrow C_I^{p,q}(\mathfrak{m}(I))$  is the projection onto the  $(p, q)$ -component with respect to  $(I, I')$  for any  $I \in S$ . Similarly we define  $\pi_I^{p,q}: C^2(V) \rightarrow C_I^{p,q}(V)$  as the projection onto the  $(p, q)$ -component with respect to  $I$ .

This motivates the definition of the following two  $G$ -submodules of  $\mathcal{R}(\mathfrak{g})$ :

$$\begin{aligned} \mathcal{R}_{\text{int}}(\mathfrak{g}) &:= \{R \in \mathcal{R}(\mathfrak{g}) \mid \pi_{\mathfrak{m}(I)}^{0,2} R = 0 \text{ for all } I \in S\}, \\ \mathcal{R}_{\text{hol}}(\mathfrak{g}) &:= \{R \in \mathcal{R}(\mathfrak{g}) \mid \pi_{\mathfrak{m}(I)}^{1,1} R = 0 \text{ for all } I \in S\}. \end{aligned}$$

We also define a  $G$ -submodule  $\mathcal{T}_{\text{int}}(\mathfrak{g}) \subset C^2(V)$  by

$$\mathcal{T}_{\text{int}}(\mathfrak{g}) := \{T \in C^2(V) \mid \pi_I^{0,2} T = 0 \text{ for all } I \in S\}.$$

**Corollary 2.** *Under the assumptions of Theorem 1, the following is true.*

(i) *The almost para-complex structure  $\mathcal{J}$  is integrable if and only if the functions  $T^{(p_0)}$  and  $R^{(p_0)}$ , associated to a local frame  $p_0$ , take values in the  $G$ -modules  $\mathcal{T}_{\text{int}}(\mathfrak{g})$  and  $\mathcal{R}_{\text{int}}(\mathfrak{g})$ , respectively.*

(ii) *The horizontal distribution is para-holomorphic if and only if  $R^{(p_0)}$  takes values in  $\mathcal{R}_{\text{hol}}(\mathfrak{g})$ .*

*Both conditions are satisfied if and only if  $\pi_{\mathfrak{m}(I)}R$  is of type  $(2, 0)$  and  $T$  is of type  $(2, 0) + (1, 1)$  for all  $I \in S$ .*

**Corollary 3.** *Under the assumptions of Theorem 1, the almost para-complex structure  $\mathcal{J}$  on the twistor space  $Z$  is integrable and the horizontal distribution  $\mathcal{H}_Z$  is para-holomorphic if for all  $x \in M$  there exists a frame  $p \in \pi^{-1}(x)$  such that the curvature  $R^{(p)} \in \mathcal{R}(\mathfrak{g})$  takes values in the  $G$ -module*

$$\mathcal{R}(\mathfrak{g})^{D^S} = \{R \in \mathcal{R}(\mathfrak{g}) \mid I \cdot R = 0 \text{ for all } I \in S\}$$

*and the torsion  $T^{(p)}$  satisfies  $\pi^{0,2}T^{(p)} = 0$ .*

Proof. This follows from (5.6) and the previous corollary. □

**Corollary 4.** *Let  $G$  be a group of para-twistor type such that  $\pi_{\mathfrak{m}(I)}\mathcal{R}(\mathfrak{g}) \subset C^{2,0}(\mathfrak{m}(I))$ , for all  $I \in S$ , for example if  $\mathcal{R}(\mathfrak{g}) = \mathcal{R}(\mathfrak{g})^{D^S}$ . Then for any  $G$ -structure  $(\pi: P \rightarrow M, \omega)$  with a torsion-free connection  $\omega$ , the almost para-complex structure  $\mathcal{J}$  on the twistor space  $Z$  is integrable and the horizontal distribution  $\mathcal{H}_Z$  is para-holomorphic.*

### 6. Integrability and holomorphicity results for the twistor spaces of a para-quaternionic Kähler manifold

**Theorem 2.** *Let  $(M, g, Q)$  be a para-quaternionic Kähler manifold and  $(Z^\epsilon, \mathcal{J}^\epsilon, \mathcal{H}_{Z^\epsilon})$  its twistor space, where  $\epsilon = \pm$ , see Sections 4 and 5.4. Then for  $\epsilon = -1$  the almost complex structure  $\mathcal{J}^\epsilon$  is integrable and the horizontal distribution is holomorphic. Similarly, for  $\epsilon = 1$  the almost para-complex structure  $\mathcal{J}^\epsilon$  is integrable and the horizontal distribution is para-holomorphic.*

Proof. By Proposition 9, the para-quaternionic Kähler structure defines a  $G$ -structure  $\pi: P \rightarrow M$ , where  $G \subset \text{GL}(\mathbb{R}^2 \otimes \mathbb{R}^{2n})$  is the normalizer of the connected Lie group  $G_0 := \text{SL}_2(\mathbb{R}) \otimes \text{Sp}(\mathbb{R}^{2n})$  in  $\text{SO}(2n, 2n)$ . Any para-quaternionic coframe  $p \in P$  defines an isometry  $p: (T_{\pi p}M, g_{\pi p}) \xrightarrow{\sim} (\mathbb{R}^2 \otimes \mathbb{R}^{2n}, g_{\text{can}})$ , which maps  $Q_{\pi p}$  to  $\mathfrak{sl}(2, \mathbb{R}) = \mathfrak{sl}(2, \mathbb{R}) \otimes \text{Id}$ , see Definition 7. The linear group  $G$  is of para-twistor type and also of twistor type,

i.e. there exists elements  $I, J \in \mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sp}(\mathbb{R}^{2n})$  such that  $I^2 = -\text{Id}$  and  $J^2 = \text{Id}$ . In fact, we can choose  $I = p \circ J_1 \circ p^{-1}$  and  $J = p \circ J_2 \circ p^{-1}$ . The symmetric space

$$\begin{aligned} \mathfrak{sl}(2, \mathbb{R}) = \mathfrak{sl}(2, \mathbb{R}) \otimes \text{Id} \supset S^- &= \text{Ad}_G(I) = G/Z_G(I) = \text{GL}_2(\mathbb{R})/Z_{\text{GL}_2(\mathbb{R})}(I) \\ &= \text{GL}_2(\mathbb{R})/\text{GL}_1(\mathbb{C}) = \text{SL}_2^\pm(\mathbb{R})/\text{SO}(2) \end{aligned}$$

is the two-sheeted hyperboloid in the three-dimensional Minkowski space  $\mathfrak{sl}(2, \mathbb{R}) \cong \mathbb{R}^{2,1}$ , whereas the symmetric space

$$\begin{aligned} \mathfrak{sl}(2, \mathbb{R}) \supset S^+ &= \text{Ad}_G(J) = G/Z_G(J) = \text{GL}_2(\mathbb{R})/Z_{\text{GL}_2(\mathbb{R})}(J) \\ &= \text{GL}_2(\mathbb{R})/\text{GL}_1(\mathbb{C}) = \text{SL}_2(\mathbb{R})/\text{SO}(1, 1) \end{aligned}$$

is the one-sheeted hyperboloid.

To finish the proof, in the case  $\epsilon = +1$  we apply Corollary 4, in the case  $\epsilon = -1$  [2] Theorem 7.3, since, by Proposition 4, the space of curvature tensors

$$\mathcal{R}(\mathfrak{g}) = \mathcal{R}(\mathfrak{g})^{\mathfrak{sl}(2, \mathbb{R})} = \mathcal{R}(\mathfrak{g})^{D_S},$$

for  $S = S^\pm$ . □

### 7. The canonical $\epsilon$ -Kähler-Einstein metric and contact structure on the twistor space $Z^\epsilon$ of a para-quaternionic Kähler manifold

DEFINITION 12. An  $\epsilon$ -Kähler manifold is a pseudo-Riemannian manifold  $(M, g)$  together with a parallel skew-symmetric  $\epsilon$ -complex structure  $J$ . An  $\epsilon$ -Kähler manifold  $(M, g, J)$  is called a *Kähler manifold* if  $\epsilon = -1$  and a *para-Kähler manifold* if  $\epsilon = +1$ . The parallel symplectic form  $\omega = g(J \cdot, \cdot)$  is called the *Kähler form*.

REMARKS. The metric of a para-Kähler manifold has signature  $(n, n)$ , since the  $\pm 1$ -eigendistributions  $T^\pm M$  of  $J$  are isotropic. Moreover, they are parallel and  $\omega$ -Lagrangian.

Conversely, a *bi-Lagrangian manifold* [8], i.e. a symplectic manifold  $(M, \omega)$  with two complementary Lagrangian integrable distributions  $T^\pm M$ , has the structure of a para-Kähler manifold, where  $J|_{T^\pm M} = \pm \text{Id}$  and  $g = \omega(J \cdot, \cdot)$ .

An integrable skew-symmetric  $\epsilon$ -complex structure on a pseudo-Riemannian manifold is *parallel*, and hence defines an  $\epsilon$ -Kähler structure, if and only if the Kähler form  $\omega$  is closed, see [10] Theorem 1.

DEFINITION 13. An  $\epsilon$ -holomorphic distribution  $\mathcal{D}$  of real codimension 2 on an  $\epsilon$ -complex manifold  $Z$  is called an  $\epsilon$ -holomorphic *contact structure* if the Frobenius form  $[\cdot, \cdot]: \bigwedge^2 \mathcal{D} \rightarrow TZ/\mathcal{D}$  is non-degenerate.

**Theorem 3.** *Let  $(Z^\epsilon, \mathcal{J}^\epsilon)$  be the  $\epsilon$ -twistor space of a para-quaternionic Kähler manifold  $(M, g, Q)$  with non-zero reduced scalar curvature  $\nu$ . Then*



- (i) *the canonical metric  $g_t = g_t^\epsilon$  on  $Z^\epsilon$  is  $\epsilon$ -Kähler-Einstein if and only if  $t = -\epsilon/\nu$ . Moreover,  $g_t$  is Einstein if and only if  $t = -\epsilon/\nu$  or  $t = -\epsilon/(\nu(n+1))$ .*
- (ii) *The horizontal distribution  $\mathcal{H}_Z \subset TZ^\epsilon$  is an  $\epsilon$ -holomorphic contact structure.*

Proof. (i) By Theorem 2 and Proposition 7 the  $\epsilon$ -complex structure  $\mathcal{J}^\epsilon$  is integrable and  $g_t$ -skew-symmetric for all  $t$ . By the above remark, to check when  $(Z^\epsilon, \mathcal{J}^\epsilon, g_t)$  is  $\epsilon$ -Kähler it is sufficient to check when the Kähler form  $\omega_t = g_t(\mathcal{J}^\epsilon \cdot, \cdot)$  is closed.

The twistor bundle  $Z^\epsilon = P/G_A \rightarrow M$ , see Proposition 9, is a bundle associated with the principal bundle

$$P' := P/Z_G(\mathrm{GL}_2) \rightarrow M = P'/\mathrm{SO}_3^\epsilon,$$

where  $\mathrm{SO}_3^\epsilon = \mathrm{SO}(2, 1)$  for  $\epsilon = +1$  and  $\mathrm{SO}_3^\epsilon = \mathrm{SO}(1, 2) \cong \mathrm{SO}(2, 1)$  for  $\epsilon = -1$ . In other words,  $P'$  is the  $\mathrm{SO}_3^\epsilon$ -principal bundle of standard bases  $p = (J_1, J_2, J_3)$  of  $\mathcal{Q}_x^\epsilon$ ,  $x \in M$ , where  $J_1^2 = \epsilon \mathrm{Id}$ ,  $J_2^2 = \mathrm{Id}$  and  $J_3^2 = -\epsilon \mathrm{Id}$ . We have a natural projection

$$\pi_{P'}: P' \rightarrow Z^\epsilon = P'/\mathrm{SO}_2^\epsilon, \quad (J_1, J_2, J_3) \mapsto J_1,$$

where  $\mathrm{SO}_2^\epsilon = \mathrm{SO}(1, 1)$  for  $\epsilon = +1$  and  $\mathrm{SO}_2^\epsilon = \mathrm{SO}(2)$  for  $\epsilon = -1$  is the stabilizer of  $(1, 0, 0)' \in \mathbb{R}^3$ .

The closure of  $\omega_t$  is equivalent to the closure of its pull back  $\omega'_t = \pi_{P'}^* \omega_t$  to  $P'$ . The two-form  $\omega'_t$  can be written as

$$(7.1) \quad \omega'_t = g'_t(\mathcal{J}_1 \cdot, \cdot), \quad g'_t = t g^v + \pi_{P'}^* g.$$

Here  $\pi_{P'}^* g$  is the pull back of the metric  $g$  on  $M$  and  $g^v$  is the metric on the vertical bundle  $T^v P'$ , which corresponds to a suitably normalized ad-invariant scalar product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{so}_3^\epsilon = \mathrm{Lie} \mathrm{SO}_3^\epsilon$ , extended by zero to the horizontal bundle  $\mathcal{H}$  associated with the Levi-Civita connection of  $M$ . The normalization of the scalar product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{so}_3^\epsilon = \mathrm{ad}(\mathfrak{sl}_2(\mathbb{R})) \cong \mathfrak{sl}_2(\mathbb{R}) = \mathrm{span}\{J_1^0, J_2^0, J_3^0\}$  is given by

$$(7.2) \quad -\epsilon \langle \mathrm{ad}_{J_\alpha^0}, \mathrm{ad}_{J_\beta^0} \rangle = -4\epsilon_\alpha \delta_{\alpha\beta} = 4 \langle J_\alpha^0, J_\beta^0 \rangle,$$

where  $(J_1^0, J_2^0, J_3^0)$  is the standard  $\epsilon$ -quaternionic basis of  $\mathfrak{sl}_2(\mathbb{R})$ , with the relations

$$(7.3) \quad (J_\alpha^0)^2 = \epsilon_\alpha \mathrm{Id}, \quad (\epsilon_1, \epsilon_2, \epsilon_3) = (\epsilon, 1, -\epsilon).$$

The above scalar product on  $\mathfrak{so}_3^\epsilon$  has signature  $(2, 1)$  if  $\epsilon = +1$  and  $(1, 2)$  if  $\epsilon = -1$ . The factor 4 is chosen such that the canonical projection  $(P', g'_t) \rightarrow (Z^\epsilon = P'/\mathrm{SO}_2^\epsilon, g_t)$

is a pseudo-Riemannian submersion. Notice that the vertical vectors

$$\text{ad}_{J_2}, \text{ad}_{J_3} \in T_p^v P' \cong \mathfrak{so}_3^\epsilon = \text{ad}(\mathfrak{sl}_2(\mathbb{R})), \quad p \in P',$$

are mapped to

$$\text{ad}_{J_2} J_1 = -2J_3, \quad \text{ad}_{J_3} J_1 = -2\epsilon J_2 \in T^v Z^\epsilon \subset Q_x^\epsilon \cong \mathfrak{sl}_2(\mathbb{R}), \quad x = \pi_{P'}(p).$$

The field  $p \mapsto (\mathcal{J}_\alpha)_p$  is defined at  $p = (J_1, J_2, J_3)$  as the following endomorphism of  $T_p P' = T^v P' \oplus \mathcal{H}_p \cong \mathfrak{so}_3^\epsilon \oplus T_x M$ ,  $x = \pi_{P'}(p)$ ,

$$\mathcal{J}_\alpha|_{\mathcal{H}_p} : T_x M \rightarrow T_x M, \quad X \mapsto J_\alpha X, \quad \mathcal{J}_\alpha|_{T^v P'} = \frac{1}{2} \text{ad}_{J_\alpha^0}.$$

It is sufficient to check  $d\omega'_t = 0$  on three vectors, each of which are horizontal or vertical. Moreover, it is sufficient to consider the fundamental vertical fields  $(V_1, V_2, V_3)$ , which correspond to  $(J_1^0, J_2^0, J_3^0)$  and basic horizontal fields  $X, Y, Z, \dots$  on  $P'$ , i.e. horizontal lifts of vector fields  $X_M, Y_M, Z_M$  on  $M$ .

**Lemma 2.** *With the above notations we have*

- (i)  $[V_1, V_2] = 2V_3, [V_3, V_1] = -2\epsilon V_2, [V_2, V_3] = -2V_1,$
- (ii) *the functions  $g'_t(V_\alpha, V_\beta)$  and  $\omega'_t(V_\alpha, V_\beta)$  are constant for all  $\alpha, \beta \in \{1, 2, 3\}$ ,*
- (iii)  $[V_\alpha, X] = 0,$
- (iv)  $[X, Y]^v = -(v/2) \sum_\alpha \epsilon_\alpha g'_t(\mathcal{J}_\alpha X, Y) V_\alpha = -(v/2) \sum_\alpha \epsilon_\alpha g(J_\alpha X_M, Y_M) V_\alpha,$  *where  $[X, Y]^v$  is evaluated at the point  $p = (J_1, J_2, J_3) \in P'$ ,*
- (v)  $\mathcal{L}_{V_\alpha} g'_t = 0, \mathcal{L}_{V_1} \mathcal{J}_1 = 0, \mathcal{L}_{V_2} \mathcal{J}_1 = -2\mathcal{J}_3, \mathcal{L}_{V_3} \mathcal{J}_1 = -2\epsilon \mathcal{J}_2$  *and*
- (vi)  $(\mathcal{L}_X g'_t)(U, V) = 0$  *for all  $U, V \in T^v P'$ .*

*Proof.* (i) follows from the  $\epsilon$ -quaternionic relations

$$[J_1^0, J_2^0] = 2J_3^0, \quad [J_3^0, J_1^0] = -2\epsilon J_3^0, \quad [J_2^0, J_3^0] = -2J_1^0.$$

(ii) Since the metric  $g^v$  corresponds to the ad-invariant scalar product (7.2), the functions

$$g'_t(V_\alpha, V_\beta) = t g^v(V_\alpha, V_\beta) = -4\epsilon t \langle J_\alpha^0, J_\beta^0 \rangle = 4\epsilon t \epsilon_\alpha \delta_{\alpha\beta}$$

are constant. Similarly, the functions  $\omega'_t(V_\alpha, V_\beta)$  are constant, because, for all fundamental vector fields  $V_\alpha$ , the vector field  $\mathcal{J}_1 V_\alpha$  is again a fundamental vector field.

(iii) The vector field  $[V_\alpha, X]$  is horizontal, since the principal action preserves the horizontal distribution. On the other hand, it is mapped to  $[0, X_M] = 0$  under the projection  $P' \rightarrow M$ . This shows that  $[V_\alpha, X] = 0$ .

(iv) follows from Proposition 4, since  $[X, Y]^v = -\Omega'(X, Y)$ , where  $\Omega'$  stands for the curvature form of the principal bundle  $P' \rightarrow M$ .

(v)  $\mathcal{L}_{V_\alpha} g'_t = 0$  follows from the ad-invariance of  $g^v$ , cf. (7.1). The remaining equations are obtained from (i) using

$$\mathcal{J}_1 V_1 = 0, \quad \mathcal{J}_1 V_2 = V_3, \quad \mathcal{J}_1 V_3 = \epsilon V_2.$$

Finally, (ii) and (iii) easily imply (vi).  $\square$

Part (i) and (ii) of the lemma, yields

$$d\omega'_t(V_1, V_2, V_3) = V_1\omega'_t(V_2, V_3) - \omega'_t([V_1, V_2], V_3) + \text{cycl.} = 0.$$

Using part (i), (ii), (v) and (vi) of the lemma, we calculate

$$\begin{aligned} d\omega'_t(V_\alpha, V_\beta, X) &= -\omega'_t([V_\beta, X], V_\alpha) - \omega'_t([X, V_\alpha], V_\beta) = -(\mathcal{L}_X \omega'_t)(V_\alpha, V_\beta) \\ &= -g'_t((\mathcal{L}_X \mathcal{J}_1)V_\alpha, V_\beta) = -g'_t([X, \mathcal{J}_1 V_\alpha] - \mathcal{J}_1[X, V_\alpha], V_\beta) \\ &= g'_t(X, [V_\beta, \mathcal{J}_1 V_\alpha]) + g'_t(X, [\mathcal{J}_1 V_\beta, V_\alpha]) = 0. \end{aligned}$$

By (iii), (iv) and (v) of the lemma, we compute

$$\begin{aligned} d\omega'_t(V_1, X, Y) &= V_1\omega'_t(X, Y) - \omega'_t([X, Y], V_1) \\ &= g'_t((\mathcal{L}_{V_1} \mathcal{J}_1)X, Y) + \frac{\nu}{2} \sum_{\alpha=1}^3 \epsilon_\alpha g(J_\alpha X_M, Y_M) \omega'_t(V_\alpha, V_1) \\ &= 0 + \frac{\nu}{2} \sum_{\alpha=1}^3 \epsilon_\alpha g(J_\alpha X_M, Y_M) g'_t(\mathcal{J}_1 V_\alpha, V_1) = 0, \end{aligned}$$

since  $\mathcal{J}_1 T^v P' = \text{span}\{V_2, V_3\}$ . Similarly, we calculate

$$\begin{aligned} d\omega'_t(V_2, X, Y) &= V_2\omega'_t(X, Y) - \omega'_t([X, Y], V_2) \\ &= g'_t((\mathcal{L}_{V_2} \mathcal{J}_1)X, Y) + \frac{\nu}{2} \sum_{\alpha=1}^3 \epsilon_\alpha g(J_\alpha X_M, Y_M) \omega'_t(V_\alpha, V_2) \\ &= -2g'_t(\mathcal{J}_3 X, Y) + \frac{\nu}{2} \sum_{\alpha=1}^3 \epsilon_\alpha g(J_\alpha X_M, Y_M) g'_t(\mathcal{J}_1 V_\alpha, V_2) \\ &= -2g(J_3 X_M, Y_M) + \frac{\nu t}{2} \epsilon_3 g(J_3 X_M, Y_M) g^v(\mathcal{J}_1 V_3, V_2) \\ &= -2g(J_3 X_M, Y_M) + \frac{\nu t}{2} (-\epsilon) g(J_3 X_M, Y_M) g^v(\epsilon V_2, V_2) \\ &= -2g(J_3 X_M, Y_M) - 2\epsilon \nu t g(J_3 X_M, Y_M), \end{aligned}$$

since  $g^v(V_2, V_2) = -4\epsilon \langle J_2^0, J_2^0 \rangle = 4\epsilon\epsilon_2 = 4\epsilon$ . In the same way, we obtain

$$\begin{aligned} d\omega'_t(V_3, X, Y) &= -2\epsilon g(J_2 X_M, Y_M) + \frac{vt}{2} g(J_2 X_M, Y_M) g^v(\mathcal{J}_1 V_2, V_3) \\ &= -2\epsilon g(J_2 X_M, Y_M) + \frac{vt}{2} g(J_2 X_M, Y_M) g^v(V_3, V_3) \\ &= -2\epsilon g(J_2 X_M, Y_M) - 2vtg(J_2 X_M, Y_M), \end{aligned}$$

since  $g^v(V_3, V_3) = -4\epsilon \langle J_3^0, J_3^0 \rangle = 4\epsilon\epsilon_3 = -4$ . This shows that  $d\omega'_t(U, X, Y) = 0$  for all vertical vector fields  $U$  if and only if  $vt = -\epsilon$ .

It remains to check that  $d\omega'_t(X_p, Y_p, Z_p)$  vanishes on three horizontal vectors

$$X_p, Y_p, Z_p \in \mathcal{H}_p, \quad p \in P'.$$

Let  $t \mapsto \tilde{c}(t) = (J_1(t), J_2(t), J_3(t)) \in P'$  be the horizontal lift of a curve  $t \mapsto c(t) \in M$  such that  $\tilde{c}(0) = p$  and  $\tilde{c}'(0) = X_p$ . Notice that the horizontality of  $\tilde{c}$  means that  $t \mapsto J_\alpha(t)$  is parallel along  $c$ .

Let  $t \mapsto Y(t) \in \mathcal{H}_{\tilde{c}(t)}$  be the horizontal lift of the vector field

$$t \mapsto Y_M(t) := \parallel_{c(0)}^{c(t)} d\pi_{P'} Y_p \in T_{c(t)} M,$$

which is parallel along the base curve  $c$ . The initial value of  $Y$  is  $Y(0) = Y_p$ . It suffices to prove that

$$(\nabla'_{X_p} \omega'_t)(Y_p, Z_p) = g'_t((\nabla'_{X_p} \mathcal{J}_1) Y_p, Z_p) = 0,$$

where  $\nabla'$  is the Levi-Civita connection of  $g'_t$ . We have to check that the horizontal component of

$$(\nabla'_{X_p} \mathcal{J}_1) Y_p = \nabla'_{X_p} (\mathcal{J}_1 Y) - \mathcal{J}_1 \nabla'_{X_p} Y$$

vanishes. Therefore, we calculate

$$\begin{aligned} d\pi_{P'}(\nabla'_{X_p} (\mathcal{J}_1 Y) - \mathcal{J}_1 \nabla'_{X_p} Y) &= \nabla_{c'(0)}(J_1(t) Y_M(t)) - J_1(0) \nabla_{c'(0)} Y_M(t) \\ &= (\nabla_{c'(0)} J_1(t)) Y_M(0) = 0. \end{aligned}$$

Here we have used two facts: first, that  $t \mapsto \mathcal{J}_1 Y(t)$  is a basic horizontal vector field along  $\tilde{c}$ , which projects onto

$$d\pi_{P'} \mathcal{J}_1 Y(t) = J_1(t) Y_M(t)$$

and, second, that  $d\pi_{P'} \nabla'_{X_p} Y = \nabla_{X_M} Y_M$  for any two basic horizontal vector fields  $X, Y$  (e.g. along a horizontal curve), where  $\nabla$  is the Levi-Civita connection in  $M$ . The latter is a standard fact about pseudo-Riemannian submersions. This proves that  $g_t$  is  $\epsilon$ -Kähler-Einstein if and only if  $t = -\epsilon/\nu$ . The above argument proves also the following proposition.

**Proposition 15.** *For any horizontal vectors  $X, Y, Z$  on  $P'$  and  $\alpha = 1, 2, 3$ , we have*

$$g'_t((\nabla_X \mathcal{J}_\alpha)Y, Z) = 0.$$

Next we study the Einstein equations for the family  $g'_t$ . We recall the definition of the O’Neill tensor and the O’Neill formulas for the covariant derivative of a pseudo-Riemannian submersion  $\pi: E \rightarrow M$  with totally geodesic fibres, see [12, 5]. The O’Neill tensor  $A \in \Omega^1(\text{End } TE)$  is a one-form with values in skew-symmetric endomorphisms. It is given by

$$(7.4) \quad A_U = 0, \quad A_X Y = -A_Y X = (\nabla_X Y)^v = \frac{1}{2}[X, Y]^v, \quad A_X U = (\nabla_X U)^h,$$

where  $U$  is a vertical vector field and  $X, Y$  are horizontal vector fields. The superscripts  $v$  and  $h$  stand for the vertical and horizontal components, respectively. If  $X$  is a basic horizontal vector field then, in addition

$$(7.5) \quad A_X U = (\nabla_X U)^h = \nabla_U X.$$

The covariant derivatives in  $E$  are given by

$$(7.6) \quad \nabla_U V = \nabla_U^F V,$$

$$(7.7) \quad \nabla_U X = (\nabla_U X)^h,$$

$$(7.8) \quad \nabla_X U = (\nabla_X U)^v + A_X U,$$

$$(7.9) \quad \nabla_X Y = A_X Y + (\nabla_X Y)^h.$$

Here  $\nabla^F$  and  $\nabla^M$  denote the covariant derivative in the fibres  $F$  and in the base  $M$ , respectively. For basic horizontal vector fields  $X, Y$ , we have  $[U, X]^h = 0$  for any vertical (and hence projectable) vector field  $U$ . Moreover, we have

$$(7.10) \quad (\nabla_X U)^v = [X, U],$$

$$(7.11) \quad \pi_* \nabla_X Y = \nabla_{\pi_* X}^M \pi_* Y.$$

In particular,  $\nabla_X Y$  is a projectable vector field on  $E$ .

**Proposition 16** (cf. [12]). *Let  $\pi: E \rightarrow M$  be a pseudo-Riemannian submersion with totally geodesic fibres  $F$ . Then the Ricci and scalar curvatures of  $E$  are given by:*

$$(7.12) \quad \text{Ric}(U, V) = \text{Ric}^F(U, V) + \sum_i \epsilon_i \langle A_{X_i} U, A_{X_i} V \rangle,$$

$$(7.13) \quad \text{Ric}(X, U) = \langle (\text{div } A)X, U \rangle = \sum_i \epsilon_i \langle (\nabla_{X_i} A)_{X_i} X, U \rangle,$$

$$(7.14) \quad \text{Ric}(X, Y) = \text{Ric}^M(\pi_*X, \pi_*Y) - 2 \sum_i \epsilon_i \langle A_X X_i, A_Y X_i \rangle,$$

$$(7.15) \quad \text{scal} = \pi^* \text{scal}^M + \text{scal}^F - \sum_{i,j} \epsilon_i \epsilon_j \langle A_{X_i} X_j, A_{X_i} X_j \rangle.$$

**Proposition 17.** *The divergence  $\text{div } A \in \Gamma(\text{End } TP')$  of the O’Neill tensor of the principal bundle  $P' \rightarrow M$  preserves the horizontal distribution. In particular,*

$$\text{Ric}(X, U) = g'_t((\text{div } A)X, U) = 0.$$

Proof. By (7.4) and Lemma 2 (iv), the value of the O’Neill tensor on two basic horizontal vector fields  $X, Y$  is given by

$$(7.16) \quad A_X Y = \frac{1}{2}[X, Y]^v = -\frac{\nu}{4} \sum_\alpha \epsilon_\alpha g'_t(\mathcal{J}_\alpha X, Y) V_\alpha.$$

It is sufficient to prove that  $g'_t((\nabla_X A)_Y Z, U) = 0$ . This follows from the remark that  $\nabla_X V_\alpha = A_X V_\alpha$  is horizontal, by (7.5), and Proposition 15. □

The skew-symmetry of  $A_X$  and (7.16) imply

$$(7.17) \quad A_X U = \frac{\nu}{4} \sum_\alpha \epsilon_\alpha g'_t(U, V_\alpha) \mathcal{J}_\alpha X.$$

In fact,

$$g'_t(A_X U, Y) = -g'_t(U, A_X Y) = \frac{\nu}{4} \sum_\alpha \epsilon_\alpha g'_t(\mathcal{J}_\alpha X, Y) g'_t(U, V_\alpha).$$

**Proposition 18.** *Let  $P'$  be the total space of the principal bundle  $P' \rightarrow M$  of admissible frames of  $Q$  over a para-quaternionic Kähler manifold  $(M, g, Q)$ . Then the Ricci curvature of the metric  $g'_t$  on  $P'$  is given by:*

$$(7.18) \quad \text{Ric}(U, V) = -\epsilon \left( \frac{1}{2t} + \nu^2 nt \right) g'_t(U, V), \quad U, V \in T^v P',$$

$$(7.19) \quad \text{Ric}(U, X) = 0,$$

$$(7.20) \quad \text{Ric}(X, Y) = \left( \nu(n+2) + \frac{3\epsilon\nu^2 t}{2} \right) g'_t(X, Y), \quad X, Y \in \mathcal{H} = (T^v P')^\perp.$$

Proof. We calculate the Ricci curvature using the formulas in Proposition 16. The fibre  $F$  is identified with the Lie group  $\text{SO}(2, 1)$  with a bi-invariant pseudo-Riemannian

metric  $g'_t$ , which is related with the Killing form  $B$  by

$$(7.21) \quad g'_t = \frac{\epsilon t}{2} B,$$

see (7.2). Therefore

$$(7.22) \quad \text{Ric}^F = -\frac{1}{4} B = \frac{-\epsilon}{2t} g'_t.$$

We compute the second term in equation (7.12) using (7.17):

$$\begin{aligned} \sum_i \epsilon_i \langle A_{X_i} U, A_{X_i} V \rangle &= \sum_i \epsilon_i \frac{\nu^2}{16} \sum_\alpha \epsilon_\alpha^2 g'_t(U, V_\alpha) g'_t(V, V_\alpha) g'_t(\mathcal{J}_\alpha X_i, \mathcal{J}_\alpha X_i) \\ &= \frac{\nu^2}{16} \sum_{i,\alpha} \epsilon_i^2 (-\epsilon_\alpha) g'_t(U, V_\alpha) g'_t(V, V_\alpha) = -\epsilon \nu^2 n t g'_t(U, V). \end{aligned}$$

This implies the first equation (7.18). The second equation (7.19) was already established in Proposition 17. Since  $M$  is an Einstein manifold with scalar curvature  $\text{scal}^M = 4n(n+2)\nu$ ,

$$(7.23) \quad \text{Ric}^M = \frac{\text{scal}}{4n} g = \nu(n+2)g.$$

We compute the second term in equation (7.14) using (7.16):

$$\begin{aligned} -2 \sum_i \epsilon_i \langle A_X X_i, A_Y X_i \rangle &= -2 \sum_{i,\alpha} \epsilon_i \frac{\nu^2}{16} g'_t(\mathcal{J}_\alpha X, X_i) g'_t(\mathcal{J}_\alpha Y, X_i) g'_t(V_\alpha, V_\alpha) \\ &= -\frac{\nu^2}{8} \sum_\alpha g'_t(\mathcal{J}_\alpha X, \mathcal{J}_\alpha Y) g'_t(V_\alpha, V_\alpha) \\ &= -\frac{\nu^2}{8} \sum_\alpha (-\epsilon_\alpha) g'_t(X, Y) (4\epsilon t \epsilon_\alpha) \\ &= \frac{3\epsilon \nu^2 t}{2} g'_t(X, Y). \end{aligned}$$

This proves the proposition. □

**Corollary 5.** *Let  $P'$  be the total space of the principal bundle  $P' \rightarrow M$  of admissible frames of  $Q$  over a para-quaternionic Kähler manifold  $(M, g, Q)$  with reduced scalar curvature  $\nu$ . Then the metric  $g'_t$  is Einstein if and only if*

$$t = \frac{-\epsilon}{\nu} \quad \text{or} \quad t = \frac{-\epsilon}{\nu(2n+3)}.$$

The corresponding Einstein constant is, respectively,

$$c = \left(n + \frac{1}{2}\right)v \quad \text{and} \quad c = \frac{4n^2 + 14n + 9}{4n + 6}v.$$

Next we calculate the Ricci curvature of the metric  $g_t^\epsilon$  on the twistor spaces  $Z^\epsilon = P'/\text{SO}_2^\epsilon$ ,  $\epsilon = \pm 1$ .

**Proposition 19.**

$$(7.24) \quad (A_X Y)_{J_1} = \frac{v}{2}(g(J_2\pi_*X, \pi_*Y)J_3 - g(J_3\pi_*X, \pi_*Y)J_2) \in T_{J_1}^v Z = \text{span}\{J_2, J_3\},$$

$$(7.25) \quad A_X J_2 = -\epsilon_2 \frac{vt}{2} \widetilde{J_3\pi_*X} = -\frac{vt}{2} \widetilde{J_3\pi_*X},$$

$$(7.26) \quad A_X J_3 = \epsilon_3 \frac{vt}{2} \widetilde{J_2\pi_*X} = -\epsilon \frac{vt}{2} \widetilde{J_2\pi_*X},$$

where  $X$  and  $Y$  are horizontal vectors and  $\tilde{X}_M \in T_{J_1} Z^\epsilon$  denotes the horizontal lift of the vector  $X_M \in T_{\pi(J_1)} M$ .

$$(7.27) \quad \text{Ric}(U, V) = -\epsilon \left(\frac{1}{t} + v^2 nt\right) g_t^\epsilon(U, V),$$

$$(7.28) \quad \text{Ric}(X, U) = 0,$$

$$(7.29) \quad \text{Ric}(X, Y) = (v(n + 2) + \epsilon v^2 t) g_t^\epsilon(X, Y),$$

where  $U$  and  $V$  are vertical vectors.

Proof. The equations (7.24)–(7.26) are obtained from (7.16), (7.17) and (7.21). We calculate the Ricci curvature using the formulas in Proposition 16. In fact, the projection  $\pi: Z^\epsilon \rightarrow M$  is a pseudo-Riemannian submersion with totally geodesic fibre  $F = \text{SO}_3^\epsilon/\text{SO}_2^\epsilon$ , where  $\text{SO}_3^\epsilon \cong \text{SO}(2, 1)$  and  $\text{SO}_2^{\epsilon=+1} = \text{SO}(1, 1)$  and  $\text{SO}_2^{\epsilon=-1} = \text{SO}(2)$ . Here  $Z_x^\epsilon \subset Q_x^\epsilon = \text{span}\{J_1, J_2, J_3\}$ , where  $(J_1, J_2, J_3)$  is an admissible basis such that  $J_\alpha^2 = \epsilon_\alpha \text{Id}$  and  $(\epsilon_1, \epsilon_2, \epsilon_3) = (\epsilon, 1, -\epsilon)$ . In both cases, the Lie algebra  $\mathfrak{so}_2^\epsilon = \mathbb{R} \text{ad}(J_1)$ . The fibre  $F$  is a two-dimensional symmetric space, with symmetric decomposition

$$\mathfrak{so}_3^\epsilon = \mathfrak{so}_2^\epsilon + \mathfrak{m}, \quad \mathfrak{m} = \mathbb{R} \text{ad}(J_2) + \mathbb{R} \text{ad}(J_3).$$

The curvature tensor is given by

$$R(\text{ad}(J_2), \text{ad}(J_3)) = -\text{ad}_{[J_2, J_3]}|_{\mathfrak{m}} = 2 \text{ad}_{J_1}|_{\mathfrak{m}}$$

and the sectional curvature of the metric  $g^F = g_t^\epsilon|_F = t g^v$  is  $-\epsilon/t$ . In particular,

$$(7.30) \quad \text{Ric}^F = -\epsilon g^v = -\frac{\epsilon}{t} g^F.$$



Next we compute the second term in equation (7.12) using (7.25):

$$\begin{aligned} \sum_i \epsilon_i g_t^\epsilon(A_{X_i} J_2, A_{X_i} J_2) &= \frac{v^2 t^2}{4} \sum_i \epsilon_i g(J_3 \pi_* X_i, J_3 \pi_* X_i) \\ &= v^2 t^2 n(-\epsilon_3) = \epsilon v^2 t^2 n \\ &= -\epsilon v^2 t n g_t^\epsilon(J_2, J_2), \end{aligned}$$

since  $g_t^\epsilon(J_2, J_2) = -t\epsilon_2 = -t$ . The same calculation for  $(U, V) = (J_2, J_3)$  and  $(U, V) = (J_3, J_3)$  shows that for any two vertical vectors  $U, V$ , we have

$$\sum_i \epsilon_i g_t^\epsilon(A_{X_i} U, A_{X_i} V) = -\epsilon v^2 t n g_t^\epsilon(U, V).$$

This proves (7.27).

Now we calculate the second term in equation (7.14) using (7.24).

$$\begin{aligned} &-2 \sum_i \epsilon_i g_t^\epsilon(A_X X_i, A_Y X_i) \\ &= -\frac{v^2}{2} \sum_i \epsilon_i g(J_2 \pi_* X, \pi_* X_i) g(J_2 \pi_* Y, \pi_* X_i) g_t^\epsilon(J_3, J_3) \\ &\quad - \frac{v^2}{2} \sum_i \epsilon_i g(J_3 \pi_* X, \pi_* X_i) g(J_3 \pi_* Y, \pi_* X_i) g_t^\epsilon(J_2, J_2) \\ &= -\frac{v^2}{2} g(J_2 \pi_* X, J_2 \pi_* Y) g_t^\epsilon(J_3, J_3) - \frac{v^2}{2} g(J_3 \pi_* X, J_3 \pi_* Y) g_t^\epsilon(J_2, J_2) \\ &= -\frac{v^2}{2} [(-\epsilon_2)(-t\epsilon_3) + (-\epsilon_3)(-t\epsilon_2)] g_t^\epsilon(X, Y) = \epsilon v^2 t g_t^\epsilon(X, Y). \end{aligned}$$

This proves (7.29).

To prove that  $\text{Ric}(X, U) = 0$ , by Proposition 16 we have to check that  $\text{div } A$  preserves the horizontal distribution  $\mathcal{H}_Z \subset TZ^\epsilon$ . It is sufficient to prove that

$$g_t^\epsilon((\nabla_X A)_Y Z, \mathcal{J}^\epsilon U) = 0$$

for all basic horizontal vector fields  $X, Y, Z$  and vertical vector fields  $U$ . We compute this using the fact that  $\nabla \mathcal{J}^\epsilon = 0$  and (7.10):

$$\begin{aligned} g_t^\epsilon((\nabla_X A)_Y Z, \mathcal{J}^\epsilon U) &= X g_t^\epsilon(A_Y Z, \mathcal{J}^\epsilon U) - g_t^\epsilon(A_Y Z, \mathcal{J}^\epsilon \nabla_X U) \\ &= X g_t^\epsilon(A_Y Z, \mathcal{J}^\epsilon U) - g_t^\epsilon(A_Y Z, \mathcal{J}^\epsilon [X, U]). \end{aligned}$$

**Lemma 3.** *For any basic horizontal vector fields  $X, Y$  and vertical vector field  $U$  we have*

$$(7.31) \quad g_t^\epsilon(A_X Y, \mathcal{J}^\epsilon U) = \frac{\epsilon \nu t}{2} g(U \pi_* X, \pi_* Y) = -\frac{1}{2} U \omega_t(X, Y),$$

where  $\omega_t = g_t^\epsilon(\mathcal{J}^\epsilon \cdot, \cdot)$  is the  $\epsilon$ -Kähler form and the value  $U_{J_1} \in T_{J_1}^\nu Z = \text{span}\{J_2, J_3\} \subset Q_x$ ,  $x = \pi(J_1)$ , of the vertical vector field  $U$  at the point  $J_1 \in Z^\epsilon$  is considered as an endomorphism of  $T_x M$ .

*Proof.* The first equation follows from (7.24) and the formulas  $\mathcal{J}^\epsilon J_2 = J_3$ ,  $\mathcal{J}^\epsilon J_3 = \epsilon J_2$ . For the second equality we use that  $[U, X]$  and  $[U, Y]$  are vertical and that  $\omega_t$  is closed:

$$\begin{aligned} \mathcal{L}_U(\omega_t(X, Y)) &= (\mathcal{L}_U \omega_t)(X, Y) = (d\iota_U \omega_t)(X, Y) \\ &= -\omega_t(U, [X, Y]) = -2g_t^\epsilon(\mathcal{J}^\epsilon U, A_X Y). \end{aligned} \quad \square$$

The following corollary finishes the proof of Theorem 3 (i).

**Corollary 6.** *Let  $Z^\epsilon$ ,  $\epsilon = \pm 1$ , be the twistor spaces of a para-quaternionic Kähler manifold. Then the metric  $g_t^\epsilon$  is Einstein if and only if*

$$t = -\frac{\epsilon}{\nu} \quad \text{or} \quad t = -\frac{\epsilon}{\nu(n+1)}.$$

The corresponding Einstein constant is, respectively,

$$c = (n+1)\nu \quad \text{and} \quad c = \frac{n^2 + 3n + 1}{n+1} \nu.$$

(ii) By Theorem 2, we know that the horizontal distribution  $\mathcal{H}_Z \subset TZ^\epsilon$  is holomorphic if  $\epsilon = -1$  and para-holomorphic if  $\epsilon = +1$ . We show that it is a para-holomorphic contact structure if  $\epsilon = +1$ . The case  $\epsilon = -1$  is similar. We have to check that the Frobenius form

$$\mathcal{H}_Z^{1,0} \times \mathcal{H}_Z^{1,0} \ni (Z, W) \mapsto ([Z, W] \bmod \mathcal{H}_Z^{1,0}) \in T^{1,0} Z^\epsilon / \mathcal{H}_Z^{1,0}$$

of  $\mathcal{H}_Z^{1,0}$  is nondegenerate.

Let  $X$  and  $Y$  be basic horizontal vector fields on  $P'$  and  $Z = X + e\mathcal{J}_1 X$  and  $W = Y + e\mathcal{J}_1 Y$  the corresponding sections of  $\mathcal{H}^{1,0} \subset \mathcal{H} \otimes C = \mathcal{H} + e\mathcal{H}$  the  $(+e)$ -eigenbundle of the  $C$ -linear extension of  $\mathcal{J}_1$  on  $\mathcal{H} \otimes C$ . Notice that  $\mathcal{J}_1^2|_{\mathcal{H}} = \epsilon \text{Id} = \text{Id}$ , since  $\epsilon = +1$ . Let us calculate, with the help of part (iv) of Lemma 2, the vertical component of  $[Z, W]$

at any point  $p = (J_1, J_2, J_3) \in P'$ :

$$\begin{aligned} [Z, W]^v &= -\frac{\nu}{2} \sum_{\alpha} \epsilon_{\alpha} (g(J_{\alpha} X_M, Y_M) + g(J_{\alpha} J_1 X_M, J_1 Y_M)) V_{\alpha} \\ &\quad - e \frac{\nu}{2} \sum_{\alpha} \epsilon_{\alpha} (g(J_{\alpha} X_M, J_1 Y_M) + g(J_{\alpha} J_1 X_M, Y_M)) V_{\alpha} \\ &= -\nu(\rho_2(X_M, Y_M) V_2 - \rho_3(X_M, Y_M) V_3) \\ &\quad + e\nu(\rho_3(X_M, Y_M) V_2 - \rho_2(X_M, Y_M) V_3) \\ &= -\nu(\rho_2(X_M, Y_M) - e\rho_3(X_M, Y_M))(V_2 + eV_3), \end{aligned}$$

where  $\rho_{\alpha} = g(J_{\alpha} \cdot, \cdot)$ . This shows that the Frobenius form of  $\mathcal{H}^{1,0} \subset TP' \otimes C$  is nondegenerate. Let us denote by  $\tilde{X}_M$  and  $\tilde{Y}_M$  the horizontal lifts of  $X_M$  and  $Y_M$  to vector fields on  $Z^{\epsilon}$ . We put  $\tilde{Z} := X_M + e\mathcal{J}^{\epsilon} X_M$  and  $\tilde{W} := Y_M + e\mathcal{J}^{\epsilon} Y_M$ . Thanks to the above formula, we can calculate the vertical component of  $[\tilde{Z}, \tilde{W}]$  at the point  $z = J_1 \in Z^{\epsilon}$ , which is the image of  $p = (J_1, J_2, J_3) \in P'$  under the natural projection  $P' \rightarrow Z^{\epsilon} = P'/\text{SO}_2^{\epsilon}$ .

$$\begin{aligned} [\tilde{Z}, \tilde{W}]^v &= -\nu(\rho_2(X_M, Y_M) - e\rho_3(X_M, Y_M))(J_2, J_1) + e[J_3, J_1] \\ &= 2\nu(\rho_2(X_M, Y_M) - e\rho_3(X_M, Y_M))(J_3 + eJ_2). \end{aligned}$$

This shows that  $\mathcal{H}_Z \subset TZ^{\epsilon}$  is a para-holomorphic contact structure if  $\epsilon = +1$ . □

**8. Twistor construction of minimal submanifolds of para-quaternionic Kähler manifolds**

**8.1. Kähler and para-Kähler submanifolds of para-quaternionic Kähler manifolds.**

DEFINITION 14. Let  $(M, g, Q)$  be a para-quaternionic Kähler manifold of dimension  $4n$ . An  $\epsilon$ -Kähler submanifold ( $\epsilon = \pm 1$ ) of  $M$  is a triple  $(N, J^{\epsilon}, g_N)$ , where  $N$  is a  $2m$ -dimensional  $g$ -nondegenerate submanifold of  $M$ ,  $g_N = g|_N$  is the induced pseudo-Riemannian metric and  $J^{\epsilon}$  is a parallel section of the para-quaternionic bundle  $Q|_N$  such that  $J^{\epsilon}TN = TN$  and  $(J^{\epsilon})^2 = \epsilon\text{Id}$ . For  $\epsilon = -1$   $(M, J^{\epsilon}, g_N)$  is called also a Kähler submanifold and for  $\epsilon = +1$  it is called a para-Kähler submanifold.

We shall include  $J^{\epsilon}$  into a local frame  $(J_1 = J^{\epsilon}, J_2, J_3 = J_1 J_2 = -J_2 J_1)$  of  $Q|_N$  such that  $J_2^2 = \text{Id}$ . Such frames  $(J_{\alpha})$  will be called adapted to the  $\epsilon$ -Kähler submanifold  $N \subset M$ .

**Proposition 20.** *Let  $(M, g, Q)$  be a para-quaternionic Kähler manifold of dimension  $4n$  with non-zero reduced scalar curvature  $\nu$  and  $N$  a  $g$ -nondegenerate submanifold*

of  $M$  endowed with a section  $J^\epsilon \in \Gamma(N, Q)$  such that  $(J^\epsilon)^2 = \epsilon \text{Id}$  and  $J^\epsilon TN = TN$ . Let  $(J_\alpha)$  be a standard local basis of  $Q$  such that  $J_1|_N = J^\epsilon$ . Then the triple  $(N, J^\epsilon, g_N)$  is an  $\epsilon$ -Kähler submanifold if and only if  $\omega_2|_N = \omega_3|_N = 0$  or, equivalently,  $J_2TN \perp TN$ . In particular, the dimension of an  $\epsilon$ -Kähler submanifold  $N \subset M$  is at most  $2n$ .

*Proof.* It is clear that  $J_1$  is parallel if and only if  $\omega_2|_N = \omega_3|_N = 0$ , see (3.2). Moreover, if  $\omega_2|_N = \omega_3|_N = 0$ , then, by the structure equation (3.3), we have that  $\rho_2|_N = \rho_3|_N = 0$ . Conversely, assume that  $J_2TN \perp TN$ , i.e.  $\rho_2|_N = \rho_3|_N = 0$ . Differentiating the structure equations for  $\rho_2$  and  $\rho_3$ , we get

$$v d\rho'_\alpha = -\epsilon_\alpha v\rho'_\beta \wedge \omega_\gamma + \epsilon_\alpha v\omega_\beta \wedge \rho_{\gamma'}.$$

Restricting this equation for  $\alpha = 2, 3$  to the the submanifold  $N$  yields

$$\rho_1 \wedge \omega_2|_N = \rho_1 \wedge \omega_3|_N = 0.$$

This shows that  $\omega_2|_N = \omega_3|_N = 0$ , i.e. that  $J^\epsilon \in \Gamma(N, Q)$  is parallel. □

**Proposition 21.** *The shape operator  $A$  of an  $\epsilon$ -Kähler submanifold  $(N, J^\epsilon, g_N)$  of a para-quaternionic Kähler manifold  $(M, g, Q)$  anticommutes with  $J := J^\epsilon|_{TN}$ .*

*Proof.* Let  $\xi$  be a normal vector field on  $N$ . Then the shape operator  $A^\xi \in \Gamma(\text{End}TN)$  is defined by

$$g(A^\xi X, Y) = -g(\nabla_X \xi, Y) = -g(\nabla_Y \xi, X) = g(\xi, \nabla_Y X).$$

Thus

$$\begin{aligned} g(A^\xi JX, Y) &= g(\xi, \nabla_Y(JX)) = g(\xi, J\nabla_Y X) = -g(J\xi, \nabla_Y X) = -g(J\xi, \nabla_X Y) \\ &= g(\xi, J\nabla_X Y) = g(\xi, \nabla_X(JY)) = g(A^\xi X, JY) = -g(JA^\xi X, Y). \end{aligned} \quad \square$$

**Corollary 7.** *Any  $\epsilon$ -Kähler submanifold of a para-quaternionic Kähler manifold is minimal.*

*Proof.* Since  $A^\xi$  anticommutes with  $J$ , we have  $A^\xi = -JA^\xi J^{-1}$ . Hence  $\text{tr} A^\xi = -\text{tr} A^\xi = 0$ . □

**8.2. Twistor construction of Kähler and para-Kähler submanifolds of para-quaternionic Kähler manifolds.** Let  $(M, g, Q)$  be a para-quaternionic Kähler manifold and  $\pi_Z: Z^\epsilon \rightarrow M$  its  $\epsilon$ -twistor space with the horizontal distribution  $\mathcal{H}_Z$ . For any  $\epsilon$ -Kähler submanifold  $(N, J^\epsilon, g_N)$  the section  $J^\epsilon: N \rightarrow Z^\epsilon \subset Q$  defines an embedding of  $N$  into  $Z^\epsilon$ . The image  $\tilde{N} = J^\epsilon(N) \subset Z^\epsilon$  is called *the canonical lift* of

$N$  in the twistor space  $Z^\epsilon$ . The following theorem gives the description of  $\epsilon$ -Kähler submanifolds of  $M$  in terms of  $\epsilon$ -complex horizontal submanifolds of  $Z^\epsilon$ , i.e. submanifolds  $L \subset Z^\epsilon$  such that  $\mathcal{J}^\epsilon TL = TL$  and  $TL \subset \mathcal{H}_Z$ .

**Theorem 4.** *Let  $(N, J^\epsilon, g_N)$  be an  $\epsilon$ -Kähler submanifold of a para-quaternionic Kähler manifold  $(M, g, Q)$  and  $\tilde{N} = J^\epsilon(N) \subset Z^\epsilon$  its canonical lift. Then*

- (i)  *$\tilde{N} \subset Z^\epsilon$  is an  $\epsilon$ -complex horizontal submanifold which is nondegenerate with respect to the canonical one-parameter family of metrics  $g_t^\epsilon$  on  $Z^\epsilon$ . Moreover, in the case  $\epsilon = +1$  the restriction of  $\mathcal{J}^\epsilon$  to  $\tilde{N}$  is a para-complex structure in the strong sense.*  
(ii) *Conversely, let  $L \subset Z^\epsilon$  be an  $\epsilon$ -complex horizontal submanifold which is nondegenerate with respect to  $g_t^\epsilon$  and such that  $\pi_Z|_L: L \rightarrow \pi_Z(L) \subset M$  is a diffeomorphism. Then its projection  $(N = \pi_Z(L), J^\epsilon, g_N)$  is a (minimal)  $\epsilon$ -Kähler submanifold of  $M$ , where*

$$J^\epsilon = d\pi_Z \circ \mathcal{J}^\epsilon \circ (d\pi_Z)^{-1}: TN \rightarrow TN, \quad g_N = g|_N.$$

*Proof.* (i) Since  $J^\epsilon$  is parallel, the submanifold  $\tilde{N} = J^\epsilon(N) \subset Z^\epsilon$ , is horizontal. Its tangent bundle  $T\tilde{N} \subset \mathcal{H}_Z$  is  $\mathcal{J}^\epsilon$ -invariant, since

$$d\pi_Z \circ \mathcal{J}^\epsilon = J^\epsilon \circ d\pi_Z,$$

on the horizontal distribution  $\mathcal{H}_Z$ , by the definition of  $\mathcal{J}^\epsilon$ , see (4.2). In the case  $\epsilon = +1$ ,  $J^\epsilon$  is a para-complex structure in the strong sense, because  $J^\epsilon$  is skew-symmetric for the metric  $g_N$ . Since  $(T_z\tilde{N}, \mathcal{J}_z^\epsilon|_{\tilde{N}}) \cong (T_xN, J_x^\epsilon)$ ,  $x = \pi_Z(z)$ ,  $\mathcal{J}^\epsilon$  restricts to a para-complex structure in the strong sense on  $\tilde{N}$ .

(ii) The  $\epsilon$ -complex structure  $J^\epsilon \in \Gamma(N, Q^\epsilon)$  is parallel, since  $L = \tilde{N}$  is horizontal. This proves that  $(N, J^\epsilon, g_N)$  is an  $\epsilon$ -Kähler submanifold of  $M$ .  $\square$

**REMARK.** The nondegeneracy assumption on the metric  $g_t^\epsilon|_L$  is essential even if we assume that  $\dim L = 2n$ . Indeed there exist  $2n$ -dimensional  $J^\epsilon$ -invariant isotropic subspaces  $U \subset T_xM$ .

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